A_∞ OBSTRUCTION THEORY AND THE STRICT ASSOCIATIVITY OF E/I

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Abstract. We prove that for a ring spectrum K with a perfect universal coeŒcient formula, the obstructions to extending the multiplication to an A_∞ multiplication lie in $Ext^{*,*}_{K_* K^{op}}(K_*, K_*)$. As a corollary, we show that if E is even and $I = (x_1, x_2, \ldots)$ is a regular sequence in E_*, then any product on E/I can be extended to an A_∞ multiplication.

1. Introduction

In [9], Alan Robinson developed an obstruction theory for extending a homotopy associative multiplication on a ring spectrum K to an A_∞ multiplication, based on Hochschild cohomology. On closer inspection, one can see that this obstruction theory requires special care if K is not homotopy commutative. To be precise, if K is not homotopy commutative, then K_* is no longer a K_*K-module, but rather a K_*K^{op}-module. Thus $HH^*_{K_*}(K_*K, K_*)$ is not even deøned. The purpose of this paper is to correct that deøciency. We show that by replacing K_*K with K_*K^{op}, Robinson's obstruction theory works out as stated. This has the added advantage that K_*K^{op} tends to have a better structure than K_*K. Also, we make the theory relative to a commutative S-algebra E. We arrive at the following theorem:

Theorem 1.1. Let K be an E-ring spectrum with a perfect universal coeføcient formula. Then the obstructions to extending the multiplication on K to an A_∞ multiplication lie in

$$Ext^{n,3-n}_{\pi_*(K \wedge_E K^{op})}(K_*, K_*),$$

for $n \geq 4$.

The obstructions to uniqueness lie in $Ext^{n,2-n}_{\pi_*(K\wedge_E K^{op})}(K_*, K_*)$, for $n \geq 3$. Actually, there is a Bousøeld-Kan spectral sequence converging to the space of A_∞ structures with E_2-term $Ext^{s,t}_{\pi_*(K\wedge_E K^{op})}(K_*, K_*)$ of the same type as considered in [8]. The groups $Ext^{n,2-n}$ give the connected components of this space.

Also, one might want to consider $Aut_E(K)$ acting on the space of A_∞ structures, and regard two A_∞ structures as equivalent if some element of $Aut_E(K)$ carries one into the other. In the last section we will see an example of this, where two A_2 structures, which can be extended to A_∞ structures, are equivalent in this way if K is regarded as an S-ring spectrum, but not as an E-ring spectrum.

If K_* is central in $\pi_*(K \wedge_E K^{op})$, then these Ext groups are the same as the Hochschild cohomology groups $HH^*_{K_*}(\pi_*(K \wedge_E K^{op}), K_*)$, and if K is homotopy commutative, then $\pi_*(K \wedge_E K^{op}) \cong \pi_*(K \wedge_E K)$, and we get back Robinson's obstruction theory.

It is also worth noting ([5, theorem IX.1.6]) that if K is A_∞, then $Ext^{*,*}_{\pi_*(K\wedge_E K^{op})}(K_*, K_*)$ is the E_2-term of a spectral sequence converging to $\pi_* THH_E(K)$, the homotopy groups of topological Hochschild cohomology of K over the ground ring E.

We use this to show that if E_* is even and $I = (x_1, x_2, \ldots)$ is a regular sequence in E_*, then any product on E/I can be extended to an A_∞ multiplication. There are many partial results in this direction in the literature; see for example [6]. As a corollary, we show that all Morava K-theories are A_∞ at any prime.

This result will also be used in [2] to study non-commutative multiplications on 2-periodic Morava K-theories.

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2. Universal coefficient and Knneth isomorphisms

We work in the category of E-modules, where E is a commutative S-algebra, as in [5]. Thus spectrum means E-module, $X \wedge Y$ means $X \wedge_E Y$, X_*Y means $\pi_*(X \wedge_E Y)$ and X^*Y means $\pi_* F_E(Y, X)$. For aesthetic reasons, we will make one exception to this rule, by representing $x \in \pi_d X$ by a map $S^d \longrightarrow X$ rather than a map $\Sigma^d E \longrightarrow X$, and smash products are over the sphere spectrum when smashing spheres, as in $S^{d_1} \wedge S^{d_2}$. We will assume that E is q-coøbrant, and that all E-modules are cell E-modules.

By a ring spectrum (E-ring spectrum) we mean a spectrum K with a multiplication $\phi : K \wedge K \longrightarrow K$ and a unit $\eta : E \longrightarrow K$ which makes K left and right unital, and associative, up to homotopy. Note that we can always promote the multiplication to a strictly unital one, in the same way as one can promote the multiplication on a homotopy unital H-space to a strictly unital one.

For a ring spectrum K, we consider K_*X as a K_*-bimodule, where the left action of K_* is the expected one and the right action involves switching K and X, i.e., for $a \in K_*X$ and $r \in K_*$, $a \otimes r$ is sent to the composite

$$S^{d_1+d_2} \cong S^{d_1} \wedge S^{d_2} \xrightarrow{a\wedge r} K \wedge X \wedge K \xrightarrow{1\wedge\tau} K \wedge K \wedge X \xrightarrow{\phi\wedge 1} K \wedge X,$$

where τ is the twist map. With $X = K$ this gives a right K_*-module structure on K_*K which is diœerent from the one considered by Adams ([1]).

We assume that K has a perfect universal coeŒcient formula, by which we mean that the following two conditions are satisøed:

(A) K_*K is projective as a left K_*-module. This implies that

$$K_*(K^{(n)}) \cong (K_*K)^{\otimes n},$$

where $K^{(n)}$ is the n-fold smash product of K and the tensor product is over K_*. We need to be explicit about this isomorphism. We always have a map $K_*X \otimes_{K_*} K_*Y \longrightarrow K_*(X \wedge Y)$ sending $a \otimes b$ to the composite

$$S^{d_1} \wedge S^{d_2} \xrightarrow{a\wedge b} K \wedge X \wedge K \wedge Y \xrightarrow{1\wedge\tau\wedge 1} K \wedge K \wedge X \wedge Y \xrightarrow{\phi\wedge 1\wedge 1} K \wedge X \wedge Y.$$

With n factors, there are as many maps $(K_*K)^{\otimes n} \longrightarrow K_*(K^{(n)})$ as there are ways to associate a word with n letters. These are all the same because K is homotopy associative, and this is the map we assume is an isomorphism.

It is perhaps also worth pointing out that $K_*X \otimes K_*Y \longrightarrow K_*(X \wedge Y)$ factors through $K_*X \otimes_{K_*} K_*Y$ because K is homotopy associative, since the two maps $(K \wedge X) \wedge K \wedge (K \wedge Y) \longrightarrow K \wedge X \wedge Y$ are homotopic.

(B) There is a universal coefficient isomorphism

$$K^*(K^{(n)}) \cong Hom_{K_*}(K_*(K^{(n)}), K_*).$$

When K_*K is projective over K_*, this condition holds if K has a universal coefficient spectral sequence. See [1, III.13] for a condition which guarantees the existence of such a spectral sequence. The isomorphism (given by the edge homomorphism in the spectral sequence if there is one) sends $f : X \longrightarrow K$ to π_* of the composite

$$K \wedge X \xrightarrow{1\wedge f} K \wedge K \xrightarrow{\phi} K.$$

This is a map of left K_*-modules, but not of right K_*-modules.

Let $R = K_*$ and $\Lambda = K_*K^{op}$, so that $\Lambda = K_*K$ additively and the ring structure on Λ is given by sending $\lambda_1 \otimes \lambda_2$ to

$$S^{d_1} \wedge S^{d_2} \xrightarrow{\lambda_1\wedge\lambda_2} K^{(4)} \xrightarrow{(1\wedge 1\wedge\tau)(1\wedge\tau\wedge 1)} K^{(4)} \xrightarrow{\phi\wedge\phi} K \wedge K.$$

We will identify $K^*(K^{(n)})$ with $Hom_\Lambda(\Lambda^{\otimes(n+1)}, R)$, after specifying the Λ-module structure on $\Lambda^{\otimes(n+1)}$. We can choose between (at least) two different Λ-module structures on $\Lambda^{\otimes(n+1)}$. We get the first one by thinking of $\Lambda^{\otimes(n+1)}$ as $\pi_*(K \wedge K^{op} \wedge K^{(n)})$, with $K \wedge K^{op}$ acting on the first two factors. Let $\Lambda^{\otimes(n+1)}_{\text{ind}}$ denote $\Lambda^{\otimes(n+1)}$ with this Λ-module structure. It is induced up from the R-module structure on $\Lambda^{\otimes n}$, so we get an isomorphism

$$\text{ind} : Hom_R(\Lambda^{\otimes n}, R) \xrightarrow{\cong} Hom_\Lambda(\Lambda^{\otimes(n+1)}_{\text{ind}}, R).$$

The other Λ-module structure on $\Lambda^{\otimes(n+1)}$ comes from thinking of $\Lambda^{\otimes(n+1)}$ as $\pi_*(K \wedge K^{(n)} \wedge K)$, with $K \wedge K^{op}$ acting on the first and last factors. Let $\Lambda^{\otimes(n+1)}_{\text{bar}}$ denote $\Lambda^{\otimes(n+1)}$ with this Λ-module structure.

Let $\sigma : K^{(n+2)} \longrightarrow K^{(n+2)}$ be the permutation which fixes the first factor and cyclically permutes the $n+1$ last factors, moving the second factor to the end. It induces an isomorphism $\sigma_* : \Lambda_{\text{ind}}^{\otimes(n+1)} \longrightarrow \Lambda_{\text{bar}}^{\otimes(n+1)}$ of Λ-modules, and thus an isomorphism

$$(\sigma^{-1})^* : Hom_\Lambda(\Lambda_{\text{ind}}^{\otimes(n+1)}, R) \xrightarrow{\cong} Hom_\Lambda(\Lambda_{\text{bar}}^{\otimes(n+1)}, R).$$

Thus we get an isomorphism

$$K^*(K^{(n)}) \cong Hom_\Lambda(\Lambda_{\text{bar}}^{\otimes(n+1)}, R), \tag{1}$$

where the isomorphism sends $f : K^{(n)} \longrightarrow K$ to π_* of the composite

$$K^{(n+2)} \xrightarrow{1\wedge f\wedge 1} K^{(3)} \xrightarrow{\phi(\phi\wedge 1)} K.$$

From now on $\Lambda^{\otimes(n+1)}$ will mean $\Lambda_{\text{bar}}^{\otimes(n+1)}$.

3. A_∞ obstruction theory

Recall the deønition of the Stasheœ associahedra $\mathcal{A}(i)$, $i \geq 0$, which form a (non-Σ) A_∞ operad, from [10]. We deøne a unital A_n structure on K as a map

$$\bigvee_{0\leq i\leq n} \mathcal{A}(i)_+ \wedge K^{(i)} \longrightarrow K$$

satisfying the usual conditions, for $n \leq \infty$. Similarly, a non-unital A_n structure is deøned as a map

$$\bigvee_{1\leq i\leq n} \mathcal{A}(i)_+ \wedge K^{(i)} \longrightarrow K$$

satisfying similar conditions.

Recall that $\mathcal{A}(n) \cong D^{n-2}$, an $n-2$ disk, and that $\mathcal{A}(n)$ has i faces of the form $\mathcal{A}(i) \times \mathcal{A}(n-i-1)$, for each $2 \leq i \leq n-3$. Given an A_{n-1} structure on K, if we want to extend it to a non-unital A_n structure, the map $\mathcal{A}(n)_+ \wedge K^{(n)} \longrightarrow K$ is already determined on $\partial\mathcal{A}(n)_+ \wedge K^{(n)} \cong \Sigma^{n-3}K^{(n)}$. If we want the A_n structure to be unital, then the map is also determined on $\mathcal{A}(n)_+ \wedge \partial K^{(n)}$, where $\partial K^{(n)}$ is the image of $\bigvee_{i=1}^n K^{(i-1)} \wedge E \wedge K^{(n-i)}$ in $K^{(n)}$. Let $\bar{\Lambda} = coker(R \longrightarrow \Lambda)$, where the map $R \longrightarrow \Lambda$ is the right unit $K_* \cong K_*E \longrightarrow K_*K$. Then $K_*(K/E) \cong \bar{\Lambda}$ and $K_*(K^{(n)}/\partial K^{(n)}) \cong \bar{\Lambda}^{\otimes n}$ as R-bimodules.

Thus elementary obstruction theory gives the following: (compare with [9, 1.5-1.7])

Lemma 3.1. For $n \geq 4$, given a (unital or non-unital) A_{n-1} structure on K, the obstruction to extending it to a non-unital A_n structure lies in

$$K^{n-3}(K^{(n)}) \cong Hom_\Lambda^{3-n}(\Lambda^{\otimes(n+1)}, R),$$

and the obstruction to extending it to a unital A_n structure (if A_{n-1} is unital) lies in

$$K^{n-3}((K/E)^{(n)}) \cong Hom_\Lambda^{3-n}(\bar{\Lambda}^{\otimes n} \otimes \Lambda, R).$$

The set of non-unital A_{n-1} structures, øxing the A_{n-2} structure, is given by

$$K^{n-3}(K^{(n-1)}) \cong Hom_\Lambda^{3-n}(\Lambda^{\otimes n}, R),$$

and the set of unital A_{n-1} structures, øxing the A_{n-2} structure, is given by

$$K^{n-3}((K/E)^{(n-1)}) \cong Hom_{\Lambda}^{3-n}(\bar{\Lambda}^{\otimes(n-1)} \otimes \Lambda, R).$$

Suppose we want to calculate $Ext_{\Lambda}^{*,*}(R,R)$. Then we need a projective resolution of R as a Λ-module, and by (A) we get one by taking homotopy groups of the two-sided bar resolution $B(K,K,K) \longrightarrow K$. (See e.g. [5, deønition XII.1.1].) Thus we get a cochain complex C^* with $C^n = Hom_{\Lambda}(\Lambda^{\otimes(n+1)}, R)$, which is isomorphic to $K^*(K^{(n)})$ by (1), calculating $Ext_{\Lambda}(R,R)$. Here $\delta : C^{n-1} \longrightarrow C^n$ is given as follows: For $g : \Lambda^{\otimes n} \longrightarrow R$, let $f : K^{(n-1)} \longrightarrow K$ be the image of g under the isomorphism (1). Then $\delta g = \sum_{i=0}^{n} (-1)^i \delta_i g$, where $\delta_i g$ is given by taking π_* of

$$B(K,K,K)_n = K^{(n+2)} \stackrel{1^i \wedge \phi \wedge 1^{n-i}}{\longrightarrow} K^{(n+1)} \stackrel{1\wedge f \wedge 1}{\longrightarrow} K^{(3)} \stackrel{\phi(\phi\wedge 1)}{\longrightarrow} K.$$

Note that we can give C^* the structure of a cosimplicial group. The codegeneracy maps are given by precomposing with the maps $K^{(n+1)} \cong K^{(i+1)} \wedge E \wedge K^{(n-i)} \longrightarrow K^{(n+2)}$ for $0 \le i \le n-1$.

Let us concentrate on the non-unital theory for now, and then see what changes we need to do to for the unital A_n structures, which is what we really care about. We need to calculate what happens to the obstruction to extending an A_{n-1} structure to a A_n structure if we change the A_{n-1} structure while øxing the A_{n-2} structure:

Proposition 3.2. Let $n \ge 4$. If we alter the A_{n-1} structure by $g : \Lambda^{\otimes n} \longrightarrow R$, then the obstruction $c_n \in Hom_{\Lambda}(\Lambda^{\otimes(n+1)}, R)$ to extending the multiplication to an A_n structure is changed by δg.

Proof. Again, let $f : K^{(n-1)} \longrightarrow K$ correspond to g under the isomorphism (1). The geometric argument in [9, 1.8] shows that the obstruction is changed by a sum of maps $K^{(n)} \longrightarrow K$ of two types, corresponding to two types of $(n-3)$-dimensional faces of $\mathcal{A}_n \cong D^{n-2}$. The maps

$$K^{(n)} \stackrel{1\wedge f}{\longrightarrow} K \wedge K \stackrel{\phi}{\longrightarrow} K \qquad (2)$$

and

$$K^{(n)} \stackrel{f\wedge 1}{\longrightarrow} K \wedge K \stackrel{\phi}{\longrightarrow} K \qquad (3)$$

give the ørst and the last term in δg, respectively, and the maps

$$K^{(n)} \stackrel{1^{i-1}\wedge\phi\wedge 1^{n-i-1}}{\longrightarrow} K^{(n-1)} \stackrel{f}{\longrightarrow} K \qquad (4)$$

for $i = 1, \ldots, n-1$ give the rest of the terms.

For example, to see that (2) gives $\delta_0 g$, consider the following homotopy commutative diagram:

$$\begin{array}{ccccc}
K^{(n+2)} & \stackrel{1\wedge 1\wedge f\wedge 1}{\longrightarrow} & K^{(4)} & \stackrel{1\wedge\phi\wedge 1}{\longrightarrow} & K^{(3)} \\
\downarrow{\scriptstyle \phi\wedge 1} & & & & \downarrow{\scriptstyle \phi(\phi\wedge 1)} \\
K^{(n+1)} & \stackrel{1\wedge f\wedge 1}{\longrightarrow} & K^{(3)} & \stackrel{\phi(\phi\wedge 1)}{\longrightarrow} & K
\end{array}$$

Applying π_* to this diagram, we see that going around clockwise gives the change to the obstruction from the $(n-3)$-cell, while going counterclockwise gives $\delta_0 g$. □

The next proposition is proved in a similar way:

Proposition 3.3. The obstruction c_n is a cocycle.

To ønish the proof of theorem 1.1, it is enough to observe that the cochain complex $\bar{C}^*$ with $\bar{C}^n = Hom^*_\Lambda(\bar{\Lambda}^{\otimes n} \otimes \Lambda, R)$ also calculates $Ext_\Lambda(R, R)$. But $\bar{C}^*$ plays the role of the normalized cochain complex, if we think of $\bar{\Lambda}^{\otimes n} \otimes \Lambda$ as $\pi_*(K \wedge (K/E)^{(n)} \wedge K)$. This makes sense because of the cosimplicial structure on C^*. This ønishes the proof of theorem 1.1.

Remark 3.4. a) The theory of A_∞ maps needs to be changed in a similar way. Given a map $L \longrightarrow K$, where L and K are A_∞ and $K^*(L^{(n)}) \cong Hom_{K_*}((K_*L)^{\otimes n}, K_*)$, we are led to study

$$Ext^{*,*}_{K_*L^{op}}(K_*, K_*).$$

Again this is forced upon us, because K is not a $K \wedge L$ module, but a $K \wedge L^{op}$ module.
b) The spectral sequence set up e.g. in [8] for calculating A_∞ structures or A_∞ maps based on Andr-Quillen cohomology has to be changed accordingly. The E_2-term has to be expressed as $D^s_{K_*}(K_*L^{op}, K_{*+t})$, derived functors of derivations of K_*L^{op} into K_* when the spectra are not homotopy commutative.

4. The strict associativity of E/I

Now suppose that E has homotopy groups only in even dimensions, and that $I = (x_1, x_2, \ldots)$ is a regular sequence in E_*, with $|x_i| = d_i$. Deøne E/x_i by the coøber sequence $\Sigma^{d_i} E \xrightarrow{x_i} E \longrightarrow E/x_i$ in the category of E-modules. Recall from [11, proposition 3.1] that each E/x_i has at least one homotopy associative multiplication, and from [11, proposition 4.8] that choosing a multiplication on each E/x_i gives a multiplication on $K = E/I$, the homotopy colimit of the spectra $E/x_1 \wedge \ldots \wedge E/x_i$. Not all multiplications on K come from smashing together multiplications on each E/x_i; we will discuss this further in [2]. It follows trivially that $(E/x_i)_* = E_*/x_i$ and $K_* = E_*/I$.

The following result has also been proved by Lazarev in [6, lemma 2.6]:

Proposition 4.1. For any homotopy associative multiplication on K we have

$$K_*K^{op} \cong \Lambda_{K_*}(\alpha_1, \alpha_2, \ldots)$$

with $|\alpha_i| = d_i + 1$.

Proof. There is a multiplicative Knneth spectral sequence (see [4])

$$E^2 = Tor^{E_*}_{*,*}(K_*, K^{op}_*) \Longrightarrow K_*K^{op}.$$

By using a Koszul resolution of $K_* = E_*/I$ it is easy to see that $E^2 = \Lambda_{K_*}(\alpha_1, \alpha_2, \ldots)$ with α_i in bidegree $(1, d_i)$. The spectral sequence collapses, so all we have to do is to show that there are no multiplicative extensions. Because $E^2_{1,*}$ is concentrated in odd total degree, it follows that $\alpha_i^2 \in K_* \otimes_{E_*} K_*^{op} \cong K_*$ in K_*K^{op}. Now there are several ways to show that $\alpha_i^2 = 0$. For example, we can use the fact that K is a $K \wedge K^{op}$-module and study the two maps $K_*K^{op} \otimes K_*K^{op} \otimes K_* \longrightarrow K_*$. One sends $\alpha_i \otimes \alpha_i \otimes 1$ to α_i^2, the other one sends it to 0. □

Note that this result does not hold for K_*K, in which case α_i might very well square to something nonzero in K_*.

We also need to observe that K satisfies conditions (A) and (B). But K_*K is projective over K_* by proposition 4.1, giving (A). For (B), inductively taking K^* of the cofiber sequence defining $E/(x_1, \ldots, x_n)$ from $E/(x_1, \ldots, x_{n-1})$ gives a long exact sequence which breaks into short exact sequences and proves that $K^*K \cong Hom_{K_*}(K_*K, K_*)$. A similar argument gives (B) for $n > 1$.

Theorem 4.2. For E even and I regular, any homotopy associative multiplication on $K = E/I$ can be extended to an A_∞ structure. Moreover, the natural map $E \longrightarrow K$ extends uniquely to a map of A_∞ ring spectra for any choice of A_∞ structure on K, making E central in K.

Proof. Recall that

$$Ext^*_{\Lambda_{K_*}(\alpha_1, \alpha_2, \ldots)}(K_*, K_*) \cong K_*[\bar{\alpha}_1, \bar{\alpha}_2, \ldots],$$

with $\bar{\alpha}_i$ in (cohomological) bidegree $(1, d_i + 1)$. Thus the Ext groups are concentrated in even total degree, and the obstructions to existence of an A_∞ structure on K vanish. The second part of the theorem is obvious. □

With $E = MU$ or $MU_{(p)}$, it follows immediately that BP, $BP\langle n \rangle$, $P(n)$ and $k(n)$ are A_∞. Using Bousfield localization, which can be done in the category of A_∞ ring spectra ([5, theorem VIII.2.1]), it follows that also $E(n), B(n)$ and $K(n)$ are A_∞. (See e.g. [11, p.4] for the homotopy type of these spectra.)

We include one more example: Let $E = \widehat{E(n)}$ be the $K(n)$-localization of $E(n)$, which is E_∞, or strictly commutative, by [3, theorem 8.2]. Let $I = (p, v_1, \ldots, v_{n-1})$, so that $K(n) = E/I$. Recall, from [12] that for p odd we have

$$\pi_* K(n) \wedge_S K(n)^{op} \cong \Lambda_{\Sigma(n)}(\alpha_0, \ldots, \alpha_{n-1}),$$

and from [7] that for $p = 2$ we have

$$\pi_* K(n) \wedge_S K(n)^{op} \cong \Sigma(n)[\alpha_0, \ldots, \alpha_{n-1}]/(\alpha_i^2 - t_{i+1}),$$

where

$$\Sigma(n) = K(n)_*[t_1, t_2, \ldots]/(t_i^{p^n} - v_n^{p^i - 1} t_i).$$

From proposition 4.1 it follows that

$$\pi_* K(n) \wedge_E K(n)^{op} \cong \Lambda_{K(n)_*}(\alpha_0, \dots, \alpha_{n-1}),$$

and the map $\pi_* K(n) \wedge_S K(n)^{op} \longrightarrow \pi_* K(n) \wedge_E K(n)^{op}$ sends each t_i to zero, because t_i exists in $\pi_* E \wedge_S E$ and is obviously sent to zero under $\pi_* E \wedge_S E \longrightarrow \pi_* E \wedge_E E \cong E_*$. (The map $\pi_* E \wedge_S E \longrightarrow E_*$ ultimately comes from the multiplication map $\pi_* MU \wedge_S MU \longrightarrow MU_*$, and the generator corresponding to t_i in $MU_* MU$ maps to zero in MU_* by [1, theorem II.11.3.ii].)

When $p = 2$, the multiplication ϕ on $K(n)$ is not homotopy commutative, but there is an automorphism of $K(n)$ as an S-module carrying ϕ to $\phi \circ \tau$. If we identify $K(n)^* K(n)$ with $Hom_{K(n)_*}(K(n)_* K(n), K(n)_*)$ this automorphism is given by sending t_n to $v_n + t_n$. However, this is not a map of E-modules, so ϕ and $\phi \circ \tau$ really are diœerent as ring structures on $K(n)$ regarded as an E-module.

However, the obstruction theory is the same for $K(n)$ as an S-ring spectrum or an E-ring spectrum, because of the following result:

Lemma 4.3. At any prime p,

$$Ext_{\pi_*(K(n) \wedge_S K(n)^{op})}(K(n)_*, K(n)_*) \cong Ext_{\pi_*(K(n) \wedge_E K(n)^{op})}(K(n)_*, K(n)_*).$$

Proof. We concentrate on the case $p = 2$, because it is slightly more complicated, and because for p odd the proof of [9, theorem 2.2] applies. Write

$$\pi_*(K(n) \wedge_S K(n)^{op}) \cong \Lambda_1 \otimes \dots \otimes \Lambda_n \otimes \Gamma_{n+1} \otimes \Gamma_{n+2} \otimes \dots,$$

where $\Lambda_i = R[\alpha_{i-1}]/(\alpha_{i-1}^{2^{n+1}} - v_n^{2^i - 1} \alpha_{i-1}^2)$ and $\Gamma_j = R[t_j]/(t_j^{2^n} - v_n^{2^j - 1} t_j)$.

Of course, v_n only serves to keep track of the grading, so the result follows from the calculations that for any ring R and any $k \geq 2$,

$$Ext^*_{R[t]/(t^k - t)}(R, R) = R$$

concentrated in degree zero and

$$Ext^*_{R[\alpha]/(\alpha^{2k} - \alpha^2)}(R, R) \cong R[\bar{\alpha}].$$

□

Thus, $K(n)$ has the same space of A_∞ structures regarded as an S-ring spectrum or an E-ring spectrum. But the group of automorphisms of $K(n)$ is larger in the ørst case, so we expect the number of non-equivalent A_∞ structures to be smaller. Indeed, we just saw that even the number of non-equivalent A_2 structures are diœerent at $p = 2$.

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