On the spectrum \(b^0 \wedge \text{tmf}\)

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Abstract

M. Mahowald, in his work on \(b^0\)-resolutions, constructed a \(b^0\)-module splitting of the spectrum \(b^0 \wedge b^0\) into a wedge of summands related to integral Brown-Gitler spectra. In this paper, a similar splitting of \(b^0 \wedge \text{tmf}\) is constructed. This splitting is then used to understand the \(b^0\_\ast\)-algebra structure of \(b^0\_\ast\text{tmf}\) and allows for a description of \(b^0\_\ast\text{tmf}\).

1 Introduction

All cohomology groups are assumed to have coefficients in \(\mathbb{F}_2\) and all spectra completed at the prime 2 unless stated otherwise. Let \(\mathcal{A}\) denote the Steenrod algebra, and \(\mathcal{A}(n)\) the subalgebra generated by \(\{Sq^1, Sq^2, \ldots, Sq^n\}\). Consider the Hopf algebra quotient \(\mathcal{A}/\mathcal{A}(n) = \mathcal{A} \otimes_{\mathcal{A}(n)} \mathbb{F}_2\). Here the right action of \(\mathcal{A}(n)\) on \(\mathcal{A}\) is induced by the inclusion and the left action on \(\mathbb{F}_2\) by the augmentation. Algebraically, one can consider the subsequent surjections

\[\mathcal{A} \to \mathcal{A}/\mathcal{A}(0) \to \mathcal{A}/\mathcal{A}(1) \to \mathcal{A}/\mathcal{A}(2) \to \mathcal{A}/\mathcal{A}(3) \to \cdots\]

and ask whether each algebra can be realized as the cohomology of some spectrum. The case \(n \geq 3\) requires the existence of a non-trivial map \(S^{2n+1-1} \to S^0\) which cannot occur due to Hopf invariant one. For \(n < 3\), however, it is now well-known that each algebra can indeed be realized by the cohomology of some spectrum:

\[H^*\mathbb{F}_2 \to H^*\mathbb{Z} \to H^*b^0 \to H^*\text{tmf}\]
There are maps realizing the above homomorphisms of cohomology groups

\[ \text{tmf} \to \text{bo} \to \mathbb{H} \to \mathbb{H}_2 \]

In particular, the spectrum tmf is at the top of a “tower” whose “lower floors” have been well studied in the literature, culminating with Mahowald’s [6] understanding of the spectrum bo \& bo and Carlsson’s [1] description of the cohomology operations [bo, bo]. More difficult questions arise: What is the structure of tmf \& tmf? What are the stable cohomology operations of tmf?

We would like to understand the spectrum bo\&tmf for a variety of reasons. First, it might serve as a nice intermediate step towards understanding the spectrum tmf \& tmf. Furthermore, determining its structure comes with an added bonus of understanding operations [tmf, bo] which may provide some insight into understanding the cohomology operations of tmf. Second, the splitting of bo \& tmf has been instrumental to the author in demonstrating the splitting of the Tate spectrum of tmf into a wedge of suspensions of bo.

Let B_1(j) denote the j\textsuperscript{th} integral Brown-Gitler spectrum, whose homology will be described as a submodule of \( H_* \mathbb{H} \). Such spectra have been studied extensively in the literature (see [2], [5], [9], for example). In particular, Mahowald [6] demonstrated the splitting of bo-module spectra bo \& bo \simeq \bigvee_{j \geq 0} \Sigma^{4j}bo \& B_1(j). \) Let \( \Omega = \bigvee_{0 \leq j \leq 1} \Sigma^{8j+4}B_1(j). \) The main theorem of this paper is the following

**Theorem 1.1.** There is a homotopy equivalence of bo-module spectra

\[ bo \& \Omega \to bo \& \text{tmf} \quad (1) \]

The splitting is analogous to that of bo \& bo of Mahowald and even MO(8) \& bo of Davis [3]. Its proof, therefore, contains ideas and results from both. Section 2 deals with demonstrating an isomorphism on the level of homotopy groups, which first requires an understanding of the left \( \mathcal{A}(1) \)-module structure of \( H^\ast \text{tmf} \). In Section 3, we construct a map of bo-module spectra realizing the isomorphism of homotopy groups. Section 4 uses this splitting along with pairings of integral Brown-Gitler spectra to explicitly determine the bo_*-algebra structure of bo_*tmf and also identifies the cohomology bo^*tmf.
2 The algebraic splitting

The $E_2$-term of the Adams spectral sequence converging to the homotopy groups of $bo \wedge tmf$ is given by

$$\text{Ext}^s_A(H^*(bo \wedge tmf), F_2) \Rightarrow \pi_{t-s}(bo \wedge tmf).$$

(2)

The Ext-group appearing in the above spectral sequence can be simplified via a change-of-rings isomorphism:

$$\text{Ext}^s_A(H^*tmf, F_2) \Rightarrow \pi_{t-s}(bo \wedge tmf).$$

(3)

Therefore, it suffices to understand the left $A(1)$-module structure of $H^*tmf$. Computations and definitions simplify upon dualizing. Indeed, the dual Steenrod algebra, $A_*$, is the graded polynomial ring $F_2[\xi_1, \xi_2, \xi_3, \ldots]$ with $|\xi_i| = 2^i - 1$. An equivalent problem after dualizing is determining the right $A(1)$-module structure of the subring $H_+tmf \subset A_*$. The homology of $tmf$ as a right $A$-module is given by Rezk [8]

$$H_+tmf \cong F_2[\zeta_1, \zeta_2, \zeta_3, \ldots].$$

(4)

The generators $\zeta_i = \chi \xi_i$, where $\chi : A_* \to A_*$ is the canonical antiautomorphism. Define a new weight on elements of $A_*$ by $\omega(\zeta_i) = 2^{i-1}$ for $i \geq 1$. For $a, b \in A_*$ define the weight on their product by $\omega(ab) = \omega(a) + \omega(b)$. Let $N_{k}^{tmf}$ denote the $F_2$-vector space inside $H_+tmf$ generated by all monomials of weight $k$ with $N_0^{tmf} = F_2$ generated by the identity.

Lemma 2.1. As right $A(2)$-modules,

$$H_+tmf \cong \bigoplus_{i \geq 0} N_{8i}^{tmf}.$$

Proof. Certainly, the two modules are isomorphic as $F_2$-vector spaces. To see there is an isomorphism of right $A(2)$-modules, note that the right action of the total square $Sq = \sum_{i \geq 0} Sq^i$ on the generators of $H_+tmf$ is given by:

$$\begin{align*}
\zeta_1^8 \cdot Sq &= \zeta_1^8 + 1; \\
\zeta_2^4 \cdot Sq &= \zeta_2^4 + \zeta_1^8 + 1; \\
\zeta_3^2 \cdot Sq &= \zeta_3^2 + \zeta_2^4 + \zeta_1^8 + 1; \\
\zeta_n \cdot Sq &= \sum_{i=0}^{n} \zeta_{8i-n}^2
\end{align*}$$
for \( n > 3 \). Since \( \omega(1) = 0 \), modulo the identity the total square preserves the weight of the generators of \( H_*\text{tmf} \). Note that \( \zeta_i^{2^{n-1}}Sq^{2^{n-1}} = 1 \), hence the total square over \( A(2) \) cannot contain a 1 in the expansion for dimensional reasons.

Consider the homomorphism \( V : A_* \to A_* \) defined on generators by

\[
V(\zeta_i) = \begin{cases} 
1, & i = 0, 1; \\
\zeta_{i-1}, & i \geq 2.
\end{cases}
\]

Restricting \( V \) to the subring \( H_*\text{tmf} \subset A_* \) clearly provides a surjection \( V_{\text{tmf}} : H_*\text{tmf} \to H_*\text{bo} \). Let \( M_{\text{bo}}(4i) \) denote the image of \( N_{\text{tmf}} \) under the homomorphism \( V_{\text{tmf}} \). It is generated by all monomials with \( \omega(\zeta^I) \leq 4i \). The following proposition is clear.

**Proposition 2.2.** As right \( A(2) \)-modules

\[
N_{\text{tmf}} \cong \sum_{i=0}^{n} M_{\text{bo}}(4i). 
\]  

**Proof.** Due to the weight requirements, \( V_{\text{tmf}} \) is injective when restricted to \( N_{\text{tmf}} \). Indeed, the exponent of \( \zeta_1 \) in each monomial is uniquely determined by the other exponents.

Additionally, if we denote by \( N_k^{\text{bo}} \) the \( \mathbb{F}_2 \)-vector space inside \( H_*\text{bo} \) generated by all elements of weight \( k \) with \( N_0^{\text{bo}} = \mathbb{F}_2 \) generated by the identity, we have a similar lemma:

**Lemma 2.3.** As right \( A(1) \)-modules,

\[
M_{\text{bo}}(4i) \cong \bigoplus_{j=0}^{n} N_{4j}^{\text{bo}}.
\]

Further restricting \( V \) to the subring \( H_*\text{bo} \) provides a surjection \( V_{\text{bo}} : H_*\text{bo} \to H_*\text{HZ} \). Let \( M_{\text{HZ}}(2j) \) denote the image of \( N_{4j}^{\text{bo}} \) under \( V \). This sub-module is generated by all monomials with \( \omega(\zeta^I) \leq 2j \). As in Proposition 2.2 we have the identification

**Proposition 2.4.** As right \( A(1) \)-modules,

\[
N_{4j}^{\text{bo}} \cong \sum_{j=0}^{j} M_{\text{HZ}}(2j).
\]
Goerss, Jones and Mahowald [5] identify the submodule $M_{HZ}(2j) \subset H_*HZ$ as the homology of the $j$th integral Brown-Gitler spectrum:

**Theorem 2.5** (Goerss, Jones, Mahowald [5]). For $j \geq 0$, there is a spectrum $B_1(j)$ and a map

$$B_1(j) \xrightarrow{g} HZ$$

such that

(i) $g_*$ sends $H_*(B_1(j))$ isomorphically onto the span of monomials of weight $\leq 2j$;

(ii) there are pairings

$$B_1(m) \wedge B_1(n) \to B_1(m+n)$$

whose homology homomorphism is compatible with the multiplication in $H_*HZ$.

**Remark 2.1.** The submodules $M_{bo}(4i)$ are the so-called $bo$-Brown-Gitler modules. There is a family of spectra with similar properties, having these modules as their homology. Proposition 2.2 demonstrates that as an $A(2)$-module, $H_*tmf$ is a direct sum of these modules. On the level of spectra, however, $tmf \wedge tmf$ does not split as a wedge of $bo$-Brown-Gitler spectra.

Combining the results of Lemmas 2.1 and 2.3 with Theorem 2.5, $H_*tmf$ as a right $A(1)$-module can be written in terms of homology of integral Brown-Gitler spectra:

**Theorem 2.6.** As right $A(1)$-modules,

$$H_*tmf \cong \bigoplus_{0 \leq j \leq i} \Sigma^{8i+4j} H_*B_1(j).$$

The $E_2$-term of the Adams spectral sequence (3) then becomes isomorphic to

$$\bigoplus_{0 \leq j \leq i} \Sigma^{8i+4j} Ext_{A(1)}^{s,t} (H^*B_1(j), F_2) \Rightarrow \pi_{t-s}(bo \wedge tmf). \quad (7)$$

This is precisely the Adams $E_2$-term converging to the homotopy of $bo \wedge \Omega$. The chart can be obtained by applying the following theorem of Davis [4] which links $bo \wedge B_1(n)$ to Adams covers of $bo$ or $bsp$, depending on the parity of $n.$
Theorem 2.7 (Davis [4]). If \( \overline{n} = (n_1, \ldots, n_s) \), let \(|n| = \sum_{i=1}^{s} n_i\) and \(\alpha(\overline{n}) = \sum_{i=1}^{s} \alpha(n_i)\), and \(B_1(\overline{n}) = \bigwedge_{i=1}^{s} B_1(n_i)\). Then there are homotopy equivalences

\[
bo \wedge B_1(\overline{n}) \simeq K \vee \begin{cases} 
bo^{2|n| - \alpha(\overline{n})}, & \text{if } |n| \text{ is even}; \\
\bsp^{2|\overline{n}| - 1 - \alpha(\overline{n})}, & \text{if } |\overline{n}| \text{ is odd};
\end{cases}
\]

where \(K\) is a wedge of suspensions of \(H\mathbb{F}_2\).

Figure 1: \(\operatorname{Ext}_{\mathcal{A}}^{s,t}(H^*(bo \wedge \text{tmf}), \mathbb{F}_2) \Rightarrow \pi_{t-s}(bo \wedge \text{tmf})\)

The charts for \(bo\) and \(bsp\) are well known. Using the above theorem along with the algebraic splitting of \(H^*\text{tmf}\), we see that Adams covers of \(bo\) begin in stems congruent to 0 mod 8 while Adams covers of \(bsp\) begin in stems congruent to 4 mod 8. The first 32 stems of the chart for \(bo \wedge \text{tmf}\) is displayed in Figure 1 modulo possible elements of order 2 in Adams filtration \(s = 0\) corresponding to free \(\mathcal{A}(1)\)'s inside \(H^*\text{tmf}\). The symbol \(\otimes\) appears in Figure 1 to reduce clutter. It is used to mark the beginning of another \(\mathbb{Z}\)-tower. In general, all \(\mathbb{Z}\)-towers are found in stems congruent to 0 mod 4 while those supporting multiplication by \(\eta\) occur in stems congruent to 4 mod 8.

Theorem 2.8. There is an isomorphism of homotopy groups

\[
\pi_*(bo \wedge \text{tmf}) \cong \pi_*(bo \wedge \Omega)
\]
Proof. The $E_2$-terms of their respective Adams spectral sequences have been shown to be isomorphic. Both spectral sequences collapse. Indeed, the classes charted in Figure 1 cannot support differentials for dimensional and naturality reasons. Each element of order two in Adams filtration $s = 0$ correspond to copies of $A(1)$ inside $H^*\text{tmf}$. These summands split off, obviating the existence of differentials. \hfill\qed

3 The topological splitting

Theorem 1.1 concerns a $\text{bo}$-module splitting of the spectrum $\text{bo} \land \text{tmf}$. The following observation will aid us in studying $\text{bo}$-module maps.

Lemma 3.1. Let $X$ and $Y$ be spectra. Then

$$[\text{bo} \land X, \text{bo} \land Y]_{\text{bo}} = [X, \text{bo} \land Y]$$

Proof. Let $u_{\text{bo}} : S^0 \to \text{bo}$ and $m_{\text{bo}} : \text{bo} \land \text{bo} \to \text{bo}$ denote the unit and the product map of $\text{bo}$, respectively. Given $f : \text{bo} \land X \to \text{bo} \land Y$ and $g : X \to \text{bo} \land Y$, the equivalence is given by the composites

$$f \mapsto f \circ (u \land 1)$$

$$g \mapsto (m_{\text{bo}} \land 1) \circ (1 \land g)$$

The spectra $(\text{bo}, m_{\text{bo}}, u_{\text{bo}})$ and $(\text{tmf}, m_{\text{tmf}}, u_{\text{tmf}})$ are both unital $E_\infty$-ring spectra [7]. This induces a unital $E_\infty$-ring structure $(\text{bo} \land \text{tmf}, m, u)$. This structure will play an important role in the proof of the main theorem. We begin by defining an increasing filtration of $\Omega$ via:

$$\Omega^n = \bigvee_{j=0}^{n} \bigvee_{i=1}^{\infty} \Sigma^{8i+4j}B_1(j)$$

Notationally, it will be convenient to let $B(j) = \Sigma^{12j}B_1(j)$, so that the filtration (8) can be rewritten as

$$\Omega^n = \bigvee_{j=0}^{n} \bigvee_{i \geq 0} \Sigma^{8i}B(j)$$

7
The proof of Theorem 1.1 will proceed inductively on \( n \). We will assume the existence of a \( \mathcal{B} \)-module map \( \varphi_{2^i-1} : \mathcal{B} \wedge \Omega^{2^i-1} \to \mathcal{B} \wedge \text{tmf} \) which is a stable \( \mathcal{A} \)-isomorphism through a certain dimension. The inductive step will be then to construct a \( \mathcal{B} \)-module map \( \varphi_{2^{i+1}-1} : \mathcal{B} \wedge \Omega^{2^{i+1}-1} \to \mathcal{B} \wedge \text{tmf} \) which is a stable \( \mathcal{A} \)-isomorphism through higher dimensions. To do this, we will employ the pairings given in Theorem 2.5(ii). Define the map

\[
g_{m,n} : \Sigma^{8n} B(m) \to \mathcal{B} \wedge \text{tmf}
\]  

(10)
to be the restriction of \( \varphi_{2^i-1} \) to the summand \( \Sigma^{8n} B(m) \). Denote by \( g_m = g_{m,0} \).

**Lemma 3.2.** Let \( m = 2^i \) and \( 0 \leq n < m \). Suppose there are \( \mathcal{B} \)-module maps \( f_m : \mathcal{B} \wedge B_1(m) \to \mathcal{B} \wedge \text{tmf} \) and \( f_n : \mathcal{B} \wedge B_1(n) \to \mathcal{B} \wedge \text{tmf} \) inducing injections on homology. Then there is a \( \mathcal{B} \)-module map

\[
f_{m+n} : \mathcal{B} \wedge B_1(m + n) \to \mathcal{B} \wedge \text{tmf}
\]

inducing an injection on homology.

**Proof.** For all \( 0 \leq n < m \), Theorem 2.7 supplies equivalences of \( \mathcal{B} \)-module spectra

\[
\mathcal{B} \wedge B_1(m) \wedge B_1(n) \simeq (\mathcal{B} \wedge B_1(m + n)) \vee K
\]

(11)
where \( K \) is a wedge of suspensions of \( \mathbb{H}F_2 \). There are no maps \([\mathbb{H}F_2, \mathcal{B} \wedge \text{tmf}]\) so that the composite \( m \circ (f_m \wedge f_n) \) lifts as a \( \mathcal{B} \)-module map to the first summand

\[
f_{m+n} : \mathcal{B} \wedge B_1(m + n) \to \mathcal{B} \wedge \text{tmf}
\]

hence is also an injection in homology. \( \Box \)

**Corollary 3.3.** Suppose there are \( \mathcal{B} \)-module maps \( \varphi_{2^i-1} : \mathcal{B} \wedge \Omega^{2^i-1} \to \mathcal{B} \wedge \text{tmf} \) and \( \varphi_{2^i} : \mathcal{B} \wedge B(2^i) \to \mathcal{B} \wedge \text{tmf} \) inducing injections on homology. Then there is a \( \mathcal{B} \)-module map

\[
\varphi_{2^{i+1}-1} : \mathcal{B} \wedge \Omega^{2^{i+1}-1} \to \mathcal{B} \wedge \text{tmf}
\]

inducing an injection on homology groups.

**Proof.** For \( 0 \leq m \leq 2^i - 1 \) and \( n \geq 0 \), there are \( \mathcal{B} \)-module maps \( \varphi_{2^i+m,n} \) inducing an injection in homology. These maps are obtained by applying Lemma 3.2 to \( \varphi_{2^i} \) and the restriction of \( \varphi_{2^i-1} \) to the summand

\[
g_{m,n} : \mathcal{B} \wedge \Sigma^{8n} B(m) \to \mathcal{B} \wedge \text{tmf}
\]

The map \( \varphi_{2^{i+1}-1} \) is the wedge of these maps. \( \Box \)
The following observation will simplify our calculations inside the Adams spectral sequence.

**Lemma 3.4.** Let $X$ and $Y$ be spectra. Suppose $F : \text{bo} \land X \to \text{bo} \land Y$ is given by the composite $(m_{\text{bo}} \land Y) \circ (\text{bo} \land f)$ for some map $f : X \to \text{bo} \land Y$. Then $F_*(r \cdot x) = r \cdot F_*(x)$ if $r \in \text{bo}_*$ and $x \in \text{bo}_* X$.

**Proof.** By construction, the composite $F$ is a bo-module map. $\square$

In particular, the bo-module map $\varrho_{2i+1-1}$ constructed in Lemma 3.2 induces a map in homotopy groups in Adams filtration $s = 0$. The above lemma allows us to apply the bo$_*$-module structure to extend the morphism into positive Adams filtrations. To complete the inductive step it suffices to construct a map $g_{2i} : B(2^i) \to \text{bo} \land \text{tmf}$ inducing an injection on homology. Indeed, we can then apply Corollary 3.3 to extend $\varrho_{2i-1}$ to a bo-module map $\varrho_{2i+1-1} : \text{bo} \land \Omega^{2^{i+1}-1} \to \text{bo} \land \text{tmf}$.

To construct $g_{2i}$, we will use $g_{2i-1}$ supplied by the inductive hypothesis. Once again we will attempt to use the pairing of integral Brown-Gitler spectra:

$$B_1(2^{i-1}) \land B_1(2^{i-1}) \to B_1(2^i)$$

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$$B_1(2^{i-1}) \land B_1(2^{i-1}) \to B_1(2^i)$$

Unfortunately, Lemma 3.2 will not apply. Indeed, the above pairings (12) are not surjective in homology since the element corresponding to $\zeta_{i+3}$ inside $H_* B_1(2^i)$ is indecomposable.

To handle this case, we turn to a lemma of Mahowald [6] made precise by Davis [3]:

**Lemma 3.5 (Davis [3]).** If $n$ is a power of 2, let $F_n = \Sigma^{8^{n-5} M_{2^i}} \land B_1(1)$. There is a map $F_n \xrightarrow{j} \text{bo} \land B_1(n) \land B_1(n)$ such that the cofibre of the composite

$$\delta : \text{bo} \land F_n \xrightarrow{1 \land j} \text{bo} \land \text{bo} \land B_1(n) \land B_1(n) \xrightarrow{\text{bo} \land m_{\text{bo}} \land 1 \land 1} \text{bo} \land B_1(n) \land B_1(n)$$

is equivalent modulo suspensions of $H \text{F}_2$ to $\text{bo} \land B_1(2n)$.

Define $m_{i-1} : \text{bo} \land B(2^{i-1}) \land B(2^{i-1}) \to \text{bo} \land \text{tmf}$ to be the bo-module map induced by the composite $m \circ (g_{2i-1} \land g_{2i-1})$. With Lemma 3.5 in mind, consider the diagram:

$$\begin{align*}
\text{bo} \land \Sigma^{2^{i+4}-5} M_{2^i} \land B_1(1) & \xrightarrow{\delta} \text{bo} \land B(2^{i-1}) \land B(2^{i-1}) \xrightarrow{m_{i-1}} \text{bo} \land B(2^{i}) \\
& \xrightarrow{\text{m}_{i-1}} \text{bo} \land \text{tmf} \\
& \xrightarrow{g_{2i}} \text{bo} \land \text{tmf}
\end{align*}$$
It suffices to show the composite $m_{i-1} \delta$ is nullhomotopic, since then $m_{i-1}$ would then extend to the desired map $g_{2i}$. The following theorem is essentially due to Davis [3, Prop. 2.8], however modified to our context.

**Theorem 3.6** (Davis, [3]). Suppose $g_{2i-1} : bo \wedge B(2^{i-1}) \to bo \wedge tmf$ induces an injection on homology. Then

$$
\pi_{2^i+4-4} (bo \wedge B(2^{i-1}) \wedge B(2^{i-1})) \cong \mathbb{Z}_2
$$

with generator $\alpha_{2^i+4-4}$ whose image under $(m_{i-1})^2$ is divisible by 2.

**Proof.** Since $bo \wedge tmf$ has the structure of an $E_\infty$-ring spectrum, the map $m_{i-1}$ factors through the quadratic construction on $B(2^{i-1})$, i.e., there is a map $j$ making the following diagram commute:

$$
\xymatrix{ & bo \wedge D_2(B(2^{i-1})) \ar[dl]_j \ar[d] \ar[dr] \ar[r] & \ar[d]_r \ar[r] & bo \wedge tmf \ar[dl]_j \ar[dr] & \\
bo \wedge B(2^{i-1}) \wedge B(2^{i-1}) & \ar[r]_{m_{i-1}} & bo \wedge tmf
}
$$

where

$$D_2(B(2^{i-1})) = S^1 \times_{\Sigma_2} (B(2^{i-1}) \wedge B(2^{i-1})).$$

Here the $\Sigma_2$-action on $S^1$ is the antipode and the action on the smash product interchanges factors. Using this factorization, it suffices to show that the induced map $j_2$ in homotopy sends the generator in dimension $2^i+4-4$ to twice an element of the homotopy of the quadratic construction. This is proved by Davis [3].

**Proof that Theorem 3.6 implies Theorem 1.1.** Let $[x_i] \in \pi_i(bo \wedge tmf)$ for $i = 0, 8, 12$ denote the classes in bidegree $(i, 0)$ in the $E_2$-term displayed in Figure 1. The class $[x_{12}]$ does not support action by $\eta$ so that $x_{12}$ extends to a map $B(1) \to bo \wedge tmf$. Upon smashing with $bo$, we get maps

$$g_0 : bo \wedge B(0) \to bo \wedge tmf$$

$$g_{0,1} : \Sigma^8 bo \wedge B(0) \to bo \wedge tmf$$

$$g_1 : bo \wedge B(1) \to bo \wedge tmf$$

inducing injections in homology. In particular, Lemma 3.2 extends these to a $bo$-module map $g_1 : bo \wedge \Omega^1 \to bo \wedge tmf$ which is also an injection on homology.
Lemma 3.4 extends this morphism to positive Adams filtrations. Figure 1 demonstrates that modulo possible order 2 elements on the zero line, this map accounts for all homotopy classes through the 23-stem. Hence, it is a stable $A$-equivalence in this range.

For the purpose of induction, assume the existence of a $bo$-module map $g_{2^i-1} : bo \wedge \Omega^{2^i-1} \to bo \wedge tmf$ inducing a stable $A$-equivalence through the $(12(2^i) - 1)$-stem. In particular, there is a map $g_{2^i-1} : bo \wedge B(2^{i-1}) \to bo \wedge tmf$ of $bo$-module spectra inducing an injection on homology groups. Define $m_i \delta$ and $\delta$ as above. We will show $m_i \delta \simeq *$.

![Figure 2: $H^*(M_{2i} \wedge B_1(1))$](image)

Figure 2 shows the cell diagram for $H^*(M_{2i} \wedge B_1(1))$. Since there are no elements of positive Adams filtration in stems congruent to $\{5, 6, 7\}$ mod 8 in the Adams spectral sequence converging to $\pi_\ast (bo \wedge tmf)$, the composite $m_i \delta$ restricts to a map $\Sigma^{2i+4-5} M_{2i} \to bo \wedge tmf$. Consider the composite

$$S^{2i+4-5} \xrightarrow{a_0} \Sigma^{2i+4-5} M_{2i} \wedge B_1(1) \xrightarrow{m_i \delta} bo \wedge tmf$$

restricting $m_i \delta$ to the bottom cell of $\Sigma^{2i+4-5} M_{2i}$. There are no elements of positive Adams filtration in stems congruent to 3 mod 8 so this restriction extends to the top cell

$$S^{2i+4-4} \xrightarrow{a_1} \Sigma^{2i+4-5} M_{2i} \wedge B_1(1) \xrightarrow{m_i \delta} bo \wedge tmf.$$ 

Theorem 3.6 indicates that the class $(m_i \delta)(\delta a_1)$ is divisible by 2. Hence, this map is nulhomotopic. Applying Corollary 3.3 gives the result. 

4 The $bo$-homology of tmf

Both $bo$ and $tmf$ have the structure of $E_\infty$-ring spectra, so that the smash product $bo \wedge tmf$ also inherits such a structure. The splitting of $bo \wedge tmf$ into
pieces involving integral Brown-Gitler spectra gives a nice description of its structure as a ring spectrum. Indeed, the pairing of the $B_1(j)$ is compatible with multiplication inside $H_*HZ$ of which $H_*tmf$ is a subring. In particular, the pairings of the integral Brown-Gitler spectra induce the ring structure of $bo\wedge tmf$. The induced structure on homotopy groups is given by the following theorem:

**Theorem 4.1.** There is an isomorphism of graded $bo_*$-algebras

$$\pi_* (bo \wedge tmf) \cong \frac{bo_*[\sigma, b_i, \mu_i \mid i \geq 0]}{(\mu b_i^2 - 8b_{i+1}, \mu b_i - 4\mu_i, \eta b_i)} \oplus F$$ (14)

where $|\sigma| = 8, |b_i| = 2^{i+4} - 4, |\mu_i| = 2^{i+4}$ and $F$ is a direct sum of $\mathbb{F}_2$ in varying dimensions.

**Proof.** Theorem 2.7 gives homotopy equivalences

$$bo \wedge B(n) \wedge B(2^i) \to K \vee (bo \wedge B(n + 2^i))$$

for all $n < 2^i$. In particular, the induced pairings

$$\pi_* (bo \wedge B(n)) \otimes \pi_* (bo \wedge B(2^i)) \to \pi_* (bo \wedge B(n + 2^i))$$

provide an isomorphism for all $n < 2^i$, modulo possible order 2 elements in Adams filtration zero corresponding to free $A(1)$ inside $H^*tmf$. Therefore, the homotopy classes inside $bo, \Sigma^8 bo$ and $bo \wedge B(2^i)$ for $i \geq 0$ generate the homotopy of $\pi_* (bo \wedge tmf)$. Hence, it suffices to examine the pairings

$$bo \wedge B(2^i) \wedge B(2^i) \to bo \wedge B(2^{i+1}).$$

Figure 3 depicts the $E_2$-term of the Adams spectral sequence converging to $bo_*B(1)$ along with its generators as a $bo_*$-module. With these generators, we can determine the decomposables inside $bo_*B(2^i)$. Indeed, Lemma 3.5 provides us with a fiber sequence

$$bo \wedge B(2^i) \wedge B(2^i) \to bo \wedge B(2^{i+1}) \to bo \wedge \Sigma^{2^{i+5} - 4} M_{2i} \wedge B_1(1)$$ (15)

inducing a long exact sequence of Ext-groups. Figure 4 shows how to use (15) to form the $E_2$-page of $bo \wedge B(2^{i+1})$. The arrows represent subsequent differentials and the dotted lines non-trivial extensions. The classes in black are those contributed by $bo \wedge B(2^i) \wedge B(2^i)$, i.e., the decomposable classes.
hit by multiplication by elements in the summand \(bo \wedge B(2^i)\). Those in red (or grey) are contributed by \(bo \wedge \Sigma^{2i+5-4}M_{2i} \wedge B_1(1)\). Denote by \(b_{i+1}\) the class found in bidegree \((2^{i+5} - 4, 0)\) and \(\mu_{i+1}\) the class in \((2^{i+4}, 1)\). These two elements are thus indecomposable in the ring \(\pi_*(bo \wedge tmf)\).

Note that the class in bidegree \((2^{i+5} - 8, 0)\) corresponds to the element \(b_i^2\). In particular, \(\mu b_i^2 = 8b_{i+1}\). Also note that \(\mu b_{i+1} = 4\mu_{i+1}\) and \(\eta b_{i+1} = 0\).

Remark 4.1. The splitting of \(bo \wedge tmf\) can also be used to give a description of the \(bo\)-cohomology of \(tmf\). Indeed, Lemma 3.1 gives that \([tmf, bo] = [bo \wedge tmf, bo]_bo\). Since Theorem 1.1 provides a splitting as \(bo\)-module spectra,
one has the following chain of equivalences of $bo^*$-comodules:

$$bo^*\text{tmf} = [\text{tmf}, bo] = [bo \wedge \text{tmf}, bo]_bo = \left[ \bigvee_{m,n \geq 0} \Sigma^{8n} bo \wedge B(m), bo \right]_bo = \left[ \bigvee_{m,n \geq 0} \Sigma^{8n} B(m), bo \right] = \bigoplus_{m,n \geq 0} \Sigma^{-8n} bo^* B(m)$$

A complete description of the summands $bo^* B(m)$ is given by Carlsson [1]. The comultiplication on $bo^*\text{tmf}$ is once again induced by the pairings of integral Brown-Gitler spectra. It would be interesting to determine the explicit generators and relations as a $bo^*$-coalgebra.

References


