NON COMMUTATIVITY OF THE GROUP OF SELF HOMOTOPY CLASSES OF CLASSICAL SIMPLE LIE GROUPS

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ABSTRACT. Let G be a topological group and let $[G, G]$ be the group of homotopy classes of maps from G into G. For a large class of simple Lei groups, we prove that the group $[G, G]$ is nonabelian. For certain Lie groups we show that $\mathrm{nil}[G, G] \geq 3$.

1. INTRODUCTION

Let G be a compact, connected topological group with multiplication $m : G \times G \to G$. There has been considerable work in homotopy theory to determine when m is homotopy-commutative [11, 12, 3, 7] that is, when m is homotopic to m^{op}, where m^{op} is defined by $m^{op}(g, g') = m(g', g)$. For a compact, connected topological group G (and more generally, a group-like space) and any space X, the set of homotopy classes $[X, G]$ inherits a group structure from G, and this group is abelian when G is homotopy-commutative. Moreover, it is easily seen that G is homotopy-commutative if and only if $[X, G]$ is abelian for *all* spaces X. The most comprehensive result on the homotopy-commutativity of finite H-spaces is the theorem of Hubbuck [7] which asserts that the only compact, connected, non-contractible topological groups G which are homotopy-commutative are tori $S^1 \times \cdots \times S^1$. Thus if G is not a torus, then $[X, G]$ is non-abelian for some space X and $\mathrm{nil}[X, G]$ is of particular interest. (Here $\mathrm{nil}\,\Gamma$ denotes the nilpotency class of the group Γ, so that $\mathrm{nil}\,\Gamma \geq 2$ if and only if Γ is not commutative.) In fact, it is a classical result of G. Whitehead [24] that

$$\mathrm{nil}\,[X, G] \leq \mathrm{cat}\,(X),$$

where $\mathrm{cat}(X)$ is the reduced Lusternik-Schnirelmann category of X. An important special case occurs when $X = G$, for then $[G, G]$ has a second binary operation obtained from composition of homotopy classes. It is a well known algebraic fact that if R is a set which satisfies all of the axioms for a ring except commutativity of addition, then addition is commutative. Now $[G, G]$ satisfies all of the axioms of a ring except commutativity of addition and one distributive law (this is sometimes called a near-ring), so it is reasonable to ask if addition in $[G, G]$ is commutative when G is a compact, connected, non-contractible topological group other than a torus. One example that immediately comes to mind is $[S^3, S^3]$ which is commutative, even though, by Hubbuck's theorem, the topological group S^3 is not homotopy-commutative and consequently the group $[S^3 \times S^3, S^3]$ is not abelian.

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More generally, we would like to know $\mathrm{nil}[G,G]$. The homotopical nilpotency $\mathrm{nil}(G)$ of a topological group G has been studied by Zabrodsky, Hopkins, Rao and others [25, 6, 22]. Since $\mathrm{nil}[G,G]$ is bounded above by $\mathrm{nil}(G)$, our results give a lower bound for $\mathrm{nil}(G)$. We have an upper bound, $\mathrm{cat}(G)$, for $\mathrm{nil}[G,G]$, but are there lower bounds? The following have been conjectured:

Conjecture 1.1. *If G is simple, then* $\mathrm{nil}[G,G] \geq \mathrm{rank}\, G$.

Conjecture 1.2. *If G is simple of* $\mathrm{rank} \geq 2$, *then* $\mathrm{nil}[G,G] \geq 2$.

If 1.1 is affirmative, then so is 1.2. Notice from Example 1.4 of [20] that these conjectures are false in general without the assumption that G be simple. It is known that $\mathrm{cat}\,(SU(n)) = \mathrm{rank}\, SU(n)$ [23], so in this case Conjecture 1.1 asserts that $\mathrm{nil}[SU(n), SU(n)] = \mathrm{rank}\, SU(n)$.

Recently two of us proved 1.2 for $SU(n)$ $(n \geq 4)$ and $Sp(n)$ $(n \geq 2)$ [1]. Recall that any classical compact connected simple Lie group is a quotient of one of the groups $SU(n)$, $Sp(n)$ or $Spin(n)$ by a central subgroup. The purpose of this note is twofold: (1) We extend the results of [A-S] on the noncommutativity of $[G,G]$ for $G = Sp(n)$ and $G = SU(n)$ to the groups $G = SU(n)/H$, $G = Sp(n)/H$ and $G = Spin(4n)/H$. This is achieved as follows: we give a simpler proof of the results of [1] and use Lemma 2.1 below to extend these results to G/H; we use a different method for $Spin(4n)/H$. (2) We obtain larger lower bounds for $\mathrm{nil}[G,G]$ in some special cases. This requires a detailed study of certain low dimensional Lie groups.

Theorem 1.3. *If the universal covering group of G is $SU(n)$ $(n \geq 3)$, $Sp(n)$ $(n \geq 2)$ or $Spin(4n)$ $(n \geq 2)$, then* $\mathrm{nil}[G,G] \geq 2$.

There are results in [16, 19, 20, 21] supporting the above conjectures: (1) $\mathrm{nil}[G,G]$ equals 1 if $G = S^3$, $SO(3)$; $\mathrm{nil}[G,G] = 2$ if $G = SU(3)$, $Sp(2)$, $S^3 \times S^3$; $\mathrm{nil}[G,G] = 3$ if $G = G_2$, $SU(4)$, $S^3 \times \cdots \times S^3$ (n-times) with $n \geq 3$; (2) $\mathrm{nil}[E_8, E_8] \geq 5$, and $\mathrm{nil}[G,G] \geq 3$ for $G = Spin(7)$, $Spin(8)$, E_6, F_4. Conjecture 1.2 remains open in the following cases: (1) $G = Spin(k)/H$ with $k \neq 4n$; (2) $E_6/\mathbb{Z}_3$ and E_7/H. We add to the evidence in support of Conjecture 1.1 by proving

Proposition 1.4. $\mathrm{nil}[SU(4)/H, SU(4)/H] \geq 3$ *for any central subgroup H of $SU(4)$.*

Proposition 1.5. *If $G = S^3 \times \cdots \times S^3$ (n-times) with $n \geq 2$ and H is a central subgroup of G, then* $\mathrm{nil}[G/H, G/H] \geq \begin{cases} 2 & n = 2 \\ 3 & n \geq 3 \end{cases}$.

We note that the conjectured inequality (1.1) and Whitehead's inequality, namely

$$\mathrm{rank}(G) \leq \mathrm{nil}[G,G] \leq \mathrm{cat}\,(G)$$

can both be strict for a simple Lie group G. For example, for the exceptional Lie group G_2 it is known that $\mathrm{rank}(G_2) = 2$, $\mathrm{nil}[G_2, G_2] = 3$ and $\mathrm{cat}\,(G_2) = 4$.

We denote by $\widetilde{G}$ the universal covering group of G. In §2 we recall some general results and fix our terminology. In §3 we prove Theorem 1.3 for $\widetilde{G} = SU(n)$ and in §4 we prove Proposition 1.4. We obtain Theorem 1.3 for $\widetilde{G} = Sp(n)$ in §5 and in §6 we establish Proposition 1.5 and complete the proof of Theorem 1.3.

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2. General results

In this paper, we do not distinguish notationally between a map and its homotopy class. We always let p denote an odd prime. The p-localization of a nilpotent group or a nilpotent space Γ is denoted by $\Gamma_{(p)}$ [5]. We write $X \simeq_p Y$ if $X_{(p)} \simeq Y_{(p)}$. Let $\pi : \widetilde{G} \to \widetilde{G}/H$ be the canonical projection for any central subgroup H of $\widetilde{G}$. Then π is a homomorphism of Lie groups and hence an H-map.

Lemma 2.1. *Let Y be a connected, homotopy associative CW H-space.*

(1) $[A, Y]_{(p)} \cong [A_{(p)}, Y_{(p)}] \cong [A, Y_{(p)}]$ *and* $\mathrm{nil}[A, Y] \geq \mathrm{nil}[A_{(p)}, Y_{(p)}]$.

(2) *If the Samelson product $\langle \alpha, \beta \rangle$ is not zero for some $\alpha \in \pi_m(Y)$ and $\beta \in \pi_n(Y)$, then* $\mathrm{nil}[S^m \times S^n, Y] \geq 2$.

(3) *If $Y \simeq_p X \times S^m \times S^n$ and the order of $\langle \alpha, \beta \rangle$ is a multiple of p for some $\alpha \in \pi_m(Y)$ and $\beta \in \pi_n(Y)$, then* $\mathrm{nil}[Y, Y]_{(p)} \geq 2$.

(4) *If H is a central subgroup of $\widetilde{G}$ such that $H_{(p)} = 0$, then $[\widetilde{G}/H, \widetilde{G}/H]_{(p)} \cong [\widetilde{G}, \widetilde{G}]_{(p)}$.*

Proof. (1) is obvious and (2) is Lemma 6.1 of [1].

Since the localizing map $e : Y \to Y_{(p)}$ is an H-map, we have $\langle e_*\alpha, e_*\beta \rangle = e_*\langle \alpha, \beta \rangle \neq 0$ under the assumption of (3). Hence $\mathrm{nil}[S^m \times S^n, Y_{(p)}] \geq 2$ by (2). Then (3) follows from (1) and the relations: $\mathrm{nil}[Y, Y]_{(p)} = \mathrm{nil}[X \times S^m \times S^n, Y_{(p)}] \geq \mathrm{nil}[S^m \times S^n, Y_{(p)}]$.

Assume $H_{(p)} = 0$. The map $\pi_{(p)} : \widetilde{G}_{(p)} \to (\widetilde{G}/H)_{(p)}$ is a weak homotopy equivalence so that it is a homotopy equivalence of H-spaces. Hence (4) follows. □

For future reference we record the centers of the classical Lie groups (Theorems 4.10, 4.14 of [18]):

$$Z(SU(n)) = \mathbb{Z}_n\{e^{2\pi\sqrt{-1}/n} I_n\}, \quad Z(Sp(n)) = \mathbb{Z}_2\{-I_n\},$$

$$Z(Spin(n)) = \begin{cases} \mathbb{Z}_4 & n \equiv 2 \pmod 4, \quad n \geq 6 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n \equiv 0 \pmod 4, \quad n \geq 4 \\ \mathbb{Z}_2 & n \equiv 1 \pmod 2, \quad n \geq 3 \end{cases}$$

where $\mathbb{Z}_n\{x\}$ is the cyclic group of order n generated by x, and I_n is the identity matrix of rank n.

Let $B_n(p)$ be the standard S^{2n+1}-bundle over $S^{2n+2p-1}$ [17] such that

$$H^*(B_n(p); \mathbb{Z}_p) = \Lambda_{\mathbb{Z}_p}(x_{2n+1}, \wp^1 x_{2n+1}), \quad |x_{2n+1}| = 2n+1.$$

A Lie group G has type $(2n_1-1, 2n_2-1, \ldots, 2n_r-1)$ if

$$H^*(G;\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_{2n_1-1}, x_{2n_2-1}, \ldots, x_{2n_r-1}), \quad |x_k| = k, \quad n_1 \le n_2 \le \cdots \le n_r.$$

The group G is *p-regular* if and only if $G \simeq_p \prod_{i=1}^r S^{2n_i-1}$; G is *quasi p-regular* if and only if $G \simeq_p \prod B_{k_i}(p) \times \prod S^{2l_i-1}$ for some $\{k_i\}$ and $\{l_i\}$. If G is simply connected, then G is p-regular if and only if $n_r \le p$ [13]. If G has no p-torsion, then $H^*(G;\mathbb{Z}_p) = \Lambda_{\mathbb{Z}_p}(x_{2n_1-1}, \ldots, x_{2n_r-1})$ and $\wp^1 x_s = x_t$ holds (up to non-zero coefficient) if and only if $t - s = 2(p-1)$ and $(s-1)/2 \not\equiv 0 \pmod p$ [17].

3. Nilpotency of the group $[SU(n), SU(n)]$

Let $b_k(k)$ be a generator of $\pi_{2k-1}(SU(k)) \cong \mathbb{Z}$ for $k \ge 2$ and write $b_k(m) = {i_{k,m}}_* b_k(k)$ for $m \ge k$, where $i_{k,m} : SU(k) \to SU(m)$ is the inclusion. Then $b_k(m)$ is a generator of $\pi_{2k-1}(SU(m)) \cong \mathbb{Z}$. Recall from [2], [13], [17] the following.

Theorem 3.1. (1) *The order of* $\langle b_k(k+l-1), b_l(k+l-1)\rangle \in \pi_{2k+2l-2}(SU(k+l-1))$ *is* $(k+l-1)!/((k-1)!(l-1)!)$.

(2) $H^*(SU(n);\mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_3, x_5, \ldots, x_{2n-1})$.

(3) $SU(n)$ *is p-regular if and only if* $n \le p$.

(4) $SU(n)$ *is quasi p-regular if and only if* $n/2 < p$.

Corollary 3.2. *For* $\max\{k,l\} \le n \le k+l-1$, *the order of* $\langle b_k(n), b_l(n)\rangle \in \pi_{2k+2l-2}(SU(n))$ *is a multiple of* $(k+l-1)!/((k-1)!(l-1)!)$.

Proof. This follows from Theorem 3.1(1), the equality ${i_{n,k+l-1}}_*\langle b_k(n), b_l(n)\rangle = \langle b_k(k+l-1), b_l(k+l-1)\rangle$ and the fact that $\pi_{2k+2l-2}(SU(n))$ is finite. □

Hence, if $n/2 < p < n$, then

$$SU(n) \simeq_p \prod_{i=1}^{n-p} B_i(p) \times \Gamma(n,p)$$

where

$$\Gamma(n,p) = \begin{cases} * & (n+1)/2 = p < n \\ S^{2p-1} & (n+2)/2 = p < n \\ \prod_{i=n-p+2}^{p} S^{2i-1} & (n+2)/2 < p < n \end{cases}.$$

Let $n > 5$ and choose p such that $(n+2)/2 < p < n$. Since the order of $\langle b_{n-p+2}(n), b_p(n)\rangle \in \pi_{2n+2}(SU(n))$ is a multiple of p by Corollary 3.2, it follows that $\mathrm{nil}[SU(n), SU(n)] \ge \mathrm{nil}[SU(n)_{(p)}, SU(n)_{(p)}] \ge 2$ by Lemma 2.1. If H is a subgroup of $Z(SU(n))$, then $H_{(p)} = 0$ since p does not divide n. Therefore $\mathrm{nil}[SU(n)/H, SU(n)/H] \ge 2$ by Lemma 2.1.

We have $SU(5) \simeq_7 S^3 \times S^5 \times S^7 \times S^9$ by 3.1(3). We proceed as in the preceding paragraph. The order of $\langle b_3(5), b_5(5)\rangle \in \pi_{14}(SU(5))$ is a multiple of 7 by Corollary 3.2. Hence $\mathrm{nil}[SU(5), SU(5)] \ge \mathrm{nil}[SU(5)_{(7)}, SU(5)_{(7)}] \ge 2$ by Lemma 2.1. Since $H_{(7)} = 0$ for any subgroup H of $Z(SU(5)) = \mathbb{Z}_5$, we have $\mathrm{nil}[SU(5)/H, SU(5)/H] \ge 2$ by Lemma 2.1.

Similar methods applied to the pairs $(SU(3),3)$ and $(SU(4),5)$ yield $\mathrm{nil}[SU(3),SU(3)] \geq 2$ and $\mathrm{nil}[SU(4)/H, SU(4)/H] \geq 2$ for any subgroup H of $Z(SU(4))$.

Since $Z(SU(3)) \cong \mathbb{Z}_3$, to complete the proof of Theorem 1.3 for $SU(n)$, $n \geq 3$, it suffices to show that $\mathrm{nil}[PSU(3), PSU(3)] \geq 2$, where $PSU(3) = SU(3)/Z(SU(3))$. Let $q : SU(3) \to S^8$ be the quotient map. Consider the following homomorphisms:

$$[PSU(3), PSU(3)] \xleftarrow{\pi_*} [PSU(3), SU(3)] \xrightarrow{\pi^*} [SU(3), SU(3)] \xleftarrow{q^*} \pi_8(SU(3)).$$

Since π_* is injective, it suffices to prove $\mathrm{nil}\,\mathrm{Im}(\pi^*) \geq 2$. Let $\psi^3 : SU(n) \to SU(n)$ and $p_n : SU(n) \to S(\mathbb{C}^n)$ be maps such that $\psi^3(x) = x^3$ and $p_n(A)$ is the first column of the matrix A, where $S(\mathbb{C}^n) = S^{2n-1}$ is the unit sphere in $\mathbb{C}^n$. Note that $\psi^3 = (id)^3$. Let $\widetilde{\psi^3} : PSU(3) \to SU(3)$ be a map such that $\widetilde{\psi^3} \circ \pi = \psi^3$. We have a commutative diagram:

$$\begin{array}{ccccccc}
SU(3) & \xrightarrow{p_3} & S^5 & \xrightarrow{3\iota_5} & S^5 & \xrightarrow{b_3(3)} & SU(3) \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} & & \| & & \\
PSU(3) & \xrightarrow{\widetilde{p_3}} & L^5(3) & \xrightarrow{q} & S^5 & &
\end{array}$$

where $L^5(3) = S(\mathbb{C}^3)/\mathbb{Z}_3$ is the mod 3 lens space of dimension 5 and the second π is the projection. Then

$$\pi^*[\widetilde{\psi^3}, b_3(3) \circ q \circ \widetilde{p_3}] = [(id)^3, (b_3(3) \circ p_3)^3],$$

where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$, the commutator of α and β. In any group, we have

$$[x, yz] = [x,y]y[x,z]y^{-1}, \quad [xy, z] = x[y,z]x^{-1}[x,z].$$

Hence

$$[x, y^m] = [x,y]y[x,y^{m-1}]y^{-1} = \cdots = ([x,y]y)^{m-1}[x,y]y^{-(m-1)},$$
$$[x^m, z] = x[x^{m-1}, z]x^{-1}[x,z] = \cdots = x^{m-1}[x,z](x^{-1}[x,z])^{m-1}$$

Set $x = id$, $y = b_3(3) \circ p_3$ and $m = 3$. It follows from [21] and [5] (Lemma 6.4 p. 91) that $[x,y] = \pm q^*\langle b_2(3), b_3(3)\rangle$ is a central element; alternatively one could observe that, since $\mathrm{nil}[SU(3), SU(3)] \leq \mathrm{cat}\,(SU(3)) = 2$ by (1.1) and [23], all commutators are central in $[SU(3), SU(3)]$. In any case, it follows that $[x, y^3] = [x^3, y] = [x,y]^3$ and

$$\pi^*[\widetilde{\psi^3}, b_3 \circ q \circ \widetilde{p_3}] = [x^3, y^3] = [x,y]^9 = \pm 9q^*\langle b_2(3), b_3(3)\rangle.$$

Since the order of the last element is 4 by p. 85 of [16], it follows that $\mathrm{nil}\,\mathrm{Im}(\pi^*) \geq 2$ as desired.

4. Proof that $\mathrm{nil}[SU(4)/H, SU(4)/H] \geq 3$

Let $H = \mathbb{Z}_m$ be a subgroup of $Z(SU(4)) = \mathbb{Z}_4$ such that $m = 2, 4$. We use the following notation in which $M(n, \mathbb{C})$ denotes the set of $n \times n$ complex matrices and $\mathbb{C}\{1, j\}$ is the division ring of quaternions.

$$c' : Sp(n) \to SU(2n), \quad c'(X + jY) = \begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix}, \quad X, Y \in M(n, \mathbb{C}),$$
$$p' : SU(4) \to SU(4)/c'(Sp(2)) = S^5, \quad \text{the projection},$$
$$i : SU(3) \to SU(4), \quad i : Sp(1) \to Sp(2), \quad i(A) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$
$$q : SU(4) \to S^{15}, \quad \text{the quotient map},$$
$$\pi_5(SU(3)) = \mathbb{Z}\{b_3(3)\}, \quad p_{3*}b_3(3) = 2\iota_5, \quad \pi_7(SU(4)) = \mathbb{Z}\{b_4(4)\}, \quad p_{4*}b_4(4) = 6\iota_7.$$

Notice that c' is a monomorphism.

In [21], we showed that the order of $[id, [b_3(4) \circ p', b_4(4) \circ p_4]] = \pm q^*\langle b_2(4), \langle b_3(4), b_4(4)\rangle\rangle \in [SU(4), SU(4)]$ is a multiple of 3. We shall show that there exist $a, b, c \in [SU(4)/H, SU(4)]$ such that

$$(id)^m = a \circ \pi, \quad (b_3(4) \circ p')^{m/2} = b \circ \pi, \quad (b_4(4) \circ p_4)^m = c \circ \pi.$$

If this is true, then $\pi^*[a, [b, c]] = [(id)^m, [(b_3(4) \circ p')^{m/2}, (b_4(4) \circ p_4)^m]]$. The homotopy commutative diagram (4.1) given below, where $G = SU(4)$, implies the last element equals $\pm\frac{m^3}{2} \cdot q^*\langle b_2(4), \langle b_3(4), b_4(4)\rangle\rangle \neq 0$. Thus $\mathrm{nil}[SU(4)/H, SU(4)] \geq 3$ and $\mathrm{nil}[SU(4)/H, SU(4)/H] \geq 3$.

$$(4.1) \qquad \begin{array}{ccccc}
G & \xrightarrow{d} & G \wedge G \wedge G & & \\
\| & & \big\downarrow{\scriptstyle id \wedge p' \wedge p_4} & & \\
G & & G \wedge S^5 \wedge S^7 & \xrightarrow{(id)^m \wedge (b_3(4))^{m/2} \wedge b_4(4)^m} & G \wedge G \wedge G \\
{\scriptstyle q}\big\downarrow & & \big\uparrow{\scriptstyle b_2(4) \wedge 1 \wedge 1} & & \big\downarrow{\scriptstyle C_3} \\
S^{15} & \underset{\simeq}{\longrightarrow} & S^3 \wedge S^5 \wedge S^7 & \underset{\langle mb_2(4), \langle (m/2)b_3(4), mb_4(4)\rangle\rangle}{\longrightarrow} & G
\end{array}$$

where d is the diagonal map and $C_3(x \wedge y \wedge z) = [x, [y, z]]$.

The existence of a is obvious. For c we use the following commutative diagram:

$$\begin{array}{ccccccc}
SU(4) & \xrightarrow{p_4} & S^7 & \xrightarrow{m\iota_7} & S^7 & \xrightarrow{b_4(4)} & SU(4) \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \| & & \\
SU(4)/H & \underset{\widetilde{p_4}}{\longrightarrow} & L^7(m) & \underset{q}{\longrightarrow} & S^7 & &
\end{array}$$

where the second π is the projection $S(\mathbb{C}^4) \to S(\mathbb{C}^4)/\mathbb{Z}_m = L^7(m)$. Then $c := b_4(4) \circ q \circ \widetilde{p_4}$ satisfies the desired property.

In the rest of the proof we show the existence of b.

Lemma 4.1. *There exists a homeomorphism* $h : SU(4)/c'(Sp(2)) \approx S^5 = S(\mathbb{C}^3)$ *which makes the following square commutative:*

$$\begin{CD} SU(4)/c'(Sp(2)) @>{h}>> S^5 \\ @V{L_{\sqrt{-1}}}VV @VV{L_{-1}}V \\ SU(4)/c'(Sp(2)) @>{h}>> S^5 \end{CD}$$

Here L_α *denotes the multiplying by* α *from the left.*

By identifying $SU(4)/c'(Sp(2))$ with S^5 by h, we have the following commutative diagram:

$$\begin{array}{ccccc} SU(4) & \longrightarrow & SU(4)/\mathbb{Z}_2 & \longrightarrow & SU(4)/\mathbb{Z}_4 \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle \hat{p'}} & & \downarrow{\scriptstyle \widetilde{p'}} \\ S^5 & = & S^5 & \longrightarrow & \mathbb{P}^5 \\ & & \| & & \downarrow{\scriptstyle q} \\ & & S^5 & \xrightarrow[2\iota_5]{} & S^5 \end{array}$$

where $\mathbb{P}^5 = S(\mathbb{C}^3)/\mathbb{Z}_2$ is the real projective space of dimension 5. When $m = 2$, let $b = b_3(4) \circ \hat{p'}$. When $m = 4$, let $b = b_3(4) \circ q \circ \widetilde{p'}$. Then these elements satisfy the desired properties.

Proof of Lemma 4.1. Let $i' : SU(2) \to SU(3)$ be a monomorphism defined by

$$i'\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} = \begin{pmatrix} a & 0 & -\overline{b} \\ 0 & 1 & 0 \\ b & 0 & \overline{a} \end{pmatrix}.$$

Then there exists a smooth map ϕ which makes the following diagram commutative:

$$\begin{CD} Sp(1) @>{i'\circ c'}>> SU(3) @>>> SU(3)/i'c'(Sp(1)) \\ @V{i}VV @VV{i}V @VV{\phi}V \\ Sp(2) @>>{c'}> SU(4) @>>> SU(4)/c'(Sp(2)) \end{CD}$$

As is easily shown, ϕ is injective. Since the isotropy group at $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in S(\mathbb{C}^3)$ of the standard action of $SU(3)$ on $\mathbb{C}^3$ is $i'c'(Sp(1))$, the map $h' : SU(3)/i'c'(Sp(1)) \to S(\mathbb{C}^3) = S^5$ defined by

$$h'(Ai'c'(Sp(1))) = Ae_2 = \vec{a_2}, \quad A = (\vec{a_1}, \vec{a_2}, \vec{a_3})$$

is a homeomorphism. Therefore ϕ is an embedding between smooth manifolds homeomorphic to S^5. Hence ϕ is a homeomorphism.

Let $\psi : SU(3)/i'c'(Sp(1)) \to SU(3)/i'c'(Sp(1))$ be defined by

$$\psi(Ai'c'(Sp(1))) = A \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} i'c'(Sp(1)).$$

The last element is $(-\vec{a_1}, -\vec{a_2}, \vec{a_3})i'c'(Sp(1))$, where $A = (\vec{a_1}, \vec{a_2}, \vec{a_3})$. The map ψ is well-defined and makes the following diagram commutative:

$$\begin{array}{ccc} S(\mathbb{C}^3) & \xrightarrow{L_{-1}} & S(\mathbb{C}^3) \\ {\scriptstyle h'}\uparrow & & \uparrow{\scriptstyle h'} \\ SU(3)/i'c'(Sp(1)) & \xrightarrow{\psi} & SU(3)/i'c'(Sp(1)) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ SU(4)/c'(Sp(2)) & \xrightarrow[L_{\sqrt{-1}}]{} & SU(4)/c'(Sp(2)) \end{array}$$

Then $h := h' \circ \phi^{-1} : SU(4)/c'(Sp(2)) \to S^5$ satisfies the desired property. □

5. Nilpotency of the group $[Sp(n), Sp(n)]$

Let $c_k(k)$ be a generator of $\pi_{4k-1}(Sp(k)) \cong \mathbb{Z}$ for $k \geq 1$ and write $c_k(m) = i_{k,m*}b_k(k)$, where $i_{k,m} : Sp(k) \to Sp(m)$ is the inclusion. Then $c_k(m)$ is a generator of $\pi_{4k-1}(Sp(m)) \cong \mathbb{Z}$. Recall from [2], [13], [17] the following.

Theorem 5.1. (1) *The order of* $\langle c_k(k+l-1), c_l(k+l-1)\rangle \in \pi_{4k+4l-2}(Sp(k+l-1))$ *is*

$$(2k+2l-1)!a(k+l)/\{(2k-1)!(2l-1)!a(k)a(l)\}$$

where $a(m)$ *is* 1 *or* 2 *according as* m *is odd or even.*

(2) $H^*(Sp(n); \mathbb{Z}) = \Lambda_{\mathbb{Z}}(x_3, x_7, \ldots, x_{4n-1})$.

(3) $Sp(n)$ *is* p*-regular if and only if* $2n \leq p$.

(4) $Sp(n)$ *is quasi* p*-regular if and only if* $n < p$.

Corollary 5.2. *For* $\max\{k, l\} \leq n \leq k+l-1$ *the order of* $\langle c_k(n), c_l(n)\rangle \in \pi_{4k+4l-2}(Sp(n))$ *is a multiple of* $(2k+2l-1)!a(k+l)/\{(2k-1)!(2l-1)!a(k)a(l)\}$.

Proof. This follows from Theorem 5.1(1), the equality $i_{n,k+l-1*}\langle c_k(n), c_l(n)\rangle = \langle c_k(k+l-1), c_l(k+l-1)\rangle$ and the fact that $\pi_{4k+4l-2}(Sp(n))$ is finite. □

If $n \geq 4$, then there exists p with $n+1 < p < 2n$. For any such p

$$Sp(n) \simeq_p \prod_{i=1}^{n-(p-1)/2} B_{2i-1}(p) \times \prod_{i=n+1-(p-1)/2}^{(p-1)/2} S^{4i-1} \tag{5.1}$$

The order of $\langle c_{n+1-(p-1)/2}(n), c_{(p-1)/2}(n)\rangle \in \pi_{4n+2}(Sp(n))$ is a multiple of p by Corollary 5.2. Hence $\mathrm{nil}[Sp(n), Sp(n)] \geq \mathrm{nil}[Sp(n)_{(p)}, Sp(n)_{(p)}] \geq 2$ by Lemma 2.1. Since $Z(Sp(n))_{(p)} = 0$, we have $\mathrm{nil}[PSp(n), PSp(n)] \geq 2$ by Lemma 2.1, where $PSp(n) = Sp(n)/Z(Sp(n))$.

If n is 2 or 3 and $p = 2n+1$, then $Sp(n) \simeq_p \prod_{i=1}^{n} S^{4i-1}$ by Theorem 5.1(3) and the order of $\langle c_1(n), c_n(n)\rangle \in \pi_{4n+2}(Sp(n))$ is a multiple of p by Corollary 5.2. Hence $\mathrm{nil}[Sp(n)/H, Sp(n)/H] \geq 2$ for every subgroup H of $Z(Sp(n)) \cong \mathbb{Z}_2$ by Lemma 2.1.

6. Nilpotency of the group $[Spin(n), Spin(n)]$

By [4], if $n \equiv 0 \pmod 2$, then we have

$$Spin(n) \simeq_p Sp(n/2-1) \times S^{n-1}.$$

Consider the case: $n = 4m$ with $m \geq 2$. Choose p such that $2m-1 < p < 4m-2$. Then, by (5.1), we have

$$Spin(4m) \simeq_p \prod_{i=1}^{2m-(p+1)/2} B_{2i-1}(p) \times \prod_{i=2m-(p-1)/2}^{(p-1)/2} S^{4i-1} \times S^{4m-1}$$

of which the last space has $S^{4m-1} \times S^{4m-1}$ as a direct factor, since $2m-(p-1)/2 \leq m \leq (p-1)/2$. Let H be a central subgroup of $Spin(4m)$. Let $\partial\iota_{4m} \in \pi_{4m-1}(SO(4m)) = \pi_{4m-1}(Spin(4m)/H)$ be the characteristic element of the bundle $SO(4m+1) \to S^{4m}$. It follows from James [9] (cf. [14], [15]) that the order of $\langle \partial\iota_{4m}, \partial\iota_{4m}\rangle \in \pi_{8m-2}(SO(4m)) = \pi_{8m-2}(Spin(4m)/H)$ is a multiple of p. Hence $\mathrm{nil}[Spin(4m)/H, Spin(4m)/H] \geq 2$ by Lemma 2.1. This completes the proof of Theorem 1.3.

Note that $Spin(4) = S^3 \times S^3$. Let $G = S^3 \times \cdots \times S^3$ (n-times) with $n \geq 2$ and let H be a central subgroup of G. Since $\pi_* : [G/H, G] = [G/H, S^3] \oplus \cdots \oplus [G/H, S^3] \to [G/H, G/H]$ is an injective homomorphism, we have $\mathrm{nil}[G/H, G/H] \geq \mathrm{nil}[G/H, S^3]$. Since $H_{(3)} = 0$, it follows from Lemma 2.1 that $\mathrm{nil}[G/H, S^3] \geq \mathrm{nil}[G, S^3]_{(3)}$. If $n = 2$, then $\mathrm{nil}[G, S^3]_{(3)} = 2$ by (1.1), p. 176 of [10] and Lemma 2.1 (cf. Proposition 3.1 of [16]), and so $\mathrm{nil}[G/H, G/H] \geq 2$. Assume $n \geq 3$. Let

$$G \xrightarrow{\pi'} S^3 \times S^3 \times S^3 \xrightarrow{pr_i} S^3$$

be defined by $\pi'(x_1, \ldots, x_n) = (x_1, x_2, x_3)$ and $pr_i(x_1, x_2, x_3) = x_i$ for $i = 1, 2, 3$. Since $\pi'^* : [S^3 \times S^3 \times S^3, S^3]_{(3)} \to [G, S^3]_{(3)}$ is an injective homomorphism, we have $\mathrm{nil}[G, S^3]_{(3)} \geq \mathrm{nil}[S^3 \times S^3 \times S^3, S^3]_{(3)}$. Since the order of $[pr_1, [pr_2, pr_3]] \in [S^3 \times S^3 \times S^3, S^3]$ is 3 by §4 of [20] (cf. §3 of [8]), it follows that $\mathrm{nil}[S^3 \times S^3 \times S^3, S^3]_{(3)} \geq 3$. Therefore $\mathrm{nil}[G, S^3]_{(3)} \geq 3$ and so $\mathrm{nil}[G/H, G/H] \geq 3$ as desired. This completes the proof of Proposition 1.5.

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