

On the spectrum $bo \wedge \text{tmf}$

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Abstract

M. Mahowald, in his work on bo -resolutions, constructed a bo -module splitting of the spectrum $bo \wedge bo$ into a wedge of summands related to integral Brown-Gitler spectra. In this paper, a similar splitting of $bo \wedge \text{tmf}$ is constructed. This splitting is then used to understand the bo_* -algebra structure of $bo_*\text{tmf}$ and allows for a description of $bo^*\text{tmf}$.

1 Introduction

All cohomology groups are assumed to have coefficients in \mathbb{F}_2 and all spectra completed at the prime 2 unless stated otherwise. Let \mathcal{A} denote the Steenrod algebra, and $\mathcal{A}(n)$ the subalgebra generated by $\{Sq^1, Sq^2, \dots, Sq^{2^n}\}$. Consider the Hopf algebra quotient $\mathcal{A}/\mathcal{A}(n) = \mathcal{A} \otimes_{\mathcal{A}(n)} \mathbb{F}_2$. Here the right action of $\mathcal{A}(n)$ on \mathcal{A} is induced by the inclusion and the left action on \mathbb{F}_2 by the augmentation. Algebraically, one can consider the subsequent surjections

$$\mathcal{A} \rightarrow \mathcal{A}/\mathcal{A}(0) \rightarrow \mathcal{A}/\mathcal{A}(1) \rightarrow \mathcal{A}/\mathcal{A}(2) \rightarrow \mathcal{A}/\mathcal{A}(3) \rightarrow \dots$$

and ask whether each algebra can be realized as the cohomology of some spectrum. The case $n \geq 3$ requires the existence of a non-trivial map $S^{2^{n+1}-1} \rightarrow S^0$ which cannot occur due to Hopf invariant one. For $n < 3$, however, it is now well-known that each algebra can indeed be realized by the cohomology of some spectrum:

$$H^*\mathbb{H}\mathbb{F}_2 \rightarrow H^*\mathbb{H}\mathbb{Z} \rightarrow H^*bo \rightarrow H^*\text{tmf}$$

There are maps realizing the above homomorphisms of cohomology groups

$$\mathrm{tmf} \rightarrow \mathrm{bo} \rightarrow \mathrm{HZ} \rightarrow \mathrm{HF}_2$$

In particular, the spectrum tmf is at the top of a “tower” whose “lower floors” have been well studied in the literature, culminating with Mahowald’s [6] understanding of the spectrum $\mathrm{bo} \wedge \mathrm{bo}$ and Carlsson’s [1] description of the cohomology operations $[\mathrm{bo}, \mathrm{bo}]$. More difficult questions arise: What is the structure of $\mathrm{tmf} \wedge \mathrm{tmf}$? What are the stable cohomology operations of tmf ?

We would like to understand the spectrum $\mathrm{bo} \wedge \mathrm{tmf}$ for a variety of reasons. First, it might serve as a nice intermediate step towards understanding the spectrum $\mathrm{tmf} \wedge \mathrm{tmf}$. Furthermore, determining its structure comes with an added bonus of understanding operations $[\mathrm{tmf}, \mathrm{bo}]$ which may provide some insight into understanding the cohomology operations of tmf . Second, the splitting of $\mathrm{bo} \wedge \mathrm{tmf}$ has been instrumental to the author in demonstrating the splitting of the Tate spectrum of tmf into a wedge of suspensions of bo .

Let $B_1(j)$ denote the j^{th} integral Brown-Gitler spectrum, whose homology will be described as a submodule of $H_*\mathrm{HZ}$. Such spectra have been studied extensively in the literature (see [2], [5], [9], for example). In particular, Mahowald [6] demonstrated the splitting of bo -module spectra $\mathrm{bo} \wedge \mathrm{bo} \simeq \bigvee_{j \geq 0} \Sigma^{4j} \mathrm{bo} \wedge B_1(j)$. Let $\Omega = \bigvee_{0 \leq j \leq i} \Sigma^{8i+4j} B_1(j)$. The main theorem of this paper is the following

Theorem 1.1. *There is a homotopy equivalence of bo -module spectra*

$$\mathrm{bo} \wedge \Omega \rightarrow \mathrm{bo} \wedge \mathrm{tmf} \tag{1}$$

The splitting is analogous to that of $\mathrm{bo} \wedge \mathrm{bo}$ of Mahowald and even $MO\langle 8 \rangle \wedge \mathrm{bo}$ of Davis [3]. Its proof, therefore, contains ideas and results from both. Section 2 deals with demonstrating an isomorphism on the level of homotopy groups, which first requires an understanding of the left $\mathcal{A}(1)$ -module structure of $H^*\mathrm{tmf}$. In Section 3, we construct a map of bo -module spectra realizing the isomorphism of homotopy groups. Section 4 uses this splitting along with pairings of integral Brown-Gitler spectra to explicitly determine the bo_* -algebra structure of $\mathrm{bo}_*\mathrm{tmf}$ and also identifies the cohomology $\mathrm{bo}^*\mathrm{tmf}$.

2 The algebraic splitting

The E_2 -term of the Adams spectral sequence converging to the homotopy groups of $bo \wedge \text{tmf}$ is given by

$$\text{Ext}_{\mathcal{A}}^{s,t}(H^*(bo \wedge \text{tmf}), \mathbb{F}_2) \Rightarrow \pi_{t-s}(bo \wedge \text{tmf}). \quad (2)$$

The Ext-group appearing in the above spectral sequence can be simplified via a change-of-rings isomorphism:

$$\text{Ext}_{\mathcal{A}(1)}^{s,t}(H^*\text{tmf}, \mathbb{F}_2) \Rightarrow \pi_{t-s}(bo \wedge \text{tmf}). \quad (3)$$

Therefore, it suffices to understand the left $\mathcal{A}(1)$ -module structure of $H^*\text{tmf}$. Computations and definitions simplify upon dualizing. Indeed, the dual Steenrod algebra, \mathcal{A}_* , is the graded polynomial ring $\mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots]$ with $|\xi_i| = 2^i - 1$. An equivalent problem after dualizing is determining the right $\mathcal{A}(1)$ -module structure of the subring $H_*\text{tmf} \subset \mathcal{A}_*$. The homology of tmf as a right \mathcal{A} -module is given by Rezk [8]

$$H_*\text{tmf} \cong \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots]. \quad (4)$$

The generators $\zeta_i = \chi \xi_i$, where $\chi : \mathcal{A}_* \rightarrow \mathcal{A}_*$ is the canonical antiautomorphism. Define a new weight on elements of \mathcal{A}_* by $\omega(\zeta_i) = 2^{i-1}$ for $i \geq 1$. For $a, b \in \mathcal{A}_*$ define the weight on their product by $\omega(ab) = \omega(a) + \omega(b)$. Let N_k^{tmf} denote the \mathbb{F}_2 -vector space inside $H_*\text{tmf}$ generated by all monomials of weight k with $N_0^{\text{tmf}} = \mathbb{F}_2$ generated by the identity.

Lemma 2.1. *As right $\mathcal{A}(2)$ -modules,*

$$H_*\text{tmf} \cong \bigoplus_{i \geq 0} N_{8i}^{\text{tmf}}$$

Proof. Certainly, the two modules are isomorphic as \mathbb{F}_2 -vector spaces. To see there is an isomorphism of right $\mathcal{A}(2)$ -modules, note that the right action of the total square $Sq = \sum_{i \geq 0} Sq^i$ on the generators of $H_*\text{tmf}$ is given by:

$$\begin{aligned} \zeta_1^8 \cdot Sq &= \zeta_1^8 + 1; \\ \zeta_2^4 \cdot Sq &= \zeta_2^4 + \zeta_1^8 + 1; \\ \zeta_3^2 \cdot Sq &= \zeta_3^2 + \zeta_2^4 + \zeta_1^8 + 1; \\ \zeta_n \cdot Sq &= \sum_{i=0}^n \zeta_{n-i}^{2^i} \end{aligned}$$

for $n > 3$. Since $\omega(1) = 0$, modulo the identity the total square preserves the weight of the generators of $H_*\text{tmf}$. Note that $\zeta_1^{2^{n-1}} Sq^{2^{n-1}} = 1$, hence the total square over $\mathcal{A}(2)$ cannot contain a 1 in the expansion for dimensional reasons. \square

Consider the homomorphism $V : \mathcal{A}_* \rightarrow \mathcal{A}_*$ defined on generators by

$$V(\zeta_i) = \begin{cases} 1, & i = 0, 1; \\ \zeta_{i-1}, & i \geq 2. \end{cases}$$

Restricting V to the subring $H_*\text{tmf} \subset \mathcal{A}_*$ clearly provides a surjection $V_{\text{tmf}} : H_*\text{tmf} \rightarrow H_*bo$. Let $M_{bo}(4i)$ denote the image of N_{8i}^{tmf} under the homomorphism V_{tmf} . It is generated by all monomials with $\omega(\zeta^I) \leq 4i$. The following proposition is clear.

Proposition 2.2. *As right $\mathcal{A}(2)$ -modules*

$$N_{8i}^{\text{tmf}} \cong \Sigma^{8i} M_{bo}(4i). \quad (5)$$

Proof. Due to the weight requirements, V_{tmf} is injective when restricted to N_{8i}^{tmf} . Indeed, the exponent of ζ_1 in each monomial is uniquely determined by the other exponents. \square

Additionally, if we denote by N_k^{bo} the \mathbb{F}_2 -vector space inside H_*bo generated by all elements of weight k with $N_0^{bo} = \mathbb{F}_2$ generated by the identity, we have a similar lemma:

Lemma 2.3. *As right $\mathcal{A}(1)$ -modules,*

$$M_{bo}(4i) \cong \bigoplus_{j=0}^i N_{4j}^{bo}.$$

Further restricting V to the subring H_*bo provides a surjection $V_{bo} : H_*bo \rightarrow H_*\mathbb{HZ}$. Let $M_{\mathbb{HZ}}(2j)$ denote the image of N_{4j}^{bo} under V . This submodule is generated by all monomials with $\omega(\zeta^I) \leq 2j$. As in Proposition 2.2 we have the identification

Proposition 2.4. *As right $\mathcal{A}(1)$ -modules,*

$$N_{4j}^{bo} \cong \Sigma^{4j} M_{\mathbb{HZ}}(2j). \quad (6)$$

Goerss, Jones and Mahowald [5] identify the submodule $M_{\mathbb{H}\mathbb{Z}}(2j) \subset H_*\mathbb{H}\mathbb{Z}$ as the homology of the j th integral Brown-Gitler spectrum:

Theorem 2.5 (Goerss, Jones, Mahowald [5]). *For $j \geq 0$, there is a spectrum $B_1(j)$ and a map*

$$B_1(j) \xrightarrow{g} \mathbb{H}\mathbb{Z}$$

such that

(i) g_* sends $H_*(B_1(j))$ isomorphically onto the span of monomials of weight $\leq 2j$;

(ii) there are pairings

$$B_1(m) \wedge B_1(n) \rightarrow B_1(m+n)$$

whose homology homomorphism is compatible with the multiplication in $H_*\mathbb{H}\mathbb{Z}$.

Remark 2.1. The submodules $M_{bo}(4i)$ are the so-called *bo*-Brown-Gitler modules. There is a family of spectra with similar properties, having these modules as their homology. Proposition 2.2 demonstrates that as an $\mathcal{A}(2)$ -module, $H_*\mathrm{tmf}$ is a direct sum of these modules. On the level of spectra, however, $\mathrm{tmf} \wedge \mathrm{tmf}$ does not split as a wedge of *bo*-Brown-Gitler spectra.

Combining the results of Lemmas 2.1 and 2.3 with Theorem 2.5, $H_*\mathrm{tmf}$ as a right $\mathcal{A}(1)$ -module can be written in terms of homology of integral Brown-Gitler spectra:

Theorem 2.6. *As right $\mathcal{A}(1)$ -modules,*

$$H_*\mathrm{tmf} \cong \bigoplus_{0 \leq j \leq i} \Sigma^{8i+4j} H_*B_1(j).$$

The E_2 -term of the Adams spectral sequence (3) then becomes isomorphic to

$$\bigoplus_{0 \leq j \leq i} \Sigma^{8i+4j} \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(H^*B_1(j), \mathbb{F}_2) \Rightarrow \pi_{t-s}(bo \wedge \mathrm{tmf}). \quad (7)$$

This is precisely the Adams E_2 -term converging to the homotopy of $bo \wedge \Omega$. The chart can be obtained by applying the following theorem of Davis [4] which links $bo \wedge B_1(n)$ to Adams covers of bo or bsp , depending on the parity of n .

Theorem 2.7 (Davis [4]). *If $\bar{n} = (n_1, \dots, n_s)$, let $|\bar{n}| = \sum_{i=1}^s n_i$ and $\alpha(\bar{n}) = \sum_{i=1}^s \alpha(n_i)$, and $B_1(\bar{n}) = \bigwedge_{i=1}^s B_1(n_i)$. Then there are homotopy equivalences*

$$bo \wedge B_1(\bar{n}) \simeq K \vee \begin{cases} bo^{2|\bar{n}|-\alpha(\bar{n})}, & \text{if } |\bar{n}| \text{ is even;} \\ bsp^{2|\bar{n}|-1-\alpha(\bar{n})}, & \text{if } |\bar{n}| \text{ is odd;} \end{cases}$$

where K is a wedge of suspensions of $\mathbf{H}\mathbb{F}_2$.

Figure 1: $\text{Ext}_{\mathcal{A}}^{s,t}(H^*(bo \wedge \text{tmf}), \mathbb{F}_2) \Rightarrow \pi_{t-s}(bo \wedge \text{tmf})$

The charts for bo and bsp are well known. Using the above theorem along with the algebraic splitting of $H^*\text{tmf}$, we see that Adams covers of bo begin in stems congruent to $0 \pmod{8}$ while Adams covers of bsp begin in stems congruent to $4 \pmod{8}$. The first 32 stems of the chart for $bo \wedge \text{tmf}$ is displayed in Figure 1 modulo possible elements of order 2 in Adams filtration $s = 0$ corresponding to free $\mathcal{A}(1)$'s inside $H^*\text{tmf}$. The symbol \odot appears in Figure 1 to reduce clutter. It is used to mark the beginning of another \mathbb{Z} -tower. In general, all \mathbb{Z} -towers are found in stems congruent to $0 \pmod{4}$ while those supporting multiplication by η occur in stems congruent to $4 \pmod{8}$.

Theorem 2.8. *There is an isomorphism of homotopy groups*

$$\pi_*(bo \wedge \text{tmf}) \cong \pi_*(bo \wedge \Omega)$$

Proof. The E_2 -terms of their respective Adams spectral sequences have been shown to be isomorphic. Both spectral sequences collapse. Indeed, the classes charted in Figure 1 cannot support differentials for dimensional and naturality reasons. Each element of order two in Adams filtration $s = 0$ correspond to copies of $\mathcal{A}(1)$ inside $H^*\mathrm{tmf}$. These summands split off, obviating the existence of differentials. \square

3 The topological splitting

Theorem 1.1 concerns a bo -module splitting of the spectrum $bo \wedge \mathrm{tmf}$. The following observation will aid us in studying bo -module maps.

Lemma 3.1. *Let X and Y be spectra. Then*

$$[bo \wedge X, bo \wedge Y]_{bo} = [X, bo \wedge Y]$$

Proof. Let $u_{bo} : S^0 \rightarrow bo$ and $m_{bo} : bo \wedge bo \rightarrow bo$ denote the unit and the product map of bo , respectively. Given $f : bo \wedge X \rightarrow bo \wedge Y$ and $g : X \rightarrow bo \wedge Y$, the equivalence is given by the composites

$$\begin{aligned} f &\mapsto f \circ (u \wedge 1) \\ g &\mapsto (m_{bo} \wedge 1) \circ (1 \wedge g) \end{aligned}$$

\square

The spectra (bo, m_{bo}, u_{bo}) and $(\mathrm{tmf}, m_{\mathrm{tmf}}, u_{\mathrm{tmf}})$ are both unital \mathcal{E}_∞ -ring spectra [7]. This induces a unital \mathcal{E}_∞ -ring structure $(bo \wedge \mathrm{tmf}, m, u)$. This structure will play an important role in the proof of the main theorem. We begin by defining an increasing filtration of Ω via:

$$\Omega^n = \bigvee_{j=0}^n \bigvee_{i=j}^{\infty} \Sigma^{8i+4j} B_1(j) \tag{8}$$

Notationally, it will be convenient to let $B(j) = \Sigma^{12j} B_1(j)$, so that the filtration (8) can be rewritten as

$$\Omega^n = \bigvee_{j=0}^n \bigvee_{i \geq 0} \Sigma^{8i} B(j). \tag{9}$$

The proof of Theorem 1.1 will proceed inductively on n . We will assume the existence of a bo -module map $\varrho_{2^i-1} : bo \wedge \Omega^{2^i-1} \rightarrow bo \wedge tmf$ which is a stable \mathcal{A} -isomorphism through a certain dimension. The inductive step will be then to construct a bo -module map $\varrho_{2^{i+1}-1} : bo \wedge \Omega^{2^{i+1}-1} \rightarrow bo \wedge tmf$ which is a stable \mathcal{A} -isomorphism through higher dimensions. To do this, we will employ the pairings given in Theorem 2.5(ii). Define the map

$$g_{m,n} : \Sigma^{8n} B(m) \rightarrow bo \wedge tmf \quad (10)$$

to be the restriction of ϱ_{2^i-1} to the summand $\Sigma^{8n} B(m)$. Denote by $g_m = g_{m,0}$.

Lemma 3.2. *Let $m = 2^i$ and $0 \leq n < m$. Suppose there are bo -module maps $f_m : bo \wedge B_1(m) \rightarrow bo \wedge tmf$ and $f_n : bo \wedge B_1(n) \rightarrow bo \wedge tmf$ inducing injections on homology. Then there is a bo -module map*

$$f_{m+n} : bo \wedge B_1(m+n) \rightarrow bo \wedge tmf$$

inducing an injection on homology.

Proof. For all $0 \leq n < m$, Theorem 2.7 supplies equivalences of bo -module spectra

$$bo \wedge B_1(m) \wedge B_1(n) \simeq (bo \wedge B_1(m+n)) \vee K \quad (11)$$

where K is a wedge of suspensions of \mathbf{HF}_2 . There are no maps $[\mathbf{HF}_2, bo \wedge tmf]$ so that the composite $\mathbf{m} \circ (f_m \wedge f_n)$ lifts as a bo -module map to the first summand

$$f_{m+n} : bo \wedge B_1(m+n) \rightarrow bo \wedge tmf$$

hence is also an injection in homology. \square

Corollary 3.3. *Suppose there are bo -module maps $\varrho_{2^i-1} : bo \wedge \Omega^{2^i-1} \rightarrow bo \wedge tmf$ and $g_{2^i} : bo \wedge B(2^i) \rightarrow bo \wedge tmf$ inducing injections on homology. Then there is a bo -module map*

$$\varrho_{2^{i+1}-1} : bo \wedge \Omega^{2^{i+1}-1} \rightarrow bo \wedge tmf$$

inducing an injection on homology groups.

Proof. For $0 \leq m \leq 2^i - 1$ and $n \geq 0$, there are bo -module maps $g_{2^i+m,n}$ inducing an injection in homology. These maps are obtained by applying Lemma 3.2 to g_{2^i} and the restriction of ϱ_{2^i-1} to the summand

$$g_{m,n} : bo \wedge \Sigma^{8n} B(m) \rightarrow bo \wedge tmf$$

The map $\varrho_{2^{i+1}-1}$ is the wedge of these maps. \square

The following observation will simplify our calculations inside the Adams spectral sequence.

Lemma 3.4. *Let X and Y be spectra. Suppose $\mathcal{F} : bo \wedge X \rightarrow bo \wedge Y$ is given by the composite $(\mathbf{m}_{bo} \wedge Y) \circ (bo \wedge f)$ for some map $f : X \rightarrow bo \wedge Y$. Then $\mathcal{F}_*(rx) = r\mathcal{F}_*(x)$ if $r \in bo_*$ and $x \in bo_*X$.*

Proof. By construction, the composite \mathcal{F} is a bo -module map. \square

In particular, the bo -module map $\varrho_{2^{i+1}-1}$ constructed in Lemma 3.2 induces a map in homotopy groups in Adams filtration $s = 0$. The above lemma allows us to apply the bo_* -module structure to extend the morphism into positive Adams filtrations. To complete the inductive step it suffices to construct a map $g_{2^i} : B(2^i) \rightarrow bo \wedge tmf$ inducing an injection on homology. Indeed, we can then apply Corollary 3.3 to extend ϱ_{2^i-1} to a bo -module map $\varrho_{2^{i+1}-1} : bo \wedge \Omega^{2^{i+1}-1} \rightarrow bo \wedge tmf$.

To construct g_{2^i} , we will use $g_{2^{i-1}}$ supplied by the inductive hypothesis. Once again we will attempt to use the pairing of integral Brown-Gitler spectra:

$$B_1(2^{i-1}) \wedge B_1(2^{i-1}) \rightarrow B_1(2^i) \quad (12)$$

to construct a map $bo \wedge B_1(2^i) \rightarrow bo \wedge tmf$. Unfortunately, Lemma 3.2 will not apply. Indeed, the above pairings (12) are not surjective in homology since the element corresponding to ζ_{i+3} inside $H_*B_1(2^i)$ is indecomposable. To handle this case, we turn to a lemma of Mahowald [6] made precise by Davis [3]:

Lemma 3.5 (Davis [3]). *If n is a power of 2, let $F_n = \Sigma^{8n-5}M_{2^i} \wedge B_1(1)$. There is a map $F_n \xrightarrow{j} bo \wedge B_1(n) \wedge B_1(n)$ such that the cofibre of the composite*

$$\delta : bo \wedge F_n \xrightarrow{1 \wedge j} bo \wedge bo \wedge B_1(n) \wedge B_1(n) \xrightarrow{\mathbf{m}_{bo \wedge 1 \wedge 1}} bo \wedge B_1(n) \wedge B_1(n)$$

is equivalent modulo suspensions of $\mathbb{H}\mathbb{F}_2$ to $bo \wedge B_1(2n)$.

Define $\mathbf{m}_{i-1} : bo \wedge B(2^{i-1}) \wedge B(2^{i-1}) \rightarrow bo \wedge tmf$ to be the bo -module map induced by the composite $\mathbf{m} \circ (g_{2^{i-1}} \wedge g_{2^{i-1}})$. With Lemma 3.5 in mind, consider the diagram:

$$\begin{array}{ccccc} bo \wedge \Sigma^{2^{i+4}-5}M_{2^i} \wedge B_1(1) & \xrightarrow{\delta} & bo \wedge B(2^{i-1}) \wedge B(2^{i-1}) & \longrightarrow & bo \wedge B(2^i) \\ & & \mathbf{m}_{i-1} \downarrow & \swarrow \text{---} g_{2^i} & \\ & & bo \wedge tmf & & \end{array}$$

It suffices to show the composite $\mathbf{m}_{i-1}\delta$ is nullhomotopic, since then \mathbf{m}_{i-1} would then extend to the desired map g_{2^i} . The following theorem is essentially due to Davis [3, Prop. 2.8], however modified to our context.

Theorem 3.6 (Davis, [3]). *Suppose $g_{2^{i-1}} : bo \wedge B(2^{i-1}) \rightarrow bo \wedge tmf$ induces an injection on homology. Then*

$$\pi_{2^{i+4}-4}(bo \wedge B(2^{i-1}) \wedge B(2^{i-1})) \cong \mathbb{Z}_{(2)} \quad (13)$$

with generator $\alpha_{2^{i+4}-4}$ whose image under $(\mathbf{m}_{i-1})_{\#}$ is divisible by 2.

Proof. Since $bo \wedge tmf$ has the structure of an \mathcal{E}_{∞} -ring spectrum, the map \mathbf{m}_{i-1} factors through the quadratic construction on $B(2^{i-1})$, i.e., there is a map j making the the following diagram commute:

$$\begin{array}{ccc} & & bo \wedge D_2(B(2^{i-1})) \\ & \nearrow j & \downarrow \\ bo \wedge B(2^{i-1}) \wedge B(2^{i-1}) & \xrightarrow{\mathbf{m}_{i-1}} & bo \wedge tmf \end{array}$$

where

$$D_2(B(2^{i-1})) = S^1 \rtimes_{\Sigma_2} (B(2^{i-1}) \wedge B(2^{i-1})).$$

Here the Σ_2 -action on S^1 is the antipode and the action on the smash product interchanges factors. Using this factorization, it suffices to show that the induced map $j_{\#}$ in homotopy sends the generator in dimension $2^{i+4} - 4$ to twice an element of the homotopy of the quadratic construction. This is proved by Davis [3]. \square

Proof that Theorem 3.6 implies Theorem 1.1. Let $[x_i] \in \pi_i(bo \wedge tmf)$ for $i = 0, 8, 12$ denote the classes in bidegree $(i, 0)$ in the E_2 -term displayed in Figure 1. The class $[x_{12}]$ does not support action by η so that x_{12} extends to a map $B(1) \rightarrow bo \wedge tmf$. Upon smashing with bo , we get maps

$$\begin{aligned} g_0 &: bo \wedge B(0) \rightarrow bo \wedge tmf \\ g_{0,1} &: \Sigma^8 bo \wedge B(0) \rightarrow bo \wedge tmf \\ g_1 &: bo \wedge B(1) \rightarrow bo \wedge tmf \end{aligned}$$

inducing injections in homology. In particular, Lemma 3.2 extends these to a bo -module map $\varrho_1 : bo \wedge \Omega^1 \rightarrow bo \wedge tmf$ which is also an injection on homology.

Lemma 3.4 extends this morphism to positive Adams filtrations. Figure 1 demonstrates that modulo possible order 2 elements on the zero line, this map accounts for all homotopy classes through the 23-stem. Hence, it is a stable \mathcal{A} -equivalence in this range.

For the purpose of induction, assume the existence of a bo -module map $\varrho_{2^i-1} : bo \wedge \Omega^{2^i-1} \rightarrow bo \wedge \text{tmf}$ inducing a stable \mathcal{A} -equivalence through the $(12(2^i) - 1)$ -stem. In particular, there is a map $g_{2^i-1} : bo \wedge B(2^{i-1}) \rightarrow bo \wedge \text{tmf}$ of bo -module spectra inducing an injection on homology groups. Define \mathbf{m}_{i-1} and δ as above. We will show $\mathbf{m}_{i-1}\delta \simeq *$.

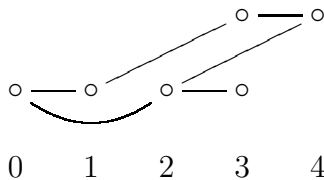


Figure 2: $H^*(M_{2^l} \wedge B_1(1))$

Figure 2 shows the cell diagram for $H^*(M_{2^l} \wedge B_1(1))$. Since there are no elements of positive Adams filtration in stems congruent to $\{5, 6, 7\} \pmod 8$ in the Adams spectral sequence converging to $\pi_*(bo \wedge \text{tmf})$, the composite $\mathbf{m}_{i-1}\delta$ restricts to a map $\Sigma^{2^{i+4}-5}M_{2^l} \rightarrow bo \wedge \text{tmf}$. Consider the composite

$$S^{2^{i+4}-5} \xrightarrow{a_0} \Sigma^{2^{i+4}-5}M_{2^l} \wedge B_1(1) \xrightarrow{\mathbf{m}_{i-1}\delta} bo \wedge \text{tmf}$$

restricting $\mathbf{m}_{i-1}\delta$ to the bottom cell of $\Sigma^{2^{i+4}-5}M_{2^l}$. There are no elements of positive Adams filtration in stems congruent to $3 \pmod 8$ so this restriction extends to the top cell

$$S^{2^{i+4}-4} \xrightarrow{a_1} \Sigma^{2^{i+4}-5}M_{2^l} \wedge B_1(1) \xrightarrow{\mathbf{m}_{i-1}\delta} bo \wedge \text{tmf}.$$

Theorem 3.6 indicates that the class $(\mathbf{m}_{i-1})_{\sharp}(\delta a_1)$ is divisible by 2. Hence, this map is nulhomotopic. Applying Corollary 3.3 gives the result. \square

4 The bo -homology of tmf

Both bo and tmf have the structure of \mathcal{E}_{∞} -ring spectra, so that the smash product $bo \wedge \text{tmf}$ also inherits such a structure. The splitting of $bo \wedge \text{tmf}$ into

pieces involving integral Brown-Gitler spectra gives a nice description of its structure as a ring spectrum. Indeed, the pairing of the $B_1(j)$ is compatible with multiplication inside $H_*\mathbb{H}\mathbb{Z}$ of which $H_*\text{tmf}$ is a subring. In particular, the pairings of the integral Brown-Gitler spectra induce the ring structure of $bo \wedge \text{tmf}$. The induced structure on homotopy groups is given by the following theorem:

Theorem 4.1. *There is an isomorphism of graded bo_* -algebras*

$$\pi_*(bo \wedge \text{tmf}) \cong \frac{bo_*[\sigma, b_i, \mu_i \mid i \geq 0]}{(\mu b_i^2 - 8b_{i+1}, \mu b_i - 4\mu_i, \eta b_i)} \oplus F \quad (14)$$

where $|\sigma| = 8$, $|b_i| = 2^{i+4} - 4$, $|\mu_i| = 2^{i+4}$ and F is a direct sum of \mathbb{F}_2 in varying dimensions.

Proof. Theorem 2.7 gives homotopy equivalences

$$bo \wedge B(n) \wedge B(2^i) \rightarrow K \vee (bo \wedge B(n + 2^i))$$

for all $n < 2^i$. In particular, the induced pairings

$$\pi_*(bo \wedge B(n)) \otimes \pi_*(bo \wedge B(2^i)) \rightarrow \pi_*(bo \wedge B(n + 2^i))$$

provide an isomorphism for all $n < 2^i$, modulo possible order 2 elements in Adams filtration zero corresponding to free $\mathcal{A}(1)$ inside $H^*\text{tmf}$. Therefore, the homotopy classes inside bo , $\Sigma^8 bo$ and $bo \wedge B(2^i)$ for $i \geq 0$ generate the homotopy of $\pi_*(bo \wedge \text{tmf})$. Hence, it suffices to examine the pairings

$$bo \wedge B(2^i) \wedge B(2^i) \rightarrow bo \wedge B(2^{i+1}).$$

Figure 3 depicts the E_2 -term of the Adams spectral sequence converging to $bo_*B(1)$ along with its generators as a bo_* -module. With these generators, we can determine the decomposables inside $bo_*B(2^i)$. Indeed, Lemma 3.5 provides us with a fiber sequence

$$bo \wedge B(2^i) \wedge B(2^i) \rightarrow bo \wedge B(2^{i+1}) \rightarrow bo \wedge \Sigma^{2^{i+5}-4} M_{2^i} \wedge B_1(1) \quad (15)$$

inducing a long exact sequence of Ext-groups. Figure 4 shows how to use (15) to form the E_2 -page of $bo \wedge B(2^{i+1})$. The arrows represent subsequent differentials and the dotted lines non-trivial extensions. The classes in black are those contributed by $bo \wedge B(2^i) \wedge B(2^i)$, i.e., the decomposable classes

Figure 3: $\text{Ext}_{\mathcal{A}(1)}^{s,t}(H^*B(1), \mathbb{F}_2)$

hit by multiplication by elements in the summand $bo \wedge B(2^i)$. Those in red (or grey) are contributed by $bo \wedge \Sigma^{2^{i+5}-4}M_{2^i} \wedge B_1(1)$. Denote by b_{i+1} the class found in bidegree $(2^{i+5}-4, 0)$ and μ_{i+1} the class in $(2^{i+4}, 1)$. These two elements are thus indecomposable in the ring $\pi_*(bo \wedge \text{tmf})$.

Note that the class in bidegree $(2^{i+5}-8, 0)$ corresponds to the element b_i^2 . In particular, $\mu b_i^2 = 8b_{i+1}$. Also note that $\mu b_{i+1} = 4\mu_{i+1}$ and $\eta b_{i+1} = 0$.

Figure 4: $\text{Ext}_{\mathcal{A}(1)}^{s,t}(H^*B(2^{i+1}), \mathbb{F}_2)$

□

Remark 4.1. The splitting of $bo \wedge \text{tmf}$ can also be used to give a description of the bo -cohomology of tmf . Indeed, Lemma 3.1 gives that $[\text{tmf}, bo] = [bo \wedge \text{tmf}, bo]_{bo}$. Since Theorem 1.1 provides a splitting as bo -module spectra,

one has the following chain of equivalences of bo^* -comodules:

$$\begin{aligned}
bo^*tmf &= [tmf, bo] \\
&= [bo \wedge tmf, bo]_{bo} \\
&= \left[\bigvee_{m,n \geq 0} \Sigma^{8n} bo \wedge B(m), bo \right]_{bo} \\
&= \left[\bigvee_{m,n \geq 0} \Sigma^{8n} B(m), bo \right] \\
&= \bigoplus_{m,n \geq 0} \Sigma^{-8n} bo^* B(m)
\end{aligned}$$

A complete description of the summands $bo^*B(m)$ is given by Carlsson [1]. The comultiplication on bo^*tmf is once again induced by the pairings of integral Brown-Gitler spectra. It would be interesting to determine the explicit generators and relations as a bo^* -coalgebra.

References

- [1] Gunnar Carlsson, *Operations in connective K-theory and associated cohomology theories*, Ph.D. thesis, Stanford, 1976.
- [2] Fred R. Cohen, Donald M. Davis, Paul G. Goerss, and Mark E. Mahowald, *Integral Brown-Gitler spectra*, Proc. Amer. Math. Soc. **103** (1988), no. 4, 1299–1304.
- [3] Donald M. Davis, *The splitting of $BO\langle 8 \rangle \wedge bo$ and $MO\langle 8 \rangle \wedge bo$* , Trans. Amer. Math. Soc. **276** (1983), no. 2, 671–683.
- [4] Donald M. Davis, Sam Gitler, and Mark Mahowald, *The stable geometric dimension of vector bundles over real projective spaces*, Trans. Amer. Math. Soc. **268** (1981), no. 1, 39–61.
- [5] Paul G. Goerss, John D. S. Jones, and Mark E. Mahowald, *Some generalized Brown-Gitler spectra*, Trans. Amer. Math. Soc. **294** (1986), no. 1, 113–132.
- [6] Mark Mahowald, *bo-resolutions*, Pacific Journal of Mathematics **92** (1981), no. 2, 365–383.

- [7] J.P. May, *Infinite loop space theory*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 456–494.
- [8] Charles Rezk, *Supplementary notes for Math 512 (ver. 0.18)*, <http://www.math.uiuc.edu/~rezk/512-spr2001-notes.pdf>, July 2007.
- [9] Don H. Shimamoto, *An integral version of the Brown-Gitler spectrum*, Trans. Amer. Math. Soc. **283** (1984), no. 2, 383–421.