

TRANSFER AND COMPLEX ORIENTED COHOMOLOGY RINGS

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1. INTRODUCTION

Let p be a prime and let $G \leq \Sigma_p$ be a subgroup of the symmetric group. In this paper we use the transfer to study homotopy orbit spaces $X_{hG}^p = EG \times_G X^p$ in complex oriented cohomology. We are particularly interested in computing the ring structure. Thus we are led to consider the relation between cup products and transfer known as Fröbenius reciprocity by analogy with representation theory

$$Tr^*(x)y = Tr^*(x\rho^*(y))$$

(formula (i) of Section 2) where $\rho : EG \times X^p \rightarrow X_{hG}^p$ is the covering projection and

$$Tr^* : E^*(X^p) \rightarrow E^*(X_{hG}^p)$$

is the associated transfer homomorphism. It is worth noting that the multiplicative structure of the cohomology groups we consider is completely determined by this formula.

In case $E = K(s)$ is Morava K -theory, G is cyclic of order p , and X is the classifying space of a finite group, Hopkins-Kuhn-Ravenel [11] have studied these cohomology groups as modules over the coefficient ring. Our paper builds on their approach by extending their notion of a *good group* to spaces. For $X = \mathbf{C}P^\infty$ we determine the algebra $K(s)^*(X_{h\Sigma_p}^p)$ for Morava K -theory; for complex cobordism we compute the ring $MU^*(X_{h\Sigma_p}^p)$ making additional use of the formal group law. This enables us to make explicit computations of the transfer in both cases. In an analogous fashion we compute the algebra $BP^*(X_{h\Sigma_p}^p)$.

The starting point and original motivation for our work comes from Quillen's famous formula for $Tr^*(1)$, the stable Euler class, for the universal Z/p covering. As explained in Section 2, our results for $\mathbf{C}P^\infty$

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provide a universal example which enable us to compute the stable Euler classes and the transfer in general for many other cases. For example universal coverings for some nonabelian p -groups, namely those with cyclic subgroups of index p and those which are semi-direct products of elementary abelian p -groups with Z/p .

The paper is organized as follows. In Section 2 we recall some preliminary results for the transfer. Section 3 is devoted to extending some results of [11] in Morava K -theory. The computation of the algebra $K(s)^*(X_{h\Sigma_p}^p)$ for $X = \mathbf{C}P^\infty$ is given as Theorem 4.5 of Section 4; analogous results are given for $X = B(Z/p)$ along with further computations of the transfer. In Section 5 we turn to the complex cobordism rings of $B(\pi \wr U(1))$. In Section 6 we derive formulas for transferred Chern classes. The ring $BP^*(X_{h\Sigma_p}^p)$ is computed in Section 7. In Section 8 stable Euler classes for the universal coverings associated with p -groups with normal cyclic subgroups of index p are computed. Similar results are given in Section 9 for the semi-direct products $(Z/p)^s \rtimes Z/p$. Finally in Section 10 we compute coefficients useful in computing the transfer for $G = Z/p$ and $G = \Sigma_p$.

2. PRELIMINARIES

We recall that a multiplicative cohomology theory E^* is called *complex oriented* if there exists a Thom class, that is, a class $u \in E^2(\mathbf{C}P^\infty)$ that restricts to a generator of the free one-dimensional E^* module $E^2(\mathbf{C}P^1)$. The universal example is complex cobordism MU^* . Then

$$E^*(\mathbf{C}P^\infty) = E^*[[x]],$$

where x is the Chern class of canonical complex line bundle ξ over $\mathbf{C}P^\infty = BU(1)$ and

$$E^*(BU(1)^p) = E^*[[x_1, \dots, x_p]],$$

$x_i = c_1(\xi_i)$ and ξ_i is the pull back bundle over $BU(1)^p$ by the projection $BU(1)^p \rightarrow BU(1)$ on the i -th factor.

Much of our paper is written in terms of transfer maps [1, 7] and formal group laws. Let us give a brief review of formal properties of the transfer. For a finite covering

$$\rho : X \rightarrow X/G$$

there is a stable transfer map

$$Tr = Tr(\rho) : X/G^+ \rightarrow X^+.$$

For any multiplicative cohomology theory E^* the induced map Tr^* is a map of $E^*(X/G)$ modules, i.e.,

$$(i) \quad Tr^*(x\rho^*(y)) = Tr^*(x)y, \quad x \in E^*(X), \quad y \in E^*(X/G).$$

For example

$$(ii) \quad Tr^*(\rho^*(y)) = Tr^*(1)y.$$

The element $Tr^*(1) \in E^0(X/G)$ is called the *index or stable Euler class* of the covering ρ . The following additional properties of the transfer will be used:

- (iii) The transfer is natural with respect to pullbacks;
- (iv) $Tr(\rho_1 \times \rho_2) = Tr(\rho_1) \wedge Tr(\rho_2)$;
- (v) If $\rho = \rho_2\rho_1$, then $Tr(\rho) = Tr(\rho_2)Tr(\rho_1)$.

More generally for a covering projection

$$\rho_{H,G} : X/H \rightarrow X/G$$

with $H \leq G$ there is a stable transfer map

$$Tr_{H,G} : X/G^+ \rightarrow X/H^+$$

For ease of notation if $H = e$, as above, we write projection and transfer in several equivalent ways $\rho = \rho_G = \rho_{e,G}$, $Tr = Tr(\rho) = Tr_G = Tr_{e,G}$.

The reverse composition to (ii) is given by

- (vi) (Double coset formula) If $K, H \leq G$ then

$$\rho_{K,G}^* Tr_{H,G}^* = \sum_x Tr_{K \cap H^x, K}^* \circ x^{-1*} \circ \rho_{K^{x^{-1}} \cap H, H}^*$$

where the sum is taken over a set of double coset representatives $x \in K \backslash G / H$. Here $H^x = xHx^{-1}$.

For a regular covering $\rho_{H,G}$, i.e. $H \trianglelefteq G$,

$$\rho_{H,G}^* Tr_{H,G}^*(x) = \sum_{g \in G/H} g^*(x).$$

In subsequent sections the reduced transfer $Tr_{H,G} : X/G \rightarrow X/H$ is used.

We recall Quillen's formula [13, 6] mentioned in the Introduction. First,

$$E^*(BZ/p) = E^*[[z]]/([p](z))$$

where z is the Chern class of the canonical complex bundle over BZ/p and $[p](z)$ is the p -series or p -fold iterated formal sum. Then

$$(1) \quad Tr_{Z/p}^*(1) = [p](z)/z,$$

where $Tr_{Z/p}^*$ is the transfer homomorphism for the universal p -covering $EZ/p \rightarrow BZ/p$. The relation $[p](z) = 0$ is equivalent to the transfer relation

$$zTr_{Z/p}^*(1) = Tr_{Z/p}^*(c_1(\mathbf{C})) = Tr_{Z/p}^*(0) = 0$$

obtained by applying (ii). Of course since transfer is natural, Quillen's formula enables us to compute the stable Euler class for any regular Z/p covering.

In this spirit, let

$$\pi \leq \Sigma_p$$

be the subgroup of cyclic permutations of order p . For a given free action of π on a space Y with a given complex line bundle $\eta \rightarrow Y$ we have an equivariant map

$$\eta_\pi = (g_1, \dots, g_p) : Y \rightarrow BU(1)^p,$$

where g_i classifies the line bundle $t^{i-1}\eta$, $t \in \pi$.

So by naturality of transfer, the computation of transferred Chern classes $Tr^*(c_1^i(\eta))$, $i \geq 1$ for cyclic coverings can be reduced to the covering

$$\rho_\pi : E\pi \times (BU(1))^p \rightarrow E\pi \times_\pi (BU(1))^p,$$

as the universal example.

Similarly for the symmetric group.

Note that by transfer property (v), $Tr(\rho_\pi)^*$ has the same value on the Chern classes x_1, \dots, x_p in $E^*(BU(1)^p) = E^*[[x_1, \dots, x_p]]$ as above: the group π permutes the x_i and $\rho_\pi t = \rho_\pi$, $t \in \pi$. Thus in computations of the transfer we sometimes write these Chern classes in an equivalent way $x, tx, \dots, t^{p-1}x$.

3. TRANSFER AND $K(s)^*(X_{h\Sigma_p}^p)$

Fix a prime p and an integer $s \geq 0$. Let X be a CW complex whose Morava K-theory $K(s)^*(X)$ is even dimensional and finitely generated as a module over $K(s)^* = \mathbf{F}_p[v_s, v_s^{-1}]$ with $|v_s| = -2(p^s - 1)$ as usual. We note there is no restriction on p : although $K(s)$ is not a commutative ring spectrum for $p = 2$ our even dimensionality assumption implies the deviation from commutativity is zero in the groups we consider by a result of Würgler [16].

In this section we study the transfer homomorphism and compute the stable Euler class. We extend some results of Hopkins-Kuhn-Ravenel [11] and Hunton [10] for the group π to Σ_p . This leads to one of our main results, the computation of the algebra $K(s)^*(X_{h\Sigma_p}^p)$ for $X =$

CP^∞ given as Theorem 4.5 and Corollary 4.6 of the next section. Here, as in the Introduction, $X_{h\Sigma_p}^p = E\Sigma_p \times_{\Sigma_p} (X)^p$ is the homotopy orbit space of Σ_p acting on X^p by permuting factors.

We recall

$$K(s)^*(B\pi) = K(s)^*[z]/(z^{p^s})$$

where $|z| = 2$. We consider the Atiyah-Hirzebruch Serre spectral sequence:

$$(2) \quad E_2^{*,*}(\pi, X) = H^*(\pi; K(s)^*X^p) \Rightarrow K(s)^*(X_{h\pi}^p)$$

By the Künneth isomorphism

$$(3) \quad K(s)^*X^p \xrightarrow{\approx} (K(s)^*X)^{\otimes p}$$

Then $K(s)^*X^p$ is a π module where π acts by permuting factors (see [11], Th. 7.3).

An element $x \in K(s)^*(X)$ is called *good* if there is a finite cover $Y \rightarrow X$ together with an Euler class $y \in K(s)^*(Y)$ such that $x = Tr^*(y)$ where $Tr^* : K(s)^*(Y) \rightarrow K(s)^*(X)$ is the transfer. The space X is called *good* if $K(s)^*(X)$ is spanned over $K(s)^*$ by good elements.

Let $\gamma = \varphi^*(z)$, where

$$\varphi : X_{h\pi}^p \rightarrow B\pi$$

is the projection and let $\{x_i\}$ be a $K(s)^*$ basis for $K(s)^*(X)$. The following result is well known in ordinary mod- p cohomology. We follow the $K(s)^*$ version for classifying spaces of [11] Th.7.3.

Proposition 3.1. *Let X be a good space.*

(i) *As a $K(s)^*$ module $K(s)^*(X_{h\pi}^p)$ is free with basis*

$$\{ \gamma^i \otimes (x_j)^{\otimes p} \mid 0 \leq i < p^s, 1 \leq j \}$$

and

$$\left\{ \sum_{(i_1, i_2, \dots, i_p) \in I} 1 \otimes x_{i_1} \otimes x_{i_2} \otimes \dots \otimes x_{i_p} \mid I \in \mathcal{P}_p \right\}$$

where $I = \{(i_1, i_2, \dots, i_p)\}$ runs over the set \mathcal{P}_p of π -equivalence classes of p -tuples of positive integers at least two of which are not equal.

(ii) $X_{h\pi}^p$ is good.

Proof: (i) By the Künneth isomorphism,

$$(4) \quad K(s)^*(X)^{\otimes p} = F \oplus T$$

as a π -module, where F is free and T is trivial. Explicitly a $K(s)^*$ basis for T is $\{(x_i)^{\otimes p}, i \geq 1\}$, while a $K(s)^*$ basis for F is $\{x_{i_1} \otimes x_{i_2} \otimes \dots \otimes x_{i_p}, i_j \geq 1\}$ where not all the factors are equal. Then

$$\begin{aligned} H^*(\pi; F) &= F^\pi & \text{if } * = 0 \\ &= 0 & \text{if } * > 0 \end{aligned}$$

and

$$H^*(\pi; T) = H^*(B\pi) \otimes T.$$

Thus $E_2^{0,*}(\pi, X) = K(s)^*(X_{h\pi}^p)^\pi = F^\pi \oplus T$.

To continue the proof we consider the covering projection

$$\rho_\pi : E\pi \times X^p \rightarrow X_{h\pi}^p$$

its associated transfer homomorphism

$$(5) \quad Tr^* = Tr_\pi^* : K(s)^*(X^p) \rightarrow K(s)^*(X_{h\pi}^p)$$

and induced homomorphism

$$\rho_\pi^* : K(s)^*(X_{h\pi}^p) \rightarrow K(s)^*(X^p).$$

Similarly for the group Σ_p . Then $\rho_\pi^* Tr^* = N$, where $N = N_\pi$ is the norm or trace map $N : K^*(X^p) \rightarrow K^*(X^p)$ defined by $N(x) = \sum_{g \in \pi} gx$. Thus we have established the following lemma.

Lemma 3.2. *If $y \in K(s)^*(X^p)$ is good then there exists a good element $z \in K(s)^*(X_{h\pi}^p)$ such that $\rho_\pi^*(z) = N(y)$. \square*

Lemma 3.3. *If $x \in K(s)^*(X)$ is good then there is a good element $z \in K(s)^*(X_{h\pi}^p)$ such that $\rho_\pi^*(z) = x^{\otimes p}$.*

Proof: By assumption there is a finite covering $f : Y \rightarrow X$ and an Euler class $e \in K(s)^*(Y)$ such that $x = Tr^*(e)$. Now consider the covering

$$\phi = f \times \dots \times f : Y^p \rightarrow X^p$$

which extends to a covering

$$1 \times \phi : Y_{h\pi}^p \rightarrow X_{h\pi}^p$$

and yields a map of coverings

$$\begin{array}{ccc} E\pi \times Y^p & \xrightarrow{\rho_\pi} & Y_{h\pi}^p \\ \downarrow \phi & & \downarrow 1 \times \phi \\ E\pi \times X^p & \xrightarrow{\rho_\pi} & X_{h\pi}^p \end{array}$$

The class $e^{\otimes p}$ is an Euler class for Y^p . Since transfer is natural and commutes with tensor products we have

$$\rho_\pi^* Tr^*(1 \otimes e^{\otimes p}) = Tr^* \rho_\pi^*(1 \otimes e^{\otimes p}) = Tr^*(e^{\otimes p}) = Tr^*(e) \otimes \dots \otimes Tr^*(e) = x^{\otimes p}$$

□

Corollary 3.4. $E_2^{0,*}(\pi, X)$ consists of permanent cycles which are good. □

Thus as differential graded $K(s)^*$ modules, there is an isomorphism of spectral sequences

$$(E_r^{*,*}(\pi, pt) \otimes_{K(s)^*} T) \oplus F^\pi \xrightarrow{\cong} E_r^{*,*}(\pi, X)$$

Thus it follows that as a $K(s)^*$ algebra, $K(s)^*(X_{h\pi}^p)$ is generated by $K(s)^*(B\pi)$, T , and F^π .

(ii) The proof of [11] Th. 7.3 carries over. This completes the proof of Proposition 3.1. □

Lemma 3.5. i) $Im(Tr^*) \cdot \gamma = 0$.

ii) $Tr^*(1) = v_s \gamma^{p^s - 1}$.

iii) If $y \in T$ then $Tr^*(y) = y \cdot Tr^*(1)$.

Proof: i) By Fröbenius reciprocity,

$$Tr^*(w) \cdot \gamma = Tr^*(w \cdot \rho_\pi^*(\gamma)) = Tr^*(w \cdot 0) = 0.$$

ii) Consider the pull back diagram of coverings

$$\begin{array}{ccc} E\pi \times X^p & \xrightarrow{\varphi'} & E\pi \\ \downarrow & & \downarrow \\ X_{h\pi}^p & \xrightarrow{\varphi} & B\pi \end{array}$$

Then $Tr^*(1) = Tr^* \varphi'^*(1) = \varphi^* Tr_{Z/p}^*(1)$ by naturality of the transfer. By Quillen's formula $Tr_{Z/p}(1) = v_s z^{p^s - 1}$. Since $\gamma = \varphi^*(z)$, the result follows.

iii) If $y \in T$ then $\rho_\pi^*(y) = y$. The result follows from transfer property (ii), $Tr^* \rho_\pi^*(y) = y \cdot Tr^*(1)$. □

Remarks : 1) From the periodicity of the cohomology of a cyclic group [4] Proposition XII, 11.1, we have isomorphisms

$$H^s(\pi; K^*(X^p)) \xrightarrow{\cdot z} H^{s+2}(\pi; K^*(X^p))$$

for $s > 0$ and

$$H^0(\pi; K^*(X^p))/Im(N) \xrightarrow{\cdot z} H^2(\pi; K^*(X^p))$$

thus multiplication by z is also injective on T at the E_2 term.

2) $\rho_\pi^* Tr^* = N$, thus modulo $ker(\rho_\pi^*)$ we have

$$(6) \quad Tr^*(x_{i_1} \otimes x_{i_2} \otimes \dots \otimes x_{i_p}) = \sum_{\sigma \in \pi} 1 \otimes x_{\sigma(i_1)} \otimes x_{\sigma(i_2)} \otimes \dots \otimes x_{\sigma(i_p)}.$$

Note that if the i_j in (6) are equal, the right hand side is zero. However

$$Tr^*(x_j^{\otimes p}) = 1 \otimes x_j^{\otimes p} \cdot Tr^*(1)$$

on the nose by Lemma 3.5 iii).

We now turn to $K(s)^*(X_{h\Sigma_p}^p)$.

Lemma 3.6. $\rho_{\pi, \Sigma_p} : B\pi \rightarrow B\Sigma_p$ induces an isomorphism of $K(s)^*$ algebras

$$\rho_{\pi, \Sigma_p}^* : K(s)^*(B\Sigma_p) \xrightarrow{\cong} \{K(s)^*(B\pi)\}^W$$

where $W = N_{\Sigma_p}(\pi)/\pi \approx \mathbf{Z}/(p-1)$. Computing invariants yields

$$K(s)^*(B\Sigma_p) = K(s)^*[y]/(y^{m_s})$$

where $\rho_{\pi, \Sigma_p}^*(y) = z^{p-1}$ and $m_s = [(p^s - 1)/(p - 1)] + 1$.

Proof: The group W has order prime to p and thus acts exactly on the AHS spectral sequence for $K(s)^*(B\pi)$ producing a new spectral sequence of invariants. It is well known that $H^*(B\Sigma_p; \mathbf{F}_p) = H^*(B\pi; \mathbf{F}_p)^W = E[a] \otimes \mathbf{F}_p[b]$ where $|a| = 2(p-1) - 1$ and $b = \beta(a)$. Therefore ρ_{π, Σ_p} induces an isomorphism of spectral sequences.

$$E_r^{*,*}(\Sigma_p, pt) \xrightarrow{\cong} E_r^{*,*}(\pi, pt)^W$$

The corresponding isomorphism on abutments yields the desired result. \square

Let $c = \varphi^*(y)$ where $\varphi : E\Sigma_p \times_{\Sigma_p} X^p \rightarrow B\Sigma_p$ is projection.

Proposition 3.7. *As a $K(s)^*$ module $K(s)^*(X_{h\Sigma_p}^p)$ is free with basis*

$$\{ c^i \otimes (x_j)^{\otimes p} \mid 0 \leq i < m_s, 1 \leq j \}$$

and

$$\left\{ \sum_{(i_1, i_2, \dots, i_p) \in I} 1 \otimes x_{i_1} \otimes x_{i_2} \otimes \dots \otimes x_{i_p} \mid I \in \mathcal{E}_p \right\}$$

where $I = \{(i_1, i_2, \dots, i_p)\}$ runs over the set \mathcal{E}_p of Σ_p -equivalence classes of p -tuples of positive integers at least two of which are not equal.

Proof: The result follows from the AHS spectral sequence since, as in Lemma 3.6, W acts exactly.

4. APPLICATIONS

1. $\mathbf{C}P^\infty$.

As an application we consider the case $X = \mathbf{C}P^\infty = BU(1)$. Then $X_{h\pi}^p = B(\pi \wr U(1))$, $X_{h\Sigma_p}^p = B(\Sigma_p \wr U(1))$. We compute the Morava K-theory algebras of these spaces as well as the transfer homomorphism (5) of Section 3 and (8) below.

We recall

$$K(s)^*(X^p) = K(s)^*[[z_1, z_2, \dots, z_p]]$$

where $|z_i| = 2$. The projection $\varphi_G : X_{hG}^p \rightarrow BG$ for $G = \pi$ or $G = \Sigma_p$ is induced by the factorisation $G \wr U(1)/U(1)^p = G$.

Proposition 4.1. *(i) For $X = \mathbf{C}P^\infty$ the cup products of $K(s)^*(X_{h\pi}^p)$, as an algebra over $K(s)^*(B\pi)$, are determined by $\text{Im}(Tr^*) \cdot \gamma = 0$.*

(ii) $B(\pi \wr U(1))$ is good.

Proof: (i) These relations hold by Lemma 3.5. By the calculation of a basis in the proof of Proposition 3.1, there are no other relations.

(ii) Since $BU(1)$ is good this follows from Proposition 3.1 (ii).

Turning to the symmetric group case we denote by $\xi^{\times p}$ the p -fold product of the canonical line bundle over X^p . This bundle extends to an p -dimensional bundle

$$(7) \quad \xi_{\Sigma_p} = E\Sigma_p \times_{\Sigma_p} \xi^{\times p}$$

on $X_{h\Sigma_p}^p$ classified by the inclusion $X_{h\Sigma_p}^p = B(\Sigma_p \wr U(1)) \hookrightarrow BU(p)$. Let $c_i = c_i(\xi_{\Sigma_p})$. Then $\rho_{\Sigma_p}^*(c_i) = c_i(\xi^{\times p}) = \sigma_i$, the i^{th} symmetric polynomial in the z_j where

$$\rho_{\Sigma_p} : E\Sigma_p \times X^p \rightarrow X_{h\Sigma_p}^p$$

is the covering projection.

Definition 4.2. *We shall also consider the classes*

$$\tilde{c}_i = Tr_{\Sigma_p}^*(z_1 z_2 \cdots z_i)$$

for $i = 1, \dots, p-1$ where

$$(8) \quad Tr_{\Sigma_p}^* : K(s)^*(E\Sigma_p \times X^p) \rightarrow K(s)^*(X_{h\Sigma_p}^p)$$

is the transfer homomorphism.

Lemma 4.3. $\rho_{\Sigma_p}^*(\tilde{c}_i) = i!(p-i)!\sigma_i$.

Proof: $\rho_{\Sigma_p}^*(\tilde{c}_i) = \rho_{\Sigma_p}^* Tr_{\Sigma_p}^*(z_1 z_2 \cdots z_i) = N_{\Sigma_p}(z_1 z_2 \cdots z_i)$. For each subset of i integers $\{j_1, j_2, \dots, j_i\}$ with $1 \leq j_k \leq p$, there are $i!$ bijections $\{1, 2, \dots, i\} \rightarrow \{j_1, j_2, \dots, j_i\}$ and $(p-i)!$ bijections $\{i+1, i+2, \dots, p\} \rightarrow \{1, 2, \dots, p\} \setminus \{j_1, j_2, \dots, j_i\}$. Thus there are $i!(p-i)!$ summands of $z_{j_1} z_{j_2} \cdots z_{j_i}$ in $N_{\Sigma_p}(z_1 z_2 \cdots z_i)$. \square

Proposition 4.4. *The $K(s)^*$ algebra $K(s)^*(X_{h\Sigma_p}^p)$ is generated by*

$$c, c_1, \dots, c_{p-1}, c_p.$$

Similarly with c_i replaced by \tilde{c}_i for $1 \leq i \leq p-1$.

Proof: We note that (4) is also a decomposition of Σ_p modules. Furthermore, $H^*(\Sigma_p; F) \subseteq H^*(\pi; F)$, and T is trivial as a Σ_p module. Thus

$$\begin{aligned} H^*(\Sigma_p; F) &= F^{\Sigma_p} \quad \text{if } * = 0 \\ &= 0 \quad \text{if } * > 0 \end{aligned}$$

and

$$H^*(\Sigma_p; T) = H^*(B\Sigma_p) \otimes T.$$

Therefore, as differential graded $K(s)^*$ modules, there is an isomorphism of spectral sequences

$$(E_r^{*,*}(\Sigma_p, pt) \otimes_{K(s)^*} T) \oplus F^{\Sigma_p} \xrightarrow{\approx} E_r^{*,*}(\Sigma_p, X)$$

It follows that as a $K(s)^*$ algebra, $K(s)^*(X_{h\Sigma_p}^p)$ is generated by $K(s)^*(B\Sigma_p)$, T , and F^{Σ_p} . However $F^{\Sigma_p} \oplus T = (F \oplus T)^{\Sigma_p}$ and by the fundamental theorem of symmetric functions

$$(F \oplus T)^{\Sigma_p} = K(s)^*[\sigma_1, \dots, \sigma_p]$$

This completes the proof since $\rho_{\Sigma_p}^*(c_i) = \sigma_i$ and for $1 \leq i \leq p-1$ $\rho_{\pi}^*(\tilde{c}_i) = \epsilon_i \sigma_i$, $\epsilon_i \neq 0$. \square

Theorem 4.5. *As a $K(s)^*(B\Sigma_p)$ algebra,*

$$K(s)^*(X_{h\Sigma_p}^p) = K(s)^*(B\Sigma_p)[[\tilde{c}_1, \dots, \tilde{c}_{p-1}, c_p]]/(c\tilde{c}_i).$$

Proof: Since $\tilde{c}_i \in \text{Im}(Tr^*)$, the relations follow from Lemma 3.5 i). That there are no other relations follows from Proposition 3.7 and the fact that c_p is represented by $x^{\otimes p} \in T$. \square

Corollary 4.6. *Over $K(s)^*$,*

$$K(s)^*(X_{h\Sigma_p}^p) = K(s)^*[[c, \tilde{c}_1, \dots, \tilde{c}_{p-1}, c_p]]/(c^{m_s}, c\tilde{c}_i).$$

Corollary 4.6 follows immediately from Corollary 3.4 and Lemma 3.5 ii).

Proposition 4.7. $Tr_{\Sigma_p}^*(1) = -v_s c^{m_s-1}$.

Proof: First we prove the formula on the level of classifying spaces, i.e. for $X = pt$, in which case it reduces to $Tr_{\Sigma_p}(1) = -v_s y^{m_s-1}$. To do this we analyze $\rho_{\pi, \Sigma_p}^* Tr_{\Sigma_p}^*(1)$. Since the transfer is transitive (v) we can factor

$$Tr_{\Sigma_p}^* : K(s)^*(Be) \xrightarrow{Tr_{\pi}^*} K(s)^*(B\pi) \xrightarrow{Tr_{\pi, \Sigma_p}^*} K(s)^*(B\Sigma_p)$$

Applying the double coset formula (transfer property (vi)) to

$$K(s)^*(B\pi) \xrightarrow{Tr_{\pi, \Sigma_p}^*} K(s)^*(B\Sigma_p) \xrightarrow{\rho_{\pi, \Sigma_p}^*} K(s)^*(B\pi)$$

we obtain

$$\sum_{g \in \pi \backslash \Sigma_p / \pi} K(s)^*(B\pi) \xrightarrow{\rho_{\pi}^* g \cap \pi} K(s)^*(B\pi^g \cap \pi) \xrightarrow{c^{g-1}} K(s)^*(B\pi^g \cap \pi) \xrightarrow{Tr_{\pi^g \cap \pi, \pi}^*} K(s)^*(B\pi)$$

Since $\pi^g \cap \pi = e$ unless $g \in N_{\Sigma_p}(\pi)/\pi$ in which case $\pi^g \cap \pi = \pi$, $\rho_{\pi^g \cap \pi}^* = Tr_{\pi^g \cap \pi, \pi}^* = id$.

Thus

$$\rho_{\pi, \Sigma_p}^* Tr_{\Sigma_p}^*(1) = \rho_{\pi, \Sigma_p}^* Tr_{\pi, \Sigma_p}^*(v_s z^{p^s-1}) = \sum_{g \in N_{\Sigma_p}(\pi)/\pi} g^*(v_s z^{p^s-1}).$$

Now $N_{\Sigma_p}(\pi)/\pi = \mathbf{Z}/(p-1)$ and $g^*(z) = gz$. Therefore

$$\rho_{\pi, \Sigma_p}^* Tr_{\Sigma_p}^*(1) = \left[\sum_{i=1}^{p-1} i^{p^s-1} \right] v_s z^{p^s-1} = -v_s z^{p^s-1}.$$

Since ρ_{π, Σ_p}^* is injective and $\rho_{\pi, \Sigma_p}^*(y) = z^{p-1}$ this proves the special case $X = pt$. The general case now follows by a pull back argument as in the proof of Lemma 3.5 (ii) using $c = \varphi^*(y)$. \square

Remark: In view of Proposition 4.7 we observe that the multiplicative relations of $K(s)^*(X_{h\Sigma_p}^p)$ are completely determined by the relation $Im(Tr_{\Sigma_p}^*) \cdot c = 0$.

Let us return to Theorem 4.5 and find the relationship between the two systems of generators c_i and \tilde{c}_i .

Corollary 4.8. *Let $\varphi_{\Sigma_p} : X_{h\Sigma_p}^p \rightarrow B\Sigma_p$ be the projection induced by factorisation $\Sigma_p \wr U(1)/U(1)^p = \Sigma_p$. There exist unique elements $\delta_{ik}^{\Sigma_p} \in K(s)^*(B\Sigma_p)$ such that in $K(s)^*(X_{h\Sigma_p}^p)$*

$$\tilde{c}_k - k!(p-k)!c_k = \sum_{i \geq 0} \varphi_{\Sigma_p}^*(\delta_{ik}^{\Sigma_p})c_p^i, \quad k = 1, \dots, p-1.$$

Proof: The existence of $\delta_{ik}^{\Sigma_p}$ follows from Lemma 4.3 and Corollary 4.6: the left hand side, as an element of $Ker\rho_{\Sigma_p}^*$, has the form of the right hand side. As for uniqueness note that multiplication by c_p is injective. \square

Definition 4.9. *Let t be a generator of π and let $F \subset K(s)^*(X^p)$ be the free π module of Proposition 3.1 (4). Then elements $\omega_k \in F$ can be defined modulo $KerN_\pi = Im(1-t)$, $t \in \pi$ by*

$$(9) \quad N_\pi(\omega_k) = \sigma_k, \quad k = 1, \dots, p-1.$$

By transfer property (v) $Tr_\pi^* = Tr_\pi^*t$, where

$$Tr_\pi^* : K(s)^*(E\pi \times BU(1)^p) \rightarrow K(s)^*(X_{h\pi}^p)$$

hence $Tr_\pi^*(\omega_k)$ is well-defined.

Remark 4.10. In order to compute the transfer homomorphism of (8) in terms of Chern classes c, c_1, \dots, c_p we recall [9], p. 44, that we can consider $K(s)^*(X^p)$, $X = CP^\infty$ as a free $K(s)^*[[\sigma_1, \dots, \sigma_p]]$ module generated by the elements $z_1^{i_1} \dots z_p^{i_p} \in F \subset K(s)^*(BU(1)^p)$, with $0 \leq i_j \leq p-j$, where F is the free π module of Definition 4.9. So by Fröbenius reciprocity it suffices to compute the transfer on these monomials. Then summed over the symmetric group $\sum g^*(z_1^{i_1} \dots z_p^{i_p})$ is a symmetric function and hence has the form

$$\sum_{\pi} \sum_{\Sigma_p/\pi} g^*(z_1^{i_1} \dots z_p^{i_p}) = \sigma_1 s_1 + \dots + \sigma_{p-1} s_{p-1} = \sum_{\pi} (\omega_1 s_1 + \dots + \omega_{p-1} s_{p-1}),$$

for the elements ω_k defined in 4.9 and some symmetric functions s_1, \dots, s_{p-1} . Hence we have modulo $\ker N = \text{Im}(1-t)$, $t \in \pi$ in F

$$\sum_{\Sigma_p/\pi} g^*(z_1^{i_1} \dots z_p^{i_p}) = \omega_1 s_1 + \dots + \omega_{p-1} s_{p-1}.$$

The left sum consists of $(p-1)!$ elements each with the same value of transfer. Also ω_k consists of $p^{-1} \binom{p}{k}$ elements with the value of transfer $\text{Tr}_{\Sigma_p}^*(z_1 \dots z_k)$. Thus the computation of the elements \tilde{c}_k and Fröbenius reciprocity is all that is needed for computing $\text{Tr}_{\Sigma_p}^*$.

Relating π and Σ_p we have a lift of ρ_{π, Σ_p}

$$\begin{array}{ccc} X_{h\pi}^p & \xrightarrow{\tilde{\rho}_{\pi, \Sigma_p}} & X_{\Sigma_p}^p \\ \varphi \downarrow & & \downarrow \varphi \\ B\pi & \xrightarrow{\rho_{\pi, \Sigma_p}} & B\Sigma_p \end{array}$$

Lemma 4.11. $\tilde{\rho}_{\pi, \Sigma_p}^*(\tilde{c}_k) = k!(p-k)! \text{Tr}_{\pi}^*(\omega_k)$.

Proof: Note that modulo $\text{Im}(1-t)$ we have $\Sigma g^*(z_1 z_2 \dots z_k) = k!(p-k)! \omega_k$ summed over Σ_p/π . Applying the double coset formula

$$\rho_{\pi, \Sigma_p}^* \tilde{c}_k = \rho_{\pi, \Sigma_p}^* \text{Tr}_{\Sigma_p}^*(z_1 z_2 \dots z_k) = \text{Tr}_{\pi}^* \sum_{g \in \Sigma_p/\pi} g^*(z_1 z_2 \dots z_k) = k!(p-k)! \text{Tr}_{\pi}^*(\omega_k).$$

□

We are grateful to D. Ravenel for supplying us with the proof of the following result which is important for calculations in Section 10.

Lemma 4.12. *For the formal group law in Morava K-theory $K(s)$ we have mod y^{p^s}*

$$F(x, y) \equiv x + y - v_s \sum_{0 < j < p} p^{-1} \binom{p}{j} (x^{p^{s-1}})^j (y^{p^{s-1}})^{p-j}.$$

Proof: This result can be derived from the recursive formula for the FGL given in 4.3.8 [14]. For the FGL in Morava K-theory it reads

$$F(x, y) = F(x + y, v_s w_1(x, y)^{p^{s-1}}, v_s^{e_2} w_2(x, y)^{p^{2(s-1)}}, \dots)$$

where w_i is a certain homogeneous polynomial of degree p^i defined by 4.3.5 and $e_i = (p^{is} - 1)/(p^s - 1)$. In particular $w_0 = x + y$ (the first term in the formula above),

$$w_1 = - \sum_{0 < j < p} p^{-1} \binom{p}{j} x^j y^{p-j},$$

and $w_i \notin (x^p, y^p)$.

For $s > 1$ we can reduce modulo the ideal $v_s^{e_2}(x^{p^{2(s-1)}}, y^{p^{2(s-1)}})$ and get

$$\begin{aligned} F(x, y) &\equiv F(x + y, v_s w_1(x, y)^{p^{s-1}}) \\ &= F(x + y + v_s w_1(x, y)^{p^{s-1}}, v_s w_1(x + y, v_s w_1(x, y)^{p^{s-1}})^{p^{s-1}}, \dots) \\ &\equiv F(x + y + v_s w_1(x, y)^{p^{s-1}}, v_s w_1(x^{p^{s-1}} + y^{p^{s-1}}, v_s^{p^{s-1}} w_1(x, y)^{p^{2(s-1)}})), \end{aligned}$$

and modulo $v_s^{1+p^{s-1}}(x^{p^{2(s-1)}}, y^{p^{2(s-1)}})$ we have

$$F(x, y) \equiv x + y + v_s w_1(x, y)^{p^{s-1}}. \quad \square$$

Let us write for short $\sigma_k = \sigma_k(x, F(x, z), \dots, F(x, (p-1)z))$.

Lemma 4.13. 1) *The following formula holds in $K(s)^*(BU(1) \times B\pi)$*

$$\sigma_k = - \sum_{0 \leq i \leq p^s} \lambda_{ik}^\pi \sigma_p^i + p^{-1} \binom{p}{k} x^k v_s z^{p^s-1},$$

where $\lambda_{ik}^\pi = \lambda_{ik}^\pi(z^{p-1})$ are polynomials in z^{p-1} . $\lambda_{0j}^\pi = 0$, $j = 1, \dots, p-2$.

2) *For $1 \leq k \leq p-1$, equating the coefficients of $x^p, \dots, x^{p^{s+1}}$ in 1) gives a system of linear equations with invertible determinant. Thus the elements λ_{ik}^π can be defined as the solution of this system.*

Proof. 1). Clearly

$$\sigma_p(x, F(x, z), \dots, F(x, (p-1)z)) = x^p + z^{p-1} A(x, z)$$

for some polynomial A . Thus in the expression for σ_k we can substitute $x^{n-p}(\sigma_p - z^{p-1}A(x, z))$ for σ_k each x^n for $n \geq p$. As a result, high powers of x become replaced either by σ_p or by terms divisible by z^{p-1} . Repeating this process gives expressions where each occurrence of powers of x exceeding $p-1$ is accompanied by higher and higher powers of z , which is nilpotent. By iteration all such powers of x become eliminated, thus giving σ_k expressed in terms of σ_p , the only remaining term containing x being $p^{-1} \binom{p}{k} x^k v_s z^{p^s-1}$.

2). Proof is elementary. □

We now consider the bundle $\xi_\pi = \tilde{\rho}_{\pi, \Sigma_p}^*(\xi_{\Sigma_p})$ over $X_{h\pi}^p$:

$$(10) \quad \xi_\pi = E\pi \times_\pi \xi^{\times p},$$

with the classifying map the inclusion $B(\pi \wr U(1)) \rightarrow BU(p)$, where $\xi_{\Sigma_p} \rightarrow X_{\Sigma_p}^p$ is the bundle (7). Then let Tr_π^* and $Tr_{\Sigma_p}^*$ be the transfer homomorphisms of (5) and (8) and $\varphi_\pi : X_{h\pi}^p \rightarrow B\pi$ and $\varphi_{\Sigma_p} : X_{\Sigma_p}^p \rightarrow B\Sigma_p$ be projections as before in this section.

Let $\lambda_{ik}^\pi = \delta_{ik}^\pi(z)$, the polynomials in z^{p-1} from $K(s)^*(B\pi) = K(s)^*[z]/([p]z)$ defined above Lemma 4.13 and $\lambda_{ik}^{\Sigma_p}$ be defined by Corollary 4.8 and

$$\tilde{\rho}_{\pi, \Sigma_p}^*(\delta_{ik}^{\Sigma_p}) = k!(p-k)!\lambda_{ik}^\pi.$$

Proposition 4.14. *There exists unique elements $\delta_{ik}^\pi \in K(s)^*(B\pi)$ such that*

1) *In $K(s)^*(X_{h\pi}^p)$ the following formula holds*

$$c_k(\xi_\pi) = Tr_\pi^*(\omega_k) - \sum_{0 \leq i \leq p^s} \varphi_\pi^*(\delta_{ik}^\pi) c_p^i(\xi_\pi).$$

2) *In $K(s)^*(X_{h\Sigma_p}^p)$ one has*

$$c_k(\xi_{\Sigma_p}) = Tr_{\Sigma_p}^*(z_1 \dots z_k) - \sum_{0 \leq i \leq p^s} \varphi_{\Sigma_p}^*(\delta_{ik}^{\Sigma_p}) c_p^i(\xi_{\Sigma_p}).$$

3) *The value of $Tr_\pi^*(c_1(\xi_\pi))$ is determined by*

$$c_1(\xi_\pi) = Tr_\pi^*(c_1(\xi_\pi)) + v_s \sum_{1 \leq i \leq s-1} \gamma^{p^s-p^i} c_p^{p^i-1}(\xi_\pi).$$

where $\gamma = \varphi_\pi^*(z)$

Proof: Theorem 5.1 gives the existence and uniqueness of the elements δ_{ik}^π .

The diagonal map $\Delta : BU(1) \rightarrow BU(1)^p$ induces an inclusion $B\pi \times BU(1) \rightarrow X_{h\pi}^p$ and the commutative diagram

$$\begin{array}{ccc} E\pi \times BU(1) & \xrightarrow{1 \times \Delta} & E\pi \times BU(1)^p \\ \pi \times 1 \downarrow & & \rho_\pi \downarrow \\ B\pi \times BU(1) & \longrightarrow & X_{h\pi}^p \end{array}$$

Then $(1 \times \Delta)^*(\omega_k) = p^{-1} \binom{p}{k} x^k$, $x = c_1(\xi)$. Hence by transfer properties (i) and (iv) we have for the transfer $Tr = Tr(\pi \times 1)$:

$$Tr^*((1 \times \Delta)^*(\omega_k)) = p^{-1} \binom{p}{k} x^k Tr^*(1) = p^{-1} \binom{p}{k} x^k v_s z^{p^s-1}.$$

On the other hand by the existence of the elements δ_{ik}^π we have

$$Tr^*((1 \times \Delta)^*(\omega_k)) =$$

$$\sigma_k(x, F(x, z), \dots, F(x, (p-1)z)) + \sum_{i \geq 0} \delta_{ik}^\pi \sigma_p^i(x, F(x, z), \dots, F(x, (p-1)z)),$$

hence by Lemma 4.13

$$\begin{aligned} & \sigma_k(x, F(x, z), \dots, F(x, (p-1)z)) = \\ & - \sum_{0 \leq i \leq p^s} \lambda_{ik}^\pi \sigma_p^i(x, F(x, z), \dots, F(x, (p-1)z)) + p^{-1} \binom{p}{k} x^k v_s z^{p^s-1}. \end{aligned}$$

This proves 1) and shows $\delta_{ik}^\pi = \lambda_{ik}^\pi$ for $0 \leq i \leq p^s$ and is zero otherwise.

2) Combining Corollary 4.8 and Lemma 4.11 this follows from 1).

Statement 3) follows from the following explicit formula for σ_1 :

Lemma 4.15. *In $K(s)^*(B\pi \times BU(1))$ one has*

$$\sigma_1 = v_s \left(z^{p^s-1} x + \sum_{i=1}^{s-1} z^{p^s-p^i} \sigma_p^{p^{i-1}} \right).$$

Proof: First note that for any prime p and any $0 < k < p$ one has

$$\sigma_k(1, 2, \dots, p-1) \equiv \begin{cases} 0 \pmod{p}, & k < p-1, \\ -1 \pmod{p}, & k = p-1 \end{cases}$$

where σ_k are the symmetric functions.

The polynomial $x^p - x$ has p roots $0, 1, \dots, p-1$ in \mathbf{F}_p , hence it equals $x(x-1)\dots(x-(p-1))$ in this field. But coefficients of the latter polynomial are exactly the symmetric functions in question.

One has

$$\begin{aligned} \sigma_1 &= x + F(x, z) + \dots + F(x, (p-1)z) = x + x + z + v_s w_1(x^{p^s-1}, z^{p^s-1}) \\ &+ \dots + x + (p-1)z + v_s w_1(x^{p^s-1}, ((p-1)z)^{p^s-1}) \\ &= px + \frac{p(p-1)}{2}z + v_s \left(\sum_{i=1}^{p-1} w_1(x, iz) \right)^{p^s-1} \\ &= v_s \left(\sum_{i=1}^{p-1} \sum_{j=1}^{p-1} -p^{-1} \binom{p}{j} i^j x^{p-j} z^j \right)^{p^s-1} \\ &= v_s \left(\sum_{j=1}^{p-1} - \left(\sum_{i=1}^{p-1} i^j \right) p^{-1} \binom{p}{j} z^j x^{p-j} \right)^{p^s-1}. \end{aligned}$$

Now $\sum_{i=1}^{p-1} i^j$ is an integral linear combination of $\sigma_k(1, 2, \dots, p-1)$ with $k \leq i$, hence by the above argument it is zero for $i < p-1$ and for $i = p-1$ it is $p-1$ as $i^{p-1} \equiv 1 \pmod{p}$ for all $i < p$.

Thus

(11)

$$\sigma_1 = -v_s \left((p-1)p^{-1} \binom{p}{p-1} z^{p-1} x^{p-(p-1)} \right)^{p^{s-1}} = v_s z^{p^s - p^{s-1}} x^{p^{s-1}}.$$

Now since $F(x, z)^p = x^p + z^p$, one has $\sigma_p^p = (x(x+z)\dots(x+(p-1)z))^p$.

But again we have $x(x+z)\dots(x+(p-1)z) = x^p - xz^{p-1}$. Substituting this one obtains

$$\begin{aligned} & v_s \left(z^{p^s-1} x + \sum_{i=1}^{s-1} z^{p^s-p^i} \sigma_p^{p^{i-1}} \right) \\ &= v_s \left(z^{p^s-1} x + z^{p^s-p} \sigma_p + \sum_{i=2}^{s-1} z^{p^s-p^i} (x^p - z^{p-1}x)^{p^{i-1}} \right) \\ &= v_s \left(z^{p^s-1} x + z^{p^s-p} \sigma_p + \sum_{i=2}^{s-1} z^{p^s-p^i} (x^{p^i} - z^{(p-1)p^{i-1}} x^{p^{i-1}}) \right). \end{aligned}$$

But

$$\begin{aligned} & \sum_{i=2}^{s-1} z^{p^s-p^i} (x^{p^i} - z^{(p-1)p^{i-1}} x^{p^{i-1}}) = \sum_{i=2}^{s-1} z^{p^s-p^i} x^{p^i} - z^{p^s-p^i+(p-1)p^{i-1}} x^{p^{i-1}} \\ &= \sum_{i=2}^{s-1} z^{p^s-p^i} x^{p^i} - z^{p^s-p^i+p^i-p^{i-1}} x^{p^{i-1}} = \sum_{i=2}^{s-1} z^{p^s-p^i} x^{p^i} - z^{p^s-p^{i-1}} x^{p^{i-1}} \\ &= z^{p^s-p^{s-1}} x^{p^{s-1}} - z^{p^s-p} x^p. \end{aligned}$$

Hence one has

$$(12) \quad v_s \left(z^{p^s-1} x + \sum_{i=1}^{s-1} z^{p^s-p^i} \sigma_p^{p^{i-1}} \right) = v_s \left(z^{p^s-1} x + z^{p^s-p} \sigma_p + z^{p^s-p^{s-1}} x^{p^{s-1}} - z^{p^s-p} x^p \right).$$

Now one has

$$z^{p^s-p} F(x, kz) = z^{p^s-p} (x + kz + v_s w_1(x^{p^{s-1}}, (kz)^{p^{s-1}})) = z^{p^s-p} (x + kz),$$

$$\text{hence } z^{p^s-p} \sigma_p = z^{p^s-p} x(x+z)\dots(x+(p-1)z) = z^{p^s-p} (x^p - z^{p-1}x)$$

$$\text{as above, i. e. } z^{p^s-p} \sigma_p = z^{p^s-p} x^p - z^{p^s-1} x. \text{ Substituting this into (12)}$$

gives $v_s \left(z^{p^s-1} x + \sum_{i=1}^{s-1} z^{p^s-p^i} \sigma_p^{p^{i-1}} \right) = v_s z^{p^s-p^{s-1}} x^{p^{s-1}}$, which is σ_1 by (11). \square

2. $\pi \wr (\mathbf{Z}/p^n)$.

We now turn to $G_n = \pi \wr (\mathbf{Z}/p^n)$ where $\pi = \mathbf{Z}/p$. Consider the AHS spectral sequence for

$$B(\mathbf{Z}/p^n)^p \rightarrow BG_n \xrightarrow{\varphi} B\pi$$

Then

$$E_2^{p,q} = H^*(B\pi; K^*(s)B((\mathbf{Z}/p^n)^p))$$

where

$$K^*(s)B(\mathbf{Z}/p^n)^p = (K(s)^*[z]/(z^{p^{ns}}))^{\otimes p} = F \oplus T$$

where F and T as above are free (resp. trivial) π modules.

Let $c = \varphi^*(z)$ where $K(s)^*(B\pi) = K(s)^*[z]/(z^{p^s})$ as above.

Proposition 4.16. *As a $K(s)^*$ module $K(s)^*(BG_n)$ is free with basis*

$$\{ c^i \otimes (z^j)^{\otimes p} \mid 0 \leq i < p^s, 0 \leq j < p^{ns} \}$$

and

$$\left\{ \sum_{(i_1, i_2, \dots, i_p) \in I} 1 \otimes z^{i_1} \otimes z^{i_2} \otimes \dots \otimes z^{i_p} \mid I \in \mathcal{P}_p(n) \right\}$$

where $I = \{(i_1, i_2, \dots, i_p)\}$ runs over the set $\mathcal{P}_p(n)$ of π -equivalence classes of p -tuples of integers $\{0 \leq i_j < p^{ns}\}$ at least two of which are not equal.

Proof: This spectral sequence computation is exactly analogous to that of Proposition 3.1. \square

Remarks: (i) Proposition 4.16 gives another derivation of Proposition 3.1. Since $\mathbf{C}P_p^\infty = [\text{colim}_n B(\mathbf{Z}/p^n)]_p^\wedge$, we have $K(s)^*(\mathbf{C}P_{h\pi}^\infty) = \text{lim}_n K(s)^*(BG_n)$.

(ii) G_n is good for $K(s)^*$ by [11] Th. 7.3.

By analogy with Lemma 3.2 we have

Lemma 4.17. *i) $\text{Im}(Tr^*) \cdot c = 0$.*

ii) $Tr^(1) = v_s c^{2^s - 1}$.*

iii) If $y \in T$ then $Tr^(y) = y \cdot Tr^*(1)$.*

Finally we consider the case $p = 2$.

Proposition 4.18. *Let $p = 2$. As a $K(s)^*(B\pi)$ algebra, $K(s)^*(BG_n)$ is generated by c, \tilde{c}_1, c_2 subject to the relations $\tilde{c}_1 \cdot c = 0$ and relations $\tilde{c}_1^{2^{ns}} = c_2^{2^{ns}} = 0$ modulo terms divisible by c .*

Proof: $\tilde{c}_1 \cdot c = 0$ by Lemma 3.5. The other relations hold in the E_∞ term of the spectral sequence. The only possible extensions are those on the fiber, involving \tilde{c}_1, c_2 . \square

Similar results hold for $\Sigma_p \wr \mathbf{Z}/p$ and $\Sigma_p \wr \Sigma_p$.

5. COMPLEX COBORDISM OF $X_{h\pi}^p$

We turn to computing the complex cobordism for $X = \mathbf{C}P^\infty$, $X_{h\pi}^p = B(\pi \wr U(1))$. Before stating the main result Theorem 5.1 we begin with some preliminary notions.

Let ξ be the canonical complex line bundle over $\mathbf{C}P^\infty = BU(1)$ and ξ_i be the pulback bundle over $BU(1)^p$ by the projection on the i -th factor as before. Then $MU^*(BU(1)^p) = MU^*[[x_1, \dots, x_p]]$, $x_i = c_1(\xi_i)$ and $x_1 \dots x_p$ is Euler class of the bundle $\xi^{\times p} = \bigoplus \xi_i$. Hence for the sphere bundle $S(\xi^{\times p})$, one has

$$MU^*(S(\xi^{\times p})) = MU^*[[x_1, \dots, x_p]]/(x_1 \dots x_p).$$

Then for the trace map N we have $\ker N = \text{Im}(1 - t)$, $t \in \pi$ in $MU^*BU(1)^p$ and after restricting N to $MU^*(S(\xi^{\times p}))$ we have the exact sequence

$$(13) \quad \dots \leftarrow MU^*(S(\xi^{\times p})) \xleftarrow{N} MU^*(S(\xi^{\times p})) \xleftarrow{1-t} MU^*(S(\xi^{\times p})) \xleftarrow{N} MU^*(S(\xi^{\times p})) \leftarrow \dots$$

Then let $\xi_\pi = E\pi \times_\pi \xi^{\times p}$ be the Atiyah transfer bundle [2] as in (10),

$$(14) \quad S(\xi_\pi) = E\pi \times_\pi S(\xi^{\times p})$$

be its sphere bundle and

$$(15) \quad D(\xi_\pi) = E\pi \times_\pi D(\xi^{\times p})$$

be the disk bundle. Then $D(\xi_\pi)$ is homotopy equivalent to $X_{h\pi}^p = B(\pi \wr U(1))$.

The cofibration $D(\xi_\pi)/S(\xi_\pi) = (X_{h\pi}^p)^{\xi_\pi}$ gives the long exact sequence

$$(16) \quad \dots \leftarrow MU^*(S(\xi_\pi)) \leftarrow MU^*(X_{h\pi}^p) \xleftarrow{\times c_p} MU^*((X_{h\pi}^p)^{\xi_\pi}) \leftarrow \dots$$

where $(X_{h\pi}^p)^{\xi_\pi}$ is the Thom space of the bundle ξ_π and the right homomorphism is multiplication by the Euler class $c_p = c_p(\xi_\pi)$.

Since the diagonal of $BU(1)^p$ is fixed under the permutation action of π the inclusion $E\pi \rightarrow E\pi \times BU(1)^p$; $x \rightarrow (x, \text{fixpoint})$ defines the inclusions $i : B\pi \rightarrow X_{h\pi}^p$ and the inclusion

$$i_0 : B\pi \rightarrow S(\xi_\pi).$$

The projection $\varphi : X_{h\pi}^p \rightarrow B\pi$ induced by $\pi \wr U(1) \rightarrow \pi$ defines the projection

$$(17) \quad \varphi_0 : S(\xi_\pi) \rightarrow B\pi$$

and the compositions $\varphi_0 i_0$, φi are the identity. We can consider φ_0 as the bundle with fiber $S(\xi^{\times p})$.

Let η be canonical line bundle over $B\pi$ and

$$\theta = \varphi^*(\eta) \rightarrow X_{h\pi}^p$$

be the pull back bundle. Thus $i^*(\theta) = \eta$ and $i^*(\xi_\pi) = \mathbf{C} + \eta + \dots + \eta^{p-1}$.

Consider the pull back diagram

$$\begin{array}{ccc} E\pi \times S(\xi^{\times p}) & \longrightarrow & E\pi \times BU(1)^p \\ \rho_0 \downarrow & & \downarrow \rho \\ S(\xi_\pi) & \longrightarrow & X_{h\pi}^p \end{array}$$

Let $Tr = Tr_\pi$ be the transfer of covering ρ , and $Tr_0 = S(\xi_\pi) \rightarrow S(\xi^{\times p})$ transfer map of ρ_0 .

Theorem 5.1. *In $MU^*(X_{h\pi}^p)$*

- a) *Annihilator of the Chern class $\gamma = c_1(\theta)$ coincides with $ImTr^*$;*
- b) *Multiplication by $c_p = c_p(\xi_\pi)$ is a monomorphism;*
- c) *Any element of $Ker(\rho^*)$ has the form $\sum_{k \geq 0} \varphi^*(\delta_k)c_p^k$, for some elements $\delta_k \in \tilde{M}U^*(B\pi)$.*
- d) *For $\pi = Z/2$,*

$$\begin{aligned} MU^*B(\pi \wr U(1)) &= MU^*[[c, c_1, c_2]]/(c_1 - c_1^*, c_2 - c_2^*) \\ &= MU^*(B\pi)[[Tr^*(x), c_2]]/(cTr^*(x)), \end{aligned}$$

where $c_i = c_i(\xi_\pi)$, $c_i^* = c_i(\xi_\pi \otimes \theta)$, $c = c_1(\theta)$ and $x \in MU^*(BU(1) = MU^*[[x, tx]])$ are Chern characteristic classes.

We need the following

Lemma 5.2. *The left homomorphism in the long sequence (16) is epimorphism and this gives the short exact sequence*

$$0 \leftarrow MU^*(S(\xi_\pi)) \leftarrow MU^*(X_{h\pi}^p) \leftarrow MU^*(X_{h\pi}^p)^{\xi_\pi} \leftarrow 0$$

Moreover we can construct a stable equivalence of the form

$$\varphi_0 \vee f_\pi : S(\xi_\pi) \rightarrow B\pi \vee X_\pi,$$

with a map f_π factoring through the following map

$$S(\xi_\pi) \rightarrow X_{h\pi}^p \xrightarrow{Tr} E\pi \times BU(1)^p.$$

and φ_0 is in (17).

Proof: Consider the Serre spectral sequence for the fibration (17)

$$S(\xi^{\times p}) \rightarrow S(\xi_\pi) \xrightarrow{\varphi_0} B\pi.$$

$E_2^{i,j} = H^i(\pi, H^j(S(\xi^{\times p}); \mathbf{F}_q))$ with the action of π on $H^*(S(\xi^{\times p}); \mathbf{F}_q)$ by permutations of the cohomological Chern classes.

When $q = p$, $E_2^{0,j} = H^j(S(\xi^{\times p}); \mathbf{F}_p)^\pi$ and $E_2^{i,0} = H^i(B\pi; \mathbf{F}_p)$.

Then note that $H^*(S(\xi^{\times p}); \mathbf{F}_q) = \mathbf{F}_q[x_1, \dots, x_p]/(x_1 \dots x_p)$ in positive dimensions is a permutation representation of π acting on monomials which have degree zero in at least one indeterminate. Since all the monomials that are fixed under this action have been factored out after taking quotient by the ideal $(x_1 \dots x_p)$, this representation is a free $\mathbf{F}_q[\pi]$ -module, hence the cohomology of π with coefficients in it is trivial in positive dimensions, i.e.

$E_2^{i,j} = 0$ when $i, j > 0$. Thus the spectral sequence collapses and we have

$$H^*(S(\xi_\pi); \mathbf{F}_p) \approx H^*(B\pi; \mathbf{F}_p) \oplus \tilde{H}^*(S(\xi^{\times p}); \mathbf{F}_p)^\pi.$$

Also if $q \neq p$ we have $H^*(S(\xi_\pi); \mathbf{F}_q) \approx H^*(S(\xi^{\times p}); \mathbf{F}_q)^\pi$.

Let X_π be a stable summand of $BU(1)^p$ defined as follows. The action of π on $BU(1)^p$ induces an action of π on stable decomposition of $BU(1)^p$ which is the wedge of all smash products of length $1, \dots, p-1$, say Y_π , and a smash product of length p . Then choose X_π such that $NX_\pi = Y_\pi$, where $N = 1 + t + \dots + t^{p-1}$. By the stable equivalence

$$S(\xi^{\times p}) \rightarrow BU(1)^p \rightarrow Y_\pi$$

we can consider X_π as a stable summand of $S(\xi^{\times p})$. For any choice of X_π , consider the composition of stable maps

$$f_\pi : S(\xi_\pi) \rightarrow X_{h\pi}^p \xrightarrow{Tr} E\pi \times BU(1)^p \rightarrow BU(1)^p \rightarrow X_\pi.$$

We have to show that the stable map $\varphi_0 \vee f_\pi$ induces an isomorphism in cohomology for any group of coefficients \mathbf{F}_q , q a prime, and hence gives us above stable equivalence by the stable Whitehead lemma. It follows from the above arguments that

$$\tilde{H}^*(S(\xi_\pi); \mathbf{F}_p) = \varphi_0^* \tilde{H}^*(B\pi; \mathbf{F}_p) \oplus Tr_0^* \tilde{H}^*(S(\xi^{\times p}); \mathbf{F}_p),$$

and $\tilde{H}^*(S(\xi_\pi); \mathbf{F}_q) = Tr_0^* \tilde{H}^*(S(\xi^{\times p}); \mathbf{F}_q)$, when $q \neq p$. The restriction of Tr_0 on X_π induces monomorphism on $Im Tr_0^*$ since by the transfer property (iv), $\rho_0^* Tr_0^* = N^*$ and the restriction of N^* on $\tilde{H}^*(X_\pi; \mathbf{F}_q)$ is monomorphism. Hence $\varphi_0^* \vee (Tr_0|_{X_\pi})^*$ is isomorphism and so is $\varphi_0^* \vee f_\pi^*$ by the commutative diagram

$$\begin{array}{ccc}
S(\xi_\pi) & \longrightarrow & X_{h\pi}^p \\
\text{Tr}_0 \downarrow & & \downarrow \text{Tr} \\
S(\xi^{\times p}) & \longrightarrow & E\pi \times BU(1)^p
\end{array}$$

This proves Lemma 5.2.

Now we are ready to prove Theorem 5.1. The statement b) of the Theorem 5.1 follows from the fact that the right homomorphism in the short exact sequence from Lemma 5.2 is multiplication by the Euler class $c_p(\xi_\pi)$. Then taking into account Lemma 5.2 and the fact that the composition $S(\xi_\pi) \rightarrow X_{h\pi}^p \xrightarrow{\varphi} B\pi$ coincides with φ_0 , we have that any element $y \in MU^*(X_{h\pi}^p)$ has the form

$$y = \varphi^*(u) + \text{Tr}^*(v) + y_1 c_p,$$

where $u \in \tilde{M}U^*(B\pi)$, $y_1 \in MU^*(X_{h\pi}^p)$ and $v \in MU^*(BU(1)^p)$ is from the direct summand $MU^*[[x_1, \dots, x_p]]/(x_1 \dots x_p)$.

Then suppose $\gamma y = 0$. We know that $\rho^*(\theta) = C$, hence $\rho^*(\gamma) = 0$ and $c\text{Tr}^*(v) = \text{Tr}^*(\rho^*(\gamma)v) = 0$ by the transfer property (i). So we have that $\gamma\varphi^*(u) + \gamma y_1 c_p = 0$. This implies that both summand are zero: i) $\gamma\varphi^*(u) = 0$ is the consequence of the transfer relation $\gamma\text{Tr}^*(1) = 0$ and ii) $\gamma y_1 c_p = 0$ which implies $\gamma y_1 = 0$ by the statement b). We have to prove i). Applying i^* we have $0 = i^*(\gamma\varphi^*(u)) + \gamma y_1 c_p = zu$ in $MU^*(B\pi) = MU^*[[z]]/([p])z$ ($i^*(c_p) = 0$ since $i^*(\xi_\pi)$ has a section as above, $i^*\varphi^*$ is the identity and $\gamma = \varphi^*(z)$). But any relation in $MU^*(B\pi)$ is the consequence of the relation $z\text{Tr}_{z/p}^*(1) = 0$ hence is $zu = 0$. Then applying φ^* we have by naturality of transfer $\gamma\varphi^*(u) = 0$ is the consequence of $\gamma\text{Tr}_\pi^*(1) = 0$. Since $\dim(y_1) = \dim(y) - p$ repetition of this argument gives us the statement a).

For the proof of c) let $y \in \text{Ker}\rho^*$. Since $\varphi\rho = *$ we have

$$0 = \rho^*(\text{Tr}^*(v)) + \rho^*(y_1 c_p).$$

If $\text{Tr}^*(v) \neq 0$ then it restricts non-trivially to $MU^*(S(\xi_\pi))$ by the definition of v . However $y_1 c_p$ restricts to zero by exactness; thus $\text{Tr}^*(v) = 0$ and so both summands are zero. Further $\rho^*(c_p) = x_1 \dots x_p$ hence $\rho^*(y_1) = 0$. So

$$y = \varphi^*(u) + y_1 c_p = \varphi^*(u) + (\varphi^*(u_1) + y_2 c_p)c_p = \varphi^*(u) + \varphi^*(u_1)c_p + y_2 c_p^2.$$

Repetition of this process proves c).

The relations d) follow from the bundle relation

$$\xi_\pi \otimes_C \theta = (\xi \otimes_C \rho^*(\theta))_\pi = \xi_\pi$$

which in turn follows from transfer property (i). So we have to prove that Chern classes c_1, c_2 with these relations are a complete system of generators and relations.

The fact that c, c_1, c_2 multiplicatively generate $MU^*B(\pi \wr U(1))$ follows from the above stable splitting of $B(\pi \wr U(1))$. Let us use the splitting principle and formal group law in complex cobordism and write formally

$$\xi_\pi = \eta_1 + \eta_2; u_1 = c_1(\eta_1); u_2 = c_1(\eta_2) \text{ and let}$$

$$F(x, y) = \sum \alpha_{ij} x^i y^j$$

be the formal group. Then the above bundle relation after applying Whitney formula for the first and second Chern classes gives two relations of the form:

$$F(u_1, c) + F(u_2, c) = c_1$$

and

$$(18) \quad F(u_1, c)F(u_2, c) = c_2;$$

or in terms of $c, c_1 = u_1 + u_2, c_2 = u_1 u_2$

$$(19) \quad c(2 + \sum \beta_{ijk} c^i c_1^j c_2^k) = 0$$

and

$$(20) \quad c(c_1 + \sum \gamma_{ijk} c^i c_1^j c_2^k) = 0,$$

for some coefficients $\beta_{ijk}, \gamma_{ijk} \in MU^*(pt)$.

We claim that relations (19) and (20) are equivalent to the following two obvious transfer relations for $Tr^* : MU^*[[x, tx]] \rightarrow MU^*(B(\pi \wr U(1)))$

$$cTr^*(1) = 0$$

and

$$cTr^*(x) = 0.$$

For the proof note that if y is annihilated by c and $\rho^*(y) = b + tb$ then $u = Tr^*(b)$. This follows from the following arguments: In $MU^*(BU(1)^2)$ we have $Ker(1 + t) = Im(1 - t)$; also by property (v) of the transfer $Tr^*(a) = Tr^*(ta)$. So if z is annihilated by c (hence $y = Tr^*(a)$ for some a by the first statement of the Theorem 5.1) and $\rho^*(y) = b + tb$ (hence $b + tb = a + ta$ by the transfer property (iv)) then $y = Tr^*(b)$ (that is $Tr^*(b) = Tr^*(a)$).

Rewrite relations (19) and (20) as follows:

$$2c + \alpha_{11}cc_1 + \sum_{k \geq 2} \alpha_{k1}(u_1^k + u_2^k)c + o(c^2) = 0;$$

$$cc_1 + 2\alpha_{11}cc_2 + \sum_{k \geq 2} \alpha_{k1}(u_1^{k-1} + u_2^{k-1})cc_2 + o(c^2) = 0.$$

By the above remark this means that

$$Tr^*(1) + \alpha_{11}Tr^*(x) + \sum_{k \geq 2} \alpha_{k1}Tr^*(x^k) = (F(u_1, c) + F(u_2, c) - c_1)/c;$$

$$Tr^*(x) + \alpha_{11}Tr^*(1)c_2 + \sum_{k \geq 2} \alpha_{k1}Tr^*(x^{k-1})c_2 = (F(u_1, c)F(u_2, c) - c_2)/c.$$

Now since

$$x^k = x^{k-1}(x + tx) - u^{k-2}(ctx),$$

transfer property (i) and computation of $Tr^*(x)$ is sufficient for the computation of $Tr^*(x^k)$, $k \geq 2$ (see Corollary 6.2). So we have

$$(21) \quad Tr^*(1)(1 + g_0) + Tr^*(x)h_0 = (F(u_1, c) + F(u_2, c) - c_1)/c,$$

and

$$(22) \quad Tr^*(1)g_1 + Tr^*(x)(1 + h_1) = (F(u_1, c)F(u_2, c) - c_2)/c,$$

where $g_0, h_0, g_1, h_1 \in MU^*(B\pi \wr U(1))$. This proves d).

This completes the proof of Theorem 5.1.

6. TRANSFERRED CHERN CLASSES

1. $p = 2$

Formula (22) for computing $Tr^*(x)$ is complicated; let us give a simpler form. Consider again (20). Note that the coefficient $\gamma_{00k} \in MU^*(pt)$ contains a factor 2: the element

$$c_1 + \sum \gamma_{ijk}c^i c_1^j c_2^k$$

as the one annihilating c , belongs to $ImTr^*$. On the other hand

$$\rho^*Tr^* = 1 + t; \rho^*(c) = 0; \rho^*(c_1) = x + tx; \rho^*(c_2) = ctx,$$

hence applying ρ^* we have that

$$x + tx + \sum \gamma_{0jk}(x + tx)^j (ctx)^k$$

belongs to $Im(1 + t)$. So $\gamma_{00k}(ctx)^k = 2\gamma_k(ctx)^k$, that is, $\gamma_{00k} = 2\gamma_k$ for some coefficient γ_k .

Recall that on the other hand $F(c, c) = 0$ that is $2c = o(c^2)$. So $\gamma_{00k}c = o(c^2)$, hence taking into account the relation $F(c, c) = 0$ we can rewrite (20) after division by

$$1 + \sum_{i,k \geq 0} \gamma_{i1k} c^i c_2^k$$

(the coefficient at cc_1) as follows

$$(23) \quad cc_1 = d_0c + d_2cc_1^2 + \dots + d_ncc_1^n + \dots,$$

where $d_k = d_k(c, c_2) \in MU^*[[c, c_2]]$ and $d_0(0, c_2) = 0$; the lower index n indicates the coefficient at cc_1^n . Since

$$\rho^*(Tr^*(x) - c_1) = 0,$$

it follows from Theorem 5.1c) that there exist some elements

$$\delta_j \in \tilde{MU}^*(B\pi)$$

such that we have

$$Tr^*(x) = c_1 + \sum_{j \geq 0} \varphi^*(\delta_j) c_2^j;$$

Since for the inclusion $i : B\pi \rightarrow B(\pi \wr U(1))$ we have that

$$i^*(c_1) = i_0^*(c); i^*Tr^*(x) = 0; i^*(c_2) = 0,$$

we obtain

$$\varphi^*(\delta_0) = -c.$$

For the calculation of other elements δ_j recall that $cTr^*(x) = 0$, hence

$$(24) \quad \begin{aligned} cc_1^n &= -c\delta^n; n \geq 1, \\ \delta &= -c + \sum_{j \geq 1} \delta_j c_2^j. \end{aligned}$$

Substituting (24) in (23) in place of cc_1^n , elements δ_j , $j > 0$ can be determined from the recurrence relations which arise from the following formula in $MU^*(B\pi)[[c_2]]$

$$\delta = d_0 + \sum_{i \geq 2} d_i \delta^i.$$

Here we act as in the proof of the above claim: By the definition the element $\delta - d_0 - \sum_{i \geq 2} d_i \delta^i$ belongs to $Ker \rho^*$. On the other hand this element is from the annihilator of c hence

$$\delta - d_0 - \sum_{i \geq 2} d_i \delta^i \in Im Tr^* \cap Ker(\rho^*) = 0.$$

So we have

Proposition 6.1. *For the constructed elements $\delta_j \in \tilde{M}U^*(BZ/2)$ the following formula holds in $MU^*B(Z/2 \wr U(1))$*

$$Tr^*(x) = c_1 - c + \sum_{j \geq 1} \varphi^*(\delta_j) c_2^j.$$

Note that Proposition 6.1 is sufficient for the calculation of $Tr^*(x^n)$, $n \geq 2$. This follows from the following simple formula

$$x^n = U_{(n)}x + U_{[n]},$$

where $U_{(n)} = U_{(n)}(x + tx, txt)$, $U_{[n]}(x + tx, txt)$ are formal series in $x + tx$ and txt :

$$U_{(1)} = 1; U_{[n]} = 0; U_{(n+1)} = (x + tx)U_{(n)} + U_{[n]}; U_{[n+1]} = -txtU_{(n)}.$$

Recall that $\rho^*(c_1) = x + tx$ and $\rho^*(c_2) = txt$ and consider formally $U_{(n)}$ and $U_{[n]}$ as a formal series in c_1 and c_2 . Then transfer property (i) gives the following recurrence formula for $Tr^*(x^n)$.

Corollary 6.2. *In $MU^*B(Z/2 \wr BU(1))$,*

$$Tr^*(x^n) = U_{(n)}(c_1, c_2)Tr^*(x) + U_{[n]}(c_1, c_2)Tr^*(1).$$

2. The case of odd prime p

Now let us elucidate the meaning of the relations

$$\xi_\pi \otimes_C \theta = \xi_\pi$$

in the general case of $B(\pi \wr U(1))$.

Again, we can use the splitting principle and write formally

$$\xi_\pi = \eta_1 + \eta_2 + \dots + \eta_p, \quad u_m = c_1(\eta_m), \quad m = 1, \dots, p.$$

Applying the Whitney formula for the relation

$$\eta_1 \otimes_C \theta + \dots + \eta_p \otimes_C \theta = \eta_1 + \dots + \eta_p,$$

and taking into account that $c_m = c_m(\xi_\pi)$ is the elementary symmetric function $\sigma_m(u_1, \dots, u_p)$ we have

$$(25) \quad \sigma_m(F(u_1, \gamma), \dots, F(u_p, \gamma)) = c_m,$$

$m = 1, \dots, p$, or in terms γ, c_1, \dots, c_p we have

$$\gamma(p + \sum \beta_{i_0, i_1, \dots, i_p}^0 \gamma^{i_0} c_1^{i_1} \dots c_p^{i_p}) = 0;$$

and

$$(26) \quad \gamma((p - k)c_k + \sum \beta_{i_0, i_1, \dots, i_p}^k \gamma^{i_0} c_1^{i_1} \dots c_p^{i_p}) = 0;$$

for $k = 1, \dots, p - 1$ and some $\beta_{i_0, i_1, \dots, i_p}^0, \beta_{i_0, i_1, \dots, i_p}^k \in MU^*(pt)$.

We claim that these relations are equivalent to the following obvious relations

$$\gamma Tr^*(1) = 0;$$

and

$$\gamma Tr^*(w_k) = 0,$$

for the elements $w_k \in MU^*(BU(1))^p$, $k = 1, \dots, p-1$ defined as follows (see Definition 4.9). The k -th Chern class of the bundle $\xi^{\times p}$ is the elementary symmetric function $\sigma_k(x_1, \dots, x_p)$ in Chern classes x_i and is the sum of $p(p-1)\dots(p-k+1)/k!$ number of elementary monomials. The action of π on the set of these monomials gives us $(p-1)\dots(p-k+1)/k!$ orbits and the transfer homomorphism on the monomials from the same orbit are equal by transfer property (v). Let $w_k = w_k(u_1, \dots, u_p)$ be the sum of representative monomials of these orbits. The value of $Tr^*(w_k)$ does not depend on the choice of w_k since w_k is defined modulo $Im(1-t^*)$ and on the elements of $Im(1-t^*)$ transfer homomorphism is zero again by (v). In other words we can take any w_k for which $Nw_k = \sigma_k(u_1, \dots, u_p)$ holds.

For the proof of our claim multiply the k -th relation from (26) by p_k , the inverse of $p-k$ in \mathbf{F}_p . Then by Theorem 5.1, *Annihilator(c)* coincides with $ImTr^*$ hence (25) implies that

$$p_k(\sigma_{k+1}(F(u_1, \gamma), \dots, F(u_p, \gamma)) - c_{k+1})/\gamma = Tr^*(a_k),$$

for some a_k which we have to find. Let us write

$$\begin{aligned} & \rho^*(p_k(\sigma_{k+1}(F(u_1, \gamma), \dots, F(u_p, \gamma)) - c_{k+1})/\gamma) = g^{(k)}(\sigma_1, \dots, \sigma_p) \\ & = \sigma_k(1 + g_k^{(k)}(\sigma_1, \dots, \sigma_p)) + \sum_{j \neq k, 1 \leq j \leq p} \sigma_j g_j^{(k)}(\sigma_j, \sigma_{j+1}, \dots, \check{\sigma}_k, \dots, \sigma_p) \\ & = N(w_k)(1 + g_k^{(k)}(\sigma_1, \dots, \sigma_p)) + \sum_{j \neq k, 1 \leq j \leq p} N(w_j) g_j^{(k)}(\sigma_j, \sigma_{j+1}, \dots, \check{\sigma}_k, \dots, \sigma_p). \end{aligned}$$

Here the symbol $\check{\sigma}_k$ indicates absence of the corresponding term. So we have

$$\begin{aligned} & p_k(\sigma_{k+1}(F(u_1, \gamma), \dots, F(u_p, \gamma)) - c_{k+1})/\gamma \\ & = Tr^*(w_k)(1 + g_k^{(k)}(c_1, \dots, c_p)) + \sum_{j \neq k, 1 \leq j \leq p} Tr^*(w_j) g_j^{(k)}(c_j, c_{j+1}, \dots, \check{c}_k, \dots, c_p), \end{aligned}$$

and

$$\begin{aligned} & (\sigma_1(F(u_1, \gamma), \dots, F(u_p, \gamma)) - c_{k+1})/\gamma \\ & = Tr^*(1)(1 + g_0^{(0)}(c_1, \dots, c_p)) + \sum_{1 \leq j \leq p} Tr^*(w_j) g_j^{(0)}(c_j, c_{j+1}, \dots, c_p). \end{aligned}$$

This proves our claim.

Now let us give a formula analogous to that of Proposition 6.1. It follows from the definition of w_k that $\rho^*(c_k - Tr^*(w_k)) = 0$. So taking into account Theorem 5.1 the difference

$$c_k - Tr^*(w_k)$$

has the form $\sum_{i \geq 0} \varphi^*(\delta_i^{(k)})c_p^i$, for some elements $\delta_i^{(k)} \in \tilde{M}U^*(B\pi)$. Using this notation we have

Proposition 6.3. *We can construct explicit elements*

$$\delta_i^{(k)} \in \tilde{M}U^*(B\pi), \quad k = 1, \dots, p-1,$$

such that the value of the transfer of the covering $\rho : BU(1)^p \rightarrow X_{h\pi}^p$ is given by

$$Tr^*(w_k) = c_k + \sum_{i \geq 0} \varphi^*(\delta_i^{(k)})c_p^i,$$

Proof: As mentioned above, existence of elements $\delta_i^{(k)}$ follows from Theorem 5.1. We start with the equations (26) and rewrite as

$$(27) \quad \gamma f_k(\gamma, c_1, \dots, c_p) = 0, \quad k = 1, \dots, p-1,$$

These are equations in a formal series algebra $MU^*(B\pi)[[e]]$, with $e = c_p$, since we know $cc_k \in cMU^*(B\pi)[[c_p]]$.

We now want to find explicitly formal series

$$(28) \quad \delta^{(k)}(e) = \sum_{i \geq 0} \delta_i^{(k)}(\gamma)e^k$$

such that

$$Tr^*(w_k) = c_k + \delta^{(k)}(c_p)$$

and hence

$$(29) \quad \gamma c_k^j = -\gamma(\delta^{(k)}(c_p))^j, \quad j \geq 1$$

For this we want to replace the equations (27) by the equations

$$(30) \quad \gamma \tilde{f}_k(\gamma, \delta^{(1)}(c_p), \dots, \delta^{(p-1)}(c_p), c_p) = 0,$$

where $\tilde{f}_k \in Ker \rho_\pi^*$ and has a form

$$\tilde{f}_k = \delta^{(k)} + \gamma g^{(k)}(\gamma)(\gamma, \delta^{(1)}, \dots, \delta^{(p-1)}, c_p),$$

where $\gamma^{(k)}(\gamma)$ some series. In fact $\tilde{f}_k = 0$ since we know that $Ann(\gamma) = Im Tr^*$ and $Ker(\rho^*) \cap Im Tr^* = 0$.

Then equating each coefficient of the resulting series

$$(31) \quad \tilde{f}_k(c_p) = \tilde{f}_k(\gamma, \delta^{(1)}(c_p), \dots, \delta^{(p-1)}(c_p), c_p) = 0$$

in the ring $MU^*(B\pi)[[c_p]]$ to zero we will obtain $p - 1$ infinite strings of equations in $MU^*(B\pi)$, relating $\delta_i^{(k)}$.

We proceed as follows: Let us look at the term $cf_k(0, 0, \dots, 0, c_p)$ in the equations (27). Note that $f_k(0, 0, \dots, 0, c_p)$ is divisible by p :

$$f_k \in \text{Ann}(\gamma) = \text{Im}Tr^* \Rightarrow \rho_\pi^* f_k \in \text{Im}N \Rightarrow f_k(0, \dots, 0, \sigma_p) \in \text{Im}N \Rightarrow f_k(0, \dots, 0, \sigma_p) \text{ is divisible by } p.$$

Next using the relation $[p]_F(\gamma) = 0$ we know that $p\gamma$ is divisible by γ^2 ; hence each occurrence of $p\gamma$ in these equations can be replaced by terms with higher powers of γ . So $\gamma f_k(0, 0, \dots, 0, c_p)$ can be replaced by the term divisible by γ^2 .

Also the k -th relation from (27) contains the term $\gamma(p - k)c_k$, and for the condition (31) we have to multiply k th equation from (27) by p_k , the inverse of $p - k$ in F_p , so that $p_k(p - k) \equiv 1 \pmod{p}$ and as above we can replace $\gamma(p - k)c_k$ by $\gamma c_k + (\text{terms divisible by } \gamma^2)$.

Then we use (29) and substitute the series $\delta_i^{(k)}$ in the resulting equations. Thus obtain (30) and hence we obtain mentioned string of equations relating $\delta_i^{(k)}$:

$$(32) \quad \delta_n^{(k)}(1 + \gamma f^{(k,n)}(\gamma)) + \gamma g^{(k,n)}(\gamma)(\gamma, (\delta_i^{(1)})_{i \leq n}, \dots, (\delta_i^{(k)})_{i < n}, \dots, (\delta_i^{(p-1)})_{i \leq n}) = 0$$

where $k = 1, \dots, p - 1$, $i \geq 0$, $f^{(k,n)}$, $g^{(k,n)}$ are some series and $\delta_0^{(k)}$ are already known as k -th Chern class of the bundle $1 + \theta + \dots + \theta^{p-1}$.

Then we have to solve (32) formally. Namely Choose an k , express using equations (32) $\delta_n^{(j)}$ for $j \neq k$ by $\delta_n^{(k)}$ and $\delta_i^{(l)}$ with $l = 1, \dots, p - 1$, $i < n$. Then substitute this into k -th equation. One obtains

$$(33) \quad \delta_n^{(k)} = \psi(\delta_n^{(k)}, (\delta_i^{(l)})_{i < n})$$

Assuming $\delta_i^{(l)}$ are already found for $i < n$. This gives

$$\delta_n^{(k)} = \tilde{\psi}(\delta_n^{(k)}),$$

where $\tilde{\psi}$ is a series with $\tilde{\psi}(0) = 0$. Solve formally $\tilde{\psi}(\delta(z)) = \delta(z)$ in formal series ring $MU^*[[z]]$, then descend to $MU^*(B\pi)$.

This gives $\delta_n^{(k)} = \delta_n^{(k)}(z) \in MU^*(B\pi)$ obviously satisfying our equations.

Remark 6.4. Note that computation for $Tr^*(w_k)$, $k = 1, \dots, p - 1$ and transfer property (i) are sufficient for the computation of $Tr^*(x^i)$, $i \geq 1$. This follows from the formulas

$$Tr^*(x) = Tr^*(w_1);$$

$$Tr^*(x^k) = \sum_{1 \leq i \leq k-1} (-1)^{i-1} Tr^*(x^{k-i})c_i + kTr^*(w_k), 2 \leq k \leq p-1;$$

$$Tr^*(x^{p+m}) = \sum_{1 \leq i \leq p} (-1)^{i-1} Tr^*(x^{p+m-i})c_i, m \geq 0.$$

Moreover, Proposition 6.3 enables us to compute the transfer homomorphism for those polynomials $a \in \tilde{M}U^*[[x_1, \dots, x_p]]$ for which $Na = a + ta + \dots + t^{p-1}a$ is symmetric in x_1, \dots, x_p . Namely if $Na = \sigma_1 a_1(\sigma_1, \dots, \sigma_p) + \dots + \sigma_{p-1} a_{p-1}(\sigma_1, \dots, \sigma_p)$, then

$$Tr^*(a) = Tr^*(w_1)a_1(c_1, \dots, c_p) + \dots + Tr^*(w_{p-1})a_{p-1}(c_1, \dots, c_p).$$

Indeed, for $\hat{a} = w_1 a_1(\sigma_1, \dots, \sigma_p) + \dots + w_{p-1} a_{p-1}(\sigma_1, \dots, \sigma_p)$, $N(a - \hat{a}) = 0$, that is, $a - \hat{a} \in Im(1 - t)$, hence $Tr^*(a) = Tr^*(\hat{a})$.

7. $BP^*(X_{h\Sigma_p}^p)$

In this section we consider the Brown-Peterson cohomology of $X_{h\Sigma_p}^p$ for $X = \mathbf{C}P^\infty$. We use notation analogous to that of the previous section, with the cyclic group replaced by the symmetric group and the bundle ξ_π over $X_{h\pi}^p$ replaced by the bundle $\xi_{\Sigma_p} = E\Sigma_p \times_{\Sigma_p} \xi^{\times p}$ defined in (7) over $X_{h\Sigma_p}^p$ which we identify with $B(\Sigma_p \wr U(1))$. Of course all Chern classes are considered in BP cohomology, in particular, there are classes \tilde{c}_i defined for BP as in Definition 4.2. In this notation one has the analog of Theorem 4.5.

Proposition 7.1. *As a $BP^*(B\Sigma_p)$ algebra*

$$BP^*(X_{h\Sigma_p}^p) = BP^*(B\Sigma_p)[[\tilde{c}_1, \dots, \tilde{c}_{p-1}, c_p]]/(\tilde{c}_i \tilde{B}P^*(B\Sigma_p)).$$

We follow the proof of Theorem 5.1. Consider the long exact sequence for the pair $(D(\xi_{\Sigma_p}), S(\xi_{\Sigma_p}))$

$$\dots \leftarrow BP^*(S(\xi_{\Sigma_p})) \leftarrow BP^*(X_{h\Sigma_p}^p) \leftarrow BP^*((X_{h\Sigma_p}^p)^{\xi_{\Sigma_p}}) \leftarrow \dots$$

Here $S(\xi_{\Sigma_p}) = E\Sigma_p \times_{\Sigma_p} S(\xi^{\times p})$ is the sphere bundle. $X_{h\Sigma_p}^p$ is homotopy equivalent to the disk bundle $D(\xi_{\Sigma_p}) = E\Sigma_p \times_{\Sigma_p} D(\xi^{\times p})$.

Then we have the obvious inclusion $i_0 : B\Sigma_p \rightarrow S(\xi_{\Sigma_p})$ and projection $\varphi_0 : S(\xi_{\Sigma_p}) \rightarrow B\Sigma_p$ with fiber $S(\xi^{\times p})$. $\varphi_0 i_0$ is the identity. Thus stably $B\Sigma_p$ is a wedge summand of $S(\xi_{\Sigma_p})$. As for the other summand let

$$X_{\Sigma_p} = \bigvee_{i=1}^{p-1} E\Sigma_i \times_{\Sigma_i} BU(1)^{\wedge i}.$$

By the standard transfer argument, localized at p , X_{Σ_p} is a stable summand of $\bigvee_{i=1}^{p-1} E\Sigma_i \times BU(1)^{\wedge i}$ and hence of $E\Sigma_p \times BU(1)^{\times p}$. From this we derive the following result.

Lemma 7.2. *One has the following short exact sequence*

$$0 \leftarrow BP^*(S(\xi_{\Sigma_p})) \leftarrow BP^*(X_{h\Sigma_p}^p) \leftarrow BP^*((X_{h\Sigma_p}^p)^{\xi_{\Sigma_p}}) \leftarrow 0$$

and a stable equivalence localized at p

$$\varphi_0 \vee f_{\Sigma_p} : S(\xi_{\Sigma_p}) \rightarrow B\Sigma_p \vee X_{\Sigma_p},$$

with f_{Σ_p} , the composition of stable maps

$$f_{\Sigma_p} : S(\xi_{\Sigma_p}) \rightarrow X_{h\Sigma_p}^p \xrightarrow{Tr_{\Sigma_p}} E\Sigma_p \times BU(1)^p \rightarrow X_{\Sigma_p}$$

Proof: All ordinary cohomology groups will be taken with simple \mathbf{F}_p coefficients. It follows from the cohomological version of Theorem 4.5 that

$$H^*(S(\xi_{\Sigma_p})) = H^*(X_{h\Sigma_p}^p)/(c_p) = H^*(B\Sigma_p)[\tilde{c}_1, \dots, \tilde{c}_{p-1}]/\tilde{c}_i \tilde{H}^*(B\Sigma_p).$$

The inclusion i_0 splits off $\varphi_0^* H^*(B\Sigma_p)$ in $H^*(S(\xi_{\Sigma_p}))$. Furthermore

$$H^*(S(\xi^{\times p})) = \mathbf{F}_p[x_1, \dots, x_p]/(\sigma_p),$$

hence

$$H^*(S(\xi^{\times p}))^{\Sigma_p} = \mathbf{F}_p[\tilde{c}_1, \dots, \tilde{c}_{p-1}].$$

Therefore there is an isomorphism

$$H^*(S(\xi_{\Sigma_p})) \xrightarrow{\rho^* \oplus i_0^*} H^*(S(\xi^{\times p}))^{\Sigma_p} \oplus H^*(B\Sigma_p)$$

where $\rho : S(\xi^{\times p}) \rightarrow S(\xi_{\Sigma_p})$ is the projection. We have to prove that the first summand is $f_{\Sigma_p}^* \tilde{H}^*(X_{\Sigma_p})$.

By naturality of the transfer we have the commutative diagram

$$\begin{array}{ccccc} S(\xi^{\times p}) & \longrightarrow & BU(1)^p & \longrightarrow & X_{\Sigma_p} \\ \uparrow Tr_0 & & \uparrow Tr_{\Sigma_p} & & \\ S(\xi_{\Sigma_p}) & \longrightarrow & X_{h\Sigma_p}^p & & \end{array}$$

Thus f_{Σ_p} coincides with \tilde{f}_{Σ_p} , the map Tr_0 followed by the horizontal maps in the above diagram. We wish to show the restriction of Tr_0^* to the image of $H^*(X_{\Sigma_p})$ is an isomorphism onto $H^*(S(\xi^{\times p}))^{\Sigma_p}$.

Now considering the transfer for the Σ_i coverings

$$E\Sigma_i \times BU(1)^{\wedge i} \rightarrow E\Sigma_i \times_{\Sigma_i} BU(1)^{\wedge i}$$

it follows from transfer properties (ii) and (vi) that $H^*(E\Sigma_i \times_{\Sigma_i} BU(1)^{\wedge i})$ is a submodule of $H^*(E\Sigma_i \times BU(1)^{\wedge i})$ generated by Σ_i norms of monomials in x_1, x_2, \dots, x_i , with non-increasing degrees. From this it is straightforward that $H^*(X_{\Sigma_p})$ and $H^*(S(\xi^{\times p}))^{\Sigma_p}$ have the same ranks in each dimension. Thus we are reduced to showing the desired map is injective.

However, for any monomial x in x_1, x_2, \dots, x_i , we have

$$Tr_0^*(N_{\Sigma_i}(x)) = i!Tr_0^*(x)$$

by naturality of the transfer. Thus the restriction of Tr_0^* to the image of $H^*(X_{\Sigma_p})$ will be a monomorphism if Tr_0^* is non-zero on polynomials consisting of monomials with non-increasing degrees. This in turn will follow if the norm N_{Σ_p} is non-zero on such polynomials. In fact we claim: 1) N_{Σ_p} is non-zero on any monomial $x_1^{i_1} \dots x_p^{i_p}$ and 2) different monomials with non-increasing degrees in $x_1, \dots, x_i, i < p$ are in different Σ_p orbits.

Claim 2) is clear. To see 1) let $x = x_1 \dots x_p$, $I = (i_1, \dots, i_p)$ and $x^I = x_1^{i_1} \dots x_p^{i_p}$. Then we will show the coefficient of x^I in $N_{\Sigma_p}(x^I)$ is prime to p . Since not all terms of I are equal, The isotropy subgroup of x^I is the finite product $\Sigma_{n_1} \times \Sigma_{n_2} \times \dots \times \Sigma_p$ where n_i is the number of terms of I equaling i . Since not all terms of I are equal this group has order $n_1!n_2! \dots$ which is prime to p . Hence $N_{\Sigma_p}(x) = (n_1!n_2! \dots)x^I +$ other monomials proving the claim.

Thus $\varphi_0 \vee f_{\Sigma_p}$ induces an isomorphism and hence is a p -local stable equivalence. So the left arrow in Lemma 7.2 is an epimorphism and hence the right arrow is a monomorphism. This proves the Lemma.

For the proof of the Proposition 7.1 we follow from that of Theorem 5.1 taking into account additionally that by Lemma 7.2 and a BP version of Remark 4.10, any element $y \in BP^*(B)$ has the form

$$y = \varphi(u) + g(\tilde{c}_1, \dots, \tilde{c}_{p-1}) + y_1 c_p$$

for some $u \in BP^*(B\Sigma_p)$, g denotes some formal power series and $y_1 \in BP^*(X_{\Sigma_p})$. \square

Before formulating the analog of the Proposition 4.14 for BP theory we recall

$$BP^*(B\pi) = BP^*[[z]]/([p]z)$$

with $|z| = 2$. The corresponding computation for $BP^*(B\Sigma_p)$ is also known [15]. For the reader's convenience we rederive the result in a form useful for our purposes.

Lemma 7.3. *As a BP^* algebra*

$$(i) \quad BP^*(B\Sigma_p) = BP^*[[y]]/(yTr_{\Sigma_p}^*(1))$$

where y and $Tr_{\Sigma_p}^*(1)$ are uniquely determined by $\rho_{\pi, \Sigma_p}^*(y) = z^{p-1}$ and

$$(ii) \quad \rho_{\pi, \Sigma_p}^*(Tr_{\Sigma_p}^*(1)) = (p-1)!Tr_{\Sigma_p}^*(1) = (p-1)![p](z)/z.$$

In particular $|y| = 2(p-1)$.

Proof: (ii): This follows directly from Section 2: the double coset formula (vi) and Quillen's result (1).

(i): The relation $yTr_{\Sigma_p}^*(1) = 0$ is a consequence of Fröbenius reciprocity. To see that it is the defining relation we recall that the cohomology of $B\Sigma_p$ with simple coefficients in $\mathbf{Z}_{(p)}$ is

$$H^*(B\Sigma_p; \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}[y]/(py)$$

where $|y| = 2(p-1)$. This follows easily from the mod- p cohomology and the Bockstein spectral sequence (see Lemma 3.6). Also $H^*(B\pi; \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}[z]/(pz)$ where $|z| = 2$. The map $\rho_{\pi, \Sigma_p} : B\pi \rightarrow B\Sigma_p$ yields $\rho_{\pi, \Sigma_p}^*(y) = z^{p-1}$. Now the AHS-spectral sequence for $BP^*(B\Sigma_p)$ is

$$E_2 = H^*(B\Sigma_p; BP^*) = BP^*[y]/(py) \implies BP^*(B\Sigma_p)$$

Since y is even dimensional, the sequence collapses $E_2 = E_\infty$. Thus $BP^*(B\Sigma_p)$ is generated by y as a BP^* algebra. By the standard transfer argument $\rho_{\pi, \Sigma_p}^* : BP^*(B\Sigma_p) \rightarrow BP^*(B\pi)$ is an injective map of BP^* algebras. Since $\rho_{\pi, \Sigma_p}^*(yTr_{\Sigma_p}^*(1)) = p!z^{p-1}$ plus terms of higher filtration, $yTr_{\Sigma_p}^*(1) = 0$ is the only relation. \square

This implies

Theorem 7.4. *As a BP^* algebra*

$$BP^*(X_{h\Sigma_p}^p) = BP^*[[c, \tilde{c}_1, \dots, \tilde{c}_{p-1}, c_p]]/(cTr_{\Sigma_p}^*(1), c\tilde{c}_i)$$

and one has the formula

$$\tilde{c}_k - k!(p-k)!c_k = \sum_{i \geq 0} \varphi^*(\delta_{ik}^{\Sigma_p})c_p^i, \quad k = 1, \dots, p-1.$$

where the elements $\delta_{ik}^\Sigma \in \tilde{BP}^*(B\Sigma_p)$ are determined by

$$\rho_{\pi, \Sigma_p}^*(\delta_{jk}^{\Sigma_p}) = k!(p-k)!\delta_{jk}^\pi, \quad j \geq 0.$$

8. p -GROUPS WITH CYCLIC SUBGROUP OF INDEX p

In this section we consider the problem of computing the stable Euler class of the universal covering associated to the class of p -groups with a (necessarily normal) cyclic subgroup of index p . It is known [3] that every p -group of this form is isomorphic to one of the groups:

- (a) Z/q ($q = p^n, n \geq 1$).
- (b) $Z/q \times Z/p$ ($q = p^n, n \geq 1$).
- (c) $Z/q \rtimes Z/p$ ($q = p^n, n \geq 2$), where the canonical generator of Z/p acts in Z/q as multiplication by $1 + p^{n-1}$. This group is called the modular group if $p \geq 3$ and the quasi-dihedral group if $p = 2, n \geq 4$.

For $p = 2$ there are three additional families.

- (d) Dihedral 2-groups $D_{2m} = Z/m \rtimes Z/2$, ($m \geq 2$), where the generator of $Z/2$ acts in Z/m as multiplication by -1 . If $m = 2^n$, D_{2m} is a 2-group. Note that D_4 belongs to (b) and D_8 belongs to (c).

- (e) Generalized quaternion 2-groups. Let H be the algebra of quaternions $R \oplus Ri \oplus Rj \oplus Rk$. For $m \geq 2$ the generalized quaternion group Q_{4m} is defined as subgroup of multiplicative group H^* generated by $x = e^{\pi i/m}$ and $y = j$. $Z/2m$ generated by x is normal and has index 2. If m is a power of 2, Q_{4m} is a 2-group. In the extension $0 \rightarrow Z/2m \rightarrow Q_{4m} \rightarrow Z/2 \rightarrow 0$ the generator of $Z/2$ acts in $Z/2m$ as -1 . In particular Q_8 is the group of quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$.

- (f) Semi-dihedral groups. $Z/q \rtimes Z/2$ ($q = p^n, n \geq 3$), where the generator of $Z/2$ acts in Z/q as multiplication by $-1 + 2^{n-1}$.

Consider now the task of computing the stable Euler class, that is, $Tr_G^*(1)$ for the universal G -covering $EG \rightarrow BG$.

For the case (a) there is the well known formula of Quillen (1)

$$Tr_{Z/q}^*(1) = [q]_F(x)/x$$

in $MU^*(BZ/q) = MU^*[[q]]/([q]_F(x))$.

For the case (b) the answer follows from the transfer property (ii): $Tr_G = Tr_{Z/q} \wedge Tr_{Z/p}$;

In cases (d), (e) and (f) Tr_G^* is the composition of two transfers $Tr_{Z/q}^*$ and $Tr_{C,G}^* : MU^*(BC) \rightarrow MU^*(BG)$, where C is the corresponding cyclic subgroup. So we have to compute $Tr_{C,G}^*(x^i)$, $i \geq 1$ and we can apply our results for $BZ/2 \wr U(1)$, namely Corollary 6.2.

Similarly for the case (c) Tr_G is the composition $Tr_{Z/q,G} Tr_{Z/q}$ and we can apply Remark 6.4.

Our task is trivial for wreath products $Z/p \wr Z/p^n$ since $Tr_{Z/p^n}^*(1)$ is symmetric in x_1, \dots, x_p in the ring

$$MU^*((BZ/p^n)^p) = MU^*[[x_1, \dots, x_p]]/([p^n](x_1), \dots, [p^n](x_p))$$

and hence invariant under Z/p action. So in this case we need only Quillen's formula.

9. OTHER EXAMPLES

Consider the semi-direct products $G = (Z/p)^s \rtimes Z/p$ where the generator α of Z/p acts on $H_s = Z/p[T]/(T^s)$ by $1 - \alpha = T$, $1 \leq s \leq p$. Then every $Z/p[Z/p]$ -module is a direct sum of the modules H_s and these semi-direct products are good in the sense of Hopkins-Kuhn-Ravenel [11], [12].

Recall that in case of the Morava K -theory $K(n)^*$ at the prime p we have the formula for p -series: $[p]_{F_{K(n)}}(x) = v_n x^{p^n}$. Then

$$K(n)^*(B(Z/p)^s) = K(n)^*[[x_1, \dots, x_s]]/(x_i^{p^n}),$$

where x_i is the Euler class of the complex line bundle θ_i . Then Z/p acts on $Z/p[[x_1, \dots, x_s]]/(x_i^{p^n})$ by

$$\alpha : x_i \rightarrow F_{K(n)}(x_i, x_{i+1}), \quad x_{s+1} := 0,$$

where $F_{K(n)}$ denotes the formal group law for Morava K -theory.

Our aim is to show that stable Euler classes are computable in terms of characteristic classes and the formal group law.

Transfer $Tr^* : K(n)^*EG \rightarrow K(n)^*BG$ is the composition of two transfers

$$Tr_1^* : K(n)^*E((Z/p)^s) \rightarrow K(n)^*B((Z/p)^s)$$

and

$$Tr_2^* : K(n)^*B((Z/p)^s) \rightarrow K(n)^*BG.$$

Recall also that

$$Tr_1^*(1) = x_1^{p^n-1} \dots x_s^{p^n-1}.$$

It is easy to see that in $K(n)^*((BZ/p)^s)$ we have

$$e^{p^n-1}(\alpha^{i_1}(\theta_1)) \dots e^{p^n-1}(\alpha^{i_s}(\theta_1)) = x_1^{p^n-1} \dots x_s^{p^n-1}$$

where $1 \leq i_1 < \dots < i_s \leq p$. Then recall elements w_s from Proposition 6.3 and let w_s^l be the sum of the same monomials after raising to a power l . Since w_s^l consist of $(p-1)\dots(p-s+1)/s!$ summands and $(p-1)\dots(p-s+1)/s! \equiv (-1)^s/s \pmod{p}$, we have that in $K(n)^*((BZ/p)^s)$

$$\eta_\pi^*(w_s^{p^n-1}) = (-1)^s x_1^{p^n-1} \dots x_s^{p^n-1}/s,$$

where the map η_π , defined in Section 2, sends $\xi_i = t^{i-1}\xi_1$ to $\alpha^{i-1}\theta_1$. Hence

$$Tr_G^*(1) = Tr_2^*(Tr_1^*(1)) = Tr_2^*((-1)^s s \eta_\pi^*(w_s^{p^n-1})) =$$

$$(-1)^s sTr_2^*(\eta_\pi^*(w_s^{p^n-1})),$$

and we have to apply Remark 6.4

10. SOME CALCULATIONS

We will now compute some of the elements δ_{ik}^π and δ_{ik}^Σ from the Proposition 4.14.

Computations rely on the key Lemma 4.12 of D. Ravenel and some Maple programs written by M. Jibladze. We are sincerely grateful to them for their help.

Of course we can descend from the elements $\delta_i \in MU^*(B\pi)$ we constructed in the Propositions 6.1 and 6.3 to BP^* , $K(s)^*$ as well to any complex oriented cohomology theory.

First recall from [8], [14] that generators for

$$\begin{array}{ccc} \pi_*BP & \subset & H_*BP \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(p)}[v_1, v_2, \dots] & \subset & \mathbf{Z}_{(p)}[m_1, m_2, \dots] \\ |v_n| = 2(p^n - 1) = |m_n| \end{array}$$

are given by

$$v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i}.$$

Given formal group law over a graded ring R_* ,

$$F(x, y) = \alpha_{ij}^R x^i y^j \in R_*[[x, y]], \quad \alpha_{ij}^R \in R_{2(i+j-1)}$$

there is a ring map $g : MU_* \rightarrow R_*$ which induces the formal group; that is $g^*(\alpha_{ij}^{MU}) = \alpha_{ij}^R$.

We use also the following well known formulas

$$F(x, y) = \exp(\log x + \log y) \quad \text{and} \quad \log x = \sum_{n \geq 0} m_n x^{n+1}$$

for computing the elements δ_i in BP theory by the algorithm of the Section 6.

Example 1. For $\delta_1 \in BP^*(BZ/2)$ we have modulo z^{16} :

$$\begin{aligned} \delta_1 = & z^{15}v_1^5v_2v_3 + z^{15}v_3v_2^2v_1^2 + z^{13}v_1^3v_2v_3 + z^{16}v_1^6v_2v_3 + v_1^4z^4 + \\ & z^8v_1^5v_2 + z^9v_1^6v_2 + z^8v_1v_3 + z^9v_1^3v_2^2 + z^6v_1^3v_2 + z^7v_2^2v_1 + z^7v_1^4v_2 + \\ & z^{10}v_1^4v_2^2 + z^{10}v_1^7v_2 + z^{10}v_2v_3 + z^{11}v_1^8v_2 + z^{12}v_1^5v_3 + z^{13}v_2^2v_3 + z^{14}v_1^7v_3 + \\ & z^{14}v_1^2v_2^4 + z^{12}v_1^9v_2 + z^{13}v_1^{10}v_2 + z^{12}v_2^4 + z^{15}v_1^9v_2^2 + z^{15}v_1^{12}v_2 + z^{16}v_1^{13}v_2 + \end{aligned}$$

$$z^{16}v_1v_4 + z^{16}v_1^7v_2^3 + z^{16}v_3^2v_1^2 + z^{16}v_1v_2^5 + z^{16}v_1^3v_2^2v_3 + z^{16}v_2^4v_1^4 + z^3v_2 + z^3v_1^3 + z^6v_1^6 + z^9v_1^2v_3 + z^7v_3 + z^{11}v_1^{11} + z^9v_2^3 + z^9v_1^9 + z^{14}v_3^2 + z^{14}v_1^{14} + z^{13}v_1^{13} + z^{15}v_2^5 + z^{15}v_4 + v_1^2z^2$$

We give results of calculations in Morava K -theories, where the formulas are more tractable. In the following examples δ_{ik}^π coincides the coefficient at σ_p^i in the expression for σ_k and $\delta_{ik}^{\Sigma^p} = k!(p-k)!\delta_{ik}^\pi$. $y = z^{p-1}$.

Example 2. $p = 3, s = 2$

$$\begin{aligned}\sigma_1 &= v_2y^3\sigma_3 + v_2y^4x \\ \sigma_2 &= 2v_2^2y^3\sigma_3^4 + 2v_2y^2\sigma_3^2 + x^2v_2y^4 + 2y\end{aligned}$$

Example 3. $p = 5, s = 3$

$$\begin{aligned}\sigma_1 &= v_3y^{25}\sigma_5^5 + v_3y^{30}\sigma_5 + v_3y^{31}x \\ \sigma_2 &= 4v_3^2y^{25}\sigma_5^{30} + 4v_3^2y^{30}\sigma_5^{26} + 3v_3y^{19}\sigma_5^{10} + v_3y^{24}\sigma_5^6 + 3v_3y^{29}\sigma_5^2 + 2v_3y^{31}x^2 \\ \sigma_3 &= 2v_3^3y^{25}\sigma_5^{55} + 2v_3^3y^{30}\sigma_5^{51} + v_3^2y^{19}\sigma_5^{35} + 2v_3^2y^{24}\sigma_5^{31} + v_3^2y^{29}\sigma_5^{27} + 2v_3y^{13}\sigma_5^{15} + v_3y^{18}\sigma_5^{11} + v_3y^{23}\sigma_5^7 + 2v_3y^{28}\sigma_5^3 + 2x^3v_3y^{31} \\ \sigma_4 &= 4v_3^4y^{25}\sigma_5^{80} + 4v_3^4y^{30}\sigma_5^{76} + 4v_3^3y^{19}\sigma_5^{60} + 3v_3^3y^{24}\sigma_5^{56} + 4v_3^3y^{29}\sigma_5^{52} + 4v_3^2y^{13}\sigma_5^{40} + 2v_3^2y^{18}\sigma_5^{36} + 2v_3^2y^{23}\sigma_5^{32} + 4v_3^2y^{28}\sigma_5^{28} + 4y^7v_3\sigma_5^{20} + y^{12}v_3\sigma_5^{16} + 4y^{17}v_3\sigma_5^{12} + y^{22}v_3\sigma_5^8 + 4y^{27}v_3\sigma_5^4 + x^4v_3y^{31} + 4y\end{aligned}$$

□

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