

The \mathcal{A} -Category and \mathcal{A} -Cone Length of a Map

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ABSTRACT. For any collection \mathcal{A} of spaces we define two numerical invariants of maps: $\mathcal{L}_{\mathcal{A}}(f)$, the \mathcal{A} -category of f , and $L_{\mathcal{A}}(f)$, the \mathcal{A} -cone length of f . These invariants are defined axiomatically, and our first results give equivalent constructive definitions in terms of mapping cone decompositions. We show that if \mathcal{A} is the collection of all spaces, then $\mathcal{L}_{\mathcal{A}}(f)$ is the category of f as defined by Fadell and Husseini and $L_{\mathcal{A}}(f)$ is the cone length of f as defined by Marcum. By specializing to the maps $* \rightarrow X$ and $X \rightarrow *$ we obtain four invariants of spaces, including $\text{cat}_{\mathcal{A}}(X) = \mathcal{L}_{\mathcal{A}}(* \rightarrow X)$ and $\text{cl}_{\mathcal{A}}(X) = L_{\mathcal{A}}(* \rightarrow X)$. Each of these four invariants has its own axiomatic and constructive definitions. We compare $\text{cat}_{\mathcal{A}}(X)$ and $\text{cl}_{\mathcal{A}}(X)$ with similar invariants defined by Scheerer and Tanré. We conclude by giving lower bounds for these invariants in terms of cohomology.

1. Introduction

The category of a topological space was introduced in the 1930's by Lusternik and Schnirelmann [L-S] and shown to be a lower bound for the number of critical points of a smooth function on a manifold. Fox collected and extended the known results on L-S category in his encyclopedic 1941 paper [Fo]. Among other things, he gave an axiomatic definition of the category of a space. Later, G. Whitehead [Wh1] introduced a homotopy invariant definition by means of deformations of the diagonal map. Subsequently, Ganea [G1] gave a characterization of category in terms of the existence of sections of certain fibrations. Since then there has been a great deal of research on category (see the surveys of James [J1, J2] and Lemaire [Le]) and on the related notion of cone length [G1, C1, C3]. Several authors have generalized the category and cone length of a space to that of a map [Mar, C1]. In addition, the category and cone length of a space with respect to a fixed collection of spaces \mathcal{A} have also been studied [S-T1].

In this paper we simultaneously generalize both of these notions by defining the category and cone length of a map f relative to a collection \mathcal{A} , denoted $\mathcal{L}_{\mathcal{A}}(f)$ and $L_{\mathcal{A}}(f)$ and called the \mathcal{A} -category and \mathcal{A} -cone length of f , respectively. We begin in Section 2 by giving two sets of simple axioms which may be satisfied by integer valued functions of maps. Then $\mathcal{L}_{\mathcal{A}}(f)$ is defined to be the maximum of

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all functions which satisfy one set of axioms and $L_{\mathcal{A}}(f)$ is similarly defined for the other set. In Section 3 we show that $\mathcal{L}_{\mathcal{A}}(f)$ and $L_{\mathcal{A}}(f)$ can be characterized as the length of the shortest appropriate \mathcal{A} -decomposition of f . Thus, for example, $L_{\mathcal{A}}(f)$ is the smallest integer n such that f has a homotopy factorization of the form $X = X_0 \xrightarrow{j_0} X_1 \xrightarrow{j_1} \cdots \xrightarrow{j_{n-1}} X_n \equiv Y$, where each j_i is part of a mapping cone sequence $L_i \rightarrow X_i \xrightarrow{j_i} X_{i+1}$ with $L_i \in \mathcal{A}$. Section 4 is devoted to proving that $\mathcal{L}_{\mathcal{A}}(f)$ is the category of a map f as defined by Fadell and Husseini [Fa, F-H] and Cornea [C2], and that $L_{\mathcal{A}}(f)$ is the cone length of the map f as defined by Marcum [Mar] when \mathcal{A} is the collection of all spaces. The latter definition of the category of a map is different from the one given by Fox [Fo] and studied by Bernstein and Ganea in [B-G]. By taking f to be the map $* \rightarrow X$, we obtain the \mathcal{A} -category of X and the \mathcal{A} -cone length of X , and by taking f to be $X \rightarrow *$ we obtain two more invariants, the \mathcal{A} -kitegory of X , which is new, and the \mathcal{A} -killing length of X , which was considered in [A-S1, A-M-S]. When \mathcal{A} is the collection of all spaces, the first two invariants coincide with the standard notions. All four of these numerical invariants of spaces have characterizations in terms of suitable \mathcal{A} -decompositions of minimal length. In addition, we give axioms which uniquely determine each of these invariants. In Section 6 we establish order relations among the \mathcal{A} -category, \mathcal{A} -cone length, \mathcal{A} -kitegory, and \mathcal{A} -killing length of a space and give examples to show that the inequalities can be strict. We end the section by comparing our notions of \mathcal{A} -category and \mathcal{A} -cone length of a space with those introduced by Scheerer and Tanré [S-T1]. We conclude the paper in Section 7 by giving some lower bounds for $\mathcal{L}_{\mathcal{A}}(f)$ and $L_{\mathcal{A}}(f)$ in terms of the cohomology rings of X and Y .

In [S-T2] Scheerer and Tanré have given a very general approach to Lusternik-Schnirelmann type invariants. Their invariants are defined for a pair of maps with the same target, and are indexed by certain constructions on maps. Our invariants $\mathcal{L}_{\mathcal{A}}(f)$ and $L_{\mathcal{A}}(f)$ can be obtained as a suitable specialization of their definition. This is discussed in some detail in Section 6.

In a subsequent paper [A-S-S] we will study the \mathcal{A} -category and \mathcal{A} -cone length of maps with regard to homotopy pushouts, products and fibrations.

For the remainder of this section, we give our notation and terminology. All topological spaces are based and connected, and have the based homotopy type of well-pointed based spaces [May, p. 56]. All maps and homotopies preserve base points. We will distinguish notationally between a map $f : X \rightarrow Y$ and its homotopy class $\alpha = [f] \in [X, Y]$. We write $f \simeq g$ to indicate that f is homotopic to g . We let $*$ denote the base point of a space or a space consisting of a single point. In addition to standard notation, we use \equiv for same homotopy type, $0 : X \rightarrow Y$ for the constant map and $\text{id} : X \rightarrow X$ for the identity map.

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2. Axioms

In this section we give the axiomatic definition of the two invariants $\mathcal{L}_{\mathcal{A}}$ and $L_{\mathcal{A}}$. We begin with some basic definitions.

DEFINITION 2.1. Given maps $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$.

- (1) We say that f' **dominates** f (or that f is a **retract** of f') if there is a homotopy commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & X' & \xrightarrow{r} & X \\ \downarrow f & & \downarrow f' & & \downarrow f \\ Y & \xrightarrow{j} & Y' & \xrightarrow{s} & Y \end{array}$$

such that $ri \simeq \text{id}$ and $sj \simeq \text{id}$.

- (2) If in addition $ir \simeq \text{id}$ and $js \simeq \text{id}$, we say that (i, j) is a **homotopy equivalence** from f to f' (and that (r, s) is a homotopy equivalence from f' to f). Then f and f' are called **homotopy equivalent**.

DEFINITION 2.2. By a **collection** \mathcal{A} is meant a collection of spaces such that (1) $*$ $\in \mathcal{A}$, (2) if $A \in \mathcal{A}$ and $A \equiv B$, then $B \in \mathcal{A}$. A collection \mathcal{A} is **closed under suspension** if $\Sigma\mathcal{A} = \{\Sigma A \mid A \in \mathcal{A}\} \subseteq \mathcal{A}$.

Examples of collections we shall consider are the following: (i) $\mathcal{A} = \{\text{all spaces}\}$, (ii) $\mathcal{A} = \mathcal{S}$, the collection consisting of all (finite or infinite) wedges of spheres (of dimension ≥ 0) and $*$, (iii) $\mathcal{A} = \mathcal{M}$, the collection consisting of all wedges of Moore spaces, (iv) $\mathcal{A} = \Sigma$, the collection of all suspensions and (v) $\mathcal{A} = \mathcal{P}$, the collection of wedges of finite complexes.

2.3 \mathcal{A} -Cone Axioms Let \mathcal{A} be a collection. A function $\ell_{\mathcal{A}}$ which assigns to each map f an integer $0 \leq \ell_{\mathcal{A}}(f) \leq \infty$ is said to satisfy the **\mathcal{A} -cone axioms** if

- (1) (Homotopy Axiom) If $f \simeq g$, then $\ell_{\mathcal{A}}(f) = \ell_{\mathcal{A}}(g)$;
- (2) (Normalization Axiom) $\ell_{\mathcal{A}}(\text{id}) = 0$;
- (3) (Composition Axiom) $\ell_{\mathcal{A}}(f \circ g) \leq \ell_{\mathcal{A}}(f) + \ell_{\mathcal{A}}(g)$;
- (4) (Mapping Cone Axiom) If $A \rightarrow X \xrightarrow{f} Y$ is a mapping cone sequence with $A \in \mathcal{A}$, then $\ell_{\mathcal{A}}(f) \leq 1$;
- (5) (Equivalence Axiom) If f is homotopy equivalent to g , then $\ell_{\mathcal{A}}(f) = \ell_{\mathcal{A}}(g)$.

2.4 \mathcal{A} -Category Axioms A function $\ell_{\mathcal{A}}$ which assigns to each map f an integer $0 \leq \ell_{\mathcal{A}}(f) \leq \infty$ is said to satisfy the **\mathcal{A} -category axioms** if (1), (2), (3) and (4) of the \mathcal{A} -cone axioms and the following axiom hold

- (5') (Domination Axiom) If f is dominated by f' , then $\ell_{\mathcal{A}}(f) \leq \ell_{\mathcal{A}}(f')$.

Clearly if $\ell_{\mathcal{A}}$ satisfies the \mathcal{A} -category axioms, then $\ell_{\mathcal{A}}$ satisfies the \mathcal{A} -cone axioms.

REMARK 2.5. It is possible to slightly change and reduce the number of \mathcal{A} -cone axioms and obtain a more streamlined set of axioms. For example, we could introduce the axiom

- (2') If h is a homotopy equivalence, then $\ell_{\mathcal{A}}(h) = 0$.

Then clearly the \mathcal{A} -cone axioms are equivalent to (1), (2'), (3) and (4). We have chosen the axioms as given in (2.3) to emphasize the relation between these axioms and the \mathcal{A} -category axioms.

DEFINITION 2.6. Let \mathcal{A} be a collection. The **\mathcal{A} -cone length** $L_{\mathcal{A}}$ is the maximum of all functions $\ell_{\mathcal{A}}$ satisfying the \mathcal{A} -cone axioms, i.e.,

$$L_{\mathcal{A}}(f) = \max\{\ell_{\mathcal{A}}(f) \mid \ell_{\mathcal{A}} \text{ satisfies (1) - (5)}\}.$$

Similarly, the \mathcal{A} -category of f is

$$\mathcal{L}_{\mathcal{A}}(f) = \max\{\ell_{\mathcal{A}}(f) \mid \ell_{\mathcal{A}} \text{ satisfies (1) – (4) and (5')}\}.$$

Clearly $L_{\mathcal{A}}$ satisfies the \mathcal{A} -cone axioms and $\mathcal{L}_{\mathcal{A}}$ satisfies the \mathcal{A} -category axioms. Since the \mathcal{A} -category axioms are stronger than the \mathcal{A} -cone axioms, we have

$$(2.7) \quad \mathcal{L}_{\mathcal{A}}(f) \leq L_{\mathcal{A}}(f)$$

for any map f . Moreover, if $\mathcal{B} \subseteq \mathcal{A}$, then

$$L_{\mathcal{A}}(f) \leq L_{\mathcal{B}}(f) \quad \text{and} \quad \mathcal{L}_{\mathcal{A}}(f) \leq \mathcal{L}_{\mathcal{B}}(f).$$

Thus $L_{\{\text{all spaces}\}}(f) \leq L_{\mathcal{A}}(f)$ and $\mathcal{L}_{\{\text{all spaces}\}}(f) \leq \mathcal{L}_{\mathcal{A}}(f)$ for any collection \mathcal{A} and any map f . In the next section we give alternate characterizations of $L_{\mathcal{A}}(f)$ and $\mathcal{L}_{\mathcal{A}}(f)$ which will be useful in computing these numbers.

3. \mathcal{A} -Decompositions

We will see that the axiomatic definitions in (2.6) are quite useful and serve to focus on the characteristic properties of $\mathcal{L}_{\mathcal{A}}$ and $L_{\mathcal{A}}$. For calculations, however, it is convenient to have a constructive definition for these invariants. We develop this approach in this section. We begin with some definitions.

DEFINITION 3.1. Given a collection \mathcal{A} and a map $f : X \rightarrow Y$. An \mathcal{A} -**cone decomposition of f of length n** is a homotopy commutative diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{j_0} & X_1 & \xrightarrow{j_1} & \cdots & \xrightarrow{j_{n-2}} & X_{n-1} & \xrightarrow{j_{n-1}} & X_n \\ \parallel & & & & & & & & \updownarrow f_n \\ X & \xrightarrow{\quad f \quad} & & & & & & & Y \end{array}$$

in which f_n is a homotopy equivalence with homotopy inverse s and each map j_i is part of a mapping cone sequence

$$A_i \longrightarrow X_i \xrightarrow{j_i} X_{i+1}$$

with $A_i \in \mathcal{A}$. Thus $f_n j_{n-1} \cdots j_0 \simeq f$, $sf \simeq j_{n-1} \cdots j_0$, $f_n s \simeq \text{id}$ and $sf_n \simeq \text{id}$. The homotopy commutative diagram above is an \mathcal{A} -**category decomposition of f of length n** if s is simply a homotopy section of f_n . Explicitly, $f_n j_{n-1} \cdots j_0 \simeq f$, $sf \simeq j_{n-1} \cdots j_0$ and $f_n s \simeq \text{id}$, but sf_n need not be homotopic to the identity.

REMARK 3.2. In an \mathcal{A} -category decomposition of a map f with notation as above, f is dominated by $j_{n-1} \cdots j_0$.

REMARK 3.3. We discuss briefly the existence of \mathcal{A} -cone decompositions (and hence of \mathcal{A} -category decompositions) for certain collections \mathcal{A} .

- (1) If (Y, X) is a finite dimensional relative CW complex [Wh2], then Y is obtained from X by repeatedly attaching cells. Thus we obtain an \mathcal{S} -cone decomposition of $X \hookrightarrow Y$. This clearly extends to arbitrary maps $f : X \rightarrow Y$.
- (2) If X and Y are simply-connected finite dimensional complexes, then we obtain an \mathcal{M} -cone decomposition of f from the homology decomposition of f [Hil].

LEMMA 3.4. If f has an \mathcal{A} -category decomposition of length n and f dominates g , then g has an \mathcal{A} -category decomposition of length n .

PROOF. We have a homotopy commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{i} & X & \xrightarrow{r} & X' \\ \downarrow g & & \downarrow f & & \downarrow g \\ Y' & \xrightarrow{j} & Y & \xrightarrow{s} & Y' \end{array}$$

with $ri \simeq \text{id}$ and $sj \simeq \text{id}$, and an \mathcal{A} -category decomposition of f with length n :

$$\begin{array}{ccccccc} L_0 & & L_1 & & & & \\ \downarrow \alpha_0 & & \downarrow \alpha_1 & & & & \\ X_0 & \xrightarrow{j_0} & X_1 & \xrightarrow{j_1} & \cdots & \xrightarrow{j_{n-1}} & X_n \\ \parallel & & & & & & \uparrow \sigma \\ X & \xrightarrow{f} & & & & & Y \end{array}$$

with $L_i \in \mathcal{A}$ and $f_n \sigma \simeq \text{id}$. Consider

$$\begin{array}{ccccc} L_0 & \xrightarrow{\alpha_0} & X_0 = X & \xrightarrow{j_0} & X_1 \\ \text{id} \downarrow & & \downarrow r & \textcircled{I} & \downarrow \theta_1 \\ L_0 & \xrightarrow{r\alpha_0} & X'_0 & \xrightarrow{j'_0} & X'_1 \end{array}$$

where $X'_0 = X'$ and X'_1 is defined as the homotopy pushout of the square I with auxilliary maps j'_0 and θ_1 . Thus X'_1 is the mapping cone of $r\alpha_0$ so we have a mapping cone sequence $L_0 \xrightarrow{r\alpha_0} X'_0 \xrightarrow{j'_0} X'_1$. Furthermore, there is a map $g_1 : X'_1 \rightarrow Y$ making the following diagram commute

$$\begin{array}{ccccc} X_0 & \xrightarrow{j_0} & X_1 & \xrightarrow{f_n j_{n-1} \cdots j_1} & Y \\ \downarrow r & \textcircled{I} & \downarrow \theta_1 & & \downarrow s \\ X'_0 & \xrightarrow{j'_0} & X'_1 & \xrightarrow{g_1} & Y' \end{array}$$

\xrightarrow{g}

For the next step, we consider

$$\begin{array}{ccccc} L_1 & \xrightarrow{\alpha_1} & X_1 & \xrightarrow{j_1} & X_2 \\ \text{id} \downarrow & & \downarrow \theta_1 & \textcircled{II} & \downarrow \theta_2 \\ L_1 & \xrightarrow{\theta_1 \alpha_1} & X'_1 & \xrightarrow{j'_1} & X'_2, \end{array}$$

where II is a homotopy pushout which defines X'_2 . We then obtain $g_2 : X'_2 \rightarrow Y'$ such that $g_2 j'_1 \simeq g_1$ and $g_2 \theta_2 \simeq s f_n j_{n-1} \cdots j_2$. We continue in this way and end up with X'_n as the mapping cone of $\theta_{n-1} \alpha_{n-1}$, a map $\theta_n : X_n \rightarrow X'_n$ and a map $g_n : X'_n \rightarrow Y'$ such that $g_n j'_{n-1} \simeq g_{n-1}$ and $g_n \theta_n \simeq s f_n$. Define a section

$\sigma' : Y' \rightarrow X'_n$ as the composition $Y' \xrightarrow{j} Y \xrightarrow{\sigma} X_n \xrightarrow{\theta_n} X'_n$. Then $g_n \sigma' \simeq \text{id}$ and $\sigma' g \simeq \theta_n \sigma j g \simeq \theta_n \sigma f_i \simeq \theta_n (j_{n-1} \cdots j_0) i \simeq (j'_{n-1} \cdots j'_0) r i \simeq j'_{n-1} \cdots j'_0$. Thus

$$\begin{array}{ccccccc} X'_0 & \xrightarrow{j'_0} & X'_1 & \xrightarrow{j'_1} & \cdots & \xrightarrow{j'_{n-2}} & X'_{n-1} & \xrightarrow{j'_{n-1}} & X'_n \\ \parallel & & & & & & & & \uparrow \downarrow \sigma' \\ & & & & & & & & g_n \\ X' & \xrightarrow{\quad g \quad} & & & & & & & Y' \end{array}$$

is an \mathcal{A} -category decomposition of g of length n . \square

LEMMA 3.5. Let $g : X \rightarrow Y$ and $f : Y \rightarrow Z$.

- (1) If g has an \mathcal{A} -category decomposition of length k and f has an \mathcal{A} -category decomposition of length l , then $f \circ g$ has an \mathcal{A} -category decomposition of length $k + l$.
- (2) If g has an \mathcal{A} -cone decomposition of length k and f has an \mathcal{A} -cone decomposition of length l , then $f \circ g$ has an \mathcal{A} -cone decomposition of length $k + l$.

PROOF. We prove the first assertion. Let

$$\begin{array}{ccc} \begin{array}{c} R_0 \\ \downarrow \alpha_0 \\ X_0 \\ \parallel \\ X \end{array} & \xrightarrow{i_0} & \begin{array}{c} R_1 \\ \downarrow \alpha_1 \\ X_1 \\ \vdots \\ \downarrow \alpha_{k-1} \\ X_{k-1} \\ \parallel \\ X \end{array} & \xrightarrow{i_1} & \cdots & \xrightarrow{i_{k-1}} & X_k \\ & & & & & & \uparrow \downarrow \sigma \\ & & & & & & g_k \\ & & & & & & Y \end{array} \quad \text{and} \quad \begin{array}{c} S_0 \\ \downarrow \beta_0 \\ Y_0 \\ \parallel \\ Y \end{array} \xrightarrow{j_0} \begin{array}{c} S_1 \\ \downarrow \beta_1 \\ Y_1 \\ \vdots \\ \downarrow \beta_{l-1} \\ Y_{l-1} \\ \parallel \\ Y \end{array} \xrightarrow{j_1} \cdots \xrightarrow{j_{l-1}} Y_l \\ \uparrow \downarrow \tau \\ f_l \\ Z \end{array}$$

be \mathcal{A} -category decompositions of g and f , respectively. Consider the diagram

$$\begin{array}{ccccc} S_0 & \xrightarrow{\beta_0} & Y_0 = Y & \xrightarrow{j_0} & Y_1 \\ \text{id} \downarrow & & \downarrow \sigma & \textcircled{I} & \downarrow \theta_1 \\ S_0 & \xrightarrow{\sigma \beta_0} & X_k & \xrightarrow{i_k} & X_{k+1} \end{array}$$

where X_{k+1} is defined as the homotopy pushout in I with auxilliary maps i_k and θ_1 . Note that there is a mapping cone sequence $S_0 \xrightarrow{\sigma \beta_0} X_k \xrightarrow{i_k} X_{k+1}$. Now define $\lambda_{k+1} : X_{k+1} \rightarrow Z$ by the diagram

$$\begin{array}{ccccc} Y_0 = Y & \xrightarrow{j_0} & Y_1 & & \\ \downarrow \sigma & & \downarrow \theta_1 & & \\ X_k & \xrightarrow{i_k} & X_{k+1} & \xrightarrow{\lambda_{k+1}} & Z \end{array}$$

$f g_k$

Continuing, we obtain a mapping cone sequence $S_{l-1} \rightarrow X_{k+l-1} \xrightarrow{i_{k+l-1}} X_{k+l}$, a homotopy pushout square which defines X_{k+l} and a map $\lambda_{k+l} : X_{k+l} \rightarrow Z$. Induction on r shows that $\lambda_{k+r} i_{k+r-1} \cdots i_k \simeq f g_k$ for $1 \leq r \leq l$. Since $f g_k i_{k-1} \cdots i_0 \simeq f g$,

these maps all fit into a decomposition

$$\begin{array}{ccccccccccc}
 X_0 & \xrightarrow{i_0} & X_1 & \xrightarrow{i_1} & \cdots & \xrightarrow{i_{k-1}} & X_k & \xrightarrow{i_k} & X_{k+1} & \xrightarrow{i_{k+1}} & \cdots & \xrightarrow{i_{k+l-1}} & X_{k+l} \\
 \parallel & & & & & & & & & & & & \lambda_{k+l} \begin{array}{c} \updownarrow \\ \rho \end{array} \\
 X & \xrightarrow{fg} & & & & & & & & & & & Z,
 \end{array}$$

where the section ρ of λ_{k+l} is the composition $Z \xrightarrow{\tau} Y_l \xrightarrow{\theta_l} X_{k+l}$.

To prove the second assertion of Lemma 3.5, we repeat the previous proof with the added assumption that σ and τ are homotopy inverses for g_k and f_l respectively. From the homotopy pushout square I, we have that θ_1 is a homotopy equivalence. We continue as before and find that each θ_i is a homotopy equivalence. Thus ρ is a homotopy equivalence. This completes the proof. \square

DEFINITION 3.6. Let \mathcal{A} be a collection and $f : X \rightarrow Y$ a map. We define an integer $\mathcal{V}_{\mathcal{A}}(f)$, $0 \leq \mathcal{V}_{\mathcal{A}}(f) \leq \infty$, as follows: If f is a homotopy equivalence, set $\mathcal{V}_{\mathcal{A}}(f) = 0$. If f has no \mathcal{A} -category decomposition, set $\mathcal{V}_{\mathcal{A}}(f) = \infty$. Otherwise, $\mathcal{V}_{\mathcal{A}}(f)$ is the integer n such that there is an \mathcal{A} -category decomposition of length n but none of length $< n$. The integer $V_{\mathcal{A}}(f)$ is similarly defined with ‘ \mathcal{A} -category decomposition’ replaced by ‘ \mathcal{A} -cone decomposition.’

We next show that the \mathcal{A} -category decomposition length of f , that is, the integer $\mathcal{V}_{\mathcal{A}}(f)$, is equal to the \mathcal{A} -category length of f , and a similar result with ‘ \mathcal{A} -cone’ replacing ‘ \mathcal{A} -category.’ These results are in the spirit of [S-T2, Thm. 2.5] in which Scheerer and Tanré characterize their invariant by axioms. Their axioms do not specialize to our axioms (2.3) and (2.4), though they are similar.

THEOREM 3.7. Let \mathcal{A} be a collection and $f : X \rightarrow Y$ a map. Then

- (1) $\mathcal{L}_{\mathcal{A}}(f) = \mathcal{V}_{\mathcal{A}}(f)$; that is, $\mathcal{L}_{\mathcal{A}}(f)$ is the smallest n such that f has an \mathcal{A} -category decomposition of length n .
- (2) $L_{\mathcal{A}}(f) = V_{\mathcal{A}}(f)$; that is, $L_{\mathcal{A}}(f)$ is the smallest n such that f has an \mathcal{A} -cone decomposition of length n .

PROOF. We prove the assertion about $\mathcal{L}_{\mathcal{A}}(f)$ by first showing that $\mathcal{L}_{\mathcal{A}}(f) \leq \mathcal{V}_{\mathcal{A}}(f)$. Suppose $\mathcal{V}_{\mathcal{A}}(f) = n < \infty$, and let

$$\begin{array}{ccccccccccc}
 X_0 & \xrightarrow{j_0} & X_1 & \xrightarrow{j_1} & \cdots & \xrightarrow{j_{n-2}} & X_{n-1} & \xrightarrow{j_{n-1}} & X_n \\
 \parallel & & & & & & & & f_n \begin{array}{c} \updownarrow \\ s \end{array} \\
 X & \xrightarrow{f} & & & & & & & Y
 \end{array}$$

be an \mathcal{A} -category decomposition of f with length n . By Remark 3.2, f is dominated by $j_{n-1} \cdots j_0$. Therefore

$$\mathcal{L}_{\mathcal{A}}(f) \leq \mathcal{L}_{\mathcal{A}}(j_{n-1} \cdots j_0) \leq \mathcal{L}_{\mathcal{A}}(j_{n-1}) + \cdots + \mathcal{L}_{\mathcal{A}}(j_0) \leq n$$

by the Domination Axiom, the Composition Axiom, and the Mapping Cone Axiom, respectively. Thus $\mathcal{L}_{\mathcal{A}}(f) \leq \mathcal{V}_{\mathcal{A}}(f)$.

Next suppose $\mathcal{L}_{\mathcal{A}}(f) = n$. Lemmas 3.5 and 3.4 imply that $\mathcal{V}_{\mathcal{A}}(f)$ satisfies the \mathcal{A} -category axioms (2.3), and hence $\mathcal{V}_{\mathcal{A}}(f) \leq n$. The proof of the second part is similar. \square

4. Comparison with Other Notions of Cone Length and Category

Many authors have studied notions of category and cone length similar to ours, so we now consider the relationships of our invariants $\mathcal{L}_{\mathcal{A}}$ and $L_{\mathcal{A}}$ to previously defined concepts. In the case $\mathcal{A} = \{\text{all spaces}\}$, Marcum has used the characterization of $L_{\mathcal{A}}$ in Theorem 3.7 as a definition of the cone length of a map [Mar]. This approach and a variation of it has also been given by Cornea [C1].

We next recall the definition of $\text{cat}(f)$, the category of a map $f : X \rightarrow Y$, as introduced by Fadell and Husseini [Fa, F-H] and redefined in terms of sections of fibrations by Cornea in [C2]. We note that these authors have denoted this as $\text{cat}(Y, X)$, but we prefer $\text{cat}(f)$ since it clearly depends on the map f and not only on the spaces X and Y . This is different from the concept described in [Fo, B-G] and [J1, Sec. 7]. One inductively defines a sequence of fibrations

$$F_n(f) \xrightarrow{i_n} G_n(f) \xrightarrow{p_n} Y$$

by the following method, known as the Ganea construction [G2]. Let $j_{n+1} : C_{i_n} \rightarrow Y$ be the obvious extension of p_n to the mapping cone C_{i_n} of i_n , and define $p_{n+1} : G_{n+1}(f) \rightarrow Y$ to be the result of converting j_{n+1} to a fibration. Thus $G_{n+1} \equiv C_{i_n}$ and $F_{n+1}(f)$ is the fiber of p_{n+1} (i.e., the homotopy fiber of j_{n+1}). The induction begins with $G_0(f) \equiv X$, p_0 the usual fiber map equivalent to f and $F_0(f)$ the fiber of p (i.e., the homotopy fiber of f). Thus there is a diagram

$$\begin{array}{ccccccc} X & \equiv & G_0(f) & \xrightarrow{j_0} & G_1(f) & \xrightarrow{j_1} & \cdots & \xrightarrow{j_{n-1}} & G_n(f) & \xrightarrow{j_n} & \cdots \\ f \downarrow & & p_0 \downarrow & & p_1 \downarrow & & & & p_n \downarrow & & \\ Y & \equiv & Y & \equiv & Y & \equiv & \cdots & \equiv & Y & \equiv & \cdots \end{array}$$

We denote the inclusion $j_{n-1} \cdots j_0$ by $\chi_n : X \rightarrow G_n(f)$ and say $\text{cat}(f) \leq n$ if there is a section $s : Y \rightarrow G_n(f)$ of p_n such that $sf \simeq \chi_n$.

PROPOSITION 4.1. If $\mathcal{A} = \{\text{all spaces}\}$, then $\mathcal{L}_{\mathcal{A}}(f) = \text{cat}(f)$ for every $f : X \rightarrow Y$.

PROOF. We first show that $\text{cat}(f)$ satisfies the \mathcal{A} -category axioms with $\mathcal{A} = \{\text{all spaces}\}$. Clearly cat is homotopy invariant and $\text{cat}(\text{id}) = 0$. We now establish the Composition Axiom. Consider $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ with $\text{cat}(f) = k$ and $\text{cat}(g) = l$. Thus the fiber map $p_k : G_k(f) \rightarrow Y$ has a section s and an inclusion $\chi_k : X \rightarrow G_k(f)$ with $sf \simeq \chi_k$; likewise, the fiber map $p'_l : G_l(g) \rightarrow Z$ has a section s' and an inclusion $\chi'_l : Y \rightarrow G_l(g)$ such that $s'g \simeq \chi'_l$. There is also a fiber map $p''_k : G_k(g \circ f) \rightarrow Z$ with inclusion $\chi''_k : X \rightarrow G_k(g \circ f)$. We obtain (by l applications of the Ganea construction) the fiber map $\hat{p}_l : G_l(p''_k) \rightarrow Z$ with inclusion $\hat{\chi}_l : G_k(g \circ f) \rightarrow G_l(p''_k)$. Clearly $G_l(p''_k) = G_{k+l}(g \circ f)$, and we have homotopy commutative diagrams

$$\begin{array}{ccc} G_l(p''_k) & \equiv & G_{k+l}(g \circ f) \\ & \searrow \hat{p}_l & \swarrow p''_{k+l} \\ & & Z \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\chi''_k} & G_k(g \circ f) \\ \downarrow \chi''_{k+l} & & \downarrow \hat{\chi}_l \\ G_{k+l}(g \circ f) & \equiv & G_l(p''_k) \end{array}$$

Thus we have the diagram

$$\begin{array}{ccccc}
 X & & & & \\
 \downarrow f & \searrow \chi_k & \xrightarrow{\chi''_k} & & \\
 Y & \xrightleftharpoons[p_k]{s} & G_k(f) & \xrightarrow{\bar{g}} & G_k(g \circ f) \\
 \downarrow g & & & \searrow \hat{\chi}_l & \downarrow p''_k \\
 Z & & & & Z \\
 \downarrow p'_l & & & & \downarrow p''_{k+l} \\
 G_l(g) & \xrightarrow{\bar{\gamma}} & G_l(p''_k) & \xlongequal{\quad} & G_{k+l}(g \circ f)
 \end{array}$$

where \bar{g} and $\bar{\gamma}$ are induced by

$$\begin{array}{ccc}
 X & \xrightarrow{\text{id}} & X \\
 \downarrow f & & \downarrow g \circ f \\
 Y & \xrightarrow{g} & Z
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccccc}
 Y & \xrightarrow{s} & G_k(f) & \xrightarrow{\bar{g}} & G_k(g \circ f) \\
 \downarrow g & & & & \downarrow p''_k \\
 Z & \xrightarrow{\text{id}} & Z & & Z
 \end{array}$$

respectively. Then a long but straightforward calculation shows that the desired section of $p''_{k+l} : G_{k+l}(g \circ f) \rightarrow Z$ is the composition $Z \xrightarrow{s} G_l(g) \xrightarrow{\bar{\gamma}} G_{k+l}(g \circ f)$. This establishes the Composition Axiom.

Next we prove the Mapping Cone Axiom for $\text{cat}(f)$. Let $A \xrightarrow{i} X \xrightarrow{f} Y$ be a mapping cone sequence. We must show that $\text{cat}(f) \leq 1$. There is a map $g : A \rightarrow F_0(f)$ such that $i_0 g = k i$:

$$\begin{array}{ccccc}
 A & \xrightarrow{i} & X & \xrightarrow{f} & Y \\
 \downarrow g & & \cong \downarrow k & & \downarrow s \\
 F_0(f) & \xrightarrow{i_0} & G_0(f) & \xrightarrow{j_0} & G_1(f)
 \end{array}$$

and hence a map $s : Y \rightarrow G_1(f)$ such that the above diagram commutes. We will see that a modification of s is the desired section of $p_1 : G_1(f) \rightarrow Y$. The mapping cone sequence $A \xrightarrow{i} X \xrightarrow{f} Y$ yields an action of $[\Sigma A, Y]$ on $[Y, Y]$ denoted by exponentiation [Hil, Ch. 15]. Now $[p_1 s] \in [Y, Y]$ and $f^*[p_1 s] = [f] = f^*[\text{id}]$. Therefore there is an $\alpha \in [\Sigma A, Y]$ such that $[p_1 s]^\alpha = [\text{id}]$. But $(p_1)_* : [\Sigma A, G_1(f)] \rightarrow [\Sigma A, Y]$ is onto since the Ganea fibration splits after looping [G2, Prop. 1.5], and so $\alpha = (p_1)_* \beta$ for some $\beta \in [\Sigma A, G_1(f)]$. Then $(p_1)_*([s]^\beta) = [p_1 s]^{(p_1)_* \beta} = [\text{id}]$, and $[s]^\beta [f] = f^*[s]^\beta = f^*[s] = [j_0]$. Therefore there is a map s' in the homotopy class $[s]^\beta$ which is a section of p_1 , and $s' f \simeq \chi_1$.

Finally we establish the Domination Axiom. Suppose g dominates f , so we have the homotopy commutative diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{i} & X' & \xrightarrow{r} & X \\
 \downarrow f & & \downarrow g & & \downarrow f \\
 Y & \xrightarrow{j} & Y' & \xrightarrow{s} & Y
 \end{array}$$

with $ri \simeq \text{id}$ and $sj \simeq \text{id}$. Suppose $\text{cat}(g) \leq n$. Then we have a homotopy commutative diagram

$$\begin{array}{ccccc}
 G_n(f) & \xrightarrow{\iota} & G_n(g) & \xrightarrow{\rho} & G_n(f) \\
 \downarrow p_n & & \sigma' \updownarrow p'_n & & \downarrow p_n \\
 Y & \xrightarrow{j} & Y' & \xrightarrow{s} & Y \\
 \uparrow f & & \uparrow g & & \uparrow f \\
 X & \xrightarrow{i} & X' & \xrightarrow{r} & X
 \end{array}$$

where σ' is the section of p'_n and ι and ρ are induced by (i, j) and (r, s) , respectively. Then $\sigma = \rho\sigma'j$ is clearly a section of p_n compatible with f . Thus there is a map $s' \simeq \sigma$ which is a section of p_n with $s'f \simeq \sigma f \simeq \chi_n$.

Since $\text{cat}(f)$ satisfies the \mathcal{A} -category axioms, $\text{cat}(f) \leq \mathcal{L}_{\mathcal{A}}(f)$ for every map f .

For the reverse inequality, we assume that $\text{cat}(f) = n$. Thus the sequence of mapping cone sequences $F_i(f) \rightarrow G_i(f) \xrightarrow{j_i} G_{i+1}(f)$, $0 \leq i < n$, is an \mathcal{A} -category decomposition of f with length n . By Theorem 3.7, $\mathcal{L}_{\mathcal{A}}(f) \leq n = \text{cat}(f)$, and this completes the proof. \square

REMARK 4.2. When $\mathcal{A} = \{\text{all spaces}\}$, Marcum has established the Composition Axiom for $L_{\mathcal{A}}(f)$ [Mar]. In this case Cornea has given the Composition and Domination Axioms as properties of $\mathcal{L}_{\mathcal{A}}(f) = \text{cat}(f)$ [C2].

The following is the analogue of a known result for spaces [Ta, Section 5].

PROPOSITION 4.3. For any collection \mathcal{A} and map $f : X \rightarrow Y$, $\mathcal{L}_{\mathcal{A}}(f) \leq n$ if and only if f is dominated by a map g such that $L_{\mathcal{A}}(g) \leq n$.

PROOF. If f is dominated by a map g with $L_{\mathcal{A}}(g) \leq n$, then $\mathcal{L}_{\mathcal{A}}(f) \leq \mathcal{L}_{\mathcal{A}}(g) \leq L_{\mathcal{A}}(g) \leq n$.

Conversely, if $\mathcal{L}_{\mathcal{A}}(f) \leq n$, then f has an \mathcal{A} -category decomposition of length n

$$\begin{array}{ccccccc}
 X_0 & \xrightarrow{i_0} & X_1 & \xrightarrow{i_1} & \cdots & \xrightarrow{i_{n-2}} & X_{n-1} & \xrightarrow{i_{n-1}} & X_n \\
 \parallel & & & & & & & & \updownarrow f_n s \\
 X & \xrightarrow{\quad\quad\quad} & & & & & & & Y
 \end{array}$$

Let $g = i_{n-1} \cdots i_0 : X_0 \rightarrow X_n$. Then the same decomposition is an \mathcal{A} -cone decomposition of g . By Remark 3.2, g dominates f . \square

The following corollary is useful to prove results about $\mathcal{L}_{\mathcal{A}}$.

COROLLARY 4.4. If $\mathcal{L}_{\mathcal{A}}(f) = n$, then f is dominated by a map g with $L_{\mathcal{A}}(g) = n$. Furthermore, we can choose g with the same domain as f .

5. Numerical Invariants of Spaces

In this section we consider $L_{\mathcal{A}}(f)$ and $\mathcal{L}_{\mathcal{A}}(f)$ for special maps f in order to obtain invariants of spaces.

DEFINITION 5.1. For any collection \mathcal{A} and space X , define

- (1) the \mathcal{A} -cone length of X by $\text{cl}_{\mathcal{A}}(X) = L_{\mathcal{A}}(* \rightarrow X)$,

- (2) the \mathcal{A} -killing length of X by $\text{kl}_{\mathcal{A}}(X) = L_{\mathcal{A}}(X \rightarrow *)$,
- (3) the \mathcal{A} -category of X by $\text{cat}_{\mathcal{A}}(X) = \mathcal{L}_{\mathcal{A}}(* \rightarrow X)$,
- (4) the \mathcal{A} -kitegory of X by $\text{kit}_{\mathcal{A}}(X) = \mathcal{L}_{\mathcal{A}}(X \rightarrow *)$.

- REMARK 5.2. (1) Let $\mathcal{A} = \{\text{all spaces}\}$. Then $\text{cat}_{\mathcal{A}}(X)$ is the ordinary (reduced) category of X and $\text{cl}_{\mathcal{A}}(X)$ is the cone length of X as studied by Ganea, Cornea, Stanley and many others, denoted $\text{cat}(X)$ and $\text{cl}(X)$, respectively. On the other hand, the mapping cone sequence $X \rightarrow X \rightarrow CX$, where $CX \equiv *$ is the cone on X , with $X \in \mathcal{A}$ shows that $\text{kl}_{\mathcal{A}}(X) = \text{kit}_{\mathcal{A}}(X) = 1$ when $\mathcal{A} = \{\text{all spaces}\}$.
- (2) Since $\mathcal{L}_{\mathcal{A}}(f) \leq L_{\mathcal{A}}(f)$, we have $\text{cat}_{\mathcal{A}}(X) \leq \text{cl}_{\mathcal{A}}(X)$ and $\text{kit}_{\mathcal{A}}(X) \leq \text{kl}_{\mathcal{A}}(X)$ for any collection \mathcal{A} . We will see in the next section that if \mathcal{A} is closed under suspension, then $\text{kl}_{\mathcal{A}}(X) \leq \text{cl}_{\mathcal{A}}(X)$ and $\text{kit}_{\mathcal{A}}(X) \leq \text{cat}_{\mathcal{A}}(X)$. (Note: the latter inequality is the source of the notation ‘kit’, since kittens are smaller than cats. The term ‘kittygory’ was used first by Freyd in another context [Fr].)
- (3) The \mathcal{A} -killing length was considered in [A-S1]. Let $Z_{\mathcal{A}}(X) \subseteq [X, X]$ be the subsemigroup consisting of all homotopy classes $\alpha = [f] \in [X, X]$ such that $f_* = 0 : [A, X] \rightarrow [A, X]$ for every $A \in \mathcal{A}$. It was proved in [A-S1, Thm. 3.3] that the nilpotency of the semigroup $Z_{\mathcal{A}}(X)$ is bounded above by $\text{kl}_{\mathcal{A}}(X)$.

We first show that each of the four invariants of spaces has a characterization in terms of decompositions.

DEFINITION 5.3. Let \mathcal{A} be a collection and consider a sequence of mapping cone sequences

$$A_i \longrightarrow X_i \xrightarrow{j_i} X_{i+1}, \quad 0 \leq i < n$$

with $A_i \in \mathcal{A}$. If $X_0 = *$ and $X_n \equiv X$, we have a $\text{cl}_{\mathcal{A}}$ -decomposition of X with length n . If $X_0 = *$ and X is dominated by X_n , we have a $\text{cat}_{\mathcal{A}}$ -decomposition of X with length n . If $X_0 = X$ and $X_n \equiv *$, we have a $\text{kl}_{\mathcal{A}}$ -decomposition of X with length n . If $X_0 = X$ and X is dominated by $*$, we have a $\text{kit}_{\mathcal{A}}$ -decomposition of X with length n .

The n mapping cone sequences in Definition 5.3 form a $\text{kit}_{\mathcal{A}}$ -decomposition of X of length n if and only if $X_0 = X$ and $j_{n-1} \cdots j_0 \simeq 0$.

DEFINITION 5.4. The **length of a minimal $\text{cat}_{\mathcal{A}}$ -decomposition of X** is defined as follows: It is 0 if $X \equiv *$ and is ∞ if X has no $\text{cat}_{\mathcal{A}}$ -decomposition. Otherwise, it is the integer n such that X has a $\text{cat}_{\mathcal{A}}$ -decomposition of length n but none of length less than n . Similar definitions hold for $\text{cl}_{\mathcal{A}}$, $\text{kl}_{\mathcal{A}}$ and $\text{kit}_{\mathcal{A}}$.

The following proposition is an immediate consequence of Theorem 3.7.

PROPOSITION 5.5. If \mathcal{A} is a collection and X is a space, then

- (1) $\text{cat}_{\mathcal{A}}(X)$ is the length of a minimal $\text{cat}_{\mathcal{A}}$ -decomposition of X ;
- (2) $\text{cl}_{\mathcal{A}}(X)$ is the length of a minimal $\text{cl}_{\mathcal{A}}$ -decomposition of X ;
- (3) $\text{kl}_{\mathcal{A}}(X)$ is the length of a minimal $\text{kl}_{\mathcal{A}}$ -decomposition of X ;
- (4) $\text{kit}_{\mathcal{A}}(X)$ is the length of a minimal $\text{kit}_{\mathcal{A}}$ -decomposition of X .

We next show that the four invariants can be characterized by axioms.

Let \mathcal{A} be a collection and let $c_{\mathcal{A}}$ be a function which assigns to every space X an integer $0 \leq c_{\mathcal{A}}(X) \leq \infty$. We consider the following axioms.

- (1) $c_{\mathcal{A}}(*) = 0$
- (2) If X and Y have the same homotopy type, then $c_{\mathcal{A}}(X) = c_{\mathcal{A}}(Y)$.
- (2') If X is dominated by Y , then $c_{\mathcal{A}}(X) \leq c_{\mathcal{A}}(Y)$.
- (3) If $A \rightarrow X \rightarrow Y$ is a mapping cone sequence with $A \in \mathcal{A}$, then $c_{\mathcal{A}}(Y) \leq c_{\mathcal{A}}(X) + 1$.
- (3') If $A \rightarrow X \rightarrow Y$ is a mapping cone sequence with $A \in \mathcal{A}$, then $c_{\mathcal{A}}(X) \leq c_{\mathcal{A}}(Y) + 1$.

PROPOSITION 5.6. (1) $\text{cl}_{\mathcal{A}}(X)$ is the maximum of all $c_{\mathcal{A}}$ satisfying Axioms (1), (2) and (3);

(2) $\text{cat}_{\mathcal{A}}(X)$ is the maximum of all $c_{\mathcal{A}}$ satisfying Axioms (1), (2') and (3);

(3) $\text{kl}_{\mathcal{A}}(X)$ is the maximum of all $c_{\mathcal{A}}$ satisfying Axioms (1), (2) and (3');

(4) $\text{kit}_{\mathcal{A}}(X)$ is the maximum of all $c_{\mathcal{A}}$ satisfying Axioms (1), (2') and (3').

The proof closely follows the arguments given in Section 3, and hence is omitted.

REMARK 5.7. An axiomatic treatment of the category and the cone length of a space relative to a collection \mathcal{A} has been given by Scheerer-Tanré [S-T1]. Their axioms are similar to ours, but are not equivalent, as we shall see in the next section.

The above axioms are useful in focusing on the basic properties of these invariants. However they can also be used to establish some elementary inequalities for these invariants. We illustrate this by giving an axiomatic proof of G. Whitehead's theorem on the nilpotency of the group $[X, G]$ when G is a group-like space. Recall that if π is a group, then $\text{nil } \pi$ is the smallest integer $n \geq 1$ such that all $(n+1)$ -fold commutators of the form $\dots[[x_1, x_2], x_3], \dots, x_{n+1}]$ are trivial. We set $\text{nil } \pi = 0$ if π is the trivial group. The following result was originally proved in [Wh1].

THEOREM 5.8. If X is any space and G is a group-like space, then $\text{nil}[X, G] \leq \text{cat}X$.

PROOF. By Proposition 5.6 it suffices to show that the function $\text{nil}[X, G]$ satisfies the cat axioms, where G is a fixed group-like space. Clearly Axioms (1) and (2') hold. We prove (3). Let $A \rightarrow X \rightarrow Y$ be a mapping cone sequence and consider part of the Puppe sequence

$$[\Sigma A, G] \xrightarrow{\delta} [Y, G] \rightarrow [X, G].$$

It is well known that the image of δ is in the center of $[Y, G]$. From this it follows that $\text{nil}[Y, G] \leq \text{nil}[X, G] + 1$. \square

6. Properties of the Invariants of Spaces

In the first part of this section we consider the ordering of the invariants $\text{cat}_{\mathcal{A}}$, $\text{cl}_{\mathcal{A}}$, $\text{kl}_{\mathcal{A}}$ and $\text{kit}_{\mathcal{A}}$. In the second part we compare $\text{cat}_{\mathcal{A}}$ and $\text{cl}_{\mathcal{A}}$ to the analogous invariants which were studied by Scheerer-Tanré [S-T1].

PROPOSITION 6.1. For any collection \mathcal{A} , $\text{kl}_{\Sigma\mathcal{A}}(X) \leq \text{cl}_{\mathcal{A}}(X)$ and $\text{kit}_{\Sigma\mathcal{A}}(X) \leq \text{cat}_{\mathcal{A}}(X)$.

PROOF. We only prove the second inequality since the first is similar. Suppose $\text{cat}_{\mathcal{A}}(X) = n$ with $\text{cat}_{\mathcal{A}}$ -decomposition $A_i \rightarrow X_i \rightarrow X_{i+1}$, $0 \leq i < n$, where $A_i \in \mathcal{A}$, $X_0 = *$ and X is dominated by X_n . For $i = 1, \dots, n$, let $k_i : X_i \rightarrow X_n$ be the composition $X_i \rightarrow X_{i+1} \rightarrow \dots \rightarrow X_n$ and define $X_{(i)}$ to be the mapping cone of k_i . Then we have a mapping cone sequence $\Sigma A_i \rightarrow X_{(i)} \rightarrow X_{(i+1)}$. Clearly $X_{(0)} = X_n$ and $X_{(n)} \equiv *$, and so $\text{kl}_{\Sigma\mathcal{A}}(X_n) \leq n$. Therefore

$$\text{kit}_{\Sigma\mathcal{A}}(X) \leq \text{kit}_{\Sigma\mathcal{A}}(X_n) \leq \text{kl}_{\Sigma\mathcal{A}}(X_n) \leq n = \text{cat}_{\mathcal{A}}(X).$$

□

The first inequality of Proposition 6.1 was proved in [**A-S1**]; a stable version was proved in [**Ch**].

Thus if \mathcal{A} is a collection which is closed under suspension, then by Remark 5.2 and Proposition 6.1 the inequalities between the four invariants can be represented by a square

$$(6.2) \quad \begin{array}{ccc} & \text{cl}_{\mathcal{A}}(X) & \\ & \swarrow \quad \searrow & \\ \text{cat}_{\mathcal{A}}(X) & & \text{kl}_{\mathcal{A}}(X) \\ & \swarrow \quad \searrow & \\ & \text{kit}_{\mathcal{A}}(X) & \end{array}$$

where the lower number is less than or equal to the higher number. We next show that each of the inequalities in (6.2) can be strict.

First observe that if X is a co-H-space which is not a suspension, then $\text{cat}(X) = 1 < 2 \leq \text{cl}(X)$. More generally, Stanley [**St**] has given examples of spaces $X(n)$ with $\text{cat}(X(n)) = n$ and $\text{cl}(X(n)) = n + 1$.

Next let \mathcal{S} be the collection of wedges of spheres and let $X = S^{n_1} \times \dots \times S^{n_r}$, $r \geq 3$. If k is the smallest integer such that $r \leq 2^k - 1$, then it was shown in [**A-M-S**, Prop. 6.2] that $\text{kl}_{\mathcal{S}}(X) = k$. The standard cone decomposition of X together with the fact that $\text{cat}_{\mathcal{S}}(X) \geq \text{cat}(X) = r$ imply that $\text{cat}_{\mathcal{S}}(X) = \text{cl}_{\mathcal{S}}(X) = r$. Thus $\text{kl}_{\mathcal{S}}(X) = k < r = \text{cl}_{\mathcal{S}}(X)$ and $\text{kit}_{\mathcal{S}}(X) \leq \text{kl}_{\mathcal{S}}(X) = k < r = \text{cat}_{\mathcal{S}}(X)$. Of course these inequalities hold if $\mathcal{A} = \{\text{all spaces}\}$ and X is any space with $\text{cat}(X) > 1$, since $\text{kit}_{\mathcal{A}}(X) \leq \text{kl}_{\mathcal{A}}(X) \leq 1$ in this case.

Finally we discuss the inequality $\text{kit}_{\mathcal{A}}(X) \leq \text{kl}_{\mathcal{A}}(X)$. Consider the collection $\mathcal{A} = \Sigma$ of all suspensions and let X be a simply-connected co-H-space which does not have the homotopy type of a suspension. Then X is a homotopy retract of some suspension X' [**Hil**, p. 209] so there are maps $i : X \rightarrow X'$ and $r : X' \rightarrow X$ such that $ri \simeq \text{id}$. Then there is a mapping cone sequence $X' \xrightarrow{r} X \xrightarrow{j} C$, with C the mapping cone of r . Clearly $j \simeq 0$, so $\text{kit}_{\Sigma}(X) \leq 1$. But if $\text{kl}_{\Sigma}(X) \leq 1$, there would be a mapping cone sequence $A \xrightarrow{r} X \xrightarrow{j} *$ with A a suspension. Hence $X \equiv A$, contradicting the fact that X does not have the homotopy type of a suspension. Thus $\text{kl}_{\Sigma}(X) > 1$, so $\text{kit}_{\Sigma}(X) < \text{kl}_{\Sigma}(X)$. As long as all of the spaces involved are simply connected, this kind of example will work for collections \mathcal{A} which are not closed under retracts.

As mentioned in the introduction, Scheerer and Tanré have developed a very general framework for category type invariants [**S-T2**]. This includes the invariants we studied here as well as those studied in [**S-T1**] as special cases, as we now

describe. If \mathcal{A} is a collection of spaces and $f : X \rightarrow Y$ is a map, then we may form the collection $\mathcal{K}_{\mathcal{A}}(f)$ consisting of all $\bar{f} : C_j \rightarrow Y$, where $j : A \rightarrow X$ is a map with $A \in \mathcal{A}$, $f \circ j \simeq 0$ and \bar{f} an extension of f . The correspondence $f \mapsto \mathcal{K}(f)$ is a construction in the sense of [S-T2], and in the notation of [S-T2], $\mathcal{L}_{\mathcal{A}}(f)$ can be identified

$$\mathcal{L}_{\mathcal{A}}(f) = (\mathcal{K}_{\mathcal{A}}, \text{id})\text{-cat}(f).$$

It can be shown that this coincides with the invariant $\mathcal{K}(\mathcal{G}_{\mathcal{A}})\text{-cat}(f)$ in [S-T2, Rem. 3.4]. Scheerer and Tanré also identify $\mathcal{A}\text{-cat}(X)$ from [S-T1] in terms of their more general setup, namely, $\mathcal{A}\text{-cat}(X) = (\mathcal{K}(\mathcal{G}_{\mathcal{A}}), G_{\mathcal{A}}(X))\text{-cat}(\text{id}_X, \text{id}_X)$. Similar considerations hold for $L_{\mathcal{A}}(f)$ and $\mathcal{A}\text{-cl}(X)$. The main theorems in [S-T2] characterize these invariants by simple properties that are similar to, but not the same as, our axioms. No relationship between our $\text{cat}_{\mathcal{A}}(X)$ and their $\mathcal{A}\text{-cat}(X)$ seems to follow immediately from the results of [S-T1, S-T2]. We obtain a relationship next, as well as a relation between $\text{cl}_{\mathcal{A}}(X)$ and $\mathcal{A}\text{-cl}(X)$.

For any collection \mathcal{A} and space X , $\mathcal{A}\text{-cat}(X)$ and $\mathcal{A}\text{-cl}(X)$ have been defined in [S-T1]. (The latter is denoted $\mathcal{A}\text{-Cat}(X)$ in [S-T1].) Let us set

$$\mathcal{A}\text{-}\widetilde{\text{cat}}(X) = \begin{cases} 0 & \text{if } X \equiv * \\ \mathcal{A}\text{-cat}(X) + 1 & \text{if } X \not\equiv *. \end{cases}$$

and define $\mathcal{A}\text{-}\widetilde{\text{cl}}(X)$ similarly. It is proved in [S-T1] that $\Sigma\text{-}\widetilde{\text{cat}}(X) = \text{cat}(X)$ and $\Sigma\text{-}\widetilde{\text{cl}}(X) = \text{cl}(X)$. That is, if $\mathcal{A} = \{\text{all spaces}\}$, then $\text{cat}_{\mathcal{A}}(X) = \Sigma\mathcal{A}\text{-}\widetilde{\text{cat}}(X)$ and $\text{cl}_{\mathcal{A}}(X) = \Sigma\mathcal{A}\text{-}\widetilde{\text{cl}}(X)$ by Proposition 4.1.

It is reasonable to then ask if these latter equalities hold for any collection \mathcal{A} .

PROPOSITION 6.3. If \mathcal{A} is a collection which is closed under suspension, then

$$\text{cat}_{\mathcal{A}}(X) \leq \Sigma\mathcal{A}\text{-}\widetilde{\text{cat}}(X) \quad \text{and} \quad \text{cl}_{\mathcal{A}}(X) \leq \Sigma\mathcal{A}\text{-}\widetilde{\text{cl}}(X).$$

PROOF. We prove the proposition for cat – the proof for cl is similar. Define

$$\widetilde{\text{cat}}_{\mathcal{A}}(X) = \begin{cases} 0 & \text{if } X \equiv *; \\ \text{cat}_{\mathcal{A}}(X) - 1 & \text{if } X \not\equiv *. \end{cases}$$

Then we must show $\widetilde{\text{cat}}_{\mathcal{A}}(X) \leq \Sigma\mathcal{A}\text{-cat}(X)$. Since $\Sigma\mathcal{A}\text{-cat}$ is the maximum of all functions which satisfy the Scheerer-Tanré axioms for $\Sigma\mathcal{A}$ [S-T1, Thm. 1, Sect. 2], it suffices to show that $\widetilde{\text{cat}}_{\mathcal{A}}$ satisfies these axioms. That is, we must show: (1) If $X \in \Sigma\mathcal{A}$, then $\widetilde{\text{cat}}_{\mathcal{A}}(X) = 0$. (2) If X is dominated by Y , then $\widetilde{\text{cat}}_{\mathcal{A}}(X) \leq \widetilde{\text{cat}}_{\mathcal{A}}(Y)$. (3) If $X \rightarrow Y \rightarrow Z$ is a mapping cone sequence with $X \in \Sigma\mathcal{A}$, then $\text{cat}_{\mathcal{A}}(Z) \leq \text{cat}_{\mathcal{A}}(Y) + 1$.

The proof of this is straightforward and hence omitted. \square

It is possible to give an example of a collection \mathcal{A} and a space X such that each inequality in Proposition 6.3 is strict, and so equality does not hold in general. We present an example only for cone length.

EXAMPLE 6.4. Let $\mathcal{A} = \{S^k, S^l \vee S^l \mid k \geq 2n-1, l \geq n-1\}$ and $X = S^n \times S^n$. We sketch the argument that $\text{cl}_{\mathcal{A}}(X) \leq 2 < \Sigma\mathcal{A}\text{-}\widetilde{\text{cl}}(X)$. The two mapping cone sequences

$$S^{n-1} \vee S^{n-1} \rightarrow C \rightarrow S^n \vee S^n \quad \text{and} \quad S^{2n-1} \xrightarrow{W} S^n \vee S^n \rightarrow S^n \times S^n,$$

where C is the cone on $S^{n-1} \vee S^{n-1}$ and W is the Whitehead product, yields $\text{cl}_{\mathcal{A}}(X) \leq 2$. Since $\Sigma\mathcal{A}\text{-}\widetilde{\text{cl}}(X) = \Sigma\mathcal{A}\text{-cl}(X) + 1$, it suffices to show $\Sigma\mathcal{A}\text{-cl}(X) > 1$. If $\Sigma\mathcal{A}\text{-cl}(X) \leq 1$ then one can show (using an argument similar to the proof of Theorem 3.7) that there is a mapping cone sequence $A \rightarrow B \rightarrow X$ with $A, B \in \Sigma\mathcal{A}$. The exact homology sequence of the mapping cone sequence implies that $H_n(B) \cong \mathbb{Z} \oplus \mathbb{Z}$, and so $B = S^n \vee S^n$. Again applying the homology sequence, we conclude that $H_{2n-1}(A) \cong \mathbb{Z}$, which is impossible. Therefore $\Sigma\mathcal{A}\text{-cl}(X) > 1$.

There is however an important case when equality does hold in Proposition 6.3. Let $\mathcal{S} = \mathcal{S}_0$ be the collection of wedges of spheres of dimension ≥ 0 and \mathcal{S}_1 be the collection of wedges of spheres of dimension ≥ 1 so that $\Sigma\mathcal{S}_0 = \mathcal{S}_1$. We first prove a lemma.

LEMMA 6.5. Suppose $\text{cl}_{\mathcal{S}_0}(X) \leq n$. Then there are mapping cone sequences $L_i \rightarrow X_i \rightarrow X_{i+1}$, $0 \leq i < n$, such that $L_0 \in \mathcal{S}_0$, $X_0 = *$, $X_n \equiv X$ and $L_i \in \mathcal{S}_1$ for $i \geq 1$.

PROOF. The proof is by induction on n . When $n = 0$ there is nothing to prove. Suppose the lemma is true for $n - 1$. Since $\text{cl}_{\mathcal{S}_0}(X) \leq n$, there are mapping cone sequences $M_i \rightarrow X'_i \rightarrow X'_{i+1}$, $0 \leq i < n$, such that $M_i \in \mathcal{S}_0$, $X'_0 = *$ and $X'_n \equiv X$. Consider the n th mapping cone sequence

$$M_{n-1} = L_{n-1} \vee S^0 \vee \cdots \vee S^0 \xrightarrow{f} X'_{n-1} \longrightarrow X'_n$$

where $L_{n-1} \in \mathcal{S}_1$. Since X'_{n-1} is connected, $f|_{S^0 \vee \cdots \vee S^0} \simeq 0$. Let $f' = f|_{L_{n-1}} : L_{n-1} \rightarrow X'_{n-1}$. Then $X'_n = C_{f'} \vee S^1 \vee \cdots \vee S^1$ and we have a mapping cone sequence

$$L_{n-1} \xrightarrow{i_1 f'} X'_{n-1} \vee S^1 \vee \cdots \vee S^1 \longrightarrow X'_n,$$

with $i_1 : X'_{n-1} \rightarrow X'_{n-1} \vee S^1 \vee \cdots \vee S^1$ the inclusion. But we have that $\text{cl}_{\mathcal{S}_0}(X'_{n-1} \vee S^1 \vee \cdots \vee S^1) \leq \text{cl}_{\mathcal{S}_0}(X'_{n-1}) \leq n - 1$. By induction there exist mapping cone sequences $L_i \rightarrow X_i \rightarrow X_{i+1}$, $0 \leq i < n - 1$, with $L_0 \in \mathcal{S}_0$, $L_i \in \mathcal{S}_1$ for $i \geq 1$, $X_0 = *$ and $X_{n-1} \equiv X'_{n-1} \vee S^1 \vee \cdots \vee S^1$. This proves the lemma. \square

PROPOSITION 6.6. $\text{cat}_{\mathcal{S}_0}(X) = \mathcal{S}_1\text{-}\widetilde{\text{cat}}(X)$ and $\text{cl}_{\mathcal{S}_0}(X) = \mathcal{S}_1\text{-}\widetilde{\text{cl}}(X)$.

PROOF. We prove the result for cone length. By Proposition 6.3 it suffices to show

$$\mathcal{S}_1\text{-}\widetilde{\text{cl}}(X) \leq \text{cl}_{\mathcal{S}_0}(X).$$

If $X \equiv *$ or if $\text{cl}_{\mathcal{S}_0}(X) = \infty$, this is clearly true. Now suppose $\text{cl}_{\mathcal{S}_0}(X) = n$ with $0 < n < \infty$. We must show $\mathcal{S}_1\text{-cl}(X) \leq n - 1$. By Lemma 6.5 there exist mapping cone sequences $L_i \rightarrow X_i \rightarrow X_{i+1}$, $0 \leq i < n$, $L_0 \in \mathcal{S}_0$, $L_i \in \mathcal{S}_1$ for $i \geq 1$, $X_0 = *$ and $X_n \equiv X$. But $L_0 \rightarrow X_0 \rightarrow X_1$ shows that $X_1 \equiv \Sigma L_0 \in \mathcal{S}_1$. By the Scheerer-Tanré axioms,

$$\mathcal{S}_1\text{-cl}(X) = \mathcal{S}_1\text{-cl}(X_n) \leq \mathcal{S}_1\text{-cl}(X_{n-1}) + 1 \leq \cdots \leq \mathcal{S}_1\text{-cl}(X_1) + n - 1 = n - 1.$$

\square

REMARK 6.7. In **[A-S-S]** we have proved that if \mathcal{A} is any collection that is closed under wedges and joins, then $\text{cat}_{\mathcal{A}}(X \times Y) \leq \text{cat}_{\mathcal{A}}(X) + \text{cat}_{\mathcal{A}}(Y)$. By Proposition 6.6 this implies $\mathcal{S}_1\text{-cat}(X \times Y) \leq \mathcal{S}_1\text{-cat}(X) + \mathcal{S}_1\text{-cat}(Y)$. This answers a question in **[S-T1, Sect. 7, Question 8]**.

7. Cohomological Lower Bounds

Recall that the **cup length** of a space X , denoted $\cup\text{-length}(X)$, is the length of the largest nonzero cup product in $\tilde{H}^*(X; R)$ for any commutative ring R . The n -fold reduced diagonal $\hat{\Delta}_n : X \rightarrow X^{(n)}$ from X to the n -fold smash product $X^{(n)}$ of X with itself is the composite of the diagonal map $\Delta_n : X \rightarrow X^n$ with the quotient map $X^n \rightarrow X^{(n)}$. The **weak category** of a space X , denoted $\text{wcat}(X)$, is the greatest integer n for which $\hat{\Delta}_n$ is not homotopic to the constant map.

In the special case $\mathcal{A} = \{\text{all spaces}\}$, the following theorem is stated without proof by Cornea [C2, p. 68], and the corresponding statement for homology can be found in [F-H]. We omit the proof.

THEOREM 7.1. Let \mathcal{A} be a collection and let $f : X \rightarrow Y$ induce the homomorphism $f^* : \tilde{H}^*(Y; R) \rightarrow \tilde{H}^*(X; R)$ of reduced cohomology with coefficients in the commutative ring R . If $\mathcal{L}_{\mathcal{A}}(f) \leq n$, then $(\ker(f^*))(\tilde{H}^*(Y; R))^n = 0$, and so

$$\min\{n \mid (\ker(f^*))(\tilde{H}^*(Y; R))^n = 0\} \leq \mathcal{L}_{\mathcal{A}}(f) \leq L_{\mathcal{A}}(f).$$

The following corollary is useful for applications.

COROLLARY 7.2. If \mathcal{A} is any collection and $f : X \rightarrow Y$, then

$$\cup\text{-length}(Y) \leq \cup\text{-length}(X) + \mathcal{L}_{\mathcal{A}}(f).$$

PROOF. Let $k = \cup\text{-length}(X)$ and $n = \mathcal{L}_{\mathcal{A}}(f)$. Suppose $y_1, \dots, y_{n+k+1} \in \tilde{H}^*(Y; R)$ are any $n+k+1$ elements. We must show $y_1 \cdots y_{n+k+1} = 0$. Now $f^*(y_1 \cdots y_{k+1}) = f^*(y_1) \cdots f^*(y_{k+1}) = 0$, so $y_1 \cdots y_{k+1} \in \ker(f^*)$. On the other hand, $y_{k+2} \cdots y_{n+k+1} \in (\tilde{H}^*(Y; R))^n$. By Theorem 7.1, $y_1 \cdots y_{n+k+1} = 0$. \square

When $f : * \rightarrow Y$, Corollary 7.2 yields the well-known inequality $\cup\text{-length}(Y) \leq \text{cat}(Y)$. However, when $f : X \rightarrow *$, we obtain no information from Corollary 7.2. To obtain lower bounds for $\text{kit}_{\mathcal{A}}$ and $\text{kl}_{\mathcal{A}}$, we prove the following.

PROPOSITION 7.3. Let \mathcal{A} be a collection with the property that each $A \in \mathcal{A}$ has $\text{wcat}(A) \leq 1$. Then

$$\cup\text{-length}(X) \leq \text{wcat}(X) \leq 2^{\text{kit}_{\mathcal{A}}(X)} - 1 \leq 2^{\text{kl}_{\mathcal{A}}(X)} - 1$$

for every space X .

PROOF. The first inequality is well-known [B-G], and the last follows from $\text{kit}_{\mathcal{A}}(X) \leq \text{kl}_{\mathcal{A}}(X)$. Therefore we only need to prove $\text{wcat}(X) \leq 2^{\text{kit}_{\mathcal{A}}(X)} - 1$.

We proceed by induction on $\text{kit}_{\mathcal{A}}(X)$. If $\text{kit}_{\mathcal{A}}(X) = 0$, then X is contractible, and the result follows. Assume that the inequality holds for all spaces with $\text{kit}_{\mathcal{A}} < m$, and that $\text{kit}_{\mathcal{A}}(X) = m$. Consider the first mapping cone sequence in a minimal $\text{kit}_{\mathcal{A}}$ -decomposition of X , $L_0 \xrightarrow{j} X \xrightarrow{p} X_1$. Then there is a homotopy commutative diagram

$$\begin{array}{ccc} L_0 & \xrightarrow{\hat{\Delta}_2} & L_0 \wedge L_0 \\ j \downarrow & & \downarrow j \wedge j \\ X & \xrightarrow{\hat{\Delta}_2} & X \wedge X. \end{array}$$

Since $L_0 \in \mathcal{A}$, it follows that $\hat{\Delta}_2 \simeq 0 : L_0 \rightarrow L_0 \wedge L_0$, and so $\hat{\Delta}_2 j \simeq 0$. Thus there is a $\delta : X_1 \rightarrow X \wedge X$ such that $\delta p \simeq \hat{\Delta}_2$. Hence we have a homotopy commutative

diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{\widehat{\Delta}_2} & X \wedge X & \xrightarrow{\widehat{\Delta}_{2^{m-1}}} & (X \wedge X)^{(2^{m-1})} \\
 & \searrow p & \uparrow \delta & & \uparrow \delta^{(2^{m-1})} \\
 & & X_1 & \xrightarrow{\widehat{\Delta}_{2^{m-1}}} & X_1^{(2^{m-1})}.
 \end{array}$$

Since $\text{kit}_{\mathcal{A}}(X_1) \leq m - 1$, we have $\widehat{\Delta}_{2^{m-1}} \simeq 0 : X_1 \rightarrow X_1^{(2^{m-1})}$. Therefore $\widehat{\Delta}_{2^m} = \widehat{\Delta}_{2^{m-1}} \widehat{\Delta}_2 \simeq 0 : X \rightarrow X^{2^m}$, and so $\text{wcat}(X) \leq 2^m - 1$. \square

The result $\text{wcat}(X) \leq 2^{\text{kl}_{\mathbb{Z}}(X)} - 1$ was proved in [A-S2] by a similar method. Since $\text{wcat}(X) \leq \text{cat}(X) \leq \text{cat}_{\mathcal{A}}(X)$, it might be conjectured that $\text{cat}_{\mathcal{A}}(X)$ is bounded above by $2^{\text{kit}_{\mathcal{A}}(X)} - 1$ for any collection \mathcal{A} as in Proposition 7.3. The following example shows that this is not the case for $\mathcal{A} = \mathcal{S}$.

EXAMPLE 7.4. Let A be a 2-dimensional, noncontractible acyclic complex. Such a space may be constructed from a perfect group G , such as the one described in [Hig], which has a presentation with the same number of generators as relations. If A is obtained from a wedge of circles by attaching a 2-cell for each relation, then $\pi_1(A) = G$ and A is acyclic. For any such space A , $A \wedge A \equiv *$, but $n \leq \text{cat}(A^n) \leq \text{cat}_{\mathcal{S}}(A^n)$ for each n [G-H, Th. 1.1]. Write $B = \bigvee_n A^n$, and conclude that $\text{cat}_{\mathcal{S}}(B)$ is infinite since B dominates A^n for each n . We claim that $\text{kit}_{\mathcal{S}}(B) = 1$.

For this, it is enough to show that $\text{kit}_{\mathcal{S}}(A^n) = 1$ for each n . We do this by induction on the intermediate wedges

$$T_k(A^n) = \{(x_1, \dots, x_n) \mid \text{at least } n - k \text{ of the } x_i = *\} \subseteq A^n.$$

When $k = 1$, we obtain the usual wedge, and when $k = n - 1$ we obtain the fat wedge.

Since A is 2-dimensional there is a mapping cone sequence $\bigvee S^1 \rightarrow A \rightarrow \bigvee S^2$. Since $[A, \bigvee S^2] \cong \bigoplus H^2(A; \mathbb{Z}) = 0$, we see, using the comment following Definition 5.3, that $\text{kit}_{\mathcal{A}}(A) = 1$, and the same is true for $T_1(A^n)$ for each n . We assume $\text{kit}_{\mathcal{S}}(T_k(A^n)) = 1$.

For each k there is a mapping cone sequence

$$T_k(A^n) \rightarrow T_{k+1}(A^n) \rightarrow \bigvee A^{(k+1)} \equiv *.$$

The argument used to prove Theorem 3.4 in [A-S1] shows that $\text{kit}_{\mathcal{A}}$ is subadditive on mapping cone sequences (see also [A-S-S]). Thus $\text{kit}_{\mathcal{S}}(T_{k+1}(A^n)) \leq \text{kit}_{\mathcal{S}}(T_k(A^n)) + \text{kit}_{\mathcal{S}}(*) = 1$. This completes the example.

The space A could be constructed equally well by removing a point from a non-simply connected manifold which is a homology 3-sphere. Also, the space B is an example of a space with infinite category and weak category 1. Felix, Halperin and Thomas have constructed a simply-connected space with this property in [F-H-T, p. 432].

Finally we mention that even in spaces with small cup length, cohomology can sometimes reveal useful lower bounds. This result is stated in [C1, p. 377] and [S-T1] for Steenrod squares.

PROPOSITION 7.5. Suppose that there are cohomology operations $\theta_1, \theta_2, \dots, \theta_n$ and an element $u \in H^*(X)$ (with any coefficients) such that $\theta_1 \circ \dots \circ \theta_n(u) \neq 0$. Then $\text{kl}_S(X) > n$.

PROOF. We work by induction on $\text{kl}_S(X)$. If $\text{kl}_S(X) = 0$, then $X \equiv \bigvee S^{n\alpha}$ and clearly every cohomology operation acts trivially on $H^*(X)$. Now assume the result is known for all spaces with $\text{kl}_S < n$ and suppose $\text{kl}_S(X) = n$. Then there is a mapping cone sequence $\bigvee S^{n\alpha} \xrightarrow{i} X \xrightarrow{q} K$ with $\text{kl}_S(K) < n$. Let $u \in H^*(X)$ and consider the class $t = \theta_n(u)$. By the inductive hypothesis, $i^*(t) = \theta_n(i^*u) = 0$, so there is a class $v \in H^*(K)$ such that $q^*(v) = t$. Therefore $\theta_1 \cdots \theta_{n-1}(t) = \theta_1 \cdots \theta_{n-1}(q^*(v)) = q^*(\theta_1 \cdots \theta_{n-1}(v)) = 0$. \square

This result generalizes in two ways. Our proof works equally well for any cohomology theory and for any collection \mathcal{A} such for any $A \in \mathcal{A}$, every operation θ acts trivially on $H^*(A)$.

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