

MINIMAL ATOMIC COMPLEXES

A.J. BAKER AND J.P. MAY

ABSTRACT. Hu, Kriz and May recently reexamined ideas implicit in Priddy's elegant homotopy theoretic construction of the Brown-Peterson spectrum at a prime p . They discussed May's notions of *nuclear complexes* and of *cores* of spaces, spectra, and commutative S -algebras. Their most striking conclusions, due to Hu and Kriz, were negative: cores are not unique up to equivalence, and BP is not a core of MU considered as a commutative S -algebra, although it is a core of MU considered as a p -local spectrum. We investigate these ideas further, obtaining much more positive conclusions. We show that nuclear complexes have several non-obviously equivalent characterizations. Up to equivalence, they are precisely the *irreducible complexes*, the *minimal atomic complexes*, and the Hurewicz complexes with trivial mod p Hurewicz homomorphism above the Hurewicz dimension, which we call *complexes with no mod p detectable homotopy*. Unlike the notion of a nuclear complex, these other notions are all invariant under equivalence. This simple and conceptual criterion for a complex to be minimal atomic allows us to prove that many familiar spectra, such as ko , eo_2 , and BoP at the prime 2, all $BP\langle n \rangle$ at any prime p , and the indecomposable wedge summands of $\Sigma^\infty \mathbb{C}P^\infty$ and $\Sigma^\infty \mathbb{H}P^\infty$ at any prime p are minimal atomic.

INTRODUCTION

Atomic spaces and spectra have long been studied. They are so tightly bound together that a self-map which induces an isomorphism on homotopy in the Hurewicz dimension must be an equivalence. Atomic spaces and spectra can often be shrunk to ones with smaller homotopy groups. Minimal ones can be shrunk no further. Clearly, these are very natural objects of study. They seem to have been first introduced in [10]. Spheres, 2-cell complexes that are not wedges, and $K(\pi, n)$'s for cyclic groups π are obviously minimal atomic, but there are many much more interesting examples.

Nuclear complexes are atomic complexes (spaces or spectra) that are built up in an especially economical way. They are minimal atomic, and we shall see that every minimal atomic complex is equivalent to a nuclear complex. We regard the invariant notion of a minimal atomic complex as the more fundamental one. The combinatorial notion of a nuclear complex provides us with a tool for proving things about minimal atomic complexes.

We think of atomic complexes as analogues of "atomic modules", namely modules for which a non-trivial self-map is an isomorphism. We think of minimal atomic complexes as analogues of irreducible modules. We give a definition of an irreducible complex that makes this analogy more transparent, and we prove that

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the irreducible complexes are precisely the minimal atomic complexes. In one direction, the implication gives a homotopical analogue of Schur's lemma. Just as in algebra, we suggest that the irreducible, or minimal atomic, complexes are more basic mathematical objects than the atomic complexes. Carrying the analogy further, we see that the spectra we consider admit (dual) composition series of infinite length, constructed in terms of irreducible complexes.

There is a different and much more elementary notion of minimality, implicitly due to Cooke [5], such that *any* complex is equivalent to a minimal complex. This notion is also combinatorial and noninvariant. We prove that a Hurewicz complex that is minimal in this sense is nuclear if and only if it has no mod p detectable homotopy. It is well known that the latter condition implies wedge indecomposability (e.g. [14, 5.4]). The fact that it implies irreducibility is much stronger. We also prove a converse, leading to the conclusion that a complex is irreducible, or equivalently minimal atomic, if and only if it has no mod p detectable homotopy. This allows us to show that a variety of familiar spectra are in fact minimal atomic.

1. DEFINITIONS AND STATEMENTS OF RESULTS

Here we give the precise definitions needed to make sense of the introduction and state our main theorems. We write things so that the stable reader can view these as statements about spectra, and the unstable reader can view them as statements about (based) spaces. We adopt the following conventions throughout. They allow us to treat spaces and spectra uniformly and to avoid repeated mention of the fact that we are working p -locally under connectivity and finite type hypotheses.

We agree once and for all that all spaces and spectra X are to be localized at a fixed prime p . Thus S^n , for example, means a p -local sphere. We also agree that all spaces and spectra are to be p -local CW spaces or spectra, so that the domains of their attaching maps are p -local spheres. Spaces are to be simply connected, and their attaching maps are to be based. Spectra are to be bounded below. In either case, we say that X has *Hurewicz dimension* n_0 if X is $(n_0 - 1)$ -connected, but not n_0 -connected. Thus $n_0 \geq 2$ in the case of spaces, and there is no real loss of generality if we take $n_0 = 0$ in the case of spectra. We may assume without loss of generality that there are no cells (except the base vertex) of dimension less than n_0 . We assume further that there are only finitely many cells in each dimension. We agree to use the ambiguous term "complex" to mean such a p -local CW space or spectrum. We say that X is a *Hurewicz complex* if it has a single cell in dimension n_0 . We write X_n for the n -skeleton of a complex X . Thus X_{n+1} is the cofiber of a map $j_n : J_n \rightarrow X_n$, where J_n is a finite wedge of (p -local) n -spheres S^n . If X is a Hurewicz complex, $X_{n_0} = S^{n_0}$. We shall use these notations generically.

By $H_*(X)$, we always understand (reduced) homology with p -local coefficients. Any $(n_0 - 1)$ -connected space or spectrum such that each $H_n(X)$ is a finitely generated $\mathbb{Z}_{(p)}$ -module is weakly equivalent to a complex in the sense that we have just specified. If, further, $H_{n_0}(X; \mathbb{F}_p) = \mathbb{F}_p$ or, equivalently, $\pi_{n_0}(X)$ is a cyclic $\mathbb{Z}_{(p)}$ -module, then X is weakly equivalent to a Hurewicz complex.

We begin with definitions of concepts that are invariant under equivalence and the statement of our main characterization theorem relating them.

Definition 1.1. Consider complexes X and Y of Hurewicz dimension n_0 . Think of Y as fixed but X as variable.

- (i) A map $f : X \rightarrow Y$ is a *monomorphism* if $f_* : \pi_{n_0}(X) \otimes \mathbb{F}_p \rightarrow \pi_{n_0}(Y) \otimes \mathbb{F}_p$ and all $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ are monomorphisms
- (ii) Y is *irreducible* if any monomorphism $f : X \rightarrow Y$ is an equivalence.
- (iii) Y is *atomic* if it is a Hurewicz complex and a self-map $f : Y \rightarrow Y$ that induces an isomorphism on π_{n_0} is an equivalence.
- (iv) Y is *minimal atomic* if it is atomic and any monomorphism $f : X \rightarrow Y$ from an atomic complex X to Y is an equivalence.
- (v) Y has *no mod p detectable homotopy* if Y is a Hurewicz complex and the mod p Hurewicz homomorphism $h : \pi_n(Y) \rightarrow H_n(Y; \mathbb{F}_p)$ is zero for all $n > n_0$.
- (vi) Y is *H^* -monogenic* if $H^*(Y; \mathbb{F}_p)$ is a cyclic algebra (in the case of spaces) or module (in the case of spectra) over the mod p Steenrod algebra A .

Remarks 1.2. We offer several comments on these notions.

- (i) The structure theory for finitely generated modules over a PID implies that if $f : X \rightarrow Y$ is a monomorphism, then $f_*(\pi_{n_0}(X))$ is a direct summand of $\pi_{n_0}(Y)$. If X is a Hurewicz complex, this summand is cyclic. If X and Y are both Hurewicz complexes, then f induces an isomorphism on π_{n_0} .
- (ii) In [10, 1.1], following [21] and other early sources, Y was defined to be irreducible if it has no non-trivial retracts. On the space level, that concept has its uses, but we think that “irreducible” is the wrong name for it. We suggest “irretractible”. On the spectrum level, irretractibility is equivalent to wedge indecomposability. However, just as in algebra, irreducibility should be stronger rather than weaker than atomic. That is, there should be implications $\text{irreducible} \implies \text{atomic} \implies \text{indecomposable}$. One could avoid the conflict with the earlier literature by using the word “simple” instead of “irreducible”, the two being synonymous in algebra, but that risks confusion with the standard use of the term “simple” in topology.
- (iii) A complex can well have more than one cell in its Hurewicz dimension and still have the property that a self-map that induces an isomorphism on π_{n_0} is an equivalence. A particularly interesting example is given in [2, §4]. It might be sensible to delete the requirement that Y be a Hurewicz complex from the definition of atomic. By Theorem 1.9 below, the notion of minimal atomic would not change.
- (iv) Since our methods are cellular, we definitely mean to consider p -local rather than p -complete spaces and spectra. However, Definition 1.1 makes just as much sense in the p -complete case as the p -local case, and it is well worth studying there. Since a finite type p -complete space or spectrum is the p -completion of a finite type p -local space or spectrum, one can easily deduce conclusions in the p -complete case from the results here. We leave the details to the interested reader.
- (v) A Hurewicz complex Y has no mod p detectable homotopy if and only if there are no permanent cycles in dimension greater than n_0 on the zeroth row of the classical (unstable or stable) mod p Adams spectral sequence for Y . It is a much more computable condition than the others.

Theorem 1.3 (The characterization theorem). *The following conditions on a complex Y are equivalent.*

- (i) Y is irreducible.

- (ii) Y is minimal atomic.
- (iii) Y has no mod p detectable homotopy.

The fact that irreducible complexes are atomic should be viewed as a homotopical analogue of Schur's lemma since its intuitive content is that a non-trivial self-map of an irreducible complex must be an equivalence. Of course, it is consistent with the analogy that not all atomic complexes are irreducible. When [10] was written, examples of minimal atomic spectra seemed hard to come by. The following result now gives us many interesting examples.

Corollary 1.4. *If Y is H^* -monogenic, then Y has no mod p detectable homotopy and is therefore minimal atomic.*

Proof. For spectra, the long exact sequence on Ext arising from the epimorphism $\Sigma^{n_0} A \rightarrow H^*(Y; \mathbb{F}_p)$ implies that the zeroth row $\text{Hom}_A^{0,*}(H^*(Y; \mathbb{F}_p), \mathbb{F}_p)$ of the Adams spectral sequence is \mathbb{F}_p concentrated in degree n_0 . Similarly for spaces. \square

Remark 1.5. The converse of Corollary 1.4 fails. For $q \geq 2$, Moore spaces and spectra $M(\mathbb{Z}/p^q, n)$ and Eilenberg-Mac Lane spaces and spectra $K(\mathbb{Z}/p^q, n)$ are elementary examples of complexes that are minimal atomic but not H^* -monogenic.

The proof of Theorem 1.3 proceeds by first showing in Theorem 1.12 that (i) and (ii) are each equivalent to a statement about one noninvariant cellular construction and then showing in Theorem 1.18 that (ii) and (iii) are each equivalent to a statement about another noninvariant cellular construction. The first noninvariant construction, codified in Theorems 1.9 and 1.11, is based on the notions of nuclear complexes and cores introduced in [10].

Definition 1.6. A *nuclear complex* is a Hurewicz complex X such that

$$(1.7) \quad \text{Ker}(j_{n*} : \pi_n(J_n) \rightarrow \pi_n(X_n)) \subset p \cdot \pi_n(J_n)$$

for each n . Observe that X is nuclear if and only if each X_n for $n \geq n_0$ is nuclear. A *core* of a complex Y is a nuclear complex X together with a monomorphism $f : X \rightarrow Y$.

This notion of a core is more general than that of [10, 1.7], where it was assumed that $\pi_{n_0}(Y)$ is cyclic; that is, the definition there was restricted to Hurewicz complexes Y . Since cores are not unique even when Y is a Hurewicz complex, the more general notion seems preferable. Of course, with our perspective that cores are analogues of irreducible sub-modules, the non-uniqueness is only to be expected. With the present language, the following results are proven in [10, 1.5, 1.6].

Proposition 1.8. *A nuclear complex is atomic.*

Theorem 1.9. *If Y has Hurewicz dimension n_0 and C is a cyclic direct summand of $\pi_{n_0}(Y)$, then there is a core $f : X \rightarrow Y$ such that $f_*(\pi_{n_0}(X)) = C$.*

This allows us to construct homotopical analogues of composition series. That is, for spectra, we can shrink homotopy groups inductively by successively taking cofibers of cores, as discussed in [10, 1.8]. This works less well for spaces, where we would have to take fibers and so gradually decrease the Hurewicz dimension. To make the analogy with algebra precise, recall that a (countably infinite) composition series of a module Y is a sequence of monomorphisms

$$Y = Y_0 \xleftarrow{i_0} Y_1 \xleftarrow{\quad} \cdots \xleftarrow{\quad} Y_n \xleftarrow{i_n} Y_{n+1} \xleftarrow{\quad} \cdots$$

such that the cokernels of the i_n are irreducible and $\lim Y_n = 0$. A *dual composition series* of Y is a sequence of epimorphisms

$$Y = Y_0 \xrightarrow{p_0} Y_1 \longrightarrow \cdots \longrightarrow Y_n \xrightarrow{p_n} Y_{n+1} \longrightarrow \cdots$$

such that the kernels of the p_n are irreducible and $\operatorname{colim} Y_n = 0$. For a spectrum Y , we construct an analogous sequence by letting $p_n : Y_n \rightarrow Y_{n+1}$ be the cofiber of a core $f_n : X_n \rightarrow Y_n$. Each p_n induces an epimorphism on all homotopy groups, we kill $\pi_{n_0}(Y)$ in finitely many steps, then kill $\pi_{n_0+1}(Y)$ in finitely many steps, and so on. If Y has non-zero homotopy groups in only finitely many dimensions, then this sequence has only finitely many terms.

Theorem 1.9 has the following immediate consequence.

Corollary 1.10. *A core of a minimal atomic complex is an equivalence, hence a minimal atomic complex is equivalent to a nuclear complex.*

The converse, which completes Proposition 1.8, was conjectured in [10, 1.12]. We will prove it in Section 3.

Theorem 1.11. *A nuclear complex is a minimal atomic complex.*

Theorem 1.12. *Conditions (i) and (ii) of Theorem 1.3 are each equivalent to the condition that any core $f : X \rightarrow Y$ of Y is an equivalence.*

Proof. Corollary 1.10 and Theorem 1.11 show that Y is minimal atomic if and only if any core of Y is an equivalence. If Y is irreducible and $f : X \rightarrow Y$ is a core, then f is an equivalence by the definition of irreducibility. Conversely, suppose that Y is minimal atomic and let $f : X \rightarrow Y$ be a monomorphism. Let $g : W \rightarrow X$ be a core of X . Then the composite $f \circ g : W \rightarrow Y$ is a core of Y and therefore an equivalence. This implies that f induces an epimorphism and hence an isomorphism on homotopy groups. Thus f is an equivalence. \square

Remark 1.13. With these implications in place, it is perhaps better to redefine the notion of core invariantly, taking X to be minimal atomic but not necessarily nuclear. There is no substantive difference.

To tie in the Hurewicz homomorphism condition (iii) of Theorem 1.3, we use another, very different, noninvariant notion of minimality for a complex X . Of course, our complexes have p -local chain complexes specified by $C_n(X) = H_n(X_n/X_{n-1})$.

Definition 1.14. A complex X is *minimal* if the differential on its mod p chain complex $C_*(X; \mathbb{F}_p)$ is zero. It is *minimal Hurewicz* if it is minimal and Hurewicz. Observe that X is minimal if and only if each X_n is minimal.

A simple inductive argument gives a homological reformulation of this notion.

Lemma 1.15. *A complex X is minimal if and only if the inclusion of skeleta $X_n \rightarrow X_{n+1}$ induces an isomorphism*

$$H_n(X_n; \mathbb{F}_p) \longrightarrow H_n(X_{n+1}; \mathbb{F}_p) = H_n(X; \mathbb{F}_p)$$

for each n .

The following two results codify our second noninvariant construction. The first is implicit in Cooke's paper [5, Theorem A], which gives an integral space level version. Cooke described the result as "a well-known, basic fact". For a recent reappearance, see [8, 4.C.1]. The proof is very easy, but we shall run through it in Section 5 in view of the importance of the result to our work.

Theorem 1.16. *For any complex Y , there is a minimal complex X and an equivalence $f : X \rightarrow Y$.*

We prove the following result in Section 4.

Theorem 1.17. *If X is a nuclear complex, then X has no mod p detectable homotopy. If X is a minimal Hurewicz complex, then X is nuclear if and only if it has no mod p detectable homotopy.*

Theorem 1.18. *Conditions (ii) and (iii) of Theorem 1.3 are each equivalent to the condition that any equivalence $X \rightarrow Y$ from a minimal complex X to Y is a core of Y ; that is, a minimal complex equivalent to Y is nuclear.*

Proof. Since a minimal atomic complex is equivalent to a nuclear complex, the second statement of Theorem 1.17 implies that (iii) is equivalent to the condition specified in the statement and that (iii) implies (ii). Similarly, the first statement of Theorem 1.17 implies that (ii) implies (iii). \square

Corollary 1.4 and, more generally, the implication (iii) implies (ii) provide a powerful tool for detecting minimal atomic complexes. We give some general results that illustrate the use of this criterion in the next section.

Restricting to spectra, we see in Section 6 that ko and eo_2 at the prime 2, $BP\langle n \rangle$ at any prime p , and the indecomposable wedge summands of $\Sigma^\infty \mathbb{C}P^\infty$ and $\Sigma^\infty \mathbb{H}P^\infty$ at any prime p are minimal atomic. We give a few other examples and remarks, but we regard this section as just a beginning. Our results imply that minimal atomic complexes exist in abundance, and something closer to a classification of them would be desirable.

In Section 7, we describe Pengelley’s 2-local spectrum BoP as a nuclear complex and thereby give it a new construction that is independent of [14]. This is in the same spirit as Priddy’s construction of BP [16] which motivated the definition of nuclear complexes and is recalled in Section 6. The key step in the proof is deferred to Section 8. A brief Appendix corrects minor errors in one of the proofs in [10].

2. CONSTRUCTIONS ON MINIMAL ATOMIC COMPLEXES

We indicate briefly how the collection of minimal atomic complexes behaves with respect to some basic topological constructions. The proofs are direct consequences of the “no mod p detectable homotopy” characterization of minimal atomic. The following triviality may help the reader see the various implications.

Lemma 2.1. *Consider a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ h \downarrow & & \downarrow h' \\ B & \xrightarrow{g} & B' \end{array}$$

of Abelian groups. If f is an epimorphism and $h = 0$, then $h' = 0$. If g is a monomorphism and $h' = 0$, then $h = 0$.

We begin by recording an immediate consequence of Theorems 1.17 and 1.18.

Proposition 2.2. *If Y is minimal atomic, then Y is equivalent to a complex X whose skeleta X_n for $n \geq n_0$ are minimal atomic.*

There is no reason to believe that the skeleta of Y itself are minimal atomic. We have a more invariant analogue for Postnikov sections, which we denote by $Y[n]$.

Proposition 2.3. *A complex Y is minimal atomic if and only if $Y[n]$ is minimal atomic for each $n \geq n_0$.*

Proof. We have the following commutative diagram.

$$\begin{array}{ccc} \pi_q(Y) & \longrightarrow & \pi_q(Y[n]) \\ h \downarrow & & \downarrow h \\ H_q(Y; \mathbb{F}_p) & \longrightarrow & H_q(Y[n]; \mathbb{F}_p) \end{array}$$

Since $\pi_q(Y[n])$ is zero for $q > n$ and the horizontal arrows are isomorphisms for $q \leq n$, the conclusion is immediate from Lemma 2.1. \square

In the following result, which is only of interest for spaces, we consider the loop and suspension functors.

Proposition 2.4. *If either ΩY or ΣY is minimal atomic, then so is Y .*

Proof. This is immediate from the following commutative diagrams.

$$\begin{array}{ccc} \pi_q(\Omega Y) & \xrightarrow{\cong} & \pi_{q+1}(Y) & \quad & \pi_q(Y) & \xrightarrow{\Sigma} & \pi_{q+1}(\Sigma Y) \\ h \downarrow & & \downarrow h & & h \downarrow & & \downarrow h \\ H_q(\Omega Y; \mathbb{F}_p) & \xrightarrow{\sigma} & H_{q+1}(Y; \mathbb{F}_p) & & H_q(Y; \mathbb{F}_p) & \xrightarrow{\cong} & H_{q+1}(\Sigma Y; \mathbb{F}_p) \quad \square \end{array}$$

This result has an analogue that relates minimal atomicity for spaces and spectra. Here, exceptionally, we must distinguish the two contexts notationally.

Proposition 2.5. *If E is a spectrum of Hurewicz dimension $n_0 \geq 2$ whose 0th space $\Omega^\infty E$ is minimal atomic, then E is minimal atomic. If Y is a simply connected space whose suspension spectrum $\Sigma^\infty Y$ is minimal atomic, then Y is minimal atomic.*

Proof. This is immediate from the following commutative diagrams.

$$\begin{array}{ccc} \pi_q(\Omega^\infty E) & \xrightarrow{\cong} & \pi_q(E) & \quad & \pi_q(Y) & \longrightarrow & \pi_q(\Sigma^\infty Y) \\ h \downarrow & & \downarrow h & & h \downarrow & & \downarrow h \\ H_q(\Omega^\infty E; \mathbb{F}_p) & \xrightarrow{\sigma} & H_q(E; \mathbb{F}_p) & & H_q(Y; \mathbb{F}_p) & \xrightarrow{\cong} & H_q(\Sigma^\infty Y; \mathbb{F}_p) \quad \square \end{array}$$

3. THE PROOF OF THEOREM 1.11

The proof in [10, 1.5] that a nuclear complex X is atomic starts with a self map $f : X \rightarrow X$ which is an isomorphism on $\pi_{n_0}(X)$ and deduces that f is an equivalence. The cited proof readily adapts to give the following analogue.

Proposition 3.1. *Let X and Y be nuclear complexes of Hurewicz dimension n_0 and let $f : X \rightarrow Y$ be a core of Y . Then f is an equivalence.*

Proof. Take $f: X \rightarrow Y$ to be cellular. Since f is a monomorphism between Hurewicz complexes, $f: X_{n_0} \rightarrow Y_{n_0}$ is an equivalence. Assume that $f: X_n \rightarrow Y_n$ is an equivalence. We must show that $f: X_{n+1} \rightarrow Y_{n+1}$ is an equivalence. The attaching maps of X and Y give rise to the following map of cofibre sequences.

$$\begin{array}{ccccc} J_n & \xrightarrow{j_n} & X_n & \longrightarrow & X_{n+1} \\ f \downarrow & & \downarrow f & & \downarrow f \\ K_n & \xrightarrow{k_n} & Y_n & \longrightarrow & Y_{n+1} \end{array}$$

Passing to homology, this gives rise to a commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{n+1}(X_{n+1}) & \longrightarrow & H_n(J_n) & \xrightarrow{j_{n*}} & H_n(X_n) & \longrightarrow & H_n(X_{n+1}) & \longrightarrow & 0 \\ & & f_* \downarrow & & \downarrow f_* & & \cong \downarrow f_* & & \downarrow f_* & & \\ 0 & \longrightarrow & H_{n+1}(Y_{n+1}) & \longrightarrow & H_n(K_n) & \xrightarrow{k_{n*}} & H_n(Y_n) & \longrightarrow & H_n(Y_{n+1}) & \longrightarrow & 0 \end{array}$$

It suffices to prove that the left and right vertical arrows are isomorphisms. By the five lemma and the Hurewicz theorem, this holds if $f_*: \pi_n(J_n) \rightarrow \pi_n(K_n)$ is an isomorphism. To see that this is so, consider the following diagram.

$$\begin{array}{ccccccc} \pi_n(J_n) & \xrightarrow{j_{n*}} & \pi_n(X_n) & \longrightarrow & \pi_n(X_{n+1}) & \longrightarrow & 0 \\ f_* \downarrow & & \downarrow f_* & & \downarrow f_* & & \\ \pi_n(K_n) & \xrightarrow{k_{n*}} & \pi_n(Y_n) & \longrightarrow & \pi_n(Y_{n+1}) & \longrightarrow & 0 \end{array}$$

The rows are exact, and a chase of the diagram shows that the right arrow f_* is an epimorphism. Now consider the following diagram.

$$\begin{array}{ccc} \pi_n(X_{n+1}) & \xrightarrow{\cong} & \pi_n(X) \\ f_* \downarrow & & \downarrow f_* \\ \pi_n(Y_{n+1}) & \xrightarrow{\cong} & \pi_n(Y) \end{array}$$

Since its right arrow f_* is a monomorphism, its left arrow f_* is a monomorphism and therefore an isomorphism. This implies that the right vertical arrow is an isomorphism in the following diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker } j_{n*} & \xrightarrow{i} & \pi_n(J_{n+1}) & \xrightarrow{\cong} & \text{Im } j_{n*} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f_* & & \downarrow \cong & & \\ 0 & \longrightarrow & \text{Ker } k_{n*} & \xrightarrow{i} & \pi_n(K_{n+1}) & \xrightarrow{\cong} & \text{Im } k_{n*} & \longrightarrow & 0 \end{array}$$

In view of (1.7), both maps i become 0 after tensoring with \mathbb{F}_p . This implies that $f_* \otimes \mathbb{F}_p$ is an isomorphism, and therefore so is f_* . \square

Proof of Theorem 1.11. Let Y be a nuclear complex of Hurewicz dimension n_0 and let $f: X \rightarrow Y$ be a monomorphism, where X is atomic. The same argument as in the last part of the proof of Theorem 1.12 shows that f is an equivalence. \square

4. THE PROOF OF THEOREM 1.17

We start with the following observation, which is implicit in Priddy [16]. It gives a homological recasting of the definition of a nuclear complex.

Lemma 4.1. *A Hurewicz complex of dimension n_0 is nuclear if and only if the mod p Hurewicz homomorphism $h : \pi_n(X_n) \rightarrow H_n(X_n; \mathbb{F}_p)$ is zero for $n > n_0$.*

Proof. Recall the defining property (1.7) of a nuclear complex. In the case of spaces, our assumption that X is simply connected allows us to quote the relative and absolute Hurewicz theorem to deduce that

$$\pi_{n+1}(X_{n+1}, X_n) \cong \pi_{n+1}(\Sigma J_n) \cong \pi_n(J_n)$$

from the trivial analogue in p -local homology. In either the space or the spectrum context, we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} \pi_{n+1}(X_n) & \longrightarrow & \pi_{n+1}(X_{n+1}) & \longrightarrow & \pi_n(J_n) & \xrightarrow{j_*} & \pi_n(X_n) \\ & & \downarrow h & & \downarrow h & & \downarrow h \\ 0 & \longrightarrow & H_{n+1}(X_{n+1}; \mathbb{F}_p) & \longrightarrow & H_n(J_n; \mathbb{F}_p) & \xrightarrow{j_*} & H_n(X_n; \mathbb{F}_p) \end{array}$$

An easy diagram chase gives that (1.7) holds for n if and only if the left arrow h is zero. The conclusion follows. \square

We prove the following reformulation of Theorem 1.17 by relating this skeletal criterion to the Hurewicz homomorphism of X itself.

Proposition 4.2. *Let X be a Hurewicz complex of dimension n_0 and consider the following conditions.*

- (i) *X has no mod p detectable homotopy.*
- (ii) *X is a minimal complex.*

If (i) and (ii) hold, then X is nuclear. Conversely, if X is nuclear, then (i) holds.

Proof. We have the following commutative diagram, where $n > n_0$.

$$\begin{array}{ccc} \pi_n(X_n) & \longrightarrow & \pi_n(X) \\ \downarrow h & & \downarrow h \\ H_n(X_n; \mathbb{F}_p) & \longrightarrow & H_n(X; \mathbb{F}_p) \end{array}$$

The conclusion is immediate from Lemmas 1.15, 2.1, and 4.1. \square

5. THE PROOF OF THEOREM 1.16

We are given a complex Y . Recall that our complexes are simply connected in the case of spaces and bounded below in the case of spectra. More fundamentally, everything is p -local. We have assumed that $H_*(Y)$ is of finite type, so that each $H_n(Y)$ is a direct sum of finitely many cyclic $\mathbb{Z}_{(p)}$ -modules $A_{n,i}$. We must construct a minimal complex X and an equivalence $f : X \rightarrow Y$, and it suffices for the latter to ensure that f induces an isomorphism on H_* . The complex X will have an n -cell $j_{n,i}$ for each free cyclic summand $A_{n,i}$ and an $(n+1)$ -cell $k_{n,i}$ with differential $q_i j_{n,i}$ for each summand $A_{n,i}$ of order q_i . Since each q_i must be a power of p , it will be immediate that the differential on $C_*(X; \mathbb{F}_p)$ is zero. The

cells $j_{n,i}$ will map to cycles that represent the generators of the $A_{n,i}$, and the cells $k_{n,i}$ will map to chains with boundary $q_i f_*(j_{n,i})$.

Assume inductively that we have constructed the n -skeleton X_n together with a (based) map $f_n : X_n \rightarrow Y$ that induces an isomorphism on homology in dimensions less than n and an epimorphism on H_n . More precisely, assume that $H_n(X_n)$ is $\mathbb{Z}_{(p)}$ -free on basis elements given by cells $j_{n,i}$ that map to chosen generators of the $A_{n,i}$. Let Cf_n be the cofiber of f_n . Then $H_m(Cf_n) = 0$ for $m \leq n$. The kernel of $f_* : H_n(X_n) \rightarrow H_n(Y)$ is free on the basis $q_i j_{n,i}$ for those i such that $A_{n,i}$ has finite order. These elements are the images of elements $k''_{n,i}$ in $H_{n+1}(Cf_n)$, and $k''_{n,i} = h(k'_{n,i})$ for unique elements $k'_{n,i}$ in $\pi_{n+1}(Cf_n)$. Similarly, the chosen generators of the $A_{n+1,i} \subset H_{n+1}(Y)$ map to elements $j''_{n+1,i} \in H_{n+1}(Cf_n)$ with $j''_{n+1,i} = h(j'_{n+1,i})$. For spectra, we have the connecting homomorphism $\pi_{n+1}(Cf_n) \rightarrow \pi_n(X_n)$. For spaces, the relative Hurewicz theorem gives $\pi_{n+1}(Mf_n, X_n) \cong \pi_{n+1}(Cf_n)$, and we have the connecting homomorphism $\pi_{n+1}(Mf_n, X_n) \rightarrow \pi_n(X_n)$. Thus in either case the elements $k'_{n,i}$ and $j'_{n+1,i}$ determine elements of $\pi_n(X_n)$. Choose maps $S^n \rightarrow X_n$ that represent these elements and use them as attaching maps for the construction of X_{n+1} from X_n by attaching cells $k_{n,i}$ and $j_{n+1,i}$.

Since the sequence $\pi_{n+1}(Cf_n) \rightarrow \pi_n(X_n) \rightarrow \pi_n(Y)$ is exact, these attaching maps become null homotopic in Y , and there is an extension $f_{n+1} : X_{n+1} \rightarrow Y$ of f_n . Thus we can construct the following map of cofiber sequences.

$$\begin{array}{ccccccc} X_n & \longrightarrow & X_{n+1} & \longrightarrow & X_{n+1}/X_n & \longrightarrow & \Sigma X_n \\ \parallel & & \downarrow f_{n+1} & & \downarrow & & \parallel \\ X_n & \xrightarrow{f_n} & Y & \longrightarrow & Cf_n & \longrightarrow & \Sigma X_n \end{array}$$

This gives rise to the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{n+1}(X_{n+1}) & \longrightarrow & H_{n+1}(X_{n+1}/X_n) & \longrightarrow & H_n(X_n) & \longrightarrow & H_n(X_{n+1}) & \longrightarrow & 0 \\ & & \downarrow (f_{n+1})_* & & \downarrow & & \parallel & & \downarrow (f_{n+1})_* & & \\ 0 & \longrightarrow & H_{n+1}(Y) & \longrightarrow & H_{n+1}(Cf_n) & \longrightarrow & H_n(X_n) & \longrightarrow & H_n(Y) & \longrightarrow & 0 \end{array}$$

Of course, the differential on $C_{n+1}(X_{n+1})$ is the composite

$$H_{n+1}(X_{n+1}/X_n) \rightarrow H_n(X_n) \rightarrow H_n(X_n/X_{n-1}),$$

where the second arrow is a monomorphism. By construction, the first arrow sends the basis elements $k_{n,i}$ to $q_i j_{n,i}$ and the basis elements $j_{n+1,i}$ to zero, so that $H_{n+1}(X_{n+1})$ is $\mathbb{Z}_{(p)}$ -free on the basis elements $j_{n+1,i}$. By construction and a chase of the diagram, the map f_{n+1} induces an isomorphism on H_n and sends the basis elements $j_{n+1,i}$ to generators of the groups $A_{n+1,i}$. This completes the inductive step in the construction of $f : X \rightarrow Y$.

6. SPECTRUM LEVEL EXAMPLES

As a first example, we revisit Priddy's construction [16] of a nuclear spectrum equivalent to BP . Although it was the motivating example for [10], it was not explicitly discussed there. We work with p -local spectra in this section. Unless otherwise stated, p is unrestricted.

Example 6.1. BP is a minimal atomic spectrum, hence the canonical monomorphism $BP \rightarrow MU$ is a core of MU .

Proof. $H^*(BP; \mathbb{F}_p)$ is a cyclic A -module, hence Corollary 1.4 applies. \square

Proposition 6.2. *Let X be the nuclear complex of [16] defined by starting with S^0 and inductively killing the homotopy groups in odd degrees. Then there is an equivalence $X \rightarrow BP$.*

Proof. A minimal complex equivalent to BP has cells only in even degrees and is nuclear. By construction, X also has cells only in even degrees and is nuclear, and its non-zero homotopy groups only occur in even degrees. Obstruction theory gives maps $f: X \rightarrow BP$ and $g: BP \rightarrow X$ that extend the identity on the bottom cell. The composites $g \circ f: X \rightarrow X$ and $f \circ g: BP \rightarrow BP$ are equivalences since X and BP are atomic. \square

Recall that, for an odd prime p , there is a splitting of ku with $BP \langle 1 \rangle$ as a wedge summand. In [10, 1.18] it is conjectured that the core of ku is $BP \langle 1 \rangle$. Here $ku = BP \langle 1 \rangle$ if $p = 2$. Since $H^*(BP \langle 1 \rangle; \mathbb{F}_p)$ is a cyclic A -module, this is now immediate.

Example 6.3. The spectrum $BP \langle 1 \rangle$ is minimal atomic, hence the canonical monomorphism $BP \langle 1 \rangle \rightarrow ku$ is a core.

More generally, $H^*(BP \langle n \rangle)$ is a cyclic A -module for all $n \geq -1$, the extreme cases being $BP \langle -1 \rangle = H\mathbb{F}_p$ and $BP \langle 0 \rangle = H\mathbb{Z}_{(p)}$.

Example 6.4. For $n \geq -1$, $BP \langle n \rangle$ is a minimal atomic spectrum.

The following example of the non-uniqueness of cores generalizes [10, 1.17].

Example 6.5. For $n \geq 0$, the canonical maps

$$BP \longrightarrow BP \wedge BP \langle n \rangle \longleftarrow BP \langle n \rangle$$

induced by the units of BP and $BP \langle n \rangle$ are both cores of BP .

Proof. The left map is a monomorphism since it factors the (p -local) Hurewicz homomorphism of BP . The right map is a monomorphism since it is split by the BP -action $BP \wedge BP \langle n \rangle \rightarrow BP \langle n \rangle$. \square

Since $H^*(ko; \mathbb{F}_2) = A//A(1)$ and $H^*(eo_2; \mathbb{F}_2) = A//A(2)$ are cyclic A -modules, we have the following complement to Proposition 6.3.

Proposition 6.6. *At $p = 2$, ko and eo_2 are minimal atomic spectra.*

Some well-known Thom complexes give further examples.

Proposition 6.7. *Let X be $\mathbb{R}P_{-1}^\infty$, $\mathbb{C}P_{-1}^\infty$, or $\mathbb{H}P_{-1}^\infty$, that is, the Thom spectrum of the negative of the canonical real, complex, or quaternionic line bundle. At $p = 2$, X is minimal atomic.*

Proof. Let $d = 1, 2$, and 4 and $P = \mathbb{R}P^\infty, \mathbb{C}P^\infty$, and $\mathbb{H}P^\infty$ in the respective cases. Then $H^*(P; \mathbb{F}_2) = \mathbb{F}_2[x]$, where $x \in H^d(P; \mathbb{F}_2)$ is the d th Stiefel-Whitney class of the canonical line bundle. Since X is a Thom spectrum, $H^*(X; \mathbb{F}_2)$ is the free $H^*(P; \mathbb{F}_2)$ -module generated by the Thom class μ in degree $-d$. A standard calculation shows that $Sq^{n^d} \mu = x^n \mu$ for $n \geq 1$, so $H^*(X; \mathbb{F}_2)$ is cyclic over A . \square

To give some examples where we must check the “no mod p homotopy” condition directly, we consider a few suspension spectra and another Thom spectrum. Let $\xi_3 \rightarrow \mathbb{H}\mathbb{P}^\infty$ be the bundle associated to the adjoint representation of S^3 and let $M\xi_3$ be its Thom complex (also known as a quaternionic quasi-projective space). It has one cell in each positive dimension congruent to 3 (mod 4).

By [9, 6, 3], for each odd prime p , there is a splitting of p -local spaces

$$\Sigma\mathbb{C}\mathbb{P}^\infty \simeq W_1 \vee W_2 \vee \cdots \vee W_{p-1},$$

where W_r has cells in all dimensions of the form $2(p-1)k + 2r + 1$ with $k \geq 0$.

Proposition 6.8. *At the prime 2, $\Sigma^\infty\mathbb{C}\mathbb{P}^\infty$, $\Sigma^\infty\mathbb{H}\mathbb{P}^\infty$ and $\Sigma^\infty M\xi_3$ are minimal atomic spectra. At an odd prime p , each $\Sigma^\infty W_r$ is minimal atomic.*

Proof. Let $a(n) = 1$ if n is even and $a(n) = 2$ if n is odd. By [17], the Hurewicz homomorphisms

$$h: \pi_{2n}(\Sigma^\infty\mathbb{C}\mathbb{P}^\infty) \rightarrow H_{2n}(\mathbb{C}\mathbb{P}^\infty) \cong \mathbb{Z} \quad \text{and} \quad h: \pi_{4n}(\Sigma^\infty\mathbb{H}\mathbb{P}^\infty) \rightarrow H_{4n}(\mathbb{H}\mathbb{P}^\infty) \cong \mathbb{Z}$$

have images of index $n!$ and $(2n)!/a(n)$, respectively. Thus, for $n > 1$, the corresponding mod 2 Hurewicz homomorphisms are trivial. By [19], the Hurewicz homomorphism

$$h: \pi_{4n+3}(\Sigma^\infty M\xi_3) \rightarrow H_{4n+3}(M\xi_3) \cong \mathbb{Z}$$

has image of index $a(n)(2n-1)!$, so for each $n \geq 1$ the associated mod 2 Hurewicz homomorphism is also trivial. The odd primary results follow similarly from the calculation of h for $\Sigma^\infty\mathbb{C}\mathbb{P}^\infty$. \square

Remark 6.9. We raise a few questions here.

- (i) There are many basic results in the literature in which interesting spaces are split p -locally into products of indecomposable factors and interesting spectra are split p -locally into wedges of indecomposable summands. (The notion of wedge indecomposability is less interesting in the case of spaces). It is a very interesting set of problems to revisit these splittings and determine which of the summands are atomic rather than just indecomposable, and which are minimal atomic rather than just atomic. The results above just give particularly elementary examples.
- (ii) The suspension spectrum of $\mathbb{R}\mathbb{P}^\infty$ presents an interesting challenge. It is a standard observation that $H^*(\mathbb{R}\mathbb{P}^\infty; \mathbb{F}_2)$ is an atomic, but not cyclic, A -module, in the sense that any A -endomorphism which is the identity on $H^1(\mathbb{R}\mathbb{P}^\infty; \mathbb{F}_2)$ is an isomorphism. This implies that $\Sigma^\infty\mathbb{R}\mathbb{P}^\infty$ is atomic. However, since the top cell of $\mathbb{R}\mathbb{P}^3$ splits off stably, the stable Hurewicz homomorphism $\pi_3(\Sigma^\infty\mathbb{R}\mathbb{P}^\infty) \rightarrow H_3(\mathbb{R}\mathbb{P}^\infty; \mathbb{F}_2)$ is non-trivial, hence $\Sigma^\infty\mathbb{R}\mathbb{P}^\infty$ cannot be minimal atomic. It would be interesting to identify a core of $\Sigma^\infty\mathbb{R}\mathbb{P}^\infty$. For an odd prime p , similar remarks apply to the $(p-1)$ wedge summands of $\Sigma^\infty B\mathbb{Z}/p$, one of which is $(B\Sigma_p)_{(p)}$.
- (iii) Fred Cohen observes that it is an open question whether or not $K(\mathbb{Z}/2, n)$ or the $p-1$ summands of $\Sigma K(\mathbb{Z}/p, n)$ are stably atomic for $n \geq 2$.

7. A CONSTRUCTION OF THE SPECTRUM BoP

In this section, all spectra are understood to be localized at 2, and $S = S^0$. Recall the spectrum BoP of Pengelley [14]. It has no mod 2 detectable homotopy [14, 5.5] and it is a retract of MSU , so we have a monomorphism $j: BoP \rightarrow MSU$.

Example 7.1. The monomorphism $j : BoP \rightarrow MSU$ is a core of MSU .

We recall a further property of BoP , proven in Pengelley [14, 6.15, 6.16].

Proposition 7.2. *There is a map $p : BoP \rightarrow ko$ that induces an epimorphism on homotopy groups in all degrees and an isomorphism in odd degrees.*

Corollary 7.3. *The odd degree homotopy groups of the fiber Fp are zero.*

We now give a description of BoP as a nuclear spectrum, thus providing a simple construction of it that is independent of [14]. Guided by Proposition 7.2, we construct a nuclear spectrum X and a map $q : X \rightarrow ko$ that induces a monomorphism on homotopy groups in odd degrees, and we prove that it induces an epimorphism on homotopy groups. That turns out to imply that X is equivalent to BoP .

We begin with $X_0 = S$, and we inductively attach even dimensional cells, letting $X_{2n} = X_{2n+1}$ for all $n \geq 0$. Suppose that we have factored the unit $\iota : S \rightarrow ko$ through a map $q_n : X_{2n-1} \rightarrow ko$. We enlarge X_{2n-1} to X_{2n} by attaching $2n$ -cells minimally, so that (1.7) is satisfied. We do this so as to kill the kernel of

$$q_{n*} : \pi_{2n-1}(X_{2n-1}) \rightarrow \pi_{2n-1}(ko).$$

Thus, in the resulting cofiber sequence

$$J_{2n-1} \rightarrow X_{2n-1} \rightarrow X_{2n},$$

$$\text{Im}(\pi_{2n-1}(J_{2n-1}) \rightarrow \pi_{2n-1}(X_{2n-1})) = \text{Ker}(\pi_{2n-1}(X_{2n-1}) \rightarrow \pi_{2n-1}(ko)).$$

Clearly q_n extends to a map

$$q_{n+1} : X_{2n} = X_{2n+1} \rightarrow ko.$$

In the limit we obtain a nuclear complex X and a map $q : X \rightarrow ko$ that induces an isomorphism on π_0 and a monomorphism on π_* in odd degrees.

Proposition 7.4. *$q : X \rightarrow ko$ induces an epimorphism on homotopy groups.*

Corollary 7.5. *The odd degree homotopy groups of the fiber Fq are zero.*

Let $\nu \in \pi_3(S)$ and $\sigma \in \pi_7(S)$ be the Hopf maps. If $x \in X$ has even degree, then νx and σx are odd degree elements of the kernel of q_* , hence they are zero. The proposition is therefore a direct consequence of the following result, which is presumably known. Since we do not know of a reference for it, we will give a proof in the next section.

Proposition 7.6. *Let X be a Hurewicz complex of dimension 0 with inclusion of the bottom cell $i : S \rightarrow X$ and let $q : X \rightarrow ko$ be a map such that the composite $S \xrightarrow{i} X \xrightarrow{q} ko$ is the unit $\iota : S \rightarrow ko$. If $\nu x = 0$ and $\sigma x = 0$ in $\pi_*(X)$ for every even degree element $x \in \pi_*(X)$, then $q_* : \pi_*(X) \rightarrow \pi_*(ko)$ is an epimorphism.*

Theorem 7.7. *There are equivalences $f : X \rightarrow BoP$ and $g : BoP \rightarrow X$ such that the following diagram is homotopy commutative.*

$$\begin{array}{ccc} X & \xrightarrow{f} & BoP & \xrightarrow{g} & X \\ & \searrow q & \downarrow p & \swarrow q & \\ & & ko & & \end{array}$$

Proof. We construct maps f and g such that the diagram is homotopy commutative. The maps f and g , hence also the composites $g \circ f$ and $f \circ g$, then induce isomorphisms on π_0 . Since X and BoP are atomic, these composites are equivalences and therefore f and g are equivalences. We may take BoP and ko to be Hurewicz complexes and take p to be the identity map on the bottom cell. Taking $f_0: X_0 = X_1 = S \rightarrow BoP$ to be the identity map on the bottom cell and h_0 to be the constant homotopy at the identity map, we assume inductively that we have a map $f_n: X_{2n-1} \rightarrow BoP$ and a homotopy $h_n: q_n \simeq p \circ f_n$. Consider the following diagram, where we implicitly precompose maps already specified with the map of cells $CJ_{2n-1} \rightarrow X_{2n+1}$ that constructs X_{2n+1} from X_{2n-1} .

$$\begin{array}{ccccc}
 J_{2n-1} & \xrightarrow{i_0} & J_{2n-1} \wedge I_+ & \xleftarrow{i_1} & J_{2n-1} \\
 \downarrow & & \downarrow & & \downarrow \\
 & & ko & \xleftarrow{p} & BoP \\
 & \nearrow q_{n+1} & & & \nwarrow f_{n+1} \\
 CJ_{2n-1} & \xrightarrow{i_0} & CJ_{2n-1} \wedge I_+ & \xleftarrow{i_1} & CJ_{2n-1} \\
 & & \downarrow & & \downarrow \\
 & & & & h_{n+1} \text{ (dashed)} \\
 & & & & f_{n+1} \text{ (dashed)}
 \end{array}$$

Since J_{2n-1} is a wedge of $(2n-1)$ -spheres and $\pi_{2n-1}(Fp) = 0$, $[J_{2n-1}, Fp] = 0$. A standard result, given in just this form in [13, Lemma 1], shows that there are maps f_{n+1} and h_{n+1} that make the diagram commute. Passing to colimits, we obtain f and a homotopy $h: q \simeq p \circ f$. Since the homology groups of BoP are concentrated in even degrees [14], we can replace it by a minimal complex, with cells only in even degrees. This allows us to reverse the roles of X and BoP to construct g . \square

A similar argument proves the following result.

Proposition 7.8. *There is a map $r: MSU \rightarrow X$ such that the following diagram is homotopy commutative.*

$$\begin{array}{ccc}
 MSU & \xrightarrow{r} & X \\
 & \searrow t & \swarrow q \\
 & & ko
 \end{array}$$

It is not clear that BoP is the only core of MSU up to equivalence, but we conjecture that it is. The following consequence of Lemma 7.6 may shed some light on this question.

Proposition 7.9. *If $Y \rightarrow MSU$ is a core, the composite $Y \rightarrow MSU \rightarrow ko$ induces an epimorphism on homotopy groups.*

Remark 7.10. It might be of interest to revisit the results of [11, 14] from our present perspective. However, it is not clear how to construct a map $X \rightarrow MSU$ that induces the identity on π_0 and how the distinguished map of [11] fits in. It might be of more interest to revisit the results of [11, 14] from the perspective of S -modules [7]. Pengelley constructs BoP by first constructing another spectrum, which he denotes by X , and then taking a fiber to kill BP summands in it. His X is obtained from MSU by using the Baas-Sullivan theory of manifolds with singularities to kill a regular sequence of elements in $\pi_*(MSU)$. We can instead use the results of [7, Ch. V] to construct X as an MSU -module together with a map

of MSU -modules $MSU \rightarrow X$. It seems plausible that the methods of [7, 18] can be used to construct BoP as a commutative MSU -ring spectrum.

8. THE PROOF OF PROPOSITION 7.6

We continue to work with spectra localized at 2. Recall that

$$(8.1) \quad \pi_*(ko) = \mathbb{Z}_{(2)}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta),$$

where $\deg \eta = 1$, $\deg \alpha = 4$, and $\deg \beta = 8$. We will describe elements of $\pi_*(X)$ that map to each of the additive generators of $\pi_*(ko)$. Note that, since we do not know that X is a ring spectrum, we cannot exploit the algebra structure of $\pi_*(ko)$. The essential point is to describe additive generators in terms of Toda brackets in $\pi_*(ko)$ that admit analogues in $\pi_*(X)$.

We are interested in Toda brackets of the form $\langle a, b, c \rangle$, where a and b are elements of $\pi_*(S)$ and c is an element of $\pi_*(Y)$ for a spectrum Y . We require $ab = 0$ and $bc = 0$, and then $\langle a, b, c \rangle$ is a coset of elements in $\pi_{|a|+|b|+|c|+1}(Y)$ with respect to the indeterminacy subgroup

$$\text{indeter } \langle a, b, c \rangle = a\pi_{|b|+|c|+1}(Y) + (\pi_{|a|+|b|+1}(S))c.$$

Such Toda brackets are natural with respect to maps $Y \rightarrow Z$.

Remark 8.2. We remark parenthetically that the theory of Toda brackets simplifies greatly if one defines them in terms of the associative smash product in one of the modern categories of spectra, such as the category of S -modules of [7]. A systematic exposition would be of value. In brief, the conclusion must be that all of the results that are catalogued in [12] for matric Massey products in the homology of DGA's carry over verbatim to S -modules.

Now take X as in Proposition 7.6. Recall that $8\nu = 0$ and $16\sigma = 0$ in $\pi_*(S)$ and that, by hypothesis, ν and σ annihilate all even degree elements of $\pi_*(X)$. Let b_0 denote $i : S \rightarrow X$ regarded as an element of $\pi_0(X)$ and choose coset representatives in iterated Toda products as follows:

$$a_1 \in \langle 8, \nu, b_0 \rangle, \quad b_k \in \langle 16, \sigma, b_{k-1} \rangle, \quad \text{and} \quad a_{k+1} \in \langle 16, \sigma, a_k \rangle,$$

where $k \geq 1$. The indeterminacies are benign for our purposes since they are

$$\begin{aligned} \text{indeter } a_1 &= (\pi_4(S))b_0 + 8\pi_4 X = 8\pi_4(X), \\ \text{indeter } b_k &= (\pi_8(S))b_{k-1} + 16\pi_8(X) \equiv 16\pi_8(X) \pmod{\text{Ker}(q_*)} \\ \text{indeter } a_k &= (\pi_8(S))a_{k-1} + 16\pi_{8k-4}(X) \equiv 16\pi_{8k-4}(X) \pmod{\text{Ker}(q_*)}. \end{aligned}$$

Here the congruences hold since $\pi_8(S)$ is 2-torsion and there are no torsion elements in the relevant degrees of $\pi_*(ko)$. For $k \geq 0$, we also have the elements

$$\mu_{8k+1}b_0 \in \pi_{8k+1}(X) \quad \text{and} \quad \mu_{8k+2}b_0 \in \pi_{8k+2}(X),$$

where μ_{8k+1} and μ_{8k+2} are the usual elements in $\pi_*(S)$. Now $q_* : \pi_*(X) \rightarrow \pi_*(ko)$ maps these elements to elements of the same form in $\pi_*(ko)$, where $b_0 \in \pi_0(ko)$ is the unit of ko . In the familiar periodic pattern $\mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$, the additive positive degree generators of $\pi_*(ko)$ are

$$\eta\beta^k = \mu_{8k+1}b_0, \quad \eta^2\beta^k = \mu_{8k+2}b_0, \quad \alpha\beta^k, \quad \text{and} \quad \beta^{k+1},$$

where $k \geq 0$. The following known result gives that $\alpha\beta^k = a_{k+1}$ and $\beta^{k+1} = b_{k+1}$ in $\pi_*(ko)$, and this completes the proof that q_* is an epimorphism.

Lemma 8.3. *In $\pi_*(ko)$,*

$$\alpha \in \langle 8, \nu, b_0 \rangle, \quad \beta^k \in \langle 16, \sigma, \beta^{k-1} \rangle, \quad \text{and} \quad \alpha\beta^k \in \langle 16, \sigma, \alpha\beta^{k-1} \rangle$$

for $k \geq 1$, where the indeterminacy is 0 mod 2 in each case.

An unstable version of the lemma is stated without proof in [20, p.64], where it is attributed to Barratt. One quick way to see the result is to use the convergence of Massey products to Toda brackets in the Adams spectral sequence and of Massey products to Toda brackets in the Adams spectral sequence, but the details would take us too far afield.

APPENDIX: ERRATA TO [10]

We take this opportunity to correct some minor errors in the proof of [10, 2.11]. In brief, the last two sentences of the cited proof should be replaced with the following two sentences. “If $p = 2$, then $Q^8(a_1) \equiv a_5 \pmod{\text{decomposables}}$, and, if $p > 2$, then $Q^{2p}(a_{p-1}) \equiv a_{(2p+1)(p-1)} \pmod{\text{decomposables}}$, by [15] or [4, II.8.1]. Here a_{p-1} is in the image of $H_*(BP)$, but $H_*(BP)$ has no indecomposable elements in degree 10 if $p = 2$ or in degree $2(2p+1)(p-1)$ if $p > 2$.”

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MATHEMATICS DEPARTMENT, UNIVERSITY OF GLASGOW, GLASGOW G12 8QW, SCOTLAND.

E-mail address: `a.baker@maths.gla.ac.uk`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637, USA.

E-mail address: `may@math.uchicago.edu`