

TOPOLOGICAL HOCHSCHILD HOMOLOGY OF CONNECTIVE COMPLEX K -THEORY

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ABSTRACT. Let ku be the connective complex K -theory spectrum, completed at an odd prime p . We present a computation of the mod (p, v_1) homotopy algebra of the topological Hochschild homology spectrum of ku .

INTRODUCTION

Since the discovery of categories of spectra with a symmetric monoidal smash product, as for instance the \mathbb{S} -modules of [EKMM], the topological Hochschild homology spectrum $THH(A)$ of a structured ring spectrum A can be defined by translating the definition of Hochschild homology of an algebra into topology, using a now standard “Algebra – Brave New Algebra” dictionary. The algebraic origin of this definition sheds light on many features of topological Hochschild homology, and has also led to more conceptual proofs of results that were based on Bökstedt’s original definition [Bö1] of topological Hochschild homology for functors with smash products, see for instance [SVW1]. As can be expected by analogy with the algebraic situation, this definition also highlights the role that topological Hochschild (co-)homology plays in the classification of \mathbb{S} -algebra extensions. See for example [SVW2], [La] or [BJ] for applications to extensions.

The aim of this paper is to exploit the advantages of such an algebraic definition to compute the mod (p, v_1) homotopy groups of $THH(ku)$ as an algebra, which we denote by $V(1)_*THH(ku)$. Here ku is the connective complex K -theory spectrum completed at an odd prime p , with a suitable \mathbb{S} -algebra structure. Let us give a succinct description of $V(1)_*THH(ku)$, referring to Theorem 8.15 for the complete structure.

0.1. Theorem. *Let p be an odd prime. The graded \mathbb{F}_p -algebra $V(1)_*THH(ku)$ contains a class μ_2 , of degree $2p^2$, that generates a polynomial subalgebra. The quotient*

$$A_* = (V(1)_*THH(ku))/(\mu_2)$$

is finite with $4(p-1)^2$ elements. Moreover A_ has a top class in dimension $2p^2 + 2p - 2$, and is self-dual in the sense that $A_n \cong A_{2p^2+2p-2-n}$ for all $n \in \mathbb{Z}$.*

The justification for performing this computation in $V(1)$ -homotopy is that $V(1)_*THH(ku)$ is a complicated but finitely presented \mathbb{F}_p -algebra. On the other

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hand, a presentation of the mod p homotopy algebra $V(0)_*THH(ku)$ requires infinitely many generators and relations. We nevertheless evaluate the additive structure of $V(0)_*THH(ku)$ in Corollary 6.8.

A first motivation for these computations is to approach the algebraic K -theory spectrum of ku . By work of Baas, Dundas and Rognes [BDR], the spectrum $K(ku)$ is conjectured to represent a cohomology theory whose zeroth group classifies equivalence classes of virtual two-vector bundles. It is also expected that the spectrum $K(ku)$ is of chromatic complexity two, which essentially means that it is suitable for studying v_2 -periodic and v_2 -torsion phenomena in stable homotopy. Thus $K(ku)$ should represent a form of elliptic cohomology with a genuine geometric content, something which has long been wished for.

Topological Hochschild homology is the target of a trace map from algebraic K -theory, which refines over the cyclotomic trace map $\mathrm{trc} : K(ku) \rightarrow TC(ku; p)$. The topological cyclic homology spectrum $TC(ku; p)$ is a very close approximation of $K(ku)_p$, since by Dundas [Du] and Hesselholt-Madsen [HM1] it sits in a cofibre sequence

$$K(ku)_p \xrightarrow{\mathrm{trc}} TC(ku; p) \rightarrow \Sigma^{-1}H\mathbb{Z}_p.$$

The spectrum $TC(ku; p)$ is built by taking the homotopy limit of a diagram whose vertices are the fixed point of $THH(ku)$ under various closed subgroups of the circle. Thus computing $THH(ku)$ is a first step in the study of $K(ku)$ by trace maps.

A second motivation for the computations presented in this paper is to pursue the exploration of the “brave new world” of ring spectra and their arithmetic. In the classical case, arithmetic properties of a ring or of a ring extension are to a large extent reflected in algebraic K -theory or its approximations, as topological cyclic homology, topological Hochschild homology or even Hochschild homology. An important example is descent in its various forms. Étale descent has been conjectured in algebraic K -theory by Lichtenbaum and Quillen, and has been proven for various classes of rings [RW], [HM2]. For Hochschild homology, Geller and Weibel [GW] proved étale descent by showing that for an étale extension $A \hookrightarrow B$ there is an isomorphism $HH_*(B) \cong B \otimes_A HH_*(A)$. A form of tamely ramified descent for topological Hochschild homology, topological cyclic homology and algebraic K -theory of discrete valuation rings has been proven by Hesselholt and Madsen [HM2], see also [Ts].

Laying the foundations of a theory of extensions for \mathbb{S} -algebras is work in progress (see for instance [Ro1], [Ro2]), and it is not known at this point to what extent such descent results can be generalized. Of course this depends also on how the various types of extensions are defined. In fact one might want to test a definition of an étale extension, or of a tamely ramified extension of \mathbb{S} -algebras, by showing that it is reflected by descent in algebraic K -theory or topological Hochschild homology. Unfortunately very few computations are available. This paper provides an interesting example of what we expect to be tamely ramified descent.

The \mathbb{S} -algebra ku has a subalgebra ℓ , called the Adams summand. The spectrum ku splits as an \mathbb{S} -module into a sum of $p - 1$ shifted copies of ℓ , namely

$$(0.2) \quad ku \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} \ell.$$

McClure and Staffeldt computed the mod p homotopy groups of $THH(\ell)$ in [MS]. This computation was then used by Rognes and the author [AR] to further evaluate the mod (p, v_1) homotopy groups of $TC(\ell)$ and $K(\ell)$. In view of the above splitting, one could expect that similar computations should follow quite easily for ku . However, we found out that a computation of $THH(ku)$ involves some surprising new features. For example the Bökstedt spectral sequence

$$E_{*,*}^2(ku) = HH_{*,*}^{\mathbb{F}_p}(H_*(ku; \mathbb{F}_p)) \implies H_*(THH(ku); \mathbb{F}_p)$$

has higher differentials than that for ℓ (Lemma 8.5), and in computing the algebra structure of $H_*(THH(ku); \mathbb{F}_p)$ we have to deal with extra multiplicative extensions (Proposition 8.9). It turns out that the multiplicative structure of $V(1)_*THH(ku)$, given in Theorem 8.15, is quite complicated and highly non-trivial. All this reflects the well known fact that the splitting (0.2) is not multiplicative. In fact, much of the added complexity of $THH(ku)$, as compared to $THH(\ell)$, can be accounted for by speculating on the extension $\ell \rightarrow ku$. We would like to think of it as the extension defined by the relation

$$(0.3) \quad ku = \ell[u]/(u^{p-1} = v_1)$$

in commutative \mathbb{S} -algebras. This formula holds on coefficients, since the homomorphism $\ell_* \rightarrow ku_*$ is the inclusion $\mathbb{Z}_p[v_1] \hookrightarrow \mathbb{Z}_p[u]$ with $v_1 = u^{p-1}$. First we notice that the prime (v_1) in ℓ ramifies, and hence the extension $\ell \rightarrow ku$ should not qualify as étale. This is confirmed by the computations of $V(1)_*THH(\ell)$ and $V(1)_*THH(ku)$, which show that

$$THH(ku) \not\cong ku \wedge_{\ell} THH(\ell)$$

(compare with the Geller-Weibel Theorem). But if we invert v_1 in ℓ and ku we obtain the periodic Adams summand L and the periodic K -theory spectrum KU . Here the ramification has vanished so the extension $L \rightarrow KU$ should be étale. And indeed we have an equivalence

$$THH(KU)_p \simeq KU \wedge_L THH(L)_p.$$

This is a consequence of McClure and Staffeldt's computation of $THH(L)_p$, which we adapt to compute $THH(KU)_p$ in Proposition 6.12.

Returning to $\ell \rightarrow ku$ and formula (0.3), we notice that the ramification index is $(p - 1)$ and that this extension ought to be tamely ramified. The behavior of topological Hochschild homology with respect to tamely ramified extensions of discrete valuations rings was studied by Hesselholt and Madsen in [HM2]. Let us assume that their results hold also in the generality of commutative \mathbb{S} -algebra. The ring ku has a maximal ideal (u) , with residue ring $H\mathbb{Z}_p$ and quotient ring KU . Following [HM2, Theorem 1.5.6] we expect to have a localization cofibre sequence in topological Hochschild homology

$$(0.4) \quad THH(H\mathbb{Z}_p) \xrightarrow{i^!} THH(ku) \xrightarrow{j} THH(ku|KU).$$

This requires that we can identify by dévissage $THH(H\mathbb{Z}_p)$ with the topological Hochschild homology spectrum of a suitable category of finite u -torsion ku -modules. The tame ramification of $\ell \rightarrow ku$ should be reflected by an equivalence

$$(0.5) \quad ku \wedge_{\ell} THH(\ell|L) \simeq THH(ku|KU)$$

Now $THH(\ell|L)$ can be computed using the localization cofibre sequence

$$THH(H\mathbb{Z}_p) \xrightarrow{i^!} THH(\ell) \xrightarrow{j} THH(\ell|L)$$

and McClure-Staffeldt's computation of $THH(\ell)$. Thus $THH(ku|KU)$ is also known, and $THH(ku)$ can be evaluated from the cofibre sequence (0.4). We elaborate more on this in Paragraph 9.4. At this point we do not know if this conceptual line of argument can be made rigorous. This would of course require a generalization of the results in [HM2] for \mathbb{S} -algebras. But promisingly, the description of $V(1)_*THH(ku)$ it provides is perfectly compatible with our computations of it given in Theorem 8.15.

The units \mathbb{Z}_p^\times act as p -adic Adams operations on ku . Let Δ be the cyclic subgroup of order $p - 1$ of \mathbb{Z}_p^\times . Then we have a homotopy equivalence

$$\ell \simeq ku^{h\Delta}$$

where $(-)^{h\Delta}$ denotes taking the homotopy fixed points. We prove the following result as Theorem 9.2.

0.6. Theorem. *Let p be an odd prime. There are homotopy equivalences of p -completed spectra*

$$\begin{aligned} THH(ku)^{h\Delta} &\simeq THH(\ell), \\ TC(ku; p)^{h\Delta} &\simeq TC(\ell; p), \quad \text{and} \\ K(ku)^{h\Delta} &\simeq K(\ell). \end{aligned}$$

We would like to interpret this Theorem as an example of tamely ramified descent for topological Hochschild homology, topological cyclic homology and algebraic K -theory.

Let us briefly review the content of the present paper. Our strategy for computing $THH(ku)$ can be summarized as follows. Taking Postnikov sections we obtain a sequence of \mathbb{S} -algebra maps $ku \rightarrow M \rightarrow H\mathbb{Z}_p$, where M is the section $ku[0, 2p-6]$. Using naturality of topological Hochschild homology we construct a sequence

$$THH(ku) \rightarrow THH(ku, M) \rightarrow THH(ku, H\mathbb{Z}_p) \rightarrow THH(H\mathbb{Z}_p).$$

We then use this sequence to interpolate from $THH(H\mathbb{Z}_p)$ to $THH(ku)$, the point being that at each step the added complexity can be handled by essentially algebraic means. In §1 we discuss some properties of ku and compute its homology. We present in §2 the computations in Hochschild homology that will be needed later on as input for various Bökstedt spectral sequences. In §3 we review the definition of topological Hochschild homology, following [MS], [EKMM] and [SVW1], and we set up the Bökstedt spectral sequence. We present in §4 a simplified computation of the mod p homotopy groups of $THH(H\mathbb{Z}_p)$, for odd primes p . We also briefly review a computation of $V(1)_*THH(\ell)$. In §5 we determine the homotopy type of the spectrum $THH(ku, H\mathbb{Z}_p)$. Its mod p homotopy groups $V(0)_*THH(ku; H\mathbb{Z}_p)$ are the input for a Bockstein spectral sequence which is computed in §6. It yields a description of $V(0)_*THH(ku)$ as a module over $V(0)_*ku$, given in Corollary 6.8. Note that §6 is not used in the later sections. In §7, we compute the mod p homology of $THH(ku, M)$. The core of this paper is §8. Here we compute the mod

p homology Bökstedt spectral sequence for $THH(ku)$, and evaluate $V(1)_*THH(ku)$ as an algebra over $V(1)_*ku$ in Theorem 8.15. Finally, in §9 we compare $THH(\ell)$ and $THH(ku)$ and elaborate on the properties of the extension $\ell \rightarrow ku$ mentioned above.

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Notations and conventions. Throughout the paper p will be a fixed odd prime, and \mathbb{Z}_p will denote the p -adic integers. For an \mathbb{F}_p -vector space V , let $E(V)$, $P(V)$ and $\Gamma(V)$ be the exterior algebra, polynomial algebra and divided power algebra on V , respectively. If V has a basis $\{x_1, \dots, x_n\}$, we write $V = \mathbb{F}_p\{x_1, \dots, x_n\}$ and $E(x_1, \dots, x_n)$, $P(x_1, \dots, x_n)$ and $\Gamma(x_1, \dots, x_n)$ for these algebras. By definition $\Gamma(x)$ is the \mathbb{F}_p -vector space $\mathbb{F}_p\{\gamma_k x | k \geq 0\}$ with product given by $\gamma_i x \cdot \gamma_j x = \binom{i+j}{i} \gamma_{i+j} x$, where $\gamma_0 x = 1$ and $\gamma_1 x = x$. Let $P_h(x) = P(x)/(x^h = 0)$ be the truncated polynomial algebra of height h .

In the description of spectral sequences, differentials are usually given up to multiplication by a unit.

We denote the mod p Moore spectrum by $V(0)$. It has a periodic v_1 -multiplication $\Sigma^{2p-2}V(0) \rightarrow V(0)$ whose cofiber is called $V(1)$. We define the mod p homotopy groups of a spectrum X by $V(0)_*X = \pi_*(V(0) \wedge X)$, and its mod (p, v_1) homotopy groups by $V(1)_*X = \pi_*(V(1) \wedge X)$. By the symbol $X \simeq_p Y$ we mean that X and Y are weakly homotopy equivalent after p -completion. We denote by β the primary mod p homology Bockstein, by $\beta_{0,r}$ the r th mod p homotopy Bockstein, and by $\beta_{1,r}$ the r th mod v_1 homotopy Bockstein.

For a ring R , let HR be the Eilenberg-Mac Lane spectrum associated to R . If A is a (-1) -connected ring spectrum, we call the ring map $A \rightarrow H\pi_0 A$ that induces the identity on π_0 the linearization map.

1. CONNECTIVE COMPLEX K-THEORY

Let ku be the p -completed connective complex K -theory spectrum, having coefficients $ku_* = \mathbb{Z}_p[u]$ with $|u| = 2$.

An E_∞ model. Since we would like to take ku as input for topological Hochschild homology, we need to specify a structured ring spectrum structure on ku . Following [MS, Section 9], we will take as model for ku the p -completion of the algebraic K -theory spectrum of a suitable field. Let q be a topological generator of \mathbb{Z}_p^\times , and let μ_{p^∞} be the set of all p th-power roots of 1 in $\overline{\mathbb{F}}_q$. We define k to be the field extension obtained by adjoining the elements of μ_{p^∞} to \mathbb{F}_q . Hence $k = \mathbb{F}_q[\mu_{p^\infty}] = \bigcup_{i \geq 0} \mathbb{F}_{q^{p^i(p-1)}}$. Quillen [Qu] proved that the Brauer lift induces a homotopy equivalence

$$K(k)_p \xrightarrow{\simeq} ku.$$

Notice that the inclusion $k \subset \overline{\mathbb{F}}_q$ also induces an equivalence $K(k)_p \simeq K(\overline{\mathbb{F}}_q)_p$, so that we do not need to go all the way up to the algebraic closure to get a model for ku . The algebraic K -theory spectrum of a commutative ring comes equipped

with a natural structure of commutative \mathbb{S} -algebra in the sense of [EKMM], and p -completion preserves this structure. In particular the Galois group

$$\mathrm{Gal}(k/\mathbb{F}_q) \cong \mathbb{Z}_p \times \mathbb{Z}/(p-1)$$

acts on $K(k)_p$ by \mathbb{S} -algebra maps. From now on ku will stand for $K(k)_p$.

The Bott element. The mod p homotopy groups of ku are given by $V(0)_*ku = P(u)$, where u is the mod p reduction of a generator of ku_2 . We call such a class u a Bott element.

The algebraic K -theory groups of k were computed by Quillen [Qu], and are given by

$$K_n(k) = \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ k^\times & \text{for } n \text{ odd } \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We have $k^\times \cong \bigoplus_{l \text{ prime } \neq q} \mathbb{Z}/l^\infty$. The universal coefficient formula for mod p homotopy implies that we have an isomorphism

$$V(0)_2K(k) \xrightarrow{\cong} \mathrm{Tor}_1^{\mathbb{Z}}(k^\times, \mathbb{F}_p) = \mathbb{F}_p.$$

Identifying $V(0)_*ku$ with $V(0)_*K(k)$, a Bott element $u \in V(0)_2ku$ corresponds under this isomorphism to a primitive p th-root of 1 in k .

1.1. The Adams summand. Let δ be a chosen generator of Δ , the cyclic subgroup of order $p-1$ of $\mathrm{Gal}(k/\mathbb{F}_q)$. Then δ permutes the primitive p -th roots of 1 in k via a cyclic permutation of order $p-1$. In particular, if $u \in V(0)_2ku$ is a chosen Bott element, then

$$\delta_* : V(0)_*ku \rightarrow V(0)_*ku$$

maps u to αu for some generator α of \mathbb{F}_p^\times . Let k' be the subfield of k fixed under the action of Δ . Then the homotopy fixed point spectrum $ku^{h\Delta} = K(k')_p$ is a commutative \mathbb{S} -algebra model for the p -completed Adams summand ℓ , with coefficients $\ell_* = \mathbb{Z}_p[v_1]$. The spectrum ku is then a commutative ℓ -algebra. It splits as an \mathbb{S} -module into

$$ku \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} \ell.$$

In $V(0)_*ku$ we have the relation $u^{p-1} = v_1$. We would like to think of ku as the extension $\ell[u]/(u^{p-1} = v_1)$ of ℓ in commutative \mathbb{S} -algebras.

The dual Steenrod algebra. Let A_* be the dual Steenrod algebra

$$A_* = P(\xi_1, \xi_2, \dots) \otimes E(\tau_0, \tau_1, \dots)$$

where ξ_i and τ_j are the generators defined by Milnor [Mi], of degree $2p^i - 2$ and $2p^j - 1$, respectively. We denote by $\bar{\xi}_i$ and $\bar{\tau}_j$ the images of ξ_i and τ_j under the canonical involution of A_* . The coproduct ψ on A_* is given by

$$\psi(\bar{\xi}_k) = \sum \bar{\xi}_j \otimes \bar{\xi}_i^{p^j} \quad \text{and} \quad \psi(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum \bar{\tau}_j \otimes \bar{\xi}_i^{p^j}$$

where by convention $\bar{\xi}_0 = 1$.

We view the mod p homology of a spectrum X as a left A_* -comodule, i.e. $H_*(X; \mathbb{F}_p) = \pi_*(H\mathbb{F}_p \wedge X)$. In particular we write

$$H_*(H\mathbb{F}_p; \mathbb{F}_p) = P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_0, \bar{\tau}_1, \dots).$$

We will denote by ν_* the coaction $H_*(X; \mathbb{F}_p) \rightarrow A_* \otimes H_*(X; \mathbb{F}_p)$. The mod p reduction map $\rho : H\mathbb{Z}_p \rightarrow H\mathbb{F}_p$ induces an injection in mod p homology, and we identify $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ with its image in $H_*(H\mathbb{F}_p; \mathbb{F}_p)$, namely

$$(1.2) \quad H_*(H\mathbb{Z}_p; \mathbb{F}_p) = P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_1, \bar{\tau}_2, \dots).$$

The homology of ku . The linearization map $j : ku \rightarrow H\mathbb{Z}_p$ has the 1-connected cover $ku[2, \infty]$ of ku as fiber. By Bott periodicity, we can identify this cover with $\Sigma^2 ku$. We assemble the iterated suspensions of the cofiber sequence

$$\Sigma^2 ku \xrightarrow{u} ku \xrightarrow{j} H\mathbb{Z}_p$$

into a diagram

$$(1.3) \quad \begin{array}{ccccc} \cdots & \xrightarrow{u} & \Sigma^4 ku & \xrightarrow{u} & \Sigma^2 ku & \xrightarrow{u} & ku \\ & & \downarrow j & & \downarrow j & & \downarrow j \\ & & \Sigma^4 H\mathbb{Z}_p & & \Sigma^2 H\mathbb{Z}_p & & H\mathbb{Z}_p. \end{array}$$

Applying $H_*(-; \mathbb{F}_p)$ we obtain an unrolled exact couple in the sense of Boardman [Bo]. Placing $\Sigma^{2s} H\mathbb{Z}_p$ in filtration degree $-2s$, it yields a spectral sequence

$$(1.4) \quad E_{s,t}^2 = (H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes P_*(x))_{(s,t)} \implies H_{s+t}(ku; \mathbb{F}_p).$$

This is a second quadrant spectral sequence where $a \in H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ has bidegree $(0, |a|)$ and x has bidegree $(-2, 4)$ and represents the image of $u \in V(0)_2 ku$ under the Hurewicz homomorphism $V(0)_* ku \rightarrow H_*(ku; \mathbb{F}_p)$.

1.5. Theorem (Adams). *There is an isomorphism of A_* -comodule algebras*

$$H_*(ku; \mathbb{F}_p) \cong H_*(\ell; \mathbb{F}_p) \otimes P_{p-1}(x)$$

where $H_*(\ell; \mathbb{F}_p) = P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_2, \bar{\tau}_3, \dots) \subset A_*$ is a sub- A_* -comodule algebra of A_* and $P_{p-1}(x)$ is spherical (hence primitive).

Proof. The unrolled exact couple given by applying $H_*(-; \mathbb{F}_p)$ to (1.3) is part of a multiplicative Cartan-Eilenberg system. Thus the spectral sequence (1.4) is a spectral sequence of differential A_* -comodule algebras. It is also strongly convergent. The E^2 -term of this spectral sequence is

$$E_{*,*}^2 = P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_1, \bar{\tau}_2, \dots) \otimes P(x).$$

There is a differential $d^{2p-2}(\bar{\tau}_1) = x^{p-1}$ (Adams [Ad], Lemma 4), after which the spectral sequence collapses for bidegree reasons, leaving

$$E_{*,*}^\infty = E_{*,*}^{2p-1} = P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_2, \bar{\tau}_2, \dots) \otimes P_{p-1}(x).$$

There are no non-trivial multiplicative extensions. For instance $x^{p-1} = 0$ because the only other possibility would be $x^{p-1} = \bar{\xi}_1$, which would contradict the fact that x is primitive. For degree reasons there cannot be any non-trivial A_* -comodule extensions. \square

This formula for $H_*(ku; \mathbb{F}_p)$ reflects the splitting $ku \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} \ell$. The class $v_1 = u^{p-1} \in V(0)_* ku$ is of Adams filtration 1 and is in the kernel of the Hurewicz homomorphism, which accounts for the relation $x^{p-1} = 0$ in $H_*(ku; \mathbb{F}_p)$.

1.6. Lemma. *Let $\delta : ku \rightarrow ku$ be the map given in 1.1. The algebra endomorphism δ_* of $H_*(ku; \mathbb{F}_p)$ is the identity on the tensor factor $H_*(\ell; \mathbb{F}_p)$, and maps x to αx for some generator α of \mathbb{F}_p^\times .*

Proof. By definition ℓ is fixed under the action of δ , and x is the Hurewicz image of a Bott element. This implies the Lemma. \square

We will also need some knowledge of the integral homology of ku in low degrees. In the mod p homology of $H\mathbb{Z}_p$, the primary Bockstein homomorphism β is given by

$$(1.7) \quad \beta : \bar{\tau}_i \mapsto \bar{\xi}_i \quad \text{for all } i \geq 1.$$

In particular $H_0(H\mathbb{Z}_p; \mathbb{Z}) = \mathbb{Z}_p$ and $pH_n(H\mathbb{Z}_p; \mathbb{Z}) = 0$ for any $n \geq 1$.

1.8. Proposition. *Consider the commutative graded algebra Λ_* over \mathbb{Z}_p defined as*

$$\Lambda_* = \mathbb{Z}_p[\tilde{x}, \tilde{\xi}_1]/(\tilde{x}^{p-1} = p\tilde{\xi}_1)$$

where \tilde{x} has degree 2 and $\tilde{\xi}_1$ has degree $2p - 2$. There exists a homomorphism of \mathbb{Z}_p -algebras

$$\lambda : \Lambda_* \rightarrow H_*(ku; \mathbb{Z})$$

such that λ is an isomorphism in degrees $\leq 2p^2 - 3$, and such that the composition

$$\Lambda_* \xrightarrow{\lambda} H_*(ku; \mathbb{Z}) \xrightarrow{\rho} H_*(ku; \mathbb{F}_p),$$

where ρ is the mod p reduction, maps \tilde{x} to x and $\tilde{\xi}_1$ to $\bar{\xi}_1$.

Proof. By comparison with the case of $H\mathbb{Z}_p$, we have primary mod p Bocksteins $\beta(\bar{\tau}_i) = \bar{\xi}_i$ in $H_*(ku; \mathbb{F}_p)$, for all $i \geq 2$. We also have $\beta(x) = 0$ for degree reasons. Hence the Bockstein spectral sequence

$$E_*^1 = H_*(ku; \mathbb{F}_p) \implies (H_*(ku; \mathbb{Z})/\text{torsion}) \otimes \mathbb{F}_p,$$

whose first differential is β , collapses at the E^2 -term, leaving

$$E_*^\infty = E_*^2 = P_{p-1}(x) \otimes P(\bar{\xi}_1).$$

Let $\tilde{x} \in H_2(ku; \mathbb{Z})$ be a lift of x and $\tilde{\xi}_1 \in H_{2p-2}(ku; \mathbb{Z})$ be a lift of $\bar{\xi}_1$. Since $H_*(ku, \mathbb{Q})$ is polynomial over \mathbb{Q}_p on one generator in degree 2, there is a multiplicative relation $\tilde{x}^{p-1} = a\tilde{\xi}_1$ in $H_*(ku; \mathbb{Z})$, for some $a \in p\mathbb{Z}_p$. The Postnikov invariant $ku[0, 2p-4] \rightarrow \Sigma^{2p-1}H\mathbb{Z}_p$ of ku is of order p , so we can choose $\tilde{\xi}_1$ such that $a = p$. We hence obtain the ring homomorphism λ . There is no torsion class in $H_*(ku; \mathbb{Z})$ of degree $\leq 2p^2 - 3$, and λ is an isomorphism in these degrees. \square

2. HOCHSCHILD HOMOLOGY

In this section we recall some properties of Hochschild homology and present elementary computations that will be needed in the later sections.

Suppose R is a graded commutative and unital ring, A a graded unital R -algebra, and M a graded A -bimodule. Let us simply write $\hat{\otimes}$ for $\hat{\otimes}_R$. The enveloping algebra

A^e of A is the graded and unital R -algebra $A \otimes A^{\text{op}}$, and it acts on A on the left and on M on the right in the usual way.

If A is flat over R , the Hochschild homology of A with coefficients in M is defined as the bigraded R -module

$$HH_{s,t}^R(A, M) = \text{Tor}_{s,t}^{A^e}(M, A).$$

Here s is the homological degree and t is the internal degree. The (two-sided) bar complex $C_*^{\text{bar}}(A)$, with $C_n^{\text{bar}}(A) = A^{\otimes(n+2)}$, is a standard resolution of A as left A^e -module, having the product $\mu : C_0^{\text{bar}}(A) = A \otimes A \rightarrow A$ of A as augmentation. The complex $M \otimes_{A^e} C_*^{\text{bar}}(A)$ is isomorphic to the Hochschild complex $C_*(A, M)$, with $C_n(A, M) = M \otimes A^{\otimes n}$, see [Lo, Chapter 1].

Suppose now that A is graded-commutative and that M is a commutative and unital A -algebra with A -bimodule structure given by forgetting part of the A -algebra structure. The standard product and coproduct on the bar resolution make the Hochschild complex into a graded differential M -bialgebra with unit and augmentation. In particular $HH_{*,*}^R(A, M)$ is a bigraded unital M -algebra, and if $HH_{*,*}^R(A, M)$ is flat over M , then $HH_{*,*}^R(A, M)$ is a bigraded unital and augmented M -bialgebra. The unit

$$\iota : M \xrightarrow{\cong} HH_{0,*}^R(A, M)$$

is given by the inclusion of the 0-cycles $M = C_0(A, M)$, and the augmentation is the projection $HH_{*,*}^R(A, M) \rightarrow HH_{0,*}^R(A, M) \cong M$. There is also an R -linear homomorphism

$$(2.1) \quad \sigma : A \rightarrow HH_{1,*}^R(A, M)$$

induced by $A \rightarrow C_1(A, M) = M \otimes A$, $a \mapsto 1 \otimes a$. It satisfies the derivation rule

$$\sigma(ab) = e(a)\sigma(b) + (-1)^{|a||b|}e(b)\sigma(a),$$

where $e : A \rightarrow M$ is the unit of M .

As usual, we write $HH_{*,*}^R(A)$ for $HH_{*,*}^R(A, A)$ and $C_*(A)$ for $C_*(A, A)$.

2.2. Proposition. *Suppose that R consists of \mathbb{F}_p concentrated in degree 0.*

(a) *Let $P(x)$ be the polynomial \mathbb{F}_p -algebra generated by x of even degree d . Then there is an isomorphism of $P(x)$ -bialgebras*

$$HH_{*,*}^{\mathbb{F}_p}(P(x)) \cong P(x) \otimes E(\sigma x)$$

with σx primitive of bidegree $(1, d)$.

(b) *Let $E(x)$ be the exterior \mathbb{F}_p -algebra generated by x of odd degree d . Then there is an isomorphism of $E(x)$ -bialgebras*

$$HH_{*,*}^{\mathbb{F}_p}(E(x)) \cong E(x) \otimes \Gamma(\sigma x)$$

with σx of bidegree $(1, d)$ and with coproduct given by

$$\Delta(\gamma_k \sigma x) = \sum_{i+j=k} \gamma_i \sigma x \otimes_{E(x)} \gamma_j \sigma x.$$

Proof. This is standard, see for instance [MS, Proposition 2.1]. \square

2.3. Proposition. *Suppose that R consists of \mathbb{F}_p concentrated in degree 0. Let $P_h(x)$ be the truncated polynomial \mathbb{F}_p -algebra of height h generated by x of even degree d , with $(p, h) = 1$. Then there is an isomorphism of $P_h(x)$ -algebras*

$$HH_{*,*}^{\mathbb{F}_p}(P_h(x)) \cong P_h(x)[z_i, y_j \mid i \geq 0, j \geq 1] / \sim$$

where z_i has bidegree $(2i+1, ihd+d)$ and y_j has bidegree $(2j, jhd+d)$. The relation \sim is generated by

$$\begin{cases} x^{h-1}z_i = x^{h-1}y_j = z_i z_k = 0 \\ z_i y_j = \binom{i+j}{i} x z_{i+j} \\ y_j y_g = \binom{j+g}{j} x y_{j+g} \end{cases}$$

for all $i, k \geq 0$ and all $j, g \geq 1$. Moreover $z_0 = \sigma x$, the generator z_i is represented in the Hochschild complex $C_*(P_h(x))$ by

$$\sum_{\substack{k_1, \dots, k_{i+1} \geq 0 \\ k_1 + \dots + k_{i+1} = i(h-1)}} (x^{k_1} \otimes x \otimes x^{k_2} \otimes x \otimes x^{k_3} \otimes x \otimes \dots \otimes x \otimes x^{k_{i+1}} \otimes x)$$

for all $i \geq 1$, and the generator y_j is represented by

$$\sum_{\substack{k_1, \dots, k_{j+1} \geq 0 \\ k_1 + \dots + k_{j+1} = j(h-1)}} (x^{k_1+1} \otimes x^{k_2} \otimes x \otimes x^{k_3} \otimes x \otimes \dots \otimes x \otimes x^{k_{j+1}} \otimes x)$$

for all $j \geq 1$.

Proof. Let $A = P_h(x)$. This proposition is proven by choosing a small differential bigraded algebra over A^e that is a projective resolution of A as left A^e -module. For example, one can take

$$X_{*,*} = A^e \otimes E(\sigma x) \otimes \Gamma(\tau),$$

where $a \in A^e$ has bidegree $(0, |a|)$, σx has bidegree $(1, d)$ and τ has bidegree $(2, dh)$. The differential d of X , of bidegree $(-1, 0)$, is given on the generators by

$$d(\sigma x) = T \quad \text{and} \quad d(\tau) = N\sigma x,$$

where $T = x \otimes 1 - 1 \otimes x \in A^e$ and $N = (x^h \otimes 1 - 1 \otimes x^h)/(x \otimes 1 - 1 \otimes x) \in A^e$. The product of A gives an augmentation $X_0 = A^e \rightarrow A$. Now $HH_{*,*}^{\mathbb{F}_p}(P_h(x))$ is isomorphic to the homology of the differential graded algebra

$$A \otimes_{A^e} X_{*,*} \cong A \otimes E(\sigma x) \otimes \Gamma(\tau)$$

with differential given by $d(\sigma x) = 0$ and $d(\tau) = hx^{h-1}\sigma x$. The class z_i is represented by the cycle $\sigma x \cdot \gamma_i \tau$ and the class y_j is represented by the cycle $x \cdot \gamma_j \tau$, for any $i \geq 0$ and $j \geq 1$. Representatives for the generators are obtained by choosing a homotopy equivalence $X_* \rightarrow C_*^{\text{bar}}(A)$, see for instance [BAC, page 55]. \square

2.4. Remark. Notice that $HH_{*,*}^{\mathbb{F}_p}(P_h(x))$ is not flat over $P_h(x)$, so that there is no coproduct in this case. Moreover there are infinitely many algebra generators. However, the set $\{z_i, y_j \mid i \geq 0, j \geq 1\}$ of given algebra generators is also a set of

$P_h(x)$ -module generators for $HH_{*,*}^{\mathbb{F}_p}(P_h(x))$. More precisely, $HH_{*,*}^{\mathbb{F}_p}(P_h(x))$ has one $P_h(x)$ -module generator in each non-negative homological degree, and is given by

$$HH_{n,*}^{\mathbb{F}_p}(P_h(x)) = \begin{cases} P_h(x) & n = 0, \\ P_{h-1}(x)\{z_{\frac{n-1}{2}}\} & n \geq 1 \text{ odd}, \\ P_{h-1}(x)\{y_{\frac{n}{2}}\} & n \geq 2 \text{ even}. \end{cases}$$

Let $B_n : HH_{n,*}^{\mathbb{F}_p}(P_h(x)) \rightarrow HH_{n+1,*}^{\mathbb{F}_p}(P_h(x))$ be Connes' operator (B_0 coincides with the operator σ given above). Then we have

$$B_{2n}(y_n) = z_n$$

up to a unit in \mathbb{F}_p , for all $i \geq 0$, where by convention we set $y_0 = x$ (see [BAC, Proposition 2.1]).

The next proposition shows that if we take Hochschild homology of $P_h(x)$ with coefficients having a lower truncation, we have both flatness and finite generation as an algebra.

2.5. Proposition. *Suppose that R consists of \mathbb{F}_p concentrated in degree 0. Let $P_h(x)$ be as above, and for $1 \leq g < h$, let $P_h(x) \rightarrow P_g(x)$ be the quotient by (x^g) . We view $P_g(x)$ as a $P_h(x)$ -algebra. Then there is an isomorphism of $P_g(x)$ -bialgebras*

$$HH_{*,*}^{\mathbb{F}_p}(P_h(x), P_g(x)) \cong P_g(x) \otimes E(\sigma x) \otimes \Gamma(y)$$

where σx has bidegree $(1, d)$ and y has bidegree $(2, dh)$ and is represented in the Hochschild complex $C_*(P_h(x), P_g(x))$ by

$$\sum_{i=0}^{g-1} x^i \otimes x^{h-i-1} \otimes x.$$

The class σx is primitive and

$$\Delta(\gamma_k y) = \sum_{i+j=k} \gamma_i y \otimes_{P_g(x)} \gamma_j y.$$

Proof. The proof is similar to that of Proposition 2.3, using the same resolution $X_{*,*}$ of $P_h(x)$. The differential algebra $X_{*,*}$ admits a coproduct, defined as follows. The class σx is primitive and the coproduct on $\Gamma(\tau)$ is given by

$$\Delta(\gamma_k \tau) = \sum_{i+j=k} \gamma_i \tau \otimes_{A^e} \gamma_j \tau.$$

By inspection this coproduct on $X_{*,*}$ is compatible with that on $C_*^{\text{bar}} P_h(x)$ under a suitable homotopy equivalence. \square

2.6. Proposition. *Consider $P(u, u^{-1})$ as an algebra over $P(v, v^{-1})$ with $v = u^h$ for some h prime to p . Then the unit*

$$P(u, u^{-1}) \rightarrow HH_{*,*}^{P(v, v^{-1})}(P(u, u^{-1}))$$

is an isomorphism.

Proof. Let $A = P(u, u^{-1})$. We have that $A = P(v, v^{-1})[u]/(u^h = v)$ is flat as $P(v, v^{-1})$ -module. The enveloping algebra is

$$A^e = P(v, v^{-1})[1 \otimes u, u \otimes 1]/((1 \otimes u)^h = (u \otimes 1)^h = v).$$

There is a two-periodic resolution of A as A^e -module

$$\cdots \xrightarrow{T} A^e \xrightarrow{N} A^e \xrightarrow{T} A^e,$$

with augmentation $A^e \rightarrow A$ given by the product of A . Here T is multiplication by $1 \otimes u - u \otimes 1$ and N is multiplication by $((1 \otimes u)^h - (u \otimes 1)^h)/(1 \otimes u - u \otimes 1)$. Applying $A \otimes_{A^e} -$ we obtain a two-periodic chain complex

$$\cdots \xrightarrow{0} A \xrightarrow{hu^{h-1}} A \xrightarrow{0} A$$

quasi-isomorphic to the Hochschild complex. Since hu^{h-1} is invertible in A the proposition follows. \square

2.7. Remark. Notice that the requirement that $hv^{\frac{h-1}{h}}$ be invertible in $P(v, v^{-1})[v^{\frac{1}{h}}]$ is equivalent to the requirement that the extension $P(v, v^{-1}) \hookrightarrow P(v, v^{-1})[v^{\frac{1}{h}}]$ be étale.

3. TOPOLOGICAL HOCHSCHILD HOMOLOGY

The category of \mathbb{S} -modules (in the sense of [EKMM]) has a symmetric monoidal smash product. Suppose that R is a unital and commutative \mathbb{S} -algebra, that A is a unital R -algebra, and that M is an A -bimodule. We denote the symmetric monoidal smash-product in the category of R -modules by \wedge_R , or simply by \wedge if $R = \mathbb{S}$. We implicitly assume in the sequel that the necessary cofibrancy conditions are satisfied. Following [MS], [EKMM], [SVW1] we define the topological Hochschild homology spectrum of the R -algebra A with coefficients in M as the realization of the simplicial spectrum $THH_{\bullet}^R(A, M)$ whose spectrum of q -simplices is

$$THH_q^R(A, M) = M \wedge_R A^{\wedge q},$$

with the usual Hochschild-type face and degeneracy maps. If $R = \mathbb{S}$ we just write THH for $THH^{\mathbb{S}}$.

From now on, we assume furthermore that A is commutative, that M is a commutative and unital A -algebra, whose A -bimodule structure is induced by the unit $e : A \rightarrow M$. Then $THH^R(A, M)$ is a unital and commutative M -algebra. The unit

$$e : M \rightarrow THH^R(A, M)$$

is given by inclusion of the 0-simplices $M = THH_0^R(A, M)$. The level-wise products $THH_q^R(A, M) \rightarrow M$ assemble into an augmentation

$$THH^R(A, M) \rightarrow M.$$

In particular M splits off from $THH^R(A, M)$.

This construction of $THH^R(A, M)$ is functorial in M . Moreover, if $L \rightarrow M \rightarrow N$ is a cofibration of A -bimodules, then there is a cofiber sequence of R -modules

$$THH^R(A, L) \rightarrow THH^R(A, M) \rightarrow THH^R(A, N).$$

We also have functoriality in A : if $A \rightarrow B \rightarrow M$ are maps of commutative R -algebras inducing the A - and B -bimodule structures on M , then there is a map of M -algebras

$$THH^R(A, M) \rightarrow THH^R(B, M).$$

The M -algebra $THH^R(A, M)$ has, in the derived category, the structure of an M -bialgebra. To construct the coproduct

$$THH^R(A, M) \rightarrow THH^R(A, M) \wedge_M THH^R(A, M)$$

in the derived category, one can use the weak equivalence $THH^R(A, M) \simeq M \wedge_{A^e} B(A)$, where $B(A)$ is the two-sided bar construction, and take advantage of the coproduct $B(A) \rightarrow B(A) \wedge_{A^e} B(A)$ defined in the homotopy category (see [MSV], [AnR]).

The Bökstedt spectral sequence. Let E be commutative \mathbb{S} -algebra, for instance $E = H\mathbb{F}_p$ or $H\mathbb{Z}_p$. Alternatively, one can take E such that E_*A is flat over E_*R . The skeletal filtration of $THH^R(A, M)$ induces in E_* -homology a conditionally convergent spectral sequence [EKMM, Th. 6.2 and 6.4]

$$(3.1) \quad E_{p,q}^2(A, M) = \mathrm{Tor}_{p,q}^{E_*(A^e)}(E_*A, E_*M) \implies E_{p+q} THH^R(A, M).$$

We call this spectral sequence the Bökstedt spectral sequence. If E_*A is flat over E_*R , we can identify the E^2 -term of this spectral sequence with

$$E_{p,q}^2(A, M) = HH_{p,q}^{E_*R}(E_*A, E_*M).$$

In good cases, the rich structure of $THH^R(A, M)$ carries over to the Bökstedt spectral sequence. Indeed, the unit, augmentation and product of $THH^R(A, M)$ are compatible with the skeletal filtration. In particular, it is a spectral sequence of differential unital and augmented E_*M -algebras. If moreover $E_{*,*}^r(A, M)$ is flat over E_*M for all r , it is also a spectral sequence of differential E_*M -bialgebras. On the E^2 -term of the Bökstedt spectral sequence, these structures coincide with the corresponding structures for Hochschild homology described in the previous section. See [AnR] for a detailed discussion of the Hopf-algebra structure on topological Hochschild homology and on the Bökstedt spectral sequence.

Finally, if E_*E is flat over E_* , then E_*A and E_*M are E_*E -comodule algebras and the Bökstedt spectral sequence is one of differential E_*E -comodules.

The map σ . There is a map $\omega : S^1_+ \wedge A \rightarrow THH^R(A)$ defined in [MS, Proposition 3.2], and which is induced by the S^1 -action on the 0-simplices. It splits in the homotopy category as the sum of the unit and a map $\sigma : \Sigma A \rightarrow THH^R(A)$. Composing the latter with the map $THH^R(A) \rightarrow THH^R(A, M)$ induced by the unit $e : A \rightarrow M$, we obtain a map

$$(3.2) \quad \sigma : \Sigma A \rightarrow THH^R(A, M).$$

Let us assume E_*A is flat over E_*R . The interplay between the homomorphisms $\sigma_* : E_*A \rightarrow E_{*+1}THH^R(A, M)$ and $\sigma : E_*A \rightarrow HH_{1,*}^{E_*R}(E_*A, E_*M)$ given above is described in the following proposition. Let us first specify a notation.

3.3. Notation. We denote by $[w]$ the class in $E_*THH^R(A, M)$ represented by w in the E^2 -term of the Bökstedt spectral sequence.

3.4. Proposition (McClure-Staffeldt). *For any $a \in E_*A$ we have $\sigma_*(a) = [\sigma a]$ in $E_{*+1}THH^R(A, M)$.*

Proof. This is Proposition 3.2 of [MS]. \square

In the case $R = \mathbb{S}$, the map σ has another useful feature : it commutes with the Dyer-Lashof operations. Let us denote by Q^i the Dyer-Lashof operation of degree $2i(p-1)$ on the mod p homology of a commutative \mathbb{S} -algebra.

3.5. Proposition (Bökstedt). *For any $a \in H_*(A; \mathbb{F}_p)$ we have*

$$Q^i \sigma_*(a) = \sigma_*(Q^i a)$$

in $H_{+2i(p-1)+1}(THH(A); \mathbb{F}_p)$.*

Proof. Bökstedt gives a proof of this proposition in [Bö2, Lemma 2.9]. His approach is to analyze the p th-reduced power of the map $S^1_+ \wedge A \rightarrow THH(A)$. Another elegant proof is presented by Angeltveit and Rognes in [AnR]. \square

4. TOPOLOGICAL HOCHSCHILD HOMOLOGY OF \mathbb{Z}_p

In a very influential but unpublished paper, Bökstedt [Bö2] computed the homotopy type of the $H\mathbb{Z}$ -module $THH(H\mathbb{Z})$. In this section we present a simplified computation of $V(0)_*THH(H\mathbb{Z}_p)$ for $p \geq 3$, since we will need this result in the sequel.

We start by computing the Bökstedt spectral sequence

$$(4.1) \quad E_{s,t}^2(H\mathbb{Z}_p) = HH_{s,t}^{\mathbb{F}_p}(H_*(H\mathbb{Z}_p; \mathbb{F}_p)) \implies H_{s+t}(THH(H\mathbb{Z}_p); \mathbb{F}_p).$$

The description of $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ given in (1.2) and Proposition 2.2 imply that the E^2 -term of this spectral sequence is

$$H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots) \otimes \Gamma(\sigma\bar{\tau}_1, \sigma\bar{\tau}_2, \dots)$$

where $a \in H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ has bidegree $(0, |a|)$, the class $\sigma\bar{\xi}_i$ has bidegree $(1, 2p^i - 2)$ and $\sigma\bar{\tau}_j$ has bidegree $(1, 2p^j - 1)$, for $i, j \geq 1$.

4.2. Lemma. *There are multiplicative relations*

$$[\sigma\bar{\tau}_i]^p = [\sigma\bar{\tau}_{i+1}]$$

in $H_(THH(H\mathbb{Z}_p); \mathbb{F}_p)$, for all $i \geq 1$.*

Proof. This follows from the Dyer-Lashof operations $Q^{p^i}\bar{\tau}_i = \bar{\tau}_{i+1}$. By Propositions 3.4 and 3.5 we have

$$[\sigma\bar{\tau}_i]^p = \sigma_*(\bar{\tau}_i)^p = Q^{p^i}\sigma_*(\bar{\tau}_i) = \sigma_*(Q^{p^i}\bar{\tau}_i) = \sigma_*(\bar{\tau}_{i+1}) = [\sigma\bar{\tau}_{i+1}]. \quad \square$$

4.3. Lemma. *In the spectral sequence (4.1) we have $d^r = 0$ for $2 \leq r \leq p-2$, and there are differentials*

$$d^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_i) = \sigma\bar{\xi}_{i+1} \cdot \gamma_k\sigma\bar{\tau}_i$$

for all $i \geq 1$ and $k \geq 0$. Taking into account the algebra structure, this leaves

$$E_{*,*}^p(H\mathbb{Z}_p) = H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1) \otimes P_p(\sigma\bar{\tau}_1, \sigma\bar{\tau}_2, \dots).$$

At this stage the spectral sequence collapses.

Proof. The E^2 -term of the spectral sequence (4.1) is flat over $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$, so this is a spectral sequence of unital augmented differential $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ -bialgebras, at least until a differential puts an end to flatness. The mod p primary Bockstein β in the mod p homology of a ring spectrum is a derivation. From the relation

$$[\sigma\bar{\xi}_i] = \sigma_*(\bar{\xi}_i) = \sigma_*(\beta(\bar{\tau}_i)) = \beta(\sigma_*(\bar{\tau}_i)) = \beta([\sigma\bar{\tau}_{i-1}]^p) = 0$$

in $H_*(THH(H\mathbb{Z}_p); \mathbb{F}_p)$ we know that each class $\sigma\bar{\xi}_i$ for $i \geq 2$ must be hit by a differential. The rich algebra structure of the spectral sequence now only leaves enough freedom for the claimed pattern of differentials. To see this, one can for example work dually. The $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ -dual of $E_{*,*}^2(H\mathbb{Z}_p)$ is given by

$$(E_{*,*}^2(H\mathbb{Z}_p))^{\#} = H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1^{\#}, \sigma\bar{\xi}_2^{\#}, \dots) \otimes P(\sigma\bar{\tau}_1^{\#}, \sigma\bar{\tau}_2^{\#}, \dots).$$

Here we used that the \mathbb{F}_p -module generators of $E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots) \otimes \Gamma(\sigma\bar{\tau}_1, \sigma\bar{\tau}_2, \dots)$ form a basis of the free module $E_{*,*}^2(H\mathbb{Z}_p)$ over $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$, and that the dual basis is generated, under the product, by the duals $\sigma\bar{\xi}_i^{\#}$ and $\sigma\bar{\tau}_j^{\#}$ of $\sigma\bar{\xi}_i$ and $\sigma\bar{\tau}_j$, respectively. By induction on $i \geq 1$ one checks that the vanishing of $\sigma\bar{\xi}_{i+1}^{\#}$ in $(E_{*,*}^{\infty}(H\mathbb{Z}_p))^{\#}$ must be accounted for by a differential

$$(d^{p-1})^{\#}(\sigma\bar{\xi}_{i+1}^{\#}) = (\sigma\bar{\tau}_i^{\#})^p.$$

In particular we also have differentials

$$(d^{p-1})^{\#}(\sigma\bar{\xi}_{i+1}^{\#} \cdot (\sigma\bar{\tau}_i^{\#})^k) = (\sigma\bar{\tau}_i^{\#})^{p+k}$$

for all $k \geq 0$. This leaves

$$(E_{*,*}^p(H\mathbb{Z}_p))^{\#} = H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1^{\#}) \otimes P_p(\sigma\bar{\tau}_1^{\#}, \sigma\bar{\tau}_2^{\#}, \dots).$$

At this stage the spectral sequence collapses for bidegree reasons. Dualizing again, we get the Lemma. \square

4.4. Remark. Bökstedt's original argument to prove this Lemma relies on a Kudo-type formula for differentials in the spectral sequence

$$HH_{s,t}^{\mathbb{F}_p}(H_*(A; \mathbb{F}_p)) \implies H_*(THH(A); \mathbb{F}_p),$$

namely

$$d^{p-1}(\gamma_{p+k}\sigma x) = \sigma(\beta Q^{\frac{n+1}{2}} x) \cdot \gamma_k\sigma x$$

whenever $x \in H_*(A; \mathbb{F}_p)$ is a class of odd degree n . See [Pö] or [Hu].

4.5. Proposition. *There is an isomorphism of A_* -comodule algebras*

$$H_*(THH(H\mathbb{Z}_p); \mathbb{F}_p) \cong H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E([\sigma\bar{\xi}_1]) \otimes P([\sigma\bar{\tau}_1]).$$

The A_* -coaction ν_* is given on the tensor factor $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ by the inclusion in the coalgebra A_* . The class $[\sigma\bar{\xi}_1]$ is primitive and

$$\nu_*([\sigma\bar{\tau}_1]) = 1 \otimes \sigma\bar{\tau}_1 + \bar{\tau}_0 \otimes [\sigma\bar{\xi}_1].$$

Proof. By Lemma 4.3 the E^∞ -term of the spectral sequence (4.1) is

$$E_{*,*}^\infty(H\mathbb{Z}_p) = H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1) \otimes P_p(\sigma\bar{\tau}_1, \sigma\bar{\tau}_2, \dots).$$

Lemma 4.2 implies that the subalgebra $P_p(\sigma\bar{\tau}_1, \sigma\bar{\tau}_2, \dots)$ of $E_{*,*}^\infty(H\mathbb{Z}_p)$ lifts as a subalgebra $P([\sigma\bar{\tau}_1])$ of $H_*(THH(H\mathbb{Z}_p); \mathbb{F}_p)$. There are no further possible multiplicative extensions. The A_* -coaction on the tensor factor $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ is determined by naturality with respect to the unit map $H\mathbb{Z}_p \rightarrow THH(H\mathbb{Z}_p)$. The values of $\nu_*(\sigma\bar{\xi}_1)$ and $\nu_*(\sigma\bar{\tau}_1)$ follow by naturality with respect to $\sigma : \Sigma H\mathbb{Z}_p \rightarrow THH(H\mathbb{Z}_p)$, which is expressed in the formula

$$(4.6) \quad \nu_*\sigma_* = (1 \otimes \sigma_*)\nu_*.$$

We use that in $A_* \otimes H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ we have

$$\nu_*(\bar{\xi}_1) = \bar{\xi}_1 \otimes 1 + 1 \otimes \bar{\xi}_1 \quad \text{and} \quad \nu_*(\bar{\tau}_1) = \bar{\tau}_1 \otimes 1 + 1 \otimes \bar{\tau}_1 + \bar{\tau}_0 \otimes \bar{\xi}_1,$$

and that σ_* is a derivation. \square

4.7. Theorem (Bökstedt). *For any prime $p \geq 3$ there is an isomorphism of \mathbb{F}_p -algebras*

$$V(0)_*THH(H\mathbb{Z}_p) \cong E(\lambda_1) \otimes P(\mu_1),$$

where $|\lambda_1| = 2p - 1$ and $|\mu_1| = 2p$.

Proof. The proof we give here is adapted from the proof of [AR, Proposition 2.6]. Since $H\mathbb{F}_p \simeq V(0) \wedge H\mathbb{Z}_p$, the spectrum $V(0) \wedge THH(H\mathbb{Z}_p)$ is an $H\mathbb{F}_p$ -module. In particular the Hurewicz homomorphism

$$V(0)_*THH(H\mathbb{Z}_p) \rightarrow H_*(V(0) \wedge THH(H\mathbb{Z}_p); \mathbb{F}_p)$$

is an injection with image the A_* -comodule primitives. Let λ_1 and μ_1 be classes that map respectively to $[\sigma\bar{\xi}_1]$ and $[\sigma\bar{\tau}_1] - \bar{\tau}_0[\sigma\bar{\xi}_1]$ under this homomorphism. By inspection these classes generate the subalgebra of A_* -comodule primitive elements in $H_*(V(0) \wedge THH(H\mathbb{Z}_p); \mathbb{F}_p)$. \square

4.8. Remark. Bökstedt proved also that there are higher mod p homotopy Bocksteins

$$\beta_{0,r}(\mu_1^{p^r}) = \mu_1^{p^r-1} \lambda_1$$

in $V(0)_*THH(H\mathbb{Z}_p)$, for all $r \geq 1$. This implies a homotopy equivalence of $H\mathbb{Z}_p$ -modules

$$THH(H\mathbb{Z}_p) \simeq_p H\mathbb{Z}_p \vee \bigvee_{k \geq 1} \Sigma^{2k-1} H\mathbb{Z}/p^{v_p(k)},$$

where v_p is the p -adic valuation.

Let ℓ be the Adams summand defined in 1.1. A very similar computation can be performed for $THH(\ell)$, yielding a description of the \mathbb{F}_p -algebra $V(1)_*THH(\ell)$

4.9. Theorem (McClure-Staffeldt). *For any prime $p \geq 5$ there are isomorphisms of \mathbb{F}_p -algebras*

$$H_*(THH(\ell); \mathbb{F}_p) \cong H_*(\ell; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P(\sigma\bar{\tau}_2)$$

and

$$V(1)_*THH(\ell) \cong E(\lambda_1, \lambda_2) \otimes P(\mu_2),$$

where $|\lambda_1| = 2p - 1$, $|\lambda_2| = 2p^2 - 1$, and $|\mu_2| = 2p^2$.

Proof. See [MS, Corollary 7.2] for a computation of $V(0)_*THH(\ell)$. The description of $V(1)_*THH(\ell)$ given here was made explicit in [AR, Proposition 2.6]. We briefly review this computation. The linearization map $\ell \rightarrow H\mathbb{Z}_p$ is injective in mod p homology and induces an injection on the E^2 -terms of the respective Bökstedt spectral sequences. By comparison this determines both the differentials and the multiplicative extensions in the Bökstedt spectral sequence for ℓ , and the description of $H_*(THH(\ell); \mathbb{F}_p)$ given follows. Now there is an equivalence $V(1) \wedge \ell \simeq \mathbb{F}_p$, so $V(1) \wedge THH(\ell)$ is an $H\mathbb{F}_p$ -module and the Hurewicz homomorphism

$$V(1)_*THH(\ell) \rightarrow H_*(V(1) \wedge THH(\ell); \mathbb{F}_p)$$

is injective with image the A_* -comodule primitives. The homotopy classes λ_1 , λ_2 and μ_2 have as image the primitive homology classes $[\sigma\bar{\xi}_1]$, $[\sigma\bar{\xi}_2]$ and $[\sigma\bar{\tau}_2] - \bar{\tau}_0[\sigma\bar{\xi}_2]$. \square

5. THE HOMOTOPY TYPE OF $THH(ku, H\mathbb{Z}_p)$

The linearization map $j : ku \rightarrow H\mathbb{Z}_p$ makes $H\mathbb{Z}_p$ into a commutative and unital ku -algebra. Our aim in this section is to determine the homotopy type of the $H\mathbb{Z}_p$ -algebra $THH(ku, H\mathbb{Z}_p)$.

We first compute its mod p homology, using the Bökstedt spectral sequence

$$(5.1) \quad E_{s,t}^2(ku, H\mathbb{Z}_p) = HH_{s,t}(H_*(ku; \mathbb{F}_p), H_*(H\mathbb{Z}_p; \mathbb{F}_p)) \implies H_{s+t}(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p).$$

The algebra homomorphism $j_* : H_*(ku; \mathbb{F}_p) \rightarrow H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ is the edge homomorphism

$$H_*(ku; \mathbb{F}_p) \rightarrow E_{0,*}^\infty \subset E_{0,*}^2 = H_*(H\mathbb{Z}_p; \mathbb{F}_p)$$

of the spectral sequence (1.4) described in the proof of Theorem 1.5. It is therefore given by

$$\begin{aligned} P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_2, \bar{\tau}_3, \dots) \otimes P_{p-1}(x) &\rightarrow P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_1, \bar{\tau}_2, \dots), \\ \bar{\xi}_i &\mapsto \bar{\xi}_i \quad \text{if } i \geq 1, \\ \bar{\tau}_i &\mapsto \bar{\tau}_i \quad \text{if } i \geq 2, \\ x &\mapsto 0. \end{aligned}$$

By Propositions 2.2 and 2.5, the E^2 -term of (5.1) is

$$H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma x, \sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots) \otimes \Gamma(y, \sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots)$$

where $a \in H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ has bidegree $(0, |a|)$, a class $\sigma\omega$ for $\omega \in H_*(ku; \mathbb{F}_p)$ has bidegree $(1, |\omega|)$, and y has bidegree $(2, 2p - 2)$.

Recall that a class $\sigma\omega$ is represented in the Hochschild complex by $1 \otimes \omega$ and y is represented by $1 \otimes \sigma^{p-2} \otimes \sigma$.

5.2. Lemma. *The classes $\sigma\bar{\xi}_1$ and y in $E_{*,*}^2(ku, H\mathbb{Z}_p)$ are permanent cycles, and in $H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p)$ there is a primary mod p Bockstein*

$$\beta([y]) = [\sigma\bar{\xi}_1].$$

Proof. To detect the mod p Bockstein claimed in this lemma we will need some knowledge of the integral homology of $THH(ku, H\mathbb{Z}_p)$. For integral computations it is more convenient to work with $THH^{\mathbb{S}^p}(ku, H\mathbb{Z}_p)$, because in this way the ground ring for the Bökstedt spectral sequence is $H_*(\mathbb{S}_p; \mathbb{Z}) = \mathbb{Z}_p$ instead of $H_*(\mathbb{S}; \mathbb{Z}) = \mathbb{Z}$ (recall that $H_*(ku, \mathbb{Z})$ and $H_*(H\mathbb{Z}_p, \mathbb{Z})$ are \mathbb{Z}_p -algebras). The natural map $THH(ku, H\mathbb{Z}_p) \rightarrow THH^{\mathbb{S}^p}(ku, H\mathbb{Z}_p)$ is an equivalence after p -completion, and induces an isomorphism of the mod p homology Bökstedt spectral sequences. It follows that the mod p homology Bockstein spectral sequences for $THH(ku, H\mathbb{Z}_p)$ and $THH^{\mathbb{S}^p}(ku, H\mathbb{Z}_p)$ are also isomorphic.

The class $\sigma\bar{\xi}_1$ is a permanent cycle for bidegree reasons. On the other hand y generates the component of total degree $2p$ in the E^2 -term of the Bökstedt spectral sequence (5.1). We claim that $[\sigma\bar{\xi}_1]$ is the mod p reduction of a class of order p in integral homology. Then $[\sigma\bar{\xi}_1]$ must be in the image of the primary mod p Bockstein, and this forces $\beta([y]) = [\sigma\bar{\xi}_1]$. In particular y is also a permanent cycle.

It remains to prove the claim that $[\sigma\bar{\xi}_1]$ is the mod p reduction of a class of order p in integral homology. Consider the commutative diagram

$$\begin{array}{ccc} H_*(ku; \mathbb{Z}) & \xrightarrow{\sigma_*} & H_{*+1}(THH^{\mathbb{S}^p}(ku, H\mathbb{Z}_p); \mathbb{Z}) \\ \downarrow \rho & & \downarrow \rho \\ H_*(ku; \mathbb{F}_p) & \xrightarrow{\sigma_*} & H_{*+1}(THH^{\mathbb{S}^p}(ku, H\mathbb{Z}_p); \mathbb{F}_p) \end{array}$$

where ρ is the mod p reduction. If $\tilde{\xi}_1 \in H_{2p-2}(ku; \mathbb{Z})$ is the class defined in 1.8, then

$$\rho\sigma_*(\tilde{\xi}_1) = \sigma_*\rho(\tilde{\xi}_1) = \sigma_*(\bar{\xi}_1) = [\sigma\bar{\xi}_1]$$

in $H_{2p-1}(THH^{\mathbb{S}^p}(ku, H\mathbb{Z}_p); \mathbb{F}_p)$. In particular $[\sigma\bar{\xi}_1]$ is the reduction of an integral class. We now prove that $pH_{2p-1}(THH^{\mathbb{S}^p}(ku, H\mathbb{Z}_p); \mathbb{Z}) = 0$, which implies the claim.

The Bökstedt spectral sequence converging to $H_*(THH^{\mathbb{S}^p}(ku, H\mathbb{Z}_p); \mathbb{Z})$ has an E^2 -term given by

$$(5.3) \quad \tilde{E}_{*,*}^2(ku, H\mathbb{Z}_p) = \mathrm{Tor}_{*,*}^{H_*(ku \wedge_{\mathbb{S}^p} ku; \mathbb{Z})} (H_*(ku; \mathbb{Z}), H_*(H\mathbb{Z}_p; \mathbb{Z})).$$

Recall the graded ring homomorphism $\lambda : \Lambda_* \rightarrow H_*(ku; \mathbb{Z})$ defined in Proposition 1.8. Since Λ_* is torsion free and λ is an isomorphism in degrees $\leq 2p^2 - 3$, the map

$$\Lambda_*^e = \Lambda_* \otimes_{\mathbb{Z}_p} \Lambda_* \rightarrow H_*(ku \wedge_{\mathbb{S}^p} ku; \mathbb{Z})$$

is also an isomorphism in this range of degrees. In particular, the Tor group of (5.3) is isomorphic to

$$HH^{\mathbb{Z}_p}(\Lambda_* \otimes_{\mathbb{Z}_p} \Lambda_*, H_*(H\mathbb{Z}_p; \mathbb{Z}))$$

in total degrees $\leq 2p^2 - 3$. Here the Λ_* -bimodule structure of $H_*(H\mathbb{Z}_p; \mathbb{Z})$ is given by the ring homomorphism $\Lambda_* \rightarrow H_*(H\mathbb{Z}_p; \mathbb{Z})$ that sends \tilde{x} to 0 and $\tilde{\xi}_1$ to a lift of $\bar{\xi}_1$ in $H_*(H\mathbb{Z}_p; \mathbb{Z})$. There is a free resolution X_* of Λ_* as Λ_*^e -module

$$0 \longrightarrow \Lambda_*^e\{w\} \xrightarrow{d_2} \Lambda_*^e\{\sigma\tilde{x}, \sigma\tilde{\xi}_1\} \xrightarrow{d_1} \Lambda_*^e$$

having as augmentation the product $\Lambda_*^e \rightarrow \Lambda_*$. The bidegree of the generators is $|\sigma\tilde{x}| = (1, 2)$, $|\sigma\tilde{\xi}_1| = (1, 2p - 2)$ and $|w| = (2, 2p - 2)$. The differential is given by

$$\begin{aligned} d_1(\sigma\tilde{x}) &= 1 \otimes \tilde{x} - \tilde{x} \otimes 1, \\ d_1(\sigma\tilde{\xi}_1) &= 1 \otimes \tilde{\xi}_1 - \tilde{\xi}_1 \otimes 1, \quad \text{and} \\ d_2(w) &= ((1 \otimes \tilde{x}^{p-1} - \tilde{x}^{p-1} \otimes 1)/(1 \otimes \tilde{x} - \tilde{x} \otimes 1))\sigma\tilde{x} - p\sigma\tilde{\xi}_1. \end{aligned}$$

In total degrees $\leq 2p^2 - 3$, the E^2 -term (5.3) is isomorphic to

$$HH_{*,*}(\Lambda_*, H_*(H\mathbb{Z}_p; \mathbb{Z})) = H_*(H_*(H\mathbb{Z}_p; \mathbb{Z}) \otimes_{\Lambda_*^e} X_*).$$

By inspection we have $\tilde{E}_{1,2p-2}^2(ku, H\mathbb{Z}_p) = \mathbb{F}_p\{\sigma\tilde{\xi}_1\}$, and the remaining groups in $\tilde{E}_{*,*}^2(ku, H\mathbb{Z}_p)$ of total degree $2p - 1$ are all trivial. For degree reasons there are no differentials affecting $\tilde{E}_{1,2p-2}^2(ku, H\mathbb{Z}_p)$. This proves that

$$H_{2p-1}(THH^{\mathbb{S}^p}(ku, H\mathbb{Z}_p); \mathbb{Z}) \cong \mathbb{F}_p\{[\sigma\tilde{\xi}_1]\}. \quad \square$$

5.4. Lemma. *There are multiplicative relations*

$$[y]^p = [\sigma\bar{\tau}_2] \quad \text{and} \quad [\sigma\bar{\tau}_i]^p = [\sigma\bar{\tau}_{i+1}]$$

in $H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p)$, for all $i \geq 2$.

Proof. The linearization $j : ku \rightarrow H\mathbb{Z}_p$ induces a map of $H\mathbb{Z}_p$ -algebras

$$j : THH(ku, H\mathbb{Z}_p) \rightarrow THH(H\mathbb{Z}_p).$$

By naturality of σ , we have $j_*([\sigma\bar{\xi}_1]) = [\sigma\bar{\xi}_1]$ and $j_*([\sigma\bar{\tau}_i]) = [\sigma\bar{\tau}_i]$ for all $i \geq 2$. The Bockstein $\beta : H_{2p}(THH(H\mathbb{Z}_p); \mathbb{F}_p) \rightarrow H_{2p-1}(THH(H\mathbb{Z}_p); \mathbb{F}_p)$ is injective and maps $[\sigma\bar{\tau}_1]$ to $[\sigma\bar{\xi}_1]$. Thus the Bockstein $\beta([y]) = [\sigma\bar{\xi}_1]$ implies that $j_*([y]) = [\sigma\bar{\tau}_1]$. Therefore the multiplicative relations for $H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p)$ follow from the ones for $H_*(THH(H\mathbb{Z}_p); \mathbb{F}_p)$ given in Lemma 4.2. \square

5.5. Lemma. *The Bökstedt spectral sequence (5.1) behaves as follows.*

(a) For $2 \leq r \leq p - 2$ we have $d^r = 0$, and there is a differential

$$d^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_i) = \sigma\bar{\xi}_{i+1} \cdot \gamma_k\sigma\bar{\tau}_i$$

for all $i \geq 2$ and $k \geq 0$. Taking into account the algebra structure, this leaves

$$E_{*,*}^p(ku, H\mathbb{Z}_p) = H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma x, \sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots) \otimes \Gamma(y).$$

(b) For $p \leq r \leq 2p - 2$ we have $d^r = 0$, and there is a differential

$$d^{2p-1}(\gamma_{p+k}y) = \sigma\bar{\xi}_2 \cdot \gamma_k y$$

for all $k \geq 0$. Taking into account the algebra structure, this leaves

$$E_{*,*}^{2p}(ku, H\mathbb{Z}_p) = H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma x, \sigma\bar{\xi}_1) \otimes P_p(y, \sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots).$$

At this stage the spectral sequence collapses.

Proof. The proof is similar to that of Lemma 4.3. Here also the spectral sequence is one of unital and augmented differential $H_*(H\mathbb{Z}_p; \mathbb{F}_p)$ -bialgebras, at least until a differential puts an end to flatness. Part (a) of the Lemma also follows by naturality with respect to $j : THH(ku, H\mathbb{Z}_p) \rightarrow THH(H\mathbb{Z}_p)$. This time the class $\sigma\bar{\xi}_2$ is a boundary because $[\sigma\bar{\xi}_2] = \beta([y]^p) = 0$. Again, an algebraic argument implies that the only possibility is a differential

$$d^{2p-1}(\gamma_p y) = \sigma\bar{\xi}_2.$$

The differential $d^{2p-1}(\gamma_{p+k} y) = \sigma\bar{\xi}_2 \cdot \gamma_k y$ are then detected using the coproduct on $\gamma_{p+k} y$. This leaves the E^{2p} -term as claimed, where all classes lie in filtration degrees $< 2p$, and the spectral sequence collapses. \square

5.6. Proposition. *There is an isomorphism of A_* -comodule algebras*

$$H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p) \cong H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E([\sigma x], [\sigma\bar{\xi}_1]) \otimes P([y])$$

where $[\sigma x]$ and $[\sigma\bar{\xi}_1]$ are A_* -comodule primitives and the coaction on $[y]$ is

$$\nu_*([y]) = \bar{\tau}_0 \otimes [\sigma\bar{\xi}_1] + 1 \otimes [y].$$

Proof. The Bökstedt spectral sequence described in Lemma 5.5 is strongly convergent and has an E^∞ -term given by

$$E_{*,*}^\infty(ku, H\mathbb{Z}_p) = H_*(H\mathbb{Z}_p; \mathbb{F}_p) \otimes E(\sigma x, \sigma\bar{\xi}_1) \otimes P_p(y, \sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots).$$

We have the multiplicative extensions $[y]^p = [\sigma\bar{\tau}_2]$ and $[\sigma\bar{\tau}_i]^p = [\sigma\bar{\tau}_{i+1}]$ established in Lemma 5.4. There are no further possible multiplicative extensions. The classes $[\sigma x]$ and $[\sigma\bar{\xi}_1]$ are comodule primitives by (4.6). The homomorphism

$$j_* : H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p) \rightarrow H_*(THH(H\mathbb{Z}_p); \mathbb{F}_p)$$

maps the class $[y]$ to $[\sigma\bar{\tau}_1]$, which has coaction $\nu_*([\sigma\bar{\tau}_1]) = \bar{\tau}_0 \otimes [\sigma\bar{\xi}_1] + 1 \otimes [\sigma\bar{\tau}_1]$. The formula for the coaction on $[y]$ follows by naturality because j_* is injective in the relevant degrees. \square

5.7. Theorem. *For any prime $p \geq 3$ there is an isomorphism of \mathbb{F}_p -algebras*

$$V(0)_* THH(ku, H\mathbb{Z}_p) \cong E(z, \lambda_1) \otimes P(\mu_1)$$

with $|z| = 3$, $|\lambda_1| = 2p - 1$ and $|\mu_1| = 2p$.

Proof. The proof is the same as for Theorem 4.7. Here z , λ_1 and μ_1 map respectively to $[\sigma x]$, $[\sigma\bar{\xi}_1]$ and $[y] - \bar{\tau}_0[\sigma\bar{\xi}_1]$ under the Hurewicz homomorphism

$$V(0)_* THH(ku, H\mathbb{Z}_p) \rightarrow H_*(V(0) \wedge THH(ku, H\mathbb{Z}_p); \mathbb{F}_p). \quad \square$$

Notice that as in 4.8 we have mod p Bocksteins

$$\beta_{0,r}(\mu_1^{p^r}) = \mu_1^{p^r-1} \lambda_1$$

in $V(0)_* THH(ku, H\mathbb{Z}_p)$.

5.8. Corollary. *For any prime $p \geq 3$, there is a homotopy equivalence*

$$THH(ku, H\mathbb{Z}_p) \simeq_p S_+^3 \wedge THH(H\mathbb{Z}_p).$$

6. THE MOD p HOMOTOPY GROUPS OF $THH(ku)$

In this section we compute $V(0)_*THH(ku)$ as a module over $P(u) = V(0)_*ku$. The strategy we use is similar to that developed by McClure and Staffeldt [MS] for computing $V(0)_*THH(\ell)$ as a $P(v_1)$ -module, except that we use the mod u Bockstein spectral sequence

$$(6.1) \quad E_*^1 = V(0)_*THH(ku, H\mathbb{Z}_p) \implies (V(0)_*THH(ku)/(u\text{-torsion})) \otimes_{P(u)} \mathbb{F}_p$$

instead of the Adams spectral sequence.

6.2. Proposition. *Let X be a connective ku -module of finite type. There is a one-column, strongly convergent spectral sequence*

$$E_*^1 = V(0)_*(H\mathbb{Z}_p \wedge_{ku} X) \implies (V(0)_*X/(u\text{-torsion})) \otimes_{P(u)} \mathbb{F}_p,$$

called the mod u Bockstein spectral sequence. Its r th differential is denoted $\beta_{u,r}$ and decreases degree by $2r + 1$. There is an isomorphism of $P(u)$ -modules

$$V(0)_*X \cong P(u) \otimes E_*^\infty \oplus \bigoplus_{r \geq 1} P_r(u) \otimes \text{im}(\beta_{u,r}).$$

Moreover, if X is a ku -algebra, then this is a spectral sequence of differential algebras.

6.3. Remark. The grading of the target group $(V(0)_*X/(u\text{-torsion})) \otimes_{P(u)} \mathbb{F}_p$ is such that a class $a \otimes_{P(u)} 1$ has the degree of its representative of minimal degree in $(V(0)_*X/(u\text{-torsion})) \times \mathbb{F}_p$

Proof. This is very similar to the mod p Bockstein spectral sequence, see for instance [Mc, Theorem 10.3]. We just sketch the proof. Consider the diagram (1.3) and prolong it to the right by desuspending. Applying $V(0)_*(- \wedge_{ku} X)$ we obtain an unrolled exact couple. Placing $V(0)_*(\Sigma^{2s} H\mathbb{Z}_p \wedge_{ku} X)$ in filtration degree $-2s$, it yields a spectral sequence

$$V(0)_*(H\mathbb{Z}_p \wedge_{ku} X) \otimes P(u, u^{-1}) \implies V(0)_*X \otimes_{P(u)} P(u, u^{-1}).$$

Here the class u represents the Bott element and has bidegree $(-2, 4)$. Strong convergence follows from the assumptions on X . This spectral sequence is one of differential $P(u, u^{-1})$ -modules. In particular all columns are isomorphic at each stage. Extracting the column of filtration 0 and taking $\beta_{u,r} = u^{-r} d^r$, we obtain the mod u Bockstein spectral sequence.

The ring $P(u)$ is a graded principal ideal domain. Since X is connective of finite type the graded module $V(0)_*X$ splits as a sum of shifted copies of $P(u)$ and its truncations (namely the quotients by an ideal generated by a homogeneous element). If there is a differential $\beta_{u,r}(a) = b$, then by definition of $\beta_{u,r}$ the class b is the image under

$$V(0)_*X \rightarrow V(0)_*(H\mathbb{Z}_p \wedge_{ku} X)$$

of a class \tilde{b} not divisible by u , such that $u^{r-1}\tilde{b} \neq 0$ and $u^r\tilde{b} = 0$. The description of the $P(u)$ -module $V(0)_*X$ given follows. Finally if X is a ku -algebra, then our

unrolled exact couple is part of a multiplicative Cartan-Eilenberg system (see [CE], XV.7) with

$$H(p, q) = V(0)_* \left((\Sigma^p ku / \Sigma^q ku) \wedge_{ku} X \right),$$

and hence this spectral sequence is one of differential algebras. \square

Let $K(1)$ be the Morava K -theory, with coefficients $K(1)_* = P(v_1, v_1^{-1})$. We compute $K(1)_* THH(ku)$, which allows us to determine the E^∞ -term of (6.1). It will then turn out that only one pattern of differentials is possible.

6.4. Proposition. *There is an isomorphism of $K(1)_*$ -algebras*

$$K(1)_* ku \cong P(u, u^{-1}) \otimes K(1)_0 \ell.$$

Here u is the Hurewicz image of the Bott element and on the right-hand side the $K(1)_*$ -module structure is given by the inclusion $K(1)_* \rightarrow P(u, u^{-1})$ with $v_1 = u^{p-1}$.

Proof. The isomorphism

$$K(1)_* \ell \cong K(1)_* \otimes K(1)_0 \ell$$

is established in [MS, Proposition 5.3.(a)]. The splitting $ku \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} \ell$ implies that the formula claimed for $K(1)_* ku$ holds additively. The multiplication-by- u map $\Sigma^2 ku \rightarrow ku$ induces an isomorphism $K(1)_{*-2} ku \cong K(1)_* ku$ since for its cofiber $H\mathbb{Z}_p$ we have $K(1)_* H\mathbb{Z}_p = 0$. Thus multiplication by u is invertible in $K(1)_* ku$. The relation $v_1 = u^{p-1}$ follows from the corresponding relation in $V(0)_* ku$. \square

6.5. Theorem. *The unit map $ku \rightarrow THH(ku)$ induces isomorphisms*

$$K(1)_* ku \xrightarrow{\cong} K(1)_* THH(ku)$$

and

$$v_1^{-1} V(0)_* ku \xrightarrow{\cong} v_1^{-1} V(0)_* THH(ku).$$

Proof. McClure and Staffeldt [MS, Th. 5.1 and Cor. 5.2] prove the corresponding statements for ℓ . Their argument extends to this case, and we just outline it, referring to [MS] for further details. By Proposition 6.4 we have an isomorphism

$$HH_{*,*}^{K(1)*} (P(u, u^{-1})) \otimes HH_{*,*}^{\mathbb{F}_p} (K(1)_0 \ell) \xrightarrow{\cong} HH_{*,*}^{K(1)*} (K(1)_* ku).$$

By Proposition 2.6 and [MS, Proposition 5.3.(c)] the unit for each of the tensor factors on the left-hand side is an isomorphism. This implies that the unit

$$K(1)_* ku \rightarrow HH_{*,*}^{K(1)*} (K(1)_* ku)$$

is an isomorphism. The isomorphism $K(1)_* ku \cong K(1)_* THH(ku)$ follows from the collapse of the Bökstedt spectral sequence

$$E_{s,t}^2 = HH_{s,t}^{K(1)*} (K(1)_* ku) \implies K(1)_{s+t} THH(ku).$$

Finally, by Lemma 5.4 of [MS] the first isomorphism claimed implies the second one. \square

6.6. Definition. Let

$$a(n) = \begin{cases} p-2 & \text{if } n = 0, \\ p^{n+1} - p^n + p^{n-1} - \cdots + p^2 - p & \text{if } n \geq 1 \text{ is odd,} \\ p^{n+1} - p^n + p^{n-1} - \cdots + p^3 - p^2 + p - 2 & \text{if } n \geq 2 \text{ is even, and} \end{cases}$$

$$b(n) = \begin{cases} 0 & \text{if } n = 0, 1, \\ p^{n-1} - p^{n-2} + \cdots + p - 1 & \text{if } n \geq 2 \text{ is even,} \\ p^{n-1} - p^{n-2} + \cdots + p^2 - p & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

We are now ready to describe the differentials in the spectral sequence (6.1). By Theorem 5.7 its E^1 -term is

$$E_*^1 = V(0)_*THH(ku, H\mathbb{Z}_p) \cong E(z, \lambda_1) \otimes P(\mu_1).$$

6.7. Theorem. *The u -Bockstein spectral sequence (6.1) for $THH(ku)$ has differentials*

$$\beta_{u,a(n)}(\mu_1^{p^n}) = \begin{cases} z\mu_1^{b(n)} & \text{if } n \geq 0 \text{ is even,} \\ \lambda_1\mu_1^{b(n)} & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

Proof. Since we have the relation $v_1 = u^{p-1}$ in $V(0)_*ku$, we know from Theorem 6.5 that $u^{-1}V(0)_*THH(ku) = P(u, u^{-1})$. In particular the E^∞ -term consists solely of a copy of \mathbb{F}_p in degree 0, and $V(0)_0THH(ku, H\mathbb{Z}_p) = \mathbb{F}_p$ is the subgroup of permanent cycles in E_*^1 . The classes z and λ_1 are infinite cycles. As can be checked by induction on n , the pattern of differentials given is the only one that leaves $E_*^\infty = \mathbb{F}_p$. \square

6.8. Corollary. *There is an isomorphism of $P(u)$ -modules*

$$V(0)_*THH(ku) \cong P(u) \oplus \bigoplus_{n \geq 0} P_{a(n)}(u) \otimes I_n,$$

where I_n is the graded \mathbb{F}_p -module

$$I_n = \begin{cases} E(\lambda_1\mu_1^{b(n+1)}) \otimes P(\mu_1^{p^{n+1}}) \otimes \mathbb{F}_p\{z\mu_1^{b(n)+p^n j} \mid j = 0, \dots, p-2\}, & n \geq 0 \text{ even,} \\ E(z\mu_1^{b(n+1)}) \otimes P(\mu_1^{p^{n+1}}) \otimes \mathbb{F}_p\{\lambda_1\mu_1^{b(n)+p^n j} \mid j = 0, \dots, p-2\}, & n \geq 1 \text{ odd.} \end{cases}$$

Proof. The $E^{a(n)}$ -term of the u -Bockstein spectral sequence is given by

$$E_*^{a(n)} = \begin{cases} P(\mu_1^{p^n}) \otimes E(z\mu_1^{b(n)}, \lambda_1\mu_1^{b(n+1)}) & \text{for } n \geq 0 \text{ even,} \\ P(\mu_1^{p^n}) \otimes E(z\mu_1^{b(n+1)}, \lambda_1\mu_1^{b(n)}) & \text{for } n \geq 1 \text{ odd.} \end{cases}$$

Using the description of $\beta_{u,a(n)}$ given above, one checks that $\text{im}(\beta_{u,a(n)}) \cong I_n$. The Corollary then follows from Proposition 6.2. \square

6.9. Remark. A computation of $V(0)_*THH(ku)$ at the prime 2 has been performed by Angeltveit and Rognes [AnR]. Their argument is similar to that of [MS] at odd primes, and involves the Adams spectral sequence.

6.10. Corollary. *In any presentation, the $P(u)$ -algebra $V(0)_*THH(ku)$ has infinitely many generators and infinitely many relations.*

Proof. This follows from Corollary 6.8 by degree and u -torsion considerations. \square

We view this Corollary as a motivation for pursuing, in the next two sections, a description of the algebra structure of $V(1)_*THH(ku)$. By analogy with the case of $THH(\ell)$ one expects it to be nicer than the structure of $V(0)_*THH(ku)$. Indeed, it will turn out that $V(1)_*THH(ku)$ admits finitely many generators and relations.

The periodic case. Let KU and L denote the periodic complex K -theory spectrum and the periodic Adams summand, both completed at p . They inherit a commutative \mathbb{S} -algebra structure as the $E(1)$ -localizations of ku and ℓ , respectively. The homotopy type of the spectrum $THH(L)_p$ was computed by McClure and Staffeldt [MS, Theorem 8.1], and is given as

$$(6.11) \quad THH(L)_p \simeq L \vee \Sigma L_{\mathbb{Q}},$$

where $L_{\mathbb{Q}}$ denotes the rationalization of the spectrum L . Their argument can be applied to compute $THH(KU)_p$, the only new ingredient being the computation of $K(1)_*THH(ku)$ given in Theorem 6.5. We therefore formulate without proof the following proposition.

6.12. Proposition. *There is an equivalence $THH(KU)_p \simeq KU \vee \Sigma KU_{\mathbb{Q}}$.*

7. COEFFICIENTS IN A POSTNIKOV SECTION

In this section we will assume that $p \geq 5$. Let M be the Postnikov section $M = ku[0, 2p - 6]$ of ku with coefficients

$$M_n = \begin{cases} ku_n & \text{if } n \leq 2p - 6, \\ 0 & \text{otherwise.} \end{cases}$$

It is known ([Ba, Theorem 8.1]) that the Postnikov sections of a commutative \mathbb{S} -algebra can be constructed within the category of commutative \mathbb{S} -algebras. We can therefore assume that the natural map $\phi : ku \rightarrow M$ is a map of commutative \mathbb{S} -algebras. In this section we compute the mod p homology groups of $THH(ku, M)$ using the Bökstedt spectral sequence. This will be useful in performing the corresponding computations for $THH(ku)$.

The mod p homology of M is given by an isomorphism of A_* -comodule algebras

$$H_*(M; \mathbb{F}_p) \cong H_*(HZ_p; \mathbb{F}_p) \otimes P_{p-2}(x),$$

where $x = \phi_*(x)$ under the map $\phi_* : H_*(ku; \mathbb{F}_p) \rightarrow H_*(M; \mathbb{F}_p)$. The proof of this statement is a variation of the proof of Theorem 1.5.

By Propositions 2.2 and 2.5 the Bökstedt spectral sequence

$$(7.1) \quad E_{*,*}^2(ku, M) = HH_{*,*}^{\mathbb{F}_p}(H_*(ku; \mathbb{F}_p), H_*(M; \mathbb{F}_p)) \implies H_*(THH(ku, M); \mathbb{F}_p)$$

has an E^2 -term given by

$$E_{*,*}^2(ku, M) \simeq H_*(HZ_p; \mathbb{F}_p) \otimes P_{p-2}(x) \otimes E_{*,*}^2(\tau_{\infty} \tau_{\infty}^{\bar{c}} \tau_{\infty}^{\bar{c}}) \otimes \Gamma_{*,*}(\tau_{\infty} \tau_{\infty}^{\bar{c}} \tau_{\infty}^{\bar{c}})$$

Let $\delta : ku \rightarrow ku$ be the operation corresponding to a chosen generator of Δ , as described in 1.1. It restricts to a map of \mathbb{S} -algebras $\delta : M \rightarrow M$ by naturality of the Postnikov section. Since topological Hochschild homology is functorial in both variables we have \mathbb{S} -algebra maps $\delta : THH(ku) \rightarrow THH(ku)$ and $\delta : THH(ku, M) \rightarrow THH(ku, M)$, inducing morphisms of spectral sequences $\delta^* : E_{*,*}^r(ku) \rightarrow E_{*,*}^r(ku)$ and $\delta^* : E_{*,*}^r(ku, M) \rightarrow E_{*,*}^r(ku, M)$. Suppose chosen a Bott element $u \in V(0)_2 ku$, and let $\alpha \in \mathbb{F}_p^\times$ be such that $\delta_*(u) = \alpha u$.

7.2. Definition. A class w in $E_{*,*}^r(ku)$ or $E_{*,*}^r(ku, M)$ has δ -weight $n \in \mathbb{Z}/(p-1)$ if $\delta^*(w) = \alpha^n w$. Similarly, a class v in $H_*(THH(ku); \mathbb{F}_p)$, $V(0)_* THH(ku)$, $V(1)_* THH(ku)$, or $H_*(THH(ku, M); \mathbb{F}_p)$ has δ -weight n if $\delta_*(v) = \alpha^n v$.

7.3. Lemma. *In $E_{*,*}^2(ku, M)$ the classes belonging to the tensor factor*

$$H_*(HZ_p, \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots) \otimes \Gamma(y, \sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots)$$

have δ -weight 0, while x and σx have δ -weight 1.

Proof. This is proven by inspection of the action of δ_* on the Hochschild complex. \square

7.4. Lemma. *There is an isomorphism of spectral sequences*

$$E_{*,*}^r(ku, M) \cong P_{p-2}(x) \otimes E_{*,*}^r(ku, HZ_p).$$

Proof. It suffices to prove by induction that for all $r \geq 2$, the following two assertions hold.

- (1) There is an isomorphism $E_{*,*}^r(ku, M) \cong P_{p-2}(x) \otimes E_{*,*}^r(ku, HZ_p)$,
- (2) For each $0 \leq i \leq p-3$ the d^r -differential maps $\mathbb{F}_p\{x^i\} \otimes E_{*,*}^r(ku, HZ_p)$ to itself.

For $r = 2$ assertion (1) holds. Each algebra generator of $E_{*,*}^2(ku, M)$ that can support a differential has δ -weight 0. Since differentials preserve the δ -weight, the first non-trivial differential maps $\mathbb{F}_p\{x^i\} \otimes E_{*,*}^2(ku, HZ_p)$ to itself. In particular it is detected by the morphism of spectral sequences

$$\varphi_{*,*}^* : E_{*,*}^2(ku, M) \rightarrow E_{*,*}^2(ku, HZ_p),$$

induced by the linearization $\varphi : M \rightarrow HZ_p$, and is given by $d^{p-1}(\gamma_p \sigma \bar{\tau}_i) = \sigma \bar{\xi}_{i+1}$ for $i \geq 2$. As in the case of $E_{*,*}^2(ku, HZ_p)$, taking into account the bialgebra structure, this leaves

$$E_{*,*}^p(ku, M) \cong P_{p-2}(x) \otimes H_*(HZ_p, \mathbb{F}_p) \otimes E(\sigma x, \sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots) \otimes \Gamma(y).$$

Again (1) holds and algebra generators that can support a differential are of δ -weight 0. We can repeat the argument until we reach $E_{*,*}^{2p}(ku, M)$, where the spectral sequence collapses for bidegree reasons. \square

7.5. Proposition. *There is an isomorphism of A_* -comodule algebras*

$$H_*(THH(ku, M); \mathbb{F}_p) \cong H_*(M; \mathbb{F}_p) \otimes E([\sigma x], [\sigma \bar{\xi}_1]) \otimes P([y])$$

where $[\sigma x]$ and $[\sigma \bar{\xi}_1]$ are A_* -comodule primitives and the coaction on $[y]$ is

$$\nu_*([y]) = \bar{\tau}_0 \otimes [\sigma \bar{\xi}_1] + 1 \otimes [y].$$

Proof. The Bökstedt spectral sequence described in Lemma 7.4 is strongly convergent and has an E^∞ -term given by

$$E_{*,*}^\infty(ku, M) = H_*(M; \mathbb{F}_p) \otimes E(\sigma x, \sigma \bar{\xi}_1) \otimes P_p(y, \sigma \bar{\tau}_2, \sigma \bar{\tau}_3, \dots).$$

The map φ^∞ is surjective and its kernel is the ideal generated by x . The multiplicative and comodule extensions are detected using the homomorphism

$$\varphi_* : H_*(THH(ku, M); \mathbb{F}_p) \rightarrow H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p). \quad \square$$

8. THE $V(1)$ HOMOTOPY GROUPS OF $THH(ku)$

In this section we compute the Bökstedt spectral sequence

$$(8.1) \quad E_{*,*}^2(ku) = HH_{*,*}^{\mathbb{F}_p}(H_*(ku; \mathbb{F}_p)) \implies H_*(THH(ku); \mathbb{F}_p)$$

and describe $V(1)_*THH(ku)$ as an algebra over $V(1)_*ku$, for primes p with $p \geq 5$. We treat the case $p = 3$ separately at the end of the section. Unless otherwise specified, we assume throughout this section that $p \geq 5$.

Recall the description of $H_*(ku; \mathbb{F}_p)$ given in Theorem 1.5, and let $P_{p-1}(x)$ be the subalgebra of $H_*(ku; \mathbb{F}_p)$ generated by $x \in H_2(ku; \mathbb{F}_p)$. Let us denote by $\Omega_{*,*}^2$ the bigraded $P_{p-1}(x)$ -algebra $HH_{*,*}(P_{p-1}(x))$. It has generators

$$\begin{cases} z_i & \text{of bidegree } (2i + 1, (2p - 2)i + 2) \text{ for } i \geq 0, \\ y_j & \text{of bidegree } (2j, (2p - 2)j + 2) \text{ for } j \geq 1, \end{cases}$$

subject to the relations given in Proposition 2.3. By Propositions 2.2 and 2.3, the E^2 -term of the Bökstedt spectral sequence (8.1) is given by

$$E_{*,*}^2(ku) = H_*(\ell; \mathbb{F}_p) \otimes E(\sigma \bar{\xi}_1, \sigma \bar{\xi}_2, \dots) \otimes \Gamma(\sigma \bar{\tau}_2, \sigma \bar{\tau}_3, \dots) \otimes \Omega_{*,*}^2.$$

Notice that $E_{*,*}^2(ku)$ is not flat over $H_*(ku; \mathbb{F}_p)$, so there is no coproduct structure on this spectral sequence.

8.2. Lemma. *Any class in the tensor factor*

$$H_*(\ell; \mathbb{F}_p) \otimes E(\sigma \bar{\xi}_1, \sigma \bar{\xi}_2, \dots) \otimes \Gamma(\sigma \bar{\tau}_2, \sigma \bar{\tau}_3, \dots)$$

of $E_{*,*}^2(ku)$ has δ -weight 0. The generators x , z_i and y_j of $\Omega_{*,*}^2$, for $i \geq 0$ and $j \geq 1$, have δ -weight 1.

Proof. This is a consequence of the action of δ_* on $H_*(ku; \mathbb{F}_p)$ which was described in Lemma 1.6. For z_i and y_j it follows from the fact that a representative for z_i or y_j in the Hochschild complex of $P_{p-1}(x)$ consists of a sum of terms having a number of factors x that is congruent to 1 modulo $(p - 1)$. \square

The \mathbb{S} -algebra map $\phi : ku \rightarrow M$ from previous section induces a map $\phi : THH(ku) \rightarrow THH(ku, M)$ and a morphism of spectral sequences $\phi^* : E_{*,*}^2(ku) \rightarrow E_{*,*}^2(ku, M)$. The term $E_{*,*}^2(ku, M)$ was given in 7.1

8.3. Lemma. *The homomorphism $\phi^2 : E_{*,*}^2(ku) \rightarrow E_{*,*}^2(ku, M)$ is characterized as follows. On the tensor factor $H_*(\ell; \mathbb{F}_p)$ it is the inclusion into $H_*(HZ_p; \mathbb{F}_p)$, on the factor $E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \dots) \otimes \Gamma(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots)$ it is the identity, and on $\Omega_{*,*}^2$ it is given by*

$$\begin{aligned}\phi^2(x) &= x, \\ \phi^2(z_i) &= \sigma x \cdot \gamma_i y \quad \text{for all } i \geq 0, \\ \phi^2(y_j) &= x \cdot \gamma_j y \quad \text{for all } j \geq 1.\end{aligned}$$

Proof. The homomorphism $\phi_* : H_*(ku; \mathbb{F}_p) \rightarrow H_*(M; \mathbb{F}_p)$ is given by the tensor product of the inclusion of $H_*(\ell; \mathbb{F}_p)$ into $H_*(HZ_p; \mathbb{F}_p)$ and the projection of $P_{p-1}(x)$ onto $P_{p-2}(x)$. This lemma follows from a computation in Hochschild homology, using the resolution given in the proof of Proposition 2.3. \square

8.4. Definition. Let $\Omega_{*,*}^\infty$ be the submodule of $\Omega_{*,*}^2$ generated by

$$\begin{cases} z_i & \text{for } 0 \leq i \leq p-1, \\ y_j & \text{for } 1 \leq j \leq p-1 \end{cases}$$

over $P_{p-1}(x)$. Then $\Omega_{*,*}^\infty$ is closed under multiplication, and hence is a subalgebra of $\Omega_{*,*}^2$.

8.5. Lemma. *The Bökstedt spectral sequence (8.1) behaves as follows.*

(a) *For $2 \leq r \leq p-2$ we have $d^r = 0$, and there are differentials*

$$d^{p-1}(\gamma_{p+k}\sigma\bar{\tau}_i) = \gamma_k\sigma\bar{\tau}_i \cdot \sigma\bar{\xi}_{i+1}$$

for all $k \geq 0$ and all $i \geq 2$. Taking into account the algebra structure, this leaves

$$E_{*,*}^p(ku) = H_*(\ell; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots) \otimes \Omega_{*,*}^2.$$

(b) *For $p \leq r \leq 2p-2$ we have $d^r = 0$, and there are differentials*

$$\begin{aligned}d^{2p-1}(z_i) &= z_{i-p} \cdot \sigma\bar{\xi}_2 \quad \text{for all } i \geq p, \\ d^{2p-1}(y_p) &= x \cdot \sigma\bar{\xi}_2, \\ d^{2p-1}(y_j) &= y_{j-p} \cdot \sigma\bar{\xi}_2 \quad \text{for all } j > p.\end{aligned}$$

Taking into account the algebra structure, this leaves

$$E_{*,*}^{2p}(ku) = H_*(\ell; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1) \otimes P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots) \otimes (\mathbb{F}_p\{\sigma\bar{\xi}_2\} \oplus \Omega_{*,*}^\infty).$$

Here $\sigma\bar{\xi}_2 \cdot \omega = 0$ for any $\omega \in \Omega_{,*}^\infty$ of positive total degree. At this stage the spectral sequence collapses.*

Proof. We use the morphism of spectral sequences

$$\phi^r : E_{*,*}^r(ku) \rightarrow E_{*,*}^r(ku, M)$$

whose description on the E^2 -term is given in Lemma 8.3. The kernel of $\phi^2 : E_{*,*}^2(ku) \rightarrow E_{*,*}^2(ku, M)$ is the ideal generated by x^{p-2} and $x^{p-3}y_j$ for all $j \geq 1$. In particular any $w \in \ker \phi^2$ is of δ -weight $p-2$. On the other hand, any algebra generator v of $E_{*,*}^2(ku)$ is of δ -weight 0 or 1. Since differentials preserve the δ -weight, this implies that the differentials of $E_{*,*}^2(ku)$ originating on algebra generators are detected by ϕ^* . Thus we obtain the claimed d^{p-1} and d^{2p-1} differential. In $E_{*,*}^{2p}(ku)$ all algebra generators lie in filtration degrees smaller than $2p$, so the spectral sequence collapses for bidegree reasons. \square

8.6. Lemma. For $0 \leq i \leq p-2$ let W_i be the \mathbb{F}_p -vector space of elements of δ -weight i in $H_*(THH(ku); \mathbb{F}_p)$. Then multiplication by $x : W_1 \rightarrow W_2$ is an isomorphism.

Proof. The map $\delta : THH(ku) \rightarrow THH(ku)$ is an isomorphism. Therefore the additive isomorphism $H_*(THH(ku); \mathbb{F}_p) \cong \text{Tot } E_{*,*}^\infty(ku)$ preserves the δ -weight, and it suffices to check the corresponding statement for $E_{*,*}^\infty(ku)$. This follows from Lemma 8.2 by inspection. \square

8.7. Lemma. Let $\phi_* : H_*(THH(ku); \mathbb{F}_p) \rightarrow H_*(THH(ku, M); \mathbb{F}_p)$ be the algebra homomorphism induced by $\phi : ku \rightarrow M$. Then $\ker(\phi_*|_{W_1 \cup W_2}) = 0$.

Proof. By Lemma 8.3, the morphism $\phi^\infty : E_{*,*}^\infty(ku) \rightarrow E_{*,*}^\infty(ku, M)$ is injective on the vector space of elements of δ -weight 1 or 2. This implies the corresponding statement for ϕ_* . \square

The $H_*(ku, \mathbb{F}_p)$ -algebra $H_*(THH(ku); \mathbb{F}_p)$ differs from its associated graded $E_{*,*}^\infty(ku)$ by multiplicative extensions. More precisely, the subalgebra

$$P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots) \otimes (\mathbb{F}_p\{\sigma\bar{\xi}_2\} \oplus \Omega_{*,*}^\infty)$$

of $E_{*,*}^\infty(ku)$ lifts to the subalgebra Ξ_* of $H_*(THH(ku); \mathbb{F}_p)$ defined as follows.

8.8. Definition. We define Ξ_* to be the (graded) commutative unital $P_{p-1}(x)$ -algebra with generators

$$\begin{cases} \bar{z}_i & 0 \leq i \leq p-1, \\ \bar{y}_j & 1 \leq j \leq p-1, \\ [\sigma\bar{\tau}_2], \end{cases}$$

and relations

$$\begin{cases} x^{p-2}\bar{z}_i = 0 & 0 \leq i \leq p-2, \\ x^{p-2}\bar{y}_j = 0 & 1 \leq j \leq p-1, \\ \bar{y}_i\bar{y}_j = x\bar{y}_{i+j} & i+j \leq p-1, \\ \bar{z}_i\bar{y}_j = x\bar{z}_{i+j} & i+j \leq p-1, \\ \bar{y}_i\bar{y}_j = x\bar{y}_{i+j-p}[\sigma\bar{\tau}_2] & i+j \geq p, \\ \bar{z}_i\bar{y}_j = x\bar{z}_{i+j-p}[\sigma\bar{\tau}_2] & i+j \geq p, \\ \bar{z}_i\bar{z}_j = 0 & 0 \leq i, j \leq p-1. \end{cases}$$

Here by convention $[\bar{y}_0] = x$, and the degree of the generators is $|\bar{z}_i| = 2pi + 3$, $|\bar{y}_j| = 2pj + 2$ and $|\sigma\bar{\tau}_2| = 2p^2$.

Beware that in Ξ_* we have $x^{p-2}\bar{z}_{p-1} \neq 0$, which accounts for an extension $x^{p-2}\bar{z}_{p-1} = [\sigma\bar{\xi}_2]$.

8.9. Proposition. There is an isomorphism of $H_*(\ell; \mathbb{F}_p)$ -algebras

$$H_*(THH(ku); \mathbb{F}_p) \cong H_*(\ell; \mathbb{F}_p) \otimes E([\sigma\bar{\xi}_1]) \otimes \Xi_*.$$

Proof. The Bökstedt spectral sequence (8.1) converges strongly and its E^∞ -term is given in Lemma 8.5.b. For $1 \leq i \leq p-1$, we define $\bar{y}_i \in H_{2pi+2}(THH(ku); \mathbb{F}_p)$ by induction on i with the following properties:

- (1) \bar{y}_i has δ -weight 1,
- (2) \bar{y}_i reduces to iy_i modulo lower filtration in $E_{*,*}^\infty(ku)$, and
- (3) $\phi_*(\bar{y}_i) = x[y]^i + ic\bar{\tau}_1[z][y]^{i-1}$ in $H_{2pi+2}(THH(ku, M); \mathbb{F}_p)$ for some $c \in \mathbb{F}_p^\times$ independent of i .

Let $\bar{y}_1 = [y_1]$. Then (1) and (2) are obviously satisfied and by Lemma 8.3 $\phi_*(\bar{y}_1) \equiv x[y]$ modulo filtration lower than 2. By the splitting of the 0 simplices in THH we deduce that $\phi_*(\bar{y}_1) \equiv x[y]$ modulo classes of filtration 1. The filtration-1 part of $H_{2p+2}(THH(ku, M); \mathbb{F}_p)$ is $\mathbb{F}_p\{\bar{\tau}_1[z]\}$. On the other hand the map

$$j_* : H_*(THH(ku); \mathbb{F}_p) \rightarrow H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p)$$

satisfies $j_*(\bar{y}_1) = c\bar{\tau}_1[z]$ for some unit $c \in \mathbb{F}_p$, because $j_*(\bar{y}_1) \neq 0$ since \bar{y}_1 is not divisible by x and $\bar{\tau}_1[z]$ generates $H_{2p+2}(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p)$. The homomorphism j_* factorizes as $\varphi_*\phi_*$, so this proves that

$$(8.10) \quad \phi_*(\bar{y}_1) = x[y] + c\bar{\tau}_1[z].$$

Assume that \bar{y}_i satisfying conditions (1), (2) and (3) has been defined for some $1 \leq i \leq p-2$. The class $\bar{y}_1\bar{y}_i$ has δ -weight 2 so is divisible by x in a unique way. Let $\bar{y}_{i+1} = x^{-1}\bar{y}_1\bar{y}_i$. Then \bar{y}_{i+1} satisfies conditions (1) and (2). By inspection there is no nonzero class w of δ -weight 1 in $H_{2p(i+1)+2}(THH(ku, M); \mathbb{F}_p)$ with $xw = 0$, so we can write

$$\phi_*(\bar{y}_{i+1}) = x^{-1}\phi_*(\bar{y}_1)\phi_*(\bar{y}_i),$$

which proves that \bar{y}_{i+1} satisfies (3).

Next, we define $\bar{z}_0 = c[z_0]$, where $c \in \mathbb{F}_p$ is the same as in condition (3) above. The class $\bar{z}_0\bar{y}_i$ has δ -weight 2, and we define $\bar{z}_i = x^{-1}\bar{z}_0\bar{y}_i$. Then \bar{z}_i has δ -weight 1, and if $i \geq 1$ it reduces to icz_i modulo lower filtration in $E_{*,*}^\infty(ku)$. Moreover we have $\phi_*(\bar{z}_i) = c[z][y]^i$ for $0 \leq i \leq p-1$.

The relations $\bar{y}_i\bar{y}_j = x\bar{y}_{i+j}$ and $\bar{z}_i\bar{y}_j = x\bar{z}_{i+j}$ for $i+j \leq p-1$ are satisfied by definition of \bar{z}_i and \bar{y}_j . It remains to check the following relations :

$$\begin{cases} x^{p-2}\bar{z}_i = 0 & 0 \leq i \leq p-2, \\ x^{p-2}\bar{y}_j = 0 & 1 \leq j \leq p-1, \\ \bar{z}_i\bar{z}_j = 0 & 0 \leq i, j \leq p-1, \\ \bar{y}_i\bar{y}_j = x\bar{y}_{i+j-p}[\sigma\bar{\tau}_2] & i+j \geq p. \\ \bar{z}_i\bar{y}_j = x\bar{z}_{i+j-p}[\sigma\bar{\tau}_2] & i+j \geq p. \end{cases}$$

The class $x^{p-2}\bar{y}_j$ is of δ -weight 0 and is in the kernel of multiplication by x . It follows that it is in the ideal generated by $[\sigma\bar{\xi}_2]$, so for degree reasons it must be zero. Similarly we have $x^{p-2}\bar{z}_i = 0$ if $0 \leq i \leq p-2$. The product $\bar{z}_i\bar{z}_j$ is of δ -weight 2 and in the kernel of ϕ_* , so must be zero. If $i+j \geq p$, we have

$$\phi_*(\bar{y}_i\bar{y}_j) = x^2[y]^{i+j} + (i+j)c\bar{\tau}_1x[z][y]^{p-i-j} = \phi_*(x\bar{y}_{i+j-p}[\sigma\bar{\tau}_2]).$$

Since both $\bar{y}_i\bar{y}_j$ and $x\bar{y}_{i+j-p}[\sigma\bar{\tau}_2]$ have δ -weight 2, they must be equal. The proof that $\bar{z}_i\bar{y}_j = x\bar{z}_{i+j-p}[\sigma\bar{\tau}_2]$ for $i+j \geq p$ is similar.

Finally, the class $[\sigma\bar{\xi}_2]$ maps to zero via

and hence must be divisible by x . The only possibility left is a multiplicative extension $x^{p-2}z_{p-1} = [\sigma\bar{\xi}_2]$ (up to multiplication by a unit). We have now established all possible multiplicative relations involving the classes x , \bar{z}_i , \bar{y}_j and $[\sigma\bar{\tau}_2]$. The associated graded of Ξ_* is isomorphic to

$$P_p(\sigma\bar{\tau}_2, \sigma\bar{\tau}_3, \dots) \otimes (\mathbb{F}_p\{\sigma\bar{\xi}_2\} \oplus \Omega_{*,*}^\infty).$$

This proves the proposition. \square

8.11. Proposition. *The A_* -coaction on $H_*(THH(ku); \mathbb{F}_p)$ is as follows :*

- on $H_*(\ell; \mathbb{F}_p)$ it is induced by inclusion into the coalgebra A_* ;
- the classes x , $[\sigma\bar{\xi}_1]$ and \bar{z}_0 are primitive ;
- on the remaining algebra generators, we have

$$\begin{aligned} \nu_*(\bar{z}_i) &= 1 \otimes \bar{z}_i + i\bar{\tau}_0 \otimes [\sigma\bar{\xi}_1]\bar{z}_{i-1} \quad \text{for } i \geq 1, \\ \nu_*(\bar{y}_1) &= 1 \otimes \bar{y}_1 + \bar{\tau}_0 \otimes (x[\sigma\bar{\xi}_1] + \bar{\xi}_1\bar{z}_0) + \bar{\tau}_1 \otimes \bar{z}_0, \\ \nu_*(\bar{y}_j) &= 1 \otimes \bar{y}_j + i\bar{\tau}_0 \otimes ([\sigma\bar{\xi}_1]\bar{y}_{j-1} + \bar{\xi}_1\bar{z}_{j-1}) + i\bar{\tau}_1 \otimes \bar{z}_{j-1} + \\ &\quad + i\bar{\tau}_0\bar{\tau}_1 \otimes [\sigma\bar{\xi}_1]\bar{z}_{j-2} \quad \text{for } j \geq 2, \\ \nu_*([\sigma\bar{\tau}_2]) &= 1 \otimes [\sigma\bar{\tau}_2] + \bar{\tau}_0 \otimes x^{p-2}\bar{z}_{p-1}. \end{aligned}$$

Proof. The class x is known to be primitive. On classes in the image of σ_* , like \bar{z}_0 , $[\sigma\bar{\xi}_1]$ and $[\sigma\bar{\tau}_2]$, the coaction is determined by (4.6).

The class \bar{y}_1 was defined such that $\phi_*(\bar{y}_1) = x[y] + c\bar{\tau}_1[z]$. By Proposition 7.5 we have

$$\nu_*\phi_*(\bar{y}_1) = 1 \otimes x[z] + \bar{\tau}_0 \otimes (x[\sigma\bar{\xi}_1] + c\bar{\xi}_1[z]) + \bar{\tau}_1 \otimes c[z]$$

in $A_* \otimes H_*(THH(ku, M); \mathbb{F}_p)$. Since ϕ_* is injective on classes of δ -weight 1 we have by naturality

$$\nu_*(\bar{y}_1) = 1 \otimes \bar{y}_1 + \bar{\tau}_0 \otimes (x[\sigma\bar{\xi}_1] + \bar{\xi}_1\bar{z}_0) + \bar{\tau}_1 \otimes \bar{z}_0.$$

The product formulas $x\bar{y}_j = \bar{y}_1\bar{y}_{j-1}$ and $x\bar{z}_i = \bar{z}_0\bar{y}_i$ allow us to compute inductively the coaction on \bar{y}_j and \bar{z}_i for $2 \leq j \leq p-1$ and $1 \leq i \leq p-1$. Again, by Lemma 8.6 there is no indeterminacy upon dividing $\nu_*(x\bar{z}_i)$ and $\nu_*(x\bar{y}_j)$ by $1 \otimes x$. \square

Recall from [Ok] that $V(1)$ is a ring spectrum if and only if $p \geq 5$. Our next aim is to describe $V(1)_*THH(ku)$ as an algebra over $V(1)_*ku$ if $p \geq 5$, and as a module over $V(1)_*ku$ if $p = 3$.

8.12. Remark. The obstruction in [Ok, Example 4.5] for $V(1)$ to be a ring spectrum at $p = 3$ vanishes when $V(1)$ is smashed with $H\mathbb{Z}_p$ or ku . In particular $V(1) \wedge THH(ku)$ is a ring spectrum at $p = 3$. However this relies on the ku -algebra structure of $THH(ku)$, and the product on $V(1)_*THH(ku)$ for $p = 3$ is not natural. For instance $TC(ku; p)$ and $K(ku)$ are not ku -algebras.

We define a $P_{p-1}(u)$ algebra Θ_* . It is the counterpart of Ξ_* in $V(1)$ -homotopy and is abstractly isomorphic to it.

8.13. Definition. Assume $p \geq 3$, and let Θ_* be the (graded) commutative unital $P_{p-1}(u)$ -algebra with generators

$$\begin{cases} a_i & 0 \leq i \leq p-1, \\ b_j & 1 \leq j \leq p-1, \\ \mu_2, \end{cases}$$

and relations

$$\begin{cases} u^{p-2}a_i = 0 & 0 \leq i \leq p-2, \\ u^{p-2}b_j = 0 & 1 \leq j \leq p-1, \\ b_i b_j = u b_{i+j} & i+j \leq p-1, \\ a_i b_j = u a_{i+j} & i+j \leq p-1, \\ b_i b_j = u b_{i+j-p} \mu_2 & i+j \geq p, \\ a_i b_j = u a_{i+j-p} \mu_2 & i+j \geq p, \\ a_i a_j = 0 & 0 \leq i, j \leq p-1. \end{cases}$$

Here by convention $b_0 = u$, and the degree of the generators is $|a_i| = 2pi + 3$, $|b_j| = 2pj + 2$ and $|\mu_2| = 2p^2$.

8.14. Remark. Let us describe the $P_{p-1}(u)$ -algebra Θ_* more explicitly. The class μ_2 generates a polynomial subalgebra $P(\mu_2) \subset \Theta_*$. The quotient algebra $Q_* = \Theta_*/(\mu_2)$ is a finite graded $P_{p-1}(u)$ -algebra with $2(p-1)^2$ elements, and is given additively as

$$Q_* = P_{p-1}(u) \oplus P_{p-2}(u)\{a_0, b_1, a_1, b_2, \dots, a_{p-2}, b_{p-1}\} \oplus P_{p-1}(u)\{a_{p-1}\}.$$

In particular Q_* has $u^{p-2}a_{p-1}$ as top-class in dimension $2p^2 - 1$, and has the remarkable property of satisfying the duality relation

$$Q_n \cong Q_{2p^2-1-n}$$

for any $0 \leq n \leq 2p^2 - 1$.

8.15. Theorem. *Let Θ_* be the $P_{p-1}(u)$ -algebra defined above. There is an isomorphism*

$$V(1)_*THH(ku) \cong E(\lambda_1) \otimes \Theta_*,$$

where λ_1 is of degree $2p-1$. If $p \geq 5$ this is an isomorphism of $P_{p-1}(u)$ -algebras. If $p = 3$ it is, at least, an isomorphism of $P_{p-1}(u)$ -modules such that $P(\mu_2)$ includes as a subalgebra in $V(1)_*THH(ku)$.

Theorem 0.1 follows from this result. The quotient A_* featured there can be identified with $Q_* \otimes E(\lambda_1)$.

Proof for $p \geq 5$. Since $V(1) \wedge \ell \simeq H\mathbb{F}_p$, the spectrum $V(1) \wedge THH(ku)$ is an $H\mathbb{F}_p$ -module and its homology is given by

$$H_*(V(1) \wedge THH(ku); \mathbb{F}_p) \cong A_* \otimes E([\sigma \bar{\xi}_1]) \otimes \Xi_*.$$

The Hurewicz map

$$V(1)_*THH(ku) \rightarrow H_*(V(1) \wedge THH(ku); \mathbb{F}_p)$$

is injective with image the A_* -comodule primitives. We identify $V(1)_*THH(ku)$ with its image (in particular $P_{p-1}(u)$ is identified with $P_{p-1}(x)$). Consider the following classes in $H_*(V(1) \wedge THH(ku); \mathbb{F}_p)$:

$$\begin{aligned} a_0 &= \bar{z}_0, \\ b_1 &= \bar{y}_1 - \bar{\tau}_0 x [\sigma \bar{\xi}_1] - \bar{\tau}_1 \bar{z}_0, \\ \lambda_1 &= [\sigma \bar{\xi}_1], \\ \mu_2 &= [\sigma \bar{\tau}_2] - \bar{\tau}_0 x^{p-2} [z_{p-1}]. \end{aligned}$$

By Proposition 8.11 these classes are comodule primitives. Lemma 8.6 also holds for $H_*(V(1) \wedge THH(ku); \mathbb{F}_p)$. We define inductively $b_{j+1} = u^{-1} b_1 b_j$, for $1 \leq j \leq p-2$, and $a_i = u^{-1} a_0 b_i$, for $1 \leq i \leq p-1$. These classes b_j and a_i are primitive by construction. By inspection, the classes a_i , b_j and μ_2 satisfy the relations over $P_{p-1}(u)$ given in Definition 8.13. There is an isomorphism

$$H_*(V(1) \wedge THH(ku); \mathbb{F}_p) \cong A_* \otimes E(\lambda_1) \otimes \Theta_*.$$

where the $P_{p-1}(u)$ -algebra $E(\lambda_1) \otimes \Theta_*$ consists of A_* -comodule primitives. \square

Proof for $p = 3$. Applying $V(1)_*THH(ku, -)$ to the diagram (1.3) we obtain an unrolled exact couple and a strongly convergent spectral sequence of algebras

$$(8.16) \quad E_{*,*}^2 = V(1)_*THH(ku, H\mathbb{Z}_p) \otimes P(u) \implies V(1)_*THH(ku)$$

analogous to the mod u spectral sequence of Proposition 6.2. Here u is of bidegree $(-2, 4)$ and represents the mod v_1 reduction of the Bott element. By Theorem 5.7 we have

$$E_{0,*}^2 = V(1)_*THH(ku, H\mathbb{Z}_p) \cong E(z, \lambda_1, \varepsilon) \otimes P(\mu_1)$$

where ε has degree $2p - 1$ with a primary v_1 Bockstein $\beta_{1,1}(\varepsilon) = 1$. We deduce from Theorem 6.7 that there is a differential

$$d^2(\mu_1) = uz.$$

The ku -module structure of $THH(ku)$ implies a differential

$$d^4(\varepsilon) = u^2.$$

At this point the spectral sequence collapses and this leaves

$$E_{*,*}^\infty = E_{*,*}^5 = E(\lambda_1) \otimes P(\mu_1^3) \otimes [E(\varepsilon) \otimes \mathbb{F}_p \{z, z\mu_1\} \oplus E(z\mu_1^2) \otimes P_2(u)].$$

Defining $\mu_2 = \mu_1^3$, $a_i = z\mu_1^i$ and $b_j = z\varepsilon\mu_1^{j-1}$ we obtain the $P_2(u)$ -module structure of $V(1)_*THH(ku)$ for $p = 3$, as claimed in Theorem 8.15. Notice that $P(\mu_2)$ is a subalgebra of $E_{*,*}^\infty$, and hence of $V(1)_*THH(ku)$. \square

8.17. Remark. The proof for $p = 3$ given above almost determines the whole product structure on $V(1)_*THH(ku)$. In fact the permanent cycles in the spectral sequence (8.16) are concentrated in filtration degrees 0 and -2 , so there is not much room for multiplicative extensions. By inspection all multiplicative relations

of Θ_* can be read from this spectral sequence except for $a_0b_2 = a_1b_1 = ua_2$ and $b_1b_2 = 0$.

8.18. Remark. The proof given for $p = 3$ is also valid for primes $p \geq 5$, and provides an easy way to determine the additive structure of $V(1)_*THH(ku)$. In that case the differentials of (8.16) are given by

$$\begin{cases} d^{2(p-2)}(\mu_1) = zu^{p-2} \\ d^{2(p-1)}(\varepsilon) = u^{p-1}. \end{cases}$$

However, the permanent cycles are now scattered through $p - 1$ filtration degrees and one is left with solving the multiplicative extensions.

At this point it is of course also possible to study the mod v_1 Bockstein spectral sequence in order to recover $V(0)_*THH(ku)$ from $V(1)_*THH(ku)$. However, because of the relation $u^{p-1} = v_1$ in $V(0)_*ku$, the mod u Bockstein spectral sequence of §6 is more appropriate. Let us just describe the primary mod v_1 Bockstein, which involves some of the generators of Θ_* . The following proposition is a consequence of Corollary 6.8.

8.19. Proposition. *Let $p \geq 3$. In $V(1)_*THH(ku)$ there are primary mod v_1 Bocksteins*

$$\beta_{1,1}(b_i) = a_{i-1}$$

for $1 \leq i \leq p - 1$.

9. ON THE EXTENSION $\ell \rightarrow ku$

In this final section we analyze the homomorphism

$$V(1)_*THH(\ell) \rightarrow V(1)_*THH(ku)$$

induced by the \mathbb{S} -algebra map $\ell \rightarrow ku$ defined in 1.1. We then interpret our computations above in terms of number-theoretic properties of the extension $\ell \rightarrow ku$.

The \mathbb{F}_p -algebras $V(1)_*THH(\ell)$ and $V(1)_*THH(ku)$ were described in Theorems 4.9 and 8.15, respectively. Let Δ be the group defined in 1.1, and recall the notion of δ -weight from Definition 7.2.

9.1. Proposition. *The classes λ_1 and μ_2 have δ -weight 0 in $V(1)_*THH(ku)$, and the classes u , a_i and b_j have δ -weight 1. The homomorphism $V(1)_*THH(\ell) \rightarrow V(1)_*THH(ku)$ is given by $\lambda_1 \mapsto \lambda_1$, $\lambda_2 \mapsto u^{p-2}a_{p-1}$ and $\mu_2 \mapsto \mu_2$. In particular it is injective with image the classes of δ -weight 0, and induces an isomorphism*

$$V(1)_*THH(\ell) = (V(1)_*THH(ku))^\Delta.$$

Proof. These statements are proven in homology, where they follow directly from the definition of the various algebra generators. \square

Let us denote by $TC(\ell; p)$ the topological cyclic homology spectrum of ℓ , and by $K(\ell)$ its algebraic K-theory

9.2. Theorem. *Let p be an odd prime. There are homotopy equivalences*

$$\begin{aligned} THH(ku)^{h\Delta} &\simeq_p THH(\ell), \\ TC(ku; p)^{h\Delta} &\simeq_p TC(\ell; p), \quad \text{and} \\ K(ku)^{h\Delta} &\simeq_p K(\ell). \end{aligned}$$

Proof. Consider the homotopy fixed point spectral sequence

$$E_{s,t}^2 = H^{-s}(\Delta, V(1)_t THH(ku)) \implies V(1)_{t+s} THH(ku)^{h\Delta}.$$

By Proposition 9.1, and since the order of Δ is prime to p , its E^2 -term is given by

$$E_{s,t}^2 = \begin{cases} V(1)_t THH(\ell) & \text{if } s = 0, \\ 0 & \text{if } s \neq 0. \end{cases}$$

Thus the spectral sequence collapses and its edge homomorphism yields an isomorphism

$$V(1)_* THH(ku)^{h\Delta} \cong V(1)_* THH(\ell).$$

The spectra $THH(ku)^{h\Delta}$ and $THH(\ell)$ are both connective. In particular their $V(1)_*$ -localization and their $V(0)_*$ -localization (or p -completion) agree. Thus we have an equivalence of p -completed spectra $THH(ku)^{h\Delta} \simeq_p THH(\ell)$.

The spectrum $TC(ku; p)$ is defined as the homotopy limit

$$TC(ku; p) = \operatorname{holim}_{F, R} THH(ku)^{C_{p^n}}$$

taken over the Frobenius and the restriction maps

$$F, R : THH(ku)^{C_{p^n}} \rightarrow THH(ku)^{C_{p^{n-1}}}$$

that are part of the cyclotomic structure of THH . In particular we have an equivalence

$$TC(ku; p)^{h\Delta} \simeq \operatorname{holim}_{F^{h\Delta}, R^{h\Delta}} (THH(ku)^{C_{p^n}})^{h\Delta}.$$

Thus the equivalence $TC(ku; p)^{h\Delta} \simeq_p TC(\ell; p)$ will follow from the claim that for each $n \geq 0$, there is an equivalence $(THH(ku)^{C_{p^n}})^{h\Delta} \simeq_p (THH(\ell)^{C_{p^n}})^{h\Delta}$. We proceed by induction on n , the case $n = 0$ having been proven above. Let $n \geq 1$ and assume that the claim has been proven for $m < n$. Consider the homotopy commutative diagram

$$\begin{array}{ccccc} THH(\ell)_{hC_{p^n}} & \xrightarrow{N} & THH(\ell)^{C_{p^n}} & \xrightarrow{R} & THH(\ell)^{C_{p^{n-1}}} \\ \downarrow & & \downarrow & & \downarrow \\ (THH(ku)_{hC_{p^n}})^{h\Delta} & \xrightarrow{N^{h\Delta}} & (THH(ku)^{C_{p^n}})^{h\Delta} & \xrightarrow{R^{h\Delta}} & (THH(ku)^{C_{p^{n-1}}})^{h\Delta} \end{array}$$

where the top line is the norm-restriction fiber sequences for $THH(\ell)$ and the bottom line is obtained by taking the Δ homotopy-fixed points of the one for $THH(ku)$. By induction hypothesis the right hand side vertical arrow is an equivalence. Since

$THH(ku)_{hC_{p^n}}$ is p -complete and Δ is of order prime to p , the homotopy norm map $(THH(ku)_{hC_{p^n}})_{h\Delta} \rightarrow (THH(ku)_{hC_{p^n}})^{h\Delta}$ is an equivalence. This implies that

$$(THH(ku)_{hC_{p^n}})^{h\Delta} \simeq (THH(ku)^{h\Delta})_{hC_{p^n}}.$$

The left hand side vertical arrow is therefore also an equivalence. Thus the middle vertical arrow is an equivalence, which completes the proof of the claim.

Finally, by [HM1] and [Du] we have natural cofibre sequences

$$K(\ell)_p \rightarrow TC(\ell; p) \rightarrow \Sigma^{-1}H\mathbb{Z}_p \quad \text{and} \quad K(ku)_p \rightarrow TC(ku; p) \rightarrow \Sigma^{-1}H\mathbb{Z}_p$$

We take the Δ -homotopy fixed points of the latter one and assemble these cofibre sequences in a commutative diagram

$$\begin{array}{ccccc} K(\ell)_p & \xrightarrow{\text{trc}} & TC(\ell; p) & \longrightarrow & \Sigma^{-1}H\mathbb{Z}_p \\ \downarrow & & \downarrow & & \downarrow = \\ K(ku)_p^{h\Delta} & \xrightarrow{\text{trc}^{h\Delta}} & TC(ku; p)^{h\Delta} & \longrightarrow & \Sigma^{-1}H\mathbb{Z}_p \end{array}$$

The middle and right hand side vertical arrows are equivalences. Thus the map $K(\ell)_p \rightarrow K(ku)_p^{h\Delta}$ is also an equivalence. \square

The computations given in this paper provide evidence for interesting speculations on the properties of the extension $\ell \rightarrow ku$, and on how these properties are reflected in topological Hochschild homology.

Let us assume that we can make sense at a spectrum level of the formula

$$(9.3) \quad ku = \ell[u]/(u^{p-1} = v_1)$$

which holds for the coefficients rings. The prime (v_1) of ℓ ramifies as $(u)^{p-1}$ in ku , so the extension $\ell \rightarrow ku$ should not qualify as an étale extension. And indeed, the computations of $V(1)_*THH(\ell)$ and $V(1)_*THH(ku)$ given in Theorems 4.9 and 8.15 imply that

$$ku \wedge_{\ell} THH(\ell) \not\cong_p THH(ku).$$

Compare with the algebraic situation, where the Geller-Weibel Theorem [GW] states that if an extension $A \rightarrow B$ of k -algebras is étale, this is reflected by an isomorphism

$$B \otimes_A HH_*^k(A) \cong HH_*^k(B)$$

in Hochschild homology.

Inverting v_1 in ℓ and ku , we obtain the periodic Adams summand L and the periodic K-theory spectrum KU (both p -completed). The map $L \rightarrow KU$ induces on coefficients the inclusion

$$L_* = \mathbb{Z}_p[v_1, v_1^{-1}] \hookrightarrow \mathbb{Z}_p[u, u^{-1}] = KU_*.$$

Now $(p-1)u^{p-2}$ is invertible in KU and we expect the extension $L \rightarrow KU$ to be étale (compare with Remark 2.7). Evidence for this is provided by the computations in topological Hochschild homology given in (6.11) and Proposition 6.12, which imply that we have an equivalence

$$KU \wedge_L THH(L) \simeq_p THH(KU).$$

9.4. Tame ramification. The extension $\ell \rightarrow ku$ is not unramified, but from formula (9.3) we nevertheless expect it to be tamely ramified. In particular we view Theorem 9.2 as an example of tamely ramified descent.

The behavior of topological Hochschild homology with respect to tamely ramified extensions of discrete valuation rings was studied by Hesselholt and Madsen in [HM2]. Their results can be used to provide an interesting, at this point very speculative explanation of the structure of $THH(ku)$. It is due to Lars Hesselholt, and I would like to thank him for sharing the ideas exposed in the remaining part of this paper.

Let us briefly recall the results of [HM2] that are relevant here. Let A be a discrete valuation ring, K its quotient field (of characteristic 0) and k its perfect residue field (of characteristic p). The localization cofibre sequence in algebraic K -theory maps via the trace map to a localization sequence in topological Hochschild homology. We have a commutative diagram

$$\begin{array}{ccccc} K(k) & \xrightarrow{i^!} & K(A) & \xrightarrow{j} & K(K) \\ \downarrow & & \downarrow & & \downarrow \\ THH(k) & \xrightarrow{i^!} & THH(A) & \xrightarrow{j} & THH(A|K) \end{array}$$

whose rows are cofibre sequences. Here the map $i^!$ is the transfer, and j is a map of ring spectra. The cofibre $THH(A|K)$ is defined in [HM2, Definition 1.5.5] as topological Hochschild homology of a suitable linear category.

Let $M = A \cap K^\times$, and consider the log ring (A, M) with pre-log structure given by the inclusion $\alpha : M \hookrightarrow A$. Then the homotopy groups $(\pi_* THH(A|K), M)$ form a log differential graded ring over (A, M) . The universal example of such a log differential graded ring is the de Rham-Witt complex with log poles $\omega_{(A, M)}^*$, and there is a canonical map $\omega_{(A, M)}^* \rightarrow \pi_* THH(A|K)$. Hesselholt and Madsen define an element $\kappa \in V(0)_* THH(A|K)$ and prove in [HM2, Theorem 2.4.1] that there is a natural isomorphism

$$(9.5) \quad \omega_{(A, M)}^* \otimes_{\mathbb{Z}} P(\kappa) \xrightarrow{\cong} V(0)_* THH(A|K).$$

Let L be a finite, tamely ramified extension of K , and B be the integral closure of A in L . It follows from [HM2, Lemma 2.2.4 and 2.2.6] that there is an isomorphism

$$(9.6) \quad B \otimes_A \omega_{(A, M_A)}^* \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\cong} \omega_{(B, M_B)}^* \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

In fact a tamely ramified extension $A \rightarrow B$ has the formal property of an étale extension in the context of log rings (i.e. it is log-étale), and this isomorphism is analogous to the Geller-Weibel Theorem mentioned above. Assembling (9.5) and (9.6) we obtain an isomorphism

$$B \otimes_A V(0)_* THH(A|K) \xrightarrow{\cong} V(0)_* THH(B|L).$$

In particular we have an equivalence

$$(9.7) \quad HB \wedge THH(A|K) \simeq THH(B|L)$$

of p -completed spectra.

Let us now optimistically assume that these results hold also in the generality of commutative \mathbb{S} -algebra. The ring ℓ has a maximal ideal (v_1) , with residue ring $H\mathbb{Z}_p$ and quotient ring L . Similarly, ku has a maximal ideal (u) , with residue ring $H\mathbb{Z}_p$ and quotient ring KU . The localization cofibre sequences in topological Hochschild homology fit into a commutative diagram

$$(9.8) \quad \begin{array}{ccccc} THH(H\mathbb{Z}_p) & \xrightarrow{i^!} & THH(\ell) & \xrightarrow{j} & THH(\ell|L) \\ \simeq \downarrow & & \downarrow & & \downarrow \\ THH(H\mathbb{Z}_p) & \xrightarrow{i^!} & THH(ku) & \xrightarrow{j} & THH(ku|KU). \end{array}$$

This requires that we can identify by dévissage $THH(H\mathbb{Z}_p)$ with the topological Hochschild homology spectrum of a suitable category of finite v_1 -torsion ℓ -modules or ku -modules. Since $\ell \rightarrow ku$ is tamely ramified, we expect that there is an equivalence

$$(9.9) \quad ku \wedge_{\ell} THH(\ell|L) \simeq THH(ku|KU)$$

analogous to (9.7). The top cofibration of (9.8) induces a long exact sequence

$$\dots \xrightarrow{\partial_*} V(1)_n THH(H\mathbb{Z}_p) \xrightarrow{i_*^!} V(1)_n THH(\ell) \xrightarrow{j_*} V(1)_n THH(\ell|L) \xrightarrow{\partial_*} \dots$$

in $V(1)$ homotopy. There are isomorphisms

$$\begin{aligned} V(1)_* THH(H\mathbb{Z}_p) &\cong E(\lambda_1, \varepsilon) \otimes P(\mu_1), \\ V(1)_* THH(\ell) &\cong E(\lambda_1, \lambda_2) \otimes P(\mu_2). \end{aligned}$$

Here ε has degree $2p - 1$ and supports a primary v_1 -Bockstein $\beta_{1,1}(\varepsilon) = 1$. From the structure of the higher v_1 -Bocksteins we know that $i_*^!(E(\lambda_1) \otimes P(\mu_1)) = 0$ and that $P(\mu_2)$ injects into $V(1)_* THH(\ell|L)$ via j_* . Thus we expect that

$$V(1)_* THH(\ell|L) = E(d, \lambda_1) \otimes P(\mu_1),$$

where $d \in V(1)_1 THH(\ell|L)$ satisfies $\partial_*(d) = 1$. We should have $\partial_*(d\lambda_1) = \lambda_1$, $\partial_*(d\mu_1^j) = \mu_1^j$, $\partial_*(\mu_1) = \varepsilon$, $j_*(\lambda_1) = \lambda_1$, $j_*(\mu_2^k) = \mu_1^{pk}$ and $i^!(\varepsilon\mu_1^{p-1}) = \lambda_2$. How the remaining classes map under ∂_* , $i_*^!$ or j_* is then forced by the grading and the exactness. We deduce from (9.9) that

$$V(1)_* THH(ku|KU) \cong P_{p-1}(u) \otimes E(\lambda_1, d) \otimes P(\mu_1).$$

Assembling these computations in diagram (9.8) and chasing, we obtain an (additive) isomorphism

$$V(1)_* THH(ku) \cong [E(\lambda_1, \lambda_2) \otimes P(\mu_2)] \oplus [E(\lambda_1, d) \otimes P(\mu_1) \otimes \mathbb{F}_p\{u, \dots, u^{p-2}\}].$$

Under the identifications $a_i = du\mu_1^i$ and $b_j = u\mu_1^j$, this is compatible with Theorem 8.15

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