ON THE ADAMS SPECTRAL SEQUENCE FOR *R*-MODULES

ANDREW BAKER & ANDREJ LAZAREV

Abstract. We consider the Adams Spectral Sequence for R-modules based on commutative localized regular quotient ring spectra of a commutative S-algebra R in the sense of Elmendorf, Kriz, Mandell, May and Strickland. The formulation of this spectral sequence is similar to the classical case, and we reduce to algebra involving the cohomology of certain 'brave new Hopf algebroids' E^R_*E. In order to work out the details we resurrect Adams' original approach to Universal Coefficient Spectral Sequences for modules over an R ring spectrum.

We show that the Adams Spectral Sequence for S_R based on $E = R/I[X^{-1}]$ converges to the homotopy of the E-nilpotent completion which has homotopy

$$\pi_* \widehat{\mathrm{L}}^R_E S_R = R_*[X^{-1}]\widehat{}_{I_*}.$$

We also show that $\widehat{\mathrm{L}}^R_E S_R$ is equivalent to $\mathrm{L}^R_E S_R$, the Bousfield localization of S_R with respect to E-theory. This seems surprising since the spectral sequence collapses at E_2, but E_r does not have a vanishing line because of the presence of polynomial generators of positive cohomological degree, thus only one of Bousfield's two standard convergence criteria applies here even though we have this equivalence. The details involve a construction of the internal I-adic tower

$$R/I \longleftarrow R/I^2 \longleftarrow \cdots \longleftarrow R/I^s \longleftarrow R/I^{s+1} \longleftarrow \cdots$$

whose homotopy limit is $\widehat{\mathrm{L}}^R_E S_R$.

Finally, we describe some examples for the case $R = MU$.

Introduction

In this note we consider the Adams Spectral Sequence for R-modules based on localized regular quotient ring spectra of a commutative S-algebra R in the sense of [9, 14] and we make systematic use of ideas and notation from those two sources. This work grew out of a preprint of the first author [3] and the related work of [5]; it is also related to ongoing collaboration with Alain Jeanneret on Bockstein operations in cohomology theories on R-modules [6].

One slightly surprising phenomenon we uncover concerns the convergence of the Adams Spectral Sequence based on $E = R/I[X^{-1}]$, a commutative localized regular quotient of a commutative S-algebra R. We show that the spectral sequence for $\pi_* S_R$ collapses at E_2, but E_r does not have a vanishing line because of the presence of polynomial generators of positive cohomological degree. Thus only one of Bousfield's two convergence criteria [8] (see Theorems 2.3 and 2.4) apply here. However we still find that the spectral sequence converges to $\pi_* \mathrm{L}^R_E S_R$, where L^R_E is the Bousfield localization functor with respect to E-theory on the category of R-modules and

$$\pi_* \mathrm{L}^R_E S_R = R_*[X^{-1}]\widehat{}_{I_*},$$

the I_*-adic completion of $R_*[X^{-1}]$, and that $\mathrm{L}^R_E S_R \simeq \widehat{\mathrm{L}}^R_E S_R$, the E-nilpotent completion.

Background assumptions, terminology and technology. We work in a good category of spectra $\mathcal{S}$ such as the category of $\mathbb{L}$-spectra of [9]. Associated to this is the subcategory of S-modules $\mathcal{M}_S$ and its derived category $\mathcal{D}_S$.

Glasgow University Mathematics Department preprint no. 01/2 [Version 12: 8/1/2001]

The first author would like to thank the University and City of Bern for providing such a hospitable environment during many visits, also Alain Jeanneret and Urs Würgler and the other participants in the Topology working seminar during spring and summer 2000.

Throughout, R will denote a commutative S-algebra in the sense of [9]. There is an associated full subcategory $\mathcal{M}_R$ of $\mathcal{M}_S$ consisting of the R-modules, and its derived homotopy category $\mathcal{D}_R$. For R-modules M and N, we set

$$M^R_*N = \pi_* M \underset{R}{\wedge} N, \quad N^*_R M = \mathcal{D}_R(M, N)^*,$$

where $\mathcal{D}_R(M,N)^n = \mathcal{D}_R(M, \Sigma^n N)$.

After Strickland [14], we will use the following terminology. If the homotopy ring $R_* = \pi_* R$ is concentrated in even degrees, a *localized quotient* of R will be an R ring spectrum of the form $R/I[X^{-1}]$. A localized quotient is *commutative* if it is a commutative R ring spectrum. A localized quotient $R/I[X^{-1}]$ is *regular* if the ideal $I_* \lhd R_*$ is generated by a regular sequence. The ideal $I_* \lhd R_*$ extends to an ideal of $R_*[X^{-1}]$ which we will again denote by I_*; then as R-modules, $R/I[X^{-1}] \simeq R[X^{-1}]/I$.

We will make use of the language and ideas of algebraic derived categories of modules over a commutative ring, mildly extended to deal with evenly graded rings and their modules. In particular, this means that chain complexes are often bigraded (or even multigraded) objects with their first grading being homological and the second and higher ones being internal.

1. Brave new Hopf algebroids and their cohomology

If E is a commutative R-ring spectrum, the smash product $E \underset{R}{\wedge} E$ is a commutative R-ring spectrum. It is also naturally an E-algebra spectrum in two ways induced from the left and right units

$$E \xrightarrow{\cong} E \underset{R}{\wedge} R \longrightarrow E \underset{R}{\wedge} E \longleftarrow E \underset{R}{\wedge} R \xleftarrow{\cong} E.$$

Theorem 1.1. *Let* E^R_*E *be flat as a left or equivalently right* E_**-module. Then*
i) (E_*, E^R_*E) *is a Hopf algebroid over* R_**;*
ii) *for any* R*-module* M*,* E^R_*M *is a left* E^R_*E*-comodule.*

Proof. This is proved using essentially the same argument as in [1, 13]. The natural map

$$E \underset{R}{\wedge} M \xrightarrow{\cong} E \underset{R}{\wedge} R \underset{R}{\wedge} M \longrightarrow E \underset{R}{\wedge} E \underset{R}{\wedge} M$$

induces the coaction

$$\psi \colon E^R_*M \longrightarrow \pi_* E \underset{R}{\wedge} E \underset{R}{\wedge} M \xrightarrow{\cong} E^R_*E \underset{E_*}{\otimes} E^R_*M,$$

which uses an isomorphism

$$\pi_* E \underset{R}{\wedge} E \underset{R}{\wedge} M \cong E^R_*E \underset{E_*}{\otimes} E^R_*M.$$

that follows from the flatness condition. □

For later use we record the following general result on the Hopf algebroids associated to commutative regular quotients. A number of examples associated with the case $R = MU$ are discussed in Section 7.

Proposition 1.2. *Let* $E = R/I$ *be a commutative regular quotient. Then as an* E_**-algebra,*

$$E^R_*E = \Lambda_{E_*}(\tau_i : i \geqslant 1).$$

Moreover, the generators τ_i *are primitive with respect to the coaction, so* E^R_*E *is a primitively generated Hopf algebra over* E_**.*

Dually, as an E_**-algebra,*

$$E^*_R E = \widehat{\Lambda}_{E_*}(Q^i : i \geqslant 1),$$

where Q^i *is the Bockstein operation dual to* τ_i *and* $\widehat{\Lambda}_{E_*}(\ )$ *indicates the completed exterior algebra generated by the anti-commuting* Q^i *elements.*

Proof. The algebra structure follows from the Künneth Spectral Sequence for R-modules [9],

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}^{R_*}_{p,q}(E_*, E_*) \Longrightarrow E^R_{p+q}E.$$

As in Proposition 5.2, with the aid of a Koszul resolution we obtain

$$\mathrm{E}^2_{*,*} = \Lambda_{E_*}(e_i : i \geqslant 1),$$

The generators have bidegree $\operatorname{bideg} e_i = (1, |u_i|)$, so the differentials on the generators e_i are trivial for dimensional reasons. By [10] Lemma 10.1, this spectral sequence is multiplicative and hence it collapses, showing that

$$E^R_* E = \Lambda_{E_*}(\tau_i : i \geqslant 1),$$

with each generator τ_i represented by e_i in the spectral sequence and having degree $\deg \tau_i = |u_i| + 1$.

For each i,

$$(R/u_i)^R_*(R/u_i) = \Lambda_{R_*/(u_i)}(\tau'_i)$$

with $\deg \tau'_i = |u_i| + 1$. Under the coproduct, τ'_i is primitive for degree reasons. By comparing the two Künneth Spectral Sequences we find that $\tau_i \in E^R_* E$ can be chosen to be the image of τ'_i under the evident ring homomorphism $(R/u_i)^R_*(R/u_i) \longrightarrow E^R_* E$, which is actually a morphism of Hopf algebroids over R_*. Hence τ_i is coaction primitive in $E^R_* E$.

For $E^*_R E$, we construct the Bockstein operation Q^i using the composition

$$R/u_i \longrightarrow \Sigma^{|u_i|+1} R \longrightarrow \Sigma^{|u_i|+1} R/u_i$$

to induce a map $E \longrightarrow \Sigma^{|u_i|+1} E$, then use the Koszul resolution to determine the Universal Coefficient Spectral sequence

$$\mathrm{E}^{p,q}_2 = \mathrm{Ext}^{p,q}_{R_*}(E_*, E_*) \Longrightarrow E^{p+q}_R E$$

which collapses at E_2. For details on the construction of these operations, see [14, 6]. □

Corollary 1.3. i) *The natural map* $E_* = E^R_* R \longrightarrow E^R_* E$ *induced by the unit* $R \longrightarrow R/I$ *is a split monomorphism of* E_**-modules.*
ii) $E^R_* E$ *is a free* E_**-module.*

Proof. An explicit splitting as in (i) is obtained using the multiplication map $E \underset{R}{\wedge} E \longrightarrow E$ which induces a homomorphism of E_*-modules $E^R_* E \longrightarrow E_*$. □

To denote the cohomology of such Hopf algebroids we will use Coext rather than the more usual Ext since we will also make heavy use of Ext groups for modules over rings; more details of the definition and calculations can be found in [1, 13]. We recall that for $E^R_* E$-comodules L_* and M_* where L_* is E_*-projective, we define $\mathrm{Coext}^{s,t}_{E^R_* E}(L_*, M_*)$ as follows. Consider a resolution

$$0 \to M_* \longrightarrow J_{0,*} \longrightarrow J_{1,*} \longrightarrow \cdots \longrightarrow J_{s,*} \longrightarrow \cdots$$

in which each $J_{s,*}$ is a summand of an extended comodule of the form

$$E^R_* E \underset{E_*}{\square} N_{s,*},$$

where $N_{s,*}$ is an E_*-module. Then the complex

$$0 \to \mathrm{Hom}^*_{E^R_* E}(L_*, J_{0,*}) \longrightarrow \mathrm{Hom}^*_{E^R_* E}(L_*, J_{1,*}) \longrightarrow \cdots \longrightarrow \mathrm{Hom}^*_{E^R_* E}(L_*, J_{s,*} \longrightarrow \cdots$$

has cohomology

$$\mathrm{H}^s(\mathrm{Hom}^*_{E^R_* E}(L_*, J_{*,*})) = \mathrm{Coext}^{s,*}_{E^R_* E}(L_*, M_*).$$

The functors $\mathrm{Coext}^{s,*}_{E^R_* E}(L_*, \ )$ are the right derived functors of the left exact functor

$$M_* \rightsquigarrow \mathrm{Hom}^*_{E^R_* E}(L_*, M_*)$$

on the category of left E^R_*E-comodules. When $L_* = E_*$, in analogy with [13], we have

$$\mathrm{Coext}^{s,*}_{E^R_*E}(E_*, M_*) = \mathrm{Cotor}^{s,*}(E_*, M_*).$$

2. The Adams Spectral Sequence for R-modules

We will describe the E-theory Adams Spectral Sequence in the homotopy category of R-module spectra. As in the classical case of $R = S$, the sphere spectrum, it turns out that the E_2-term is built up from the Coext functors.

Let L, M be R-modules and E a commutative R-ring spectrum with E^R_*E flat as a left or right E_*-module.

Theorem 2.1. *If E^R_*L is projective as an E_*-module, there is an Adams Spectral Sequence with*

$$\mathrm{E}_2^{s,t}(L, M) = \mathrm{Coext}^{s,t}_{E^R_*E}(E^R_*L, E^R_*M).$$

Proof. The proof follows that of Adams [1], replacing the sphere spectrum S with $S_R \simeq R$ and working in the derived category $\mathcal{D}_R$ throughout. The Adams resolution of M is built up in the usual way by splicing together cofibre triangles as in the following diagram.

$$\begin{tikzcd}
M \arrow[dr] & & \overline{E}\underset{R}{\wedge}M \arrow[ll] \arrow[dr] & & \overline{E}\underset{R}{\wedge}\overline{E}\underset{R}{\wedge}M \arrow[ll] \arrow[dr] & & \cdots \arrow[ll] \\
& E\underset{R}{\wedge}M \arrow[ur] & & E\underset{R}{\wedge}\overline{E}\underset{R}{\wedge}M \arrow[ur] & & {} &
\end{tikzcd}$$

The algebraic identification of the E_2-term proceeds as in [1]. □

In the rest of this paper we will have $L = S_R \simeq R$, and set

$$\mathrm{E}_2^{s,t}(M) = \mathrm{Coext}^{s,t}_{E^R_*E}(E_*, E^R_*M).$$

We will refer to this spectral sequence as the Adams Spectral Sequence based on E for the R-module M.

To understand convergence of such a spectral sequence we use a criterion of Bousfield [8, 12]. For an R-module M, let D_sM $(s \geqslant 0)$ be the R-modules defined by $D_0M = M$ and taking D_sM to be the fibre of the natural map

$$D_{s-1}M \cong R\underset{R}{\wedge}D_{s-1}M \longrightarrow E\underset{R}{\wedge}D_{s-1}M.$$

Also for $s \geqslant 0$ let K_sM be the cofibre of the natural map $D_sM \longrightarrow M$. Then the homotopy limit

$$\widehat{\mathrm{L}}^R_E M = \operatorname*{holim}_s K_sM$$

is called the *E-nilpotent completion* of M.

Remark 2.2. It is easy to see that if $M \longrightarrow N$ is a map of R-modules which is an E-equivalence, then for each s, there is an equivalence $K_sM \longrightarrow K_sN$, hence

$$\widehat{\mathrm{L}}^R_E M \simeq \widehat{\mathrm{L}}^R_E N.$$

Theorem 2.3. *If for each pair (s, t) there is an r_0 for which $\mathrm{E}_r^{s,t}(M) = \mathrm{E}_\infty^{s,t}(M)$ whenever $r \geqslant r_0$, then the Adams Spectral Sequence for M based on E converges to $\pi_*\widehat{\mathrm{L}}^R_E M$.*

Note that although there is a natural map $\mathrm{L}^R_E M \longrightarrow \widehat{\mathrm{L}}^R_E M$, it is not a weak equivalence in general; this condition is guaranteed by another result due to Bousfield [8].

Theorem 2.4. *Suppose that there is an r_1 such that for every R-module N there is an s_1 for which*

$$\mathrm{E}_r^{s,t}(N) = 0$$

whenever $r \geqslant r_1$ *and* $s \geqslant s_1$*. Then for every* R*-module* M *the Adams Spectral Sequence for* M *based on* E *converges to* $\pi_* \mathrm{L}^R_E M$ *and*

$$\mathrm{L}^R_E M \simeq \widehat{\mathrm{L}}^R_E M.$$

3. The Universal Coefficient Spectral Sequence for regular quotients

Let R be a commutative S-algebra and $E = R/I$ a commutative regular quotient of R, where $u_1, u_2, \ldots$ is a regular sequence generating $I_* \lhd R_*$.

We will discuss the existence of the Universal Coefficient Spectral Sequence

$$E^2_{r,s} = \mathrm{Ext}^{r,s}_{E_*}(E^R_* M, N_*) \Longrightarrow N^*_R M, \tag{3.1}$$

where M and N are R-modules and N is also an E-module spectrum in $\mathcal{M}_R$. The classical prototype of this was of course described by Adams [1] and used in setting up the E-theory Adams Spectral Sequence. It is straightforward to verify that Adams' approach can be followed in $\mathcal{D}_R$. The existence of such a spectral sequence depends on the following criterion.

Criterion 3.1. E is a homotopy colimit of finite cell R-modules E_α whose R-Spanier Whitehead duals $\mathrm{D}_R E_\alpha = \mathcal{F}_R(E_\alpha, R)$ satisfy the conditions
(A) $E^R_* \mathrm{D}_R E_\alpha$ is E_*-projective;
(B) the natural map

$$N^*_R M \longrightarrow \mathrm{Hom}_{E_*}(E^R_* M, N_*)$$

is an isomorphism.

Theorem 3.2. *For a commutative regular quotient* $E = R/I$ *of* R*,* E *can be expressed as a homotopy colimit of finite cell* R*-modules satisfying the conditions of* Criterion 3.1*. In fact we can take* $E^R_* \mathrm{D}_R E_\alpha$ *to be* E_**-free.*

The proof will use the following Lemma.

Lemma 3.3. *Let* $u \in R_{2d}$ *be non-zero divisor in* R_**. Suppose that* P *is an* R*-module for which* $E^R_* P$ *is* E_**-projective and for an* E*-module* R*-spectrum* N*,*

$$N^*_R P \cong \mathrm{Hom}_{E_*}(E^R_* P, N_*).$$

Then $E^R_* P \underset{R}{\wedge} R/u$ *is* E_**-projective and*

$$N^*_R P \underset{R}{\wedge} R/u \cong \mathrm{Hom}_{E_*}(E^R_* P \underset{R}{\wedge} R/u, N_*).$$

Proof. Smashing $E \underset{R}{\wedge} P$ with the cofibre sequence (3.2) and taking homotopy, we obtain an exact triangle

$$\begin{tikzcd}
E^R_* P \arrow[rr, "u"] & & E^R_* P \arrow[dl] \\
& E^R_* P \underset{R}{\wedge} R/u \arrow[ul] &
\end{tikzcd}$$

As multiplication by u induces the trivial map in E^R-homology, this is actually a short exact sequence of E_*-modules,

$$0 \to E^R_* P \longrightarrow E^R_* P \underset{R}{\wedge} R/u \longrightarrow E^R_* P \to 0$$

which clearly splits, so $E^R_* P \underset{R}{\wedge} R/u$ is E_*-projective.

In the evident diagram of exact triangles

$$\begin{tikzcd}
N_R^*P \arrow[rr] \arrow[dd] & & N_R^*P \arrow[dl] \arrow[dd] \\
& N_R^*P\underset{R}{\wedge}R/u \arrow[ul] \arrow[dd] & \\
\mathrm{Hom}_{E_*}(E_*^RP, N_*) \arrow[rr] & & \mathrm{Hom}_{E_*}(E_*^RP, N_*) \arrow[dl] \\
& \mathrm{Hom}_{E_*}(E_*^RP\underset{R}{\wedge}R/u, N_*) \arrow[ul] &
\end{tikzcd}$$

the map $N_R^*P \longrightarrow \mathrm{Hom}_{E_*}(E_*^RP, N_*)$ is an isomorphism, so

$$N_R^*P\underset{R}{\wedge}R/u \longrightarrow \mathrm{Hom}_{E_*}(E_*^RP\underset{R}{\wedge}R/u, N_*)$$

is also an isomorphism by the Five Lemma. □

Proof of Theorem 3.2. Recall from [14] that if $u_1, u_2, \ldots$ is a regular sequence generating $I_* \triangleleft R_*$, then

$$E = \underset{k}{\mathrm{hocolim}}\, R/u_1\underset{R}{\wedge}R/u_2\underset{R}{\wedge}\cdots\underset{R}{\wedge}R/u_k,$$

and we use the notation $R/u = R/(u)$. For $u \in R_{2d}$ a non-zero divisor, the R_*-free resolution

$$0 \to R_* \longrightarrow R_* \xrightarrow{u} R_*/(u) \to 0$$

corresponds to an R-cell structure on R/u with one cell in each of the dimensions 0 and $2d+1$. There is an associated cofibre sequence

$$\cdots \longrightarrow \Sigma^{2d}R \xrightarrow{u} R \longrightarrow R/u \longrightarrow \Sigma^{2d+1}R \longrightarrow \cdots, \tag{3.2}$$

for which the induced long exact sequence in E^R-homology shows that E_*^RR/u is E_*-free. The dual D_RR/u is equivalent to $\Sigma^{-(2d+1)}R/u$, so R/u is essentially self dual.

For an E-module spectrum N in $\mathcal{D}_R$, there are two exact triangles and morphisms between them,

$$\begin{tikzcd}
N_R^*R \arrow[rr] \arrow[dd] & & N_R^*R \arrow[dl] \arrow[dd] \\
& N_R^*R/u \arrow[ul] \arrow[dd] & \\
\mathrm{Hom}_{E_*}(E_*, N_*) \arrow[rr] & & \mathrm{Hom}_{E_*}(E_*, N_*) \arrow[dl] \\
& \mathrm{Hom}_{E_*}(E_*^RR/u, N_*) \arrow[ul] &
\end{tikzcd}$$

The identifications

$$N_* \cong N_R^*R \cong \mathrm{Hom}_{E_*}(E_*, N_*),$$

and the Five Lemma imply that

$$N_R^*R/u \cong \mathrm{Hom}_{E_*}(E_*^RR/u, N_*).$$

Lemma 3.3 now implies that each of the spectra $R/u_1\underset{R}{\wedge}R/u_2\underset{R}{\wedge}\cdots\underset{R}{\wedge}R/u_k$ satisfies conditions (A) and (B). □

4. THE ADAMS SPECTRAL SEQUENCE BASED ON A REGULAR QUOTIENT

For an R-module M, let $M^{(s)}$ denote the s-fold R-smash power of M,

$$M^{(s)} = M \underset{R}{\wedge} M \underset{R}{\wedge} \cdots \underset{R}{\wedge} M.$$

If M is an $R[X^{-1}]$-module, then

$$M^{(s)} = M \underset{R[X^{-1}]}{\wedge} M \underset{R[X^{-1}]}{\wedge} \cdots \underset{R[X^{-1}]}{\wedge} M.$$

Let $E = R/I[X^{-1}]$ be a localized regular quotient and $u_1, u_2, \ldots$ a regular sequence generating I_*. We will discuss the Adams Spectral Sequence based on E. By Remark 2.2, we can work in the category of $R[X^{-1}]$-modules and replace the Adams Spectral Sequence of S_R by that of $S_{R[X^{-1}]}$. For notational simplicity, from now on we will replace R by $R[X^{-1}]$ and therefore assume that $E = R/I$ is a regular quotient of R.

First we identify the canonical Adams resolution giving rise to the Adams Spectral Sequence based on the regular quotient $E = R/I$. We will relate this to a tower described by Lazarev [10]; note that his notation for $I^{(s)}$ is I^s which we will use to denote a different spectrum.

There is a fibre sequence $I \longrightarrow R \longrightarrow R/I$ and a tower of maps of R-modules

$$R \longleftarrow I \longleftarrow I^{(2)} \longleftarrow \cdots \longleftarrow I^{(s)} \longleftarrow I^{(s+1)} \longleftarrow \cdots$$

in which $I^{(s+1)} \longrightarrow I^{(s)}$ is the evident composite

$$I^{(s+1)} \longrightarrow R \underset{R}{\wedge} I^{(s)} = I^{(s)}.$$

Setting $R/I^{(s)} = \text{cofibre}(I^{(s)} \longrightarrow R)$, we obtain a tower

$$R/I \longleftarrow R/I^{(2)} \longleftarrow \cdots \longleftarrow R/I^{(s)} \longleftarrow R/I^{(s+1)} \longleftarrow \cdots$$

which we will refer to as the *external I-adic tower*. The following is immediate from the definitions.

Proposition 4.1. *We have*

$$D_0 S_R = R, \qquad D_s S_R = I^{(s)}, \quad (s \geqslant 1),$$

and

$$K_s S_R = R/I^{(s+1)} \quad (s \geqslant 0).$$

It is not immediately clear how to determine the limit

$$\widehat{\mathrm{L}}^R_E S_R = \operatorname*{holim}_s R/I^{(s)}.$$

Instead of doing this directly, we will adopt an approach suggested by Bousfield [8], making use of another E-nilpotent resolution, associated with the *internal I-adic tower* which we describe below.

Before doing this, we need to understand convergence. We will show that the condition of Theorem 2.3 is satisfied for a commutative regular quotient $E = R/I$.

Proposition 4.2. *The* E_2*-term of the E-theory Adams Spectral Sequence for $\pi_* S_R$ is*

$$\mathrm{E}_2^{s,t}(S_R) = \mathrm{Coext}^{s,t}_{E^R_* E}(E_*, E_*) = E_*[U_i : i \geqslant 1],$$

where $\operatorname{bideg} U_i = (1, |u_i| + 1)$. *So this spectral sequence collapses at* $\mathrm{E}_\infty^{*,*}(S_R) = \mathrm{E}_2^{*,*}(S_R)$ *and converges to* $\pi_* \widehat{\mathrm{L}}^R_E S_R$.

Proof. By Proposition 1.2,

$$E^R_*E = \Lambda_{R_*}(\tau_i : i \geqslant 1),$$

with generators τ_i which are primitive with respect to the Hopf algebroid coproduct. The determination of $\mathrm{Coext}^{*,*}_{E^R_*E}(E_*, E_*)$ is now standard and the differentials are trivial for degree reasons. □

Induction on the number of cells now gives

Corollary 4.3. *For a finite cell R-module M, the E-theory Adams Spectral Sequence for π_*M converges to $\pi_*\widehat{\mathrm{L}}^R_E M$.*

5. The internal I-adic tower

Suppose that $I_* \lhd R_*$ is generated by a regular sequence $u_1, u_2, \ldots$. We will often indicate a monomial in the u_i by writing $u_{(i_1,\ldots,i_k)} = u_{i_1} \cdots u_{i_k}$. We will write $E = R/I$ and make use of algebraic results from [4] which we now recall in detail.

For $s \geqslant 0$, we define the R-module I^s/I^{s+1} to be the wedge of copies of E indexed on the distinct monomials of degree s in the generators u_i. For an explanation of this, see Corollary 5.4.

We will show that there is a tower of R-modules

$$R/I \longleftarrow R/I^2 \longleftarrow \cdots \longleftarrow R/I^s \longleftarrow R/I^{s+1} \longleftarrow \cdots$$

with fibre sequences of the form

$$R/I^s \longleftarrow R/I^{s+1} \longleftarrow I^s/I^{s+1}.$$

This is the (*internal*) *I-adic tower.* On setting $I^s = \mathrm{fibre}(R \longrightarrow R/I^s)$, there is another tower

$$R/I \longleftarrow I^2 \longleftarrow \cdots \longleftarrow I^s \longleftarrow I^{s+1} \longleftarrow \cdots$$

which is analogous to the external version of [10]. This extension corresponds to a certain element of

$$\mathrm{Ext}^1_{R_*}(R_*/I^s_*, I^s_*/I^{s+1}_*)$$

via the Universal Coefficient Spectral Sequence converging to $\mathcal{D}_R(R/I^s, I^s/I^{s+1})^*$. A related construction appeared in [2, 7] for the case of $R = \widehat{E(n)}$ (which we showed admitted a not necessarily commutative S-algebra structure) and $I = I_n$.

Underlying our work is the classical *Koszul resolution*

$$\mathbf{K}_{*,*} \longrightarrow R_*/I_* \to 0,$$

where

$$\mathbf{K}_{*,*} = \Lambda_{R_*}(e_i : i \geqslant 1),$$

which has grading given by $\deg e_i = |u_i| + 1$ and differential

$$\begin{aligned} \mathrm{d}\, e_i &= u_i, \\ \mathrm{d}(xy) &= (\mathrm{d}\, x)y + (-1)^r x\, \mathrm{d}\, y \quad (x \in \mathbf{K}_{r,*},\ y \in \mathbf{K}_{s,*}). \end{aligned}$$

Hence $(\mathbf{K}_{*,*}, \mathrm{d})$ a R_*-free resolution of R_*/I_* which is a differential graded R_*-algebra. Tensoring with R_*/I_* and taking homology leads to a well known result.

Proposition 5.1. *As an R_*/I_*-algebra,*

$$\mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, R_*/I_*) = \Lambda_{R_*/I_*}(e_i : i \geqslant 1).$$

Corollary 5.2. $\mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, R_*/I_*)$ *is a free R_*/I_*-module.*

This is of course closely related to the topological result Proposition 1.2.

Returning now to our algebraic discussion, we recall the following standard result.

Lemma 5.3 ([11], Theorem 16.2)**.** *For $s \geqslant 0$, I^s_*/I^{s+1}_* is a free R_*/I_*-module with a basis consisting of residue classes of the distinct monomials $u_{(i_1,\ldots,i_s)}$ of degree s.*

Corollary 5.4. *For $s \geqslant 0$, there is an isomorphism of R_*-modules*

$$\pi_* I^s/I^{s+1} = I^s_*/I^{s+1}_*.$$

Hence $\pi_* I^s/I^{s+1}$ is a free R_*/I_*-module with a basis indexed on the distinct monomials $u_{(i_1,\ldots,i_s)}$ of degree s.

Let $\mathrm{U}^{(s)}_*$ be the free R_*-module on a basis indexed on the distinct monomials of degree s in the u_i. For $s \geqslant 0$, set

$$\mathbf{Q}^{(s)}_{*,*} = \mathbf{K}_{*,*} \underset{R_*}{\otimes} \mathrm{U}^{(s)}_*, \quad \mathrm{d}^{(s)}_{\mathbf{Q}} = \mathrm{d} \otimes 1,$$

and also for $x \in \mathbf{K}_{*,*}$ write

$$x\widetilde{u}_{(i_1,\ldots,i_s)} = x \otimes u_{(i_1,\ldots,i_s)}.$$

There is an obvious augmentation

$$\mathbf{Q}^{(s)}_{0,*} \longrightarrow I^s_*/I^{s+1}_*.$$

Lemma 5.5. *For $s \geqslant 1$,*

$$\mathbf{Q}^{(s)}_{*,*} \xrightarrow{\varepsilon^{(s)}} I^s_*/I^{s+1}_* \to 0$$

is a resolution by free R_*-modules.

Given a complex $(\mathbf{C}_{*,*}, \mathrm{d}_{\mathbf{C}})$, the k-shifted complex $(\mathbf{C}[k]_{*,*}, \mathrm{d}_{\mathbf{C}[k]})$ is defined by

$$\mathbf{C}[k]_{n,*} = \mathbf{C}_{n+k,*}, \quad \mathrm{d}_{\mathbf{C}[k]} = (-1)^k \, \mathrm{d}_{\mathbf{C}}\,.$$

There is a morphism of chain complexes

$$\partial^{(s+1)} \colon \mathbf{Q}^{(s)}_{*,*} \longrightarrow \mathbf{Q}^{(s+1)}[-1]_{*,*}; \quad \partial^{(s+1)} e_{i_1} \cdots e_{i_r} \widetilde{u}_{(j_1,\ldots,j_s)} = \sum_{k=1}^{r} (-1)^k e_{i_1} \cdots \widehat{e}_{i_k} \cdots e_{i_r} \widetilde{u}_{(j_1,\ldots,j_s)}.$$

Using the identification $\mathbf{Q}^{(s+1)}[-1]_{n,*} = \mathbf{Q}^{(s+1)}_{n-1,*}$, we will often view $\partial^{(s+1)}$ as a homomorphism

$$\partial^{(s+1)} \colon \mathbf{Q}^{(s)}_{*,*} \longrightarrow \mathbf{Q}^{(s+1)}_{*,*}$$

of bigraded R_*-modules of degree -1.

There are also external pairings

$$\mathbf{Q}^{(r)}_{*,*} \underset{R_*}{\otimes} \mathbf{Q}^{(s)}_{*,*} \longrightarrow \mathbf{Q}^{(r+s)}_{*,*}; \quad x\widetilde{u}_{(i_1,\ldots,i_s)} \otimes y\widetilde{u}_{(j_1,\ldots,j_s)} \longmapsto xy\widetilde{u}_{(i_1,\ldots,i_s,j_1,\ldots,j_s)} \quad (x, y \in \mathbf{K}_{*,*}).$$

In particular, each $\mathbf{Q}^{(r)}_{*,*}$ is a differential module over the differential graded R_*-algebra $\mathbf{K}^{(0)}_{*,*}$ and $\partial^{(s+1)}$ is a $\mathbf{K}^{(0)}_{*,*}$-derivation.

Theorem 5.6. *For $s \geqslant 1$, there is a resolution*

$$\mathbf{K}^{(s-1)}_{*,*} \xrightarrow{\varepsilon^{(s-1)}} R_*/I^s_* \to 0,$$

by free R_*-modules, where

$$\mathbf{K}^{(s-1)}_{*,*} = \mathbf{Q}^{(0)}_{*,*} \oplus \mathbf{Q}^{(1)}_{*,*} \oplus \cdots \oplus \mathbf{Q}^{(s-1)}_{*,*},$$

and the differential is

$$\mathrm{d}^{(s-1)} = (\mathrm{d}^{(0)}_{\mathbf{Q}}, \partial^{(1)} + \mathrm{d}^{(1)}_{\mathbf{Q}}, \partial^{(2)} + \mathrm{d}^{(2)}_{\mathbf{Q}}, \ldots, \partial^{(s-1)} + \mathrm{d}^{(s-1)}_{\mathbf{Q}}).$$

In fact $(\mathbf{K}^{(s-1)}_{*,*}, \mathrm{d}^{(s-1)})$ is a differential graded R_*-algebra which provides a multiplicative resolution of R_*/I^s, with the augmentation given by

$$\varepsilon^{(s-1)}(x_0, x_1\widetilde{u}_{\mathbf{i}_1}, \ldots, x_{s-1}\widetilde{u}_{\mathbf{i}_{s-1}}) = x_0 + x_1 u_{\mathbf{i}_1} + \cdots + x_{s-1} u_{\mathbf{i}_{s-1}}.$$

The algebraic extension of R_*-modules

$$0 \leftarrow R_*/I_*^s \longleftarrow R_*/I_*^{s+1} \longleftarrow I_*^s/I_*^{s+1} \leftarrow 0$$

is classified by an element of

$$\mathrm{Ext}^1_{R_*}(R_*/I_*^s, I_*^s/I_*^{s+1}) = \mathrm{Hom}_{\mathcal{D}_{R_*}}(R_*/I_*^s, I_*^s/I_*^{s+1}[-1]),$$

where $\mathrm{Hom}_{\mathcal{D}_{R_*}}$ denotes morphisms in the derived category $\mathcal{D}_{R_*}$ of the ring R_* [15]. This element is represented by the composite

$$\widetilde{\partial}_*^{(s)} \colon \mathbf{K}_{*,*}^{(s-1)} \xrightarrow{\mathrm{proj}} \mathbf{Q}_{*,*}^{(s-1)} \xrightarrow{\partial^{(s)}} \mathbf{Q}^{(s)}[-1]_{*,*}. \tag{5.1}$$

The next result was proved in [4] for ungraded rings. The version we require involves a grading but the proof is easily adapted to this case.

Proposition 5.7. *For each $s \geqslant 2$, the following complex is exact:*

$$\mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, R_*/I_*) \xrightarrow{\partial_*^{(1)}} \mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, I_*/I_*^2) \xrightarrow{\partial_*^{(2)}} \cdots \xrightarrow{\partial_*^{(s-1)}} \mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, I_*^{s-1}/I_*^s).$$

Theorem 5.8. *For $s \geqslant 2$,*

$$\mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, R_*/I_*^s) = R_*/I_* \oplus \operatorname{coker} \partial_*^{(s-1)}.$$

This is a free R_*/I_*-module and with its natural R_*/I_*-algebra structure, $\mathrm{Tor}^{R_*}_{*,*}(R_*/I_*, R_*/I_*^s)$ has trivial products.

Given this algebraic background, we can now construct the I-adic tower.

Theorem 5.9. *There is a tower of R-modules*

$$R/I \longleftarrow R/I^2 \longleftarrow \cdots \longleftarrow R/I^s \longleftarrow R/I^{s+1} \longleftarrow \cdots$$

whose maps define fibre sequences

$$R/I^s \longleftarrow R/I^{s+1} \longleftarrow I^s/I^{s+1}$$

which in homotopy realise the exact sequences of R_*-modules

$$0 \leftarrow R_*/I_*^s \longleftarrow R_*/I_*^{s+1} \longleftarrow I_*^s/I_*^{s+1} \leftarrow 0.$$

Furthermore, the following conditions are satisfied for each $s \geqslant 1$.
(i) *$E_*^R R/I^s$ is a free E_*-module and the unit induces a splitting*

$$E_*^R R/I^s = E_* \oplus (\ker \colon E_*^R R/I^s \longrightarrow E_*);$$

(ii) *the projection map $R/I^{s+1} \longrightarrow R/I^s$ induces the zero map*

$$(\ker \colon E_*^R R/I^{s+1} \longrightarrow E_*) \longrightarrow (\ker \colon E_*^R R/I^s \longrightarrow E_*);$$

(iii) *the inclusion map $j_s \colon I^s/I^{s+1} \longrightarrow R/I^{s+1}$ induces an exact sequence*

$$E_*^R I^{s-1}/I^s \xrightarrow{\partial_*^{(s)}} E_*^R I^s/I^{s+1} \xrightarrow{j_{s*}} (\ker \colon E_*^R R/I^{s+1} \longrightarrow E_*) \to 0.$$

Proof. The proof is by induction on s. Assuming that R/I^s exists with the asserted properties, we will define a suitable map $\delta_s \colon R/I^s \longrightarrow \Sigma I^s/I^{s+1}$ which induces a fibre sequence

$$R/I^s \longleftarrow X^{(s+1)} \longleftarrow I^s/I^{s+1}$$

with $\pi_* X^{(s+1)} = R_*/I_*^{s+1}$ as an R_*-module.

If M is an R-module which is an E module spectrum, Theorem 3.2 provides a Universal Coefficient Spectral Sequence

$$\mathrm{E}_2^{*,*} = \mathrm{Ext}^{p,q}_{E_*}(E_*^R R/I^s, M_*) \Longrightarrow \mathcal{D}_R(R/I^s, M)^{p+q}.$$

Since $E_*^R R/I^s$ is E_*-free, this spectral sequence collapses to give

$$\mathcal{D}_R(R/I^s, M)^* = \mathrm{Hom}^*_{E_*}(E_*^R R/I^s, M_*).$$

In particular, for $M = I^s/I^{s+1}$,

$$\mathcal{D}_R(R/I^s, I^s/I^{s+1})^n = \mathrm{Hom}^n_{E_*}(E^R_* R/I^s, I^s_*/I^{s+1}_*).$$

By (5.1) and Theorem 5.6, there is an element

$$\widetilde{\partial}^{(s)}_* \in \mathrm{Hom}^0_{E_*}(E^R_* R/I^s, I^s_*/I^{s+1}_*[-1]) = \mathrm{Hom}^1_{E_*}(E^R_* R/I^s, I^s_*/I^{s+1}_*),$$

corresponding to an element $\delta_s \colon R/I^s \longrightarrow \Sigma I^s/I^{s+1}$ inducing a fibre sequence

$$R/I^s \longleftarrow X^{(s+1)} \longleftarrow I^s/I^{s+1}.$$

We still need to verify that $\pi_* X^{(s+1)} = R_*/I^{s+1}_*$ as an R_*-module.

For this, we will use the resolutions $\mathbf{K}^{(s-1)}_{*,*} \longrightarrow R_*/I^s_* \to 0$ and $\mathbf{K}_{*,*} \longrightarrow R_*/I_* \to 0$. These free resolutions give rise to cell R-module structures on R/I^s and E. By [9], the R-module $E\underset{R}{\wedge}R/I^s$ admits a cell structure with cells in one-one correspondence with the elements of the obvious tensor product basis of $\mathbf{K}_{*,*}\underset{R_*}{\otimes}\mathbf{K}^{(s-1)}_{*,*}$. Hence there is a resolution by free R_*-modules

$$\mathbf{K}_{*,*}\underset{R_*}{\otimes}\mathbf{K}^{(s-1)}_{*,*} \longrightarrow E^R_* R/I^s \to 0.$$

There are morphisms of chain complexes

$$\mathbf{K}^{(s-1)}_{*,*} \xrightarrow{\rho_s} \mathbf{K}_{*,*}\underset{R_*}{\otimes}\mathbf{K}^{(s-1)}_{*,*} \xrightarrow{\widetilde{\delta}_s} \mathbf{Q}^{(s)}_{*,*}[-1],$$

where ρ_s is the obvious inclusion and $\widetilde{\delta}_s$ is a chain map lifting $\widetilde{\partial}^{(s)}_*$ which can be chosen so that

$$\widetilde{\delta}_s(e_i \otimes x) = 0.$$

Considering the composite $\widetilde{\delta}_s\rho_s$, we find its effect on the generator $e_i\widetilde{u}_{(j_1,\ldots,j_{s-1})} \in \mathbf{K}^{(s-1)}_{1,*}$ to be

$$\widetilde{\partial}^{(s)}_* e_i\widetilde{u}_{(j_1,\ldots,j_{s-1})} = \widetilde{u}_{(i,j_1,\ldots,j_{s-1})},$$

while the elements of form $e_i \otimes \widetilde{u}_{(j_1,\ldots,j_{k-1})}$ with $k < s$ are annihilated. The composite homomorphism

$$\mathbf{K}^{(s-1)}_{1,*} \xrightarrow{\widetilde{\delta}_s\rho_s} \mathbf{Q}^{(s)}_{0,*}[-1] \xrightarrow{\varepsilon_1} I^s_*/I^{s+1}_*[-1]$$

is a cocycle. There is a morphism of exact sequences

$$\begin{tikzcd}
0 & R_*/I^s_* \arrow[l] \arrow[d, equal] & \mathbf{K}^{(s-1)}_{0,*} \arrow[l] \arrow[d, "\alpha_0"'] & \mathbf{K}^{(s-1)}_{1,*} \arrow[l] \arrow[d, "\alpha_1"'] & \mathbf{K}^{(s-1)}_{2,*} \arrow[l] \arrow[d] \\
0 & R_*/I^s_* \arrow[l] & R_*/I^{s+1}_* \arrow[l] & I^s_*/I^{s+1}_* \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where the cohomology class

$$[\alpha_1] \in \mathrm{Ext}^{1,*}_{R_*}(R_*/I^s_*, I^s_*/I^{s+1}_*)$$

represents the extension of R_*-modules on the bottom row. It is easy to see that $[\alpha_1] = [\varepsilon_1\widetilde{\delta}_s\rho_s]$, hence this class also represents the extension

$$0 \leftarrow R_*/I^s_* \longleftarrow \pi_* X^{s+1} \longleftarrow I^s_*/I^{s+1}_* \leftarrow 0.$$

There is a diagram of cofibre triangles

$$\begin{tikzcd}
R/I^{s+1} \arrow[d] & I/I^{s+1} \arrow[l] \arrow[d] & I^2/I^{s+1} \arrow[l] \arrow[d] & \cdots \arrow[l] & I^{s-1}/I^{s+1} \arrow[l] \arrow[d] & I^s/I^{s+1} \arrow[l] \arrow[d, "="] \\
R/I \arrow[ur] & I/I^2 \arrow[ur] & I^2/I^3 \arrow[ur] & {} \arrow[ur] & I^{s-1}/I^s \arrow[ur] & I^s/I^{s+1}
\end{tikzcd}$$

and applying $E^R_*(\ )$ we obtain a spectral sequence converging to E^R_*R/I^{s+1} whose E_2-term is the homology of the complex

$$0 \to E^R_*R/I \xrightarrow{\partial^{(1)}_*} E^R_*I/I^2 \xrightarrow{\partial^{(2)}_*} E^R_*I^2/I^3 \longrightarrow \cdots \xrightarrow{\partial^{(s)}_*} E^R_*I^s/I^{s+1} \to 0,$$

where the $\partial^{(k)}_*$ are essentially the maps of [4] used in computing $\mathrm{Tor}^{R_*}_*(R_*/I_*, R_*/I^{s+1}_*)$. By Proposition 5.7 and Theorem 5.8, this complex is exact except at the ends, where we have $\ker \partial^{(1)}_* = E_*$. □

Corollary 5.10. *For any E-module spectrum N and $s \geqslant 1$,*

$$N^*_R R/I^s \cong \mathrm{Hom}_{E_*}(E^R_*R/I^s, N_*).$$

Proof. This follows from Theorem 5.9(i). □

We will also use the following result.

Corollary 5.11. *For $s \geqslant 1$, the natural map*

$$E^R_*R/I^{s+1} \longrightarrow E^R_*R/I^s,$$

has image equal to $E_* = E^R_*R$.

Proof. This follows from Theorem 5.9(ii). □

Corollary 5.12. *For any E-module spectrum N and $s \geqslant 1$,*

$$\operatorname*{colim}_s N^*_R R/I^s \cong N^*_R R \cong N_*.$$

Proof. This is immediate from Corollaries 5.10 and 5.11 since

$$\operatorname*{colim}_s \mathrm{Hom}_{E_*}(E^R_*R/I^s, N_*) \cong \mathrm{Hom}_{E_*}(E_*, N_*).$$

□

6. The I-adic tower and the Adams Spectral Sequence

Continuing with the notation of Section 5, the first substantial result of this section is

Theorem 6.1. *The I-adic tower*

$$R/I \longleftarrow R/I^2 \longleftarrow \cdots \longleftarrow R/I^s \longleftarrow R/I^{s+1} \longleftarrow \cdots$$

has homotopy limit

$$\operatorname*{holim}_s R/I^s \simeq \widehat{\mathrm{L}}^R_E S_R.$$

Our approach follows ideas of Bousfield [8] where it is shown that the following Lemma implies Theorem 6.1.

Lemma 6.2. *Let $E = R/I$.*
(i) *Each R/I^s is E-nilpotent.*
(ii) *For each E-nilpotent R-module M,*

$$\operatorname*{colim}_s \mathcal{D}_R(R/I^s, M)^* = M_{-*}.$$

Proof. (i) is proved by an easy induction on $s \geqslant 1$.
(ii) is a consequence of Corollary 5.12. □

By Bousfield [8], we now obtain

$$\operatorname*{holim}_s R/I^s \simeq \widehat{\mathrm{L}}^R_E S_R.$$

Since the maps $R_*/I_*^{s+1} \longrightarrow R_*/I_*^s$ are surjective, from the standard exact sequence for $\pi_*(\ )$ of a homotopy limit we obtain

$$\pi_*\widehat{\mathrm{L}}^R_E S_R = \lim_s R_*/I_*^s. \tag{6.1}$$

We can generalize this to the case where E is a commutative localized regular quotient.

Theorem 6.3. *Let $E = R/I[X^{-1}]$ be a commutative localized regular quotient of R. Then*

$$\pi_*\widehat{\mathrm{L}}^R_E S_R = R_*[X^{-1}]\widehat{{}_{I_*}} = \lim_s R_*[X^{-1}]/I_*^s.$$

The natural map $S_R \longrightarrow \widehat{\mathrm{L}}^R_E S_R$ is an E-equivalence, hence

$$\mathrm{L}^R_E S_R \simeq \widehat{\mathrm{L}}^R_E S_R,$$
$$\pi_*\mathrm{L}^R_E S_R = R_*[X^{-1}]\widehat{{}_{I_*}}.$$

Proof. The first statement is easy to verify.

To simplify notation, by Remark 2.2 we may as well replace R by $R[X^{-1}]$ and so assume that $E = R/I$ is a commutative regular quotient of R.

Using the Koszul complex $(\Lambda_{R_*}(e_j : j), \mathrm{d})$, we see that $\mathrm{Tor}^{R_*}_{*,*}(E_*, (R_*)\widehat{{}_{I_*}})$ as the homology of the complex

$$\Lambda_{R_*}(e_j : j) \underset{R_*}{\otimes} (R_*)\widehat{{}_{I_*}} = \Lambda_{(R_*)\widehat{{}_{I_*}}}(e_j : j)$$

with differential $\mathrm{d}' = \mathrm{d} \otimes 1$. Since the sequence u_j remains regular in $(R_*)\widehat{{}_{I_*}}$, this complex provides a free resolution of $E_* = R_*/I_*$ as an $(R_*)\widehat{{}_{I_*}}$-module. So

$$\mathrm{Tor}^{R_*}_{*,*}(E_*, (R_*)\widehat{{}_{I_*}}) = \mathrm{Tor}^{(R_*)\widehat{{}_{I_*}}}_{*,*}(E_*, (R_*)\widehat{{}_{I_*}}) = E_*.$$

To calculate $E^R_*\widehat{\mathrm{L}}^R_E S_R$ we may use the Künneth Spectral Sequence of [9],

$$\mathrm{E}_2^{s,t} = \mathrm{Tor}^{R_*}_{s,t}(E_*, \widehat{\mathrm{L}}^R_E S_R) \Longrightarrow E^R_{s+t}\widehat{\mathrm{L}}^R_E S_R.$$

By Theorem 6.3, the E_2-term is

$$\mathrm{Tor}^{R_*}_{*,*}(E_*, (R_*)\widehat{{}_{I_*}}) = E_* = E^R_* R.$$

Hence the natural homomorphism

$$E^R_* S_R \longrightarrow E^R_*\widehat{\mathrm{L}}^R_E S_R$$

is an isomorphism. □

A generalization of this result is easily proved by induction on the number of cells.

Theorem 6.4. *Let E be a commutative localized regular quotient of R. Then for any finite cell R-module M, the natural map $M \longrightarrow \widehat{\mathrm{L}}^R_E M$ is an E-equivalence, hence*

$$\mathrm{L}^R_E M \simeq \widehat{\mathrm{L}}^R_E M,$$
$$\pi_*\mathrm{L}^R_E M = M_*[X^{-1}]\widehat{{}_{I_*}} = R_*[X^{-1}]\widehat{{}_{I_*}} \underset{R_*}{\otimes} M_*.$$

The alert reader may wonder if the following conjecture is true, the algebraic issue being that it does not appear to be true that for a commutative ring A, the extension $A \longrightarrow A\widehat{{}_J}$ is always flat for an ideal $J \lhd A$, a Noetherian condition normally being required to establish such a result.

Conjecture 6.5. The conclusion of Theorem 6.4 holds when E is any commutative localized quotient of R.

7. Some localized regular quotients of MU

An obvious source of commutative localized regular quotients is the commutative S-algebra $R = MU$ and we will describe some important examples. It would appear to be algebraically simpler to work with BP at a prime p in place of MU, but it is not yet known if BP admits a commutative S-algebra structure.

Example A: $MU \longrightarrow H\mathbb{F}_p$.

Let p be a prime. By considering the Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$ as a commutative MU-algebra [9], we can form $H\mathbb{F}_p \underset{MU}{\wedge} H\mathbb{F}_p$. The Künneth Spectral Sequence gives

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{MU_*}_{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow H\mathbb{F}_p{}^{MU}_{s+t} H\mathbb{F}_p.$$

Using a Koszul complex over MU_*, it is straightforward to see that

$$\mathrm{E}^2_{*,*} = \Lambda_{\mathbb{F}_p}(\tau_j : j \geqslant 0),$$

the exterior algebra over $\mathbb{F}_p$ with generators $\tau_j \in \mathrm{E}^2_{1,2j}$.

Taking $R = MU$ and $E = H\mathbb{F}_p$, we obtain a spectral sequence

$$\mathrm{E}_2^{s,t}(MU) = \mathrm{Coext}^{s,t}_{\Lambda_{\mathbb{F}_p}(\tau_j : j \geqslant 0)}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_{s+t}\, \mathrm{L}^{MU}_{H\mathbb{F}_p} S_{MU},$$

where $I_\infty \lhd MU_*$ is generated by p together with all positive degree elements, so $MU_*/I_\infty = \mathbb{F}_p$. Also,

$$\pi_*\, \mathrm{L}^{MU}_{H\mathbb{F}_p} S_{MU} = (MU_*)\widehat{{}_{I_\infty}}.$$

More generally, for a finite cell MU-module M, the Adams Spectral Sequence has the form

$$\mathrm{E}_2^{s,t}(M) = \mathrm{Coext}^{s,t}_{\Lambda_{\mathbb{F}_p}(\tau_j : j \geqslant 0)}(\mathbb{F}_p, H\mathbb{F}_p{}^{MU}_* M) \Longrightarrow \pi_{s+t}\, \mathrm{L}^{MU}_{H\mathbb{F}_p} M,$$

where

$$\pi_*\, \mathrm{L}^{MU}_{H\mathbb{F}_p} M = (M_*)\widehat{{}_{I_\infty}}.$$

Example B: $MU \longrightarrow E(n)$.

By [9, 14], the Johnson-Wilson spectrum $E(n)$ is a commutative MU-ring spectrum and we can form $E(n) \underset{MU}{\wedge} E(n)$. There is a Künneth Spectral Sequence,

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{MU_*}_{s,t}(E(n)_*, E(n)_*) \Longrightarrow E(n)^{MU}_{s+t} E(n).$$

By using a Koszul complex over MU_* for $MU\langle n\rangle_*$ and localizing at v_n, we find that

$$\mathrm{E}^2_{*,*} = \Lambda_{E(n)_*}(\tau_j : j \geqslant 1 \text{ and } j \neq p^k - 1 \text{ with } 1 \leqslant k \leqslant n),$$

where Λ denotes an exterior algebra and $\tau_j \in \mathrm{E}^2_{1,2j}$. So as an $E(n)_*$-algebra,

$$E(n)^{MU}_* E(n) = \Lambda_{E(n)_*}(\tau_j : j \geqslant 1 \text{ and } j \neq p^k - 1 \text{ with } 1 \leqslant k \leqslant n).$$

Taking $R = MU$ and $E = E(n)$, we obtain a spectral sequence

$$\mathrm{E}_2^{s,t}(MU) = \mathrm{Coext}^{s,t}_{\Lambda_{E(n)_*}(\tau_j : j \geqslant n+1)}(E(n)_*, E(n)_*) \Longrightarrow \pi_{s+t}\, \mathrm{L}^{MU}_{E(n)} MU,$$

where

$$\pi_*\, \mathrm{L}^{MU}_{E(n)} MU = (MU_*)_{(p)}[v_n^{-1}]\widehat{{}_{J_{n+1}}}$$

and

$$J_{n+1} = (\ker\colon (MU_*)_{(p)}[v_n^{-1}] \longrightarrow E(n)_*) \lhd MU_*[v_n^{-1}].$$

In the E_2-term we have

$$\mathrm{E}_2^{s,t}(MU) = E(n)_*[U_j : 0 \leqslant j \neq p^k - 1 \text{ for } 0 \leqslant k \leqslant n],$$

with generator $U_j \in \mathrm{E}_2^{1,2j+1}(MU)$ corresponding to an exterior generator in $E(n)^{MU}_* E(n)$ associated with a polynomial generator of MU_* in degree $2j$ lying in $\ker MU_* \longrightarrow E(n)_*$.

More generally, for a finite cell MU-module M,

$$\mathrm{E}_2^{s,t}(M) = \mathrm{Coext}^{s,t}_{\Lambda_{E(n)_*}(\tau_j : j \geqslant n+1)}(E(n)_*, E(n)^{MU}_* M), \Longrightarrow \pi_{s+t} \mathrm{L}^{MU}_{E(n)} M,$$

where

$$\pi_* \mathrm{L}^{MU}_{E(n)} M = M\widehat{}_{J_{n+1}} = (MU_*)_{(p)}[v_n^{-1}]\widehat{}_{J_{n+1}} \underset{MU}{\otimes} M.$$

Example C: $MU \longrightarrow K(n)$.

By [14], Morava K-theory is represented by a commutative MU-ring spectrum $K(n)$. There is a Künneth Spectral Sequence

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{MU_*}_{s,t}(K(n)_*, K(n)_*) \Longrightarrow K(n)^{MU}_{s+t} K(n).$$

We find that

$$\mathrm{E}^2_{*,*} = \Lambda_{K(n)_*}(\tau_j : 0 \leqslant j \neq p^n - 1).$$

Taking $R = MU$ and $E = K(n)$, we obtain a spectral sequence

$$\mathrm{E}_2^{s,t}(MU) = \mathrm{Coext}^{s,t}_{\Lambda_{K(n)_*}(\tau_j : 0 \leqslant j \neq n)}(K(n)_*, K(n)_*) \Longrightarrow \pi_{s+t} \mathrm{L}^{MU}_{K(n)} MU,$$

where

$$\pi_* \mathrm{L}^{MU}_{K(n)} MU = (MU_*)\widehat{}_{I_{n,\infty}}$$

with $I_{n,\infty} = \ker MU_* \longrightarrow K(n)_*$. In the E_2-term we have

$$\mathrm{E}_2^{s,t}(MU) = E(n)_*[U_j : 0 \leqslant j \neq p^n - 1],$$

with generator $U_j \in \mathrm{E}_2^{1,2j+1}(MU)$ corresponding to an exterior generator in $E(n)^{MU}_* E(n)$ associated with a polynomial generator of MU_* in degree $2k$ lying in $\ker MU_* \longrightarrow E(n)_*$ (or when $j = 0$, associated with p).

More generally, for a finite cell MU-module M,

$$\mathrm{E}_2^{s,t}(M) = \mathrm{Coext}^{s,t}_{\Lambda_{K(n)_*}(\tau_j : 0 \leqslant j \neq n)}(K(n)_*, K(n)^{MU}_* M) \Longrightarrow \pi_{s+t} \mathrm{L}^{MU}_{K(n)} M,$$

where

$$\pi_* \mathrm{L}^{MU}_{K(n)} M = (M_*)\widehat{}_{I_{n,\infty}} = (MU_*)\widehat{}_{I_{n,\infty}} \underset{MU_*}{\otimes} M_*.$$

Concluding remarks

There are several outstanding issues raised by our work.

Apart from the question of whether it is possible to weaken the assumptions from (commutative) regular quotients to a more general class, it seems reasonable to ask whether the internal I-adic tower is one of R ring spectra. Since $\operatorname{holim}_s R/I^s = \mathrm{L}^R_E R$, the localization theory of [9, 16] shows that this can be realized as a commutative R-algebra. However, showing that each R/I^s is an R ring spectrum or even an R algebra seem to involve far more intricate calculations. We expect that this will turn out to be true and even that the tower is one of R-algebras. This should involve techniques similar to those of [10, 5]. It is also worth noting that our proofs make no distinction between the cases where $I_* \lhd R_*$ is infinitely or finitely generated. There are a number of algebraic simplifications possible in the latter case, however we have avoided using them since the most interesting examples we know are associated with infinitely generated regular ideals in MU_*. The spectra E_n of Hopkins, Miller et al., have Noetherian homotopy rings and there are towers based on powers of their maximal ideals similar to those in the first author's older work [2, 7].

We also hope that our preliminary exploration of Adams Spectral Sequences for R-modules will lead to further work on this topic, particularly in the case $R = MU$ and related examples. A more ambitious project would be to investigate the commutative S-algebra MSp from this point of view, perhaps reworking the results of Vershinin, Gorbounov and Botvinnik in the context of MSp-modules.

References

[1] J. F. Adams, Stable Homotopy and Generalised Homology, University of Chicago Press (1974).

[2] A. Baker, A_∞ stuctures on some spectra related to Morava K-theory, Quart. J. Math. Oxf. **42** (1991), 403–419.

[3] ———, Brave new Hopf algebroids and the Adams spectral sequence for R-modules, Glasgow University Mathematics Department preprint 00/12.

[4] ———, On the homology of regular quotients, Glasgow University Mathematics Department preprint 01/1.

[5] A. Baker & A. Jeanneret, Brave new Hopf algebroids and extensions of MU-algebras, Glasgow University Mathematics Department preprint 00/18.

[6] ———, Brave new Bockstein operations, in preparation.

[7] A. Baker & U. Würgler, Bockstein operations in Morava K-theory, Forum Math. **3** (1991), 543–60.

[8] A. K. Bousfield, The localization of spectra with respect to homology, Topology **18** (1979), 257–281.

[9] A. Elmendorf, I. Kriz, M. Mandell & J. P. May, Rings, modules, and algebras in stable homotopy theory, Mathematical Surveys and Monographs, **47** (1999).

[10] A. Lazarev, Homotopy theory of A_∞ ring spectra and applications to MU-modules, preprint 2000.

[11] H. Matsumura, Commutative Ring Theory, Cambridge University Press, (1986).

[12] D. C. Ravenel, Localization with respect to certain periodic homology theories, Amer. J. Math. **106** (1984), 351–414.

[13] ———, Complex Cobordism and the Stable Homotopy Groups of Spheres, Academic Press (1986).

[14] N. P. Strickland, Products on MU-modules, Trans. Amer. Math. Soc. **351** (1999), 2569–2606.

[15] C. A. Weibel, An Introduction to Homological Algebra, Cambridge University Press (1994).

[16] J. J. Wolbert, Classifying modules over K-theory spectra, J. Pure Appl. Algebra **124** (1998), 289–323.

Department of Mathematics, University of Glasgow, Glasgow G12 8QW, Scotland.
E-mail address: `a.baker@maths.gla.ac.uk`
URL: `http://www.maths.gla.ac.uk/∼ajb`

Department of Mathematics, University of Bristol, Bristol BS8 1TW, England.
E-mail address: `A.Lazarev@bris.ac.uk`
URL: `http://www.maths.bris.ac.uk/∼pure/staff/maxal/maxal`