THE MITCHELL-RICHTER FILTRATION OF LOOPS ON STIEFEL MANIFOLDS STABLY SPLITS

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ABSTRACT. We prove that the Mitchell-Richter filtration of the space of loops on complex Stiefel manifolds stably splits. The result is obtained as a special case of a more general splitting theorem. Another special case is H. Miller's splitting of Stiefel manifolds. The proof uses the theory of orthogonal calculus developed by M. Weiss. The argument is inspired by an old argument of Goodwillie for a different, but related, general splitting result.

1. STATEMENT OF MAIN RESULTS

Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{U}$ be an infinite-dimensional vector space over $\mathbb{F}$ with a positive-definite inner product. Let $\mathcal{J}$ be the category of finite-dimensional vector subspaces of $\mathcal{U}$ with morphisms being linear maps respecting the inner product. We will use the letters U, V, W to denote objects of $\mathcal{J}$. Let $\mathrm{Aut}(n)$ be $\mathrm{O}(n)$ or $\mathrm{U}(n)$ if $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ respectively. For V an object of $\mathcal{J}$, let S^{nV} be the one-point compactification of $\mathbb{F}^n \otimes V$. S^{nV} is a sphere with a natural action of $\mathrm{Aut}(n)$.

Here is our main theorem:

Theorem 1.1. *Let $F : \mathcal{J} \to \mathrm{Spaces}_*$ be a continuous functor from $\mathcal{J}$ to based spaces. Suppose that there exists a filtration of F by sub-functors F_n such that $F_0(V) \equiv *$, and for all $n \geq 1$ the functor*

$$V \mapsto F_n(V)/F_{n-1}(V) := \text{ homotopy cofiber of the map } F_{n-1}(V) \to F_n(V)$$

is (up to a natural weak equivalence) of the form

$$V \mapsto (X_n \wedge S^{nV})_{h\,\mathrm{Aut}(n)} := (X_n \wedge S^{nV} \wedge \mathrm{E}\,\mathrm{Aut}(n)_+)_{\mathrm{Aut}(n)}$$

where X_n is a based space equipped with a based action of $\mathrm{Aut}(n)$. Then the filtration stably splits, i.e., there is a natural stable equivalence

$$F \simeq \bigvee_{n=1}^{\infty} F_n/F_{n-1}$$

One immediate consequence of theorem 1.1 is H. Miller's stable splitting of Stiefel manifolds [C86, Mi85]. To recall what Miller's splitting is, let V be a fixed object of $\mathcal{J}$, and consider the functor

$$W \mapsto \mathrm{Mor}(V, V \oplus W)$$

where Mor stands for the space of morphisms in $\mathcal{J}$, i.e., linear isometric inclusions. Following [C86], elements of $\mathrm{Mor}(V, V \oplus W)$ will be written as pairs (g, h) with $g \in \hom(V, V)$ and $h \in \hom(V, W)$. The Stiefel manifold is filtered by the subspaces

$$R^n(V; W) = \{(g, h) \in \mathrm{Mor}(V, V \oplus W) \mid \dim\, (\ker(g-1))^{\perp} \leq n\}$$

where $n = 0, \ldots, \dim(V)$. It is well known (see [C86, page 42], for instance) that the maps $R^{n-1}(V; W) \to R^n(V; W)$ are cofibrations for all $n = 1, \ldots, \dim(V)$ and

Received by the editors June 3, 1999.
1991 *Mathematics Subject Classification.* 55P35.
Key words and phrases. stable splitting, Stiefel manifolds, Weiss' calculus.
The author is partially supported by the NSF.

that the n-th subquotient of this filtration $R^n(V;W)/R^{n-1}(V;W)$ is homotopy equivalent (in fact, homeomorphic) to the Thom space

$$G_n(V)^{\mathrm{Ad}_n \oplus \hom(\mathbb{F}^n, W)}$$

where $G_n(V)$ is the grassmanian of n-dimensional subspaces of V, Ad_n is the adjoint representation of $\mathrm{Aut}(n)$ (considered as a vector bundle over $G_n(V)$). Clearly, another way to write this Thom space is

$$(S^{\mathrm{Ad}_n} \wedge \mathrm{Mor}(\mathbb{F}^n, V)_+ \wedge S^{nW})_{h\,\mathrm{Aut}(n)}$$

and it is easy to verify that all the identifications in sight are functorial in W. It follows that the filtration of $\mathrm{Mor}(V, V \oplus W)$ by the subspaces $R^n(V;W)$ satisfies the hypothesis of theorem 1.1 and therefore stably splits. Thus one obtains Miller's splitting ([C86, theorem 1.16]).

Another consequence of theorem 1.1 is a stable splitting of the Mitchell-Richter filtration of the space of loops on *complex* Stiefel manifolds. Until the end of the section, $\mathbb{F} = \mathbb{C}$ and $\mathcal{J}$ is the category of complex inner-product spaces. The Mitchell-Richter filtration of the space $\Omega\,\mathrm{Mor}(V, V \oplus W)$ is a filtration by subspaces $S^n(V;W)$ described in [C86, page 50] (strictly speaking, the spaces $S^n(V;W)$ do not filter $\Omega\,\mathrm{Mor}(V, V \oplus W)$, but a certain space of algebraic loops that is weakly equivalent to it if $\dim(W) > 0$. If $W = 0$ then the space $\Omega\,\mathrm{Mor}(V, V)$ is the group completion of the space of algebraic loops). The maps $S^{n-1}(V;W) \to S^n(V;W)$ are closed cofibrations, and the quotient $S^n(V;W)/S^{n-1}(V;W)$ is naturally homeomorphic to a certain Thom space ([C86, proposition 2.20]) $S_n(V)^{\hom(\mathbb{C}^n, W)}$. Here $S_n(V) \hookrightarrow G_n(V)$ is a certain subspace of the Grassmanian, and the bundle over $S_n(V)$ is the pullback of the obvious $\hom(\mathbb{C}^n, W)$-bundle over $G_n(V)$. Let $\tilde{S}_n(V)$ be the pullback of the diagram

$$S_n(V) \hookrightarrow G_n(V) \leftarrow \mathrm{Mor}(\mathbb{C}^n, V)$$

The space $\tilde{S}_n(V)$ as a free action of $\mathrm{U}(n)$, and it is clear that there is a natural homotopy equivalence

$$S_n(V)^{\hom(\mathbb{C}^n, W)} \simeq (\tilde{S}_n(V)_+ \wedge S^{nW})_{h\,\mathrm{U}(n)}$$

Again, it is quite obvious that all the constructions and identifications that we used are functorial in W. It follows that the Mitchell-Richter filtration satisfies the hypothesis of theorem 1.1 and therefore stably splits. We obtained the following theorem (originally conjectured by M. Mahowald):

Theorem 1.2. *The Mitchell-Richter filtration of* $\Omega\,\mathrm{Mor}(V, V \oplus W)$ *stably splits. Thus there is a stable equivalence*

$$\Omega\,\mathrm{Mor}(V, V \oplus W) \simeq \bigvee_{n=1}^{\infty} S^n(V;W)/S^{n-1}(V;W)$$

To the best of our knowledge, theorem 1.2 is new when $\dim(W) > 1$, and this may be the main justification for writing this note. When $\dim(W) = 1$ then there is a homeomorphism $\mathrm{Mor}(V, V \oplus W) \cong \mathrm{SU}(V \oplus W)$ and thus $\Omega\,\mathrm{Mor}(V, V \oplus W)$ can be identified with the space of loops on the special unitary group. So, in the case $\dim(W) = 1$ theorem 1.2 recovers the Mitchell-Richter splitting of $\Omega\mathrm{SU}(n)$.

2. Proof of theorem 1.1

It is well known that there exists a large family of functorial filtrations, of which the May-Milgram filtration of $\Omega^k\Sigma^k X$ is the prime example, that stably split. The most general published result in this direction is probably due to Cohen, May and Taylor [CMT78]. In fact, it turns out that all these splitting results are immediate consequences of the existence of Goodwillie's calculus of homotopy functors. This was known to Goodwillie for many years. The argument, unfortunately, remains unpublished, but it can be found in some versions of [G96].

It also turns out that theorem 1.1 follows from the existence of Weiss' orthogonal calculus [W95] in almost exactly the same way as the Cohen-May-Taylor theorem follows from Goodwillie's homotopy calculus. Our argument is, therefore, nothing but a straightforward adaptation of an old (and unpublished) argument of Goodwillie. Thus, there are no new ideas in this paper (but there is a new result, which we hope justifies writing it).

Let us recall some definitions and results from [W95].

Definition 2.1. Let F be a continuous functor from $\mathcal{J}$ to based spaces. F is polynomial of degree $\leq n$ if the natural map

$$F(V) \to \operatorname{holim}\{F(U \oplus V) \mid 0 \neq U \subset \mathbb{F}^{n+1}\}$$

is a homotopy equivalence for all V.

Note that the indexing category for the homotopy limit, i.e., the category of non-zero vector subspaces of $\mathbb{F}^{n+1}$, is a topological category, and the homotopy limit depends on this topology in an obvious way. The following proposition is obvious from the definition

Proposition 2.2. *Let $F_1 \to F_2$ be a natural transformation of two polynomial functors of degree $\leq n$. Then the homotopy fiber of this transformation is a polynomial functor of degree $\leq n$.*

Next we introduce an important class of polynomial functors.

Theorem 2.3. *Let Θ be a spectrum with an action of $\mathrm{Aut}(n)$. Define $F(V)$ by*

$$F(V) := \Omega^\infty\left(\left(\Theta \wedge S^{nV}\right)_{h\,\mathrm{Aut}(n)}\right)$$

The functor F is polynomial of degree n.

Proof. This is the content of example 5.7 in [W95]. Only the real case is considered there, but it is clear that the same analysis works in the complex case. In fact, the functor F is *homogeneous* of degree n, as will be explained below. □

In particular, if X_n is a based space with an action of $\mathrm{Aut}(n)$ then the functor $V \mapsto \Omega^\infty\Sigma^\infty\left(\left(X_n \wedge S^{nV}\right)_{h\,\mathrm{Aut}(n)}\right)$ is polynomial of degree n.

Now suppose F satisfies the hypothesis of theorem 1.1. Consider the functor $\Omega^\infty\Sigma^\infty(F_n)$. We claim that it is polynomial of degree n. In fact, for any $k \geq 0$, the functor $\Omega^\infty\Sigma^{\infty+k}(F_n)$ is polynomial of degree n. We prove it by induction on

n. For $n=0$ there is nothing to prove. Suppose it is true for $n-1$. Consider the fibration sequence of functors

$$\Omega^\infty\Sigma^{\infty+k}(F_n)\to\Omega^\infty\Sigma^{\infty+k}\left(\left(X_n\wedge S^{nV}\right)_{h\,\mathrm{Aut}(n)}\right)\to\Omega^\infty\Sigma^{\infty+k+1}(F_{n-1})$$

The last two functors are polynomial of degree $\leq n$ by the remark following theorem 2.3 and by the induction hypothesis. It follows, by proposition 2.2, that $\Omega^\infty\Sigma^{\infty+k}(F_n)$ is polynomial of degree n.

The next step is to recall that given a functor F, there exists, in some sense, a best possible approximation of F by a polynomial functor of degree n (the n-th "Taylor polynomial" of F). Given a functor F, define the functor T_nF by

$$T_nF(V)=\mathrm{holim}\{F(V\oplus U)\mid 0\neq U\subset\mathbb{F}^{n+1}\}$$

the functor T_nF comes equipped with a canonical natural transformation $F\to T_nF$. Define the functor P_nF to be the homotopy colimit

$$T_nF\to T_nT_nF\to\cdots\to T_n^kF\to\cdots$$

For a general functor F, the functor P_nF is polynomial of degree n, and it is, in some sense, the best possible approximation of F by a polynomial functor of degree n. But we will not need this. We do need the observation that if F is polynomial of degree $\leq n$ then, by the very definitions, the map $F(V)\to T_nF(V)$ (and therefore also the map $F(V)\to P_nF(V)$) is an equivalence for all V. We will also need the fact if F is as in theorem 2.3 then F is homogeneous of degree n, i.e., $P_{n-1}F\simeq *$. This is proved in [W95] in the real case, and the proof for the complex case is similar. We will also need the following proposition, whose proof is obvious (similar to the proof of proposition 2.2)

Proposition 2.4. *Let $F_1\to F_2\to F_3$ be a fibration sequence of functors. It induces fibration sequences $T_nF_1\to T_nF_2\to T_nF_3$ and $P_nF_1\to P_nF_2\to P_nF_3$.*

Now suppose again that F satisfies the hypothesis of theorem 1.1. Consider the fibration sequence of functors

$$\Omega^\infty\Sigma^\infty F_{n-1}\to\Omega^\infty\Sigma^\infty F_n\to\Omega^\infty\Sigma^\infty F_n/F_{n-1}$$

We saw that the first functor in this sequence is polynomial of degree $n-1$, the second one is polynomial of degree n, and the third one is homogeneous of degree n. Consider the following diagram

$$\begin{array}{ccccc}\Omega^\infty\Sigma^\infty F_{n-1} & \to & \Omega^\infty\Sigma^\infty F_n & \to & \Omega^\infty\Sigma^\infty F_n/F_{n-1}\\ \downarrow\simeq & & \downarrow & & \downarrow\\ P_{n-1}\Omega^\infty\Sigma^\infty F_{n-1} & \stackrel{\simeq}{\to} & P_{n-1}\Omega^\infty\Sigma^\infty F_n & \to & P_{n-1}\Omega^\infty\Sigma^\infty F_n/F_{n-1}\simeq *\end{array}$$

In this diagram, the rows are fibration sequences, the bottom right space is contractible, and the arrows marked with $\simeq$ are weak equivalences. It follows that $\Omega^\infty\Sigma^\infty F_{n-1}$ is a homotopy retract of $\Omega^\infty\Sigma^\infty F_n$. It is easy to see from the definitions that the retraction is an infinite loop map, and thus

$$\Sigma^\infty F_n\simeq\Sigma^\infty F_{n-1}\vee F_n/F_{n-1}$$

The proof of theorem 1.1 is completed by induction on n.

Presumably, it is possible to write explicit splitting maps, by writing explicit models for the functors P_iF_n, but it seems more trouble than it is worth, at this stage.

Acknowledgement: I would like to express warmest thanks to Bill Richter for many helpful emails and for his enthusiasm for the questions discussed here. Bill declined to be a co-author, but without him this paper (for better or for worse) would not have been written.

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