STABLE GEOMETRIC DIMENSION OF VECTOR BUNDLES OVER EVEN-DIMENSIONAL REAL PROJECTIVE SPACES

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Abstract. In 1981, Davis, Gitler, and Mahowald determined the geometric dimension of stable vector bundles of order 2^e over RP^{2n} if n is sufficiently large and $e \geq 75$. In this paper, we use the Bendersky-Davis computation of $v_1^{-1}\pi_*(SO(m))$ to determine this geometric dimension for all values of e (still provided that n is sufficiently large). The same formula that worked for $e \geq 75$ works for $e \geq 6$, but for $e < 6$ the formula is different due to anomalies in the formula for $v_1^{-1}\pi_*(SO(m))$ when $m \leq 10$.

1. Statement of results

The geometric dimension $\mathrm{gd}(\theta)$ of a stable vector bundle θ over a space X is the smallest integer m such that θ is stably equivalent to an m-plane bundle. Equivalently, $\mathrm{gd}(\theta)$ is the smallest m such that the classifying map $X \xrightarrow{\theta} BO$ factors through $BO(m)$. The group $\widetilde{KO}(P^n)$ of equivalence classes of stable vector bundles over real projective space is a finite cyclic 2-group generated by the Hopf line bundle ξ_n. Many papers (e.g., [1], [22], [23]) have been devoted to computing the geometric dimension of multiples $k\xi_n$ of the Hopf bundles, in part because certain cases are equivalent to determining whether P^n can be immersed in a certain Euclidean space. (e.g., [11])

For n even, there exists a map $P^{n+8} \xrightarrow{\phi} P^n$ which induces a monomorphism in $\widetilde{KO}(-)$. This map may be obtained as a compression of multiplication by 16 of the suspension spectra. Hence, as was observed in [12], the geometric dimension of vector bundles of order 2^e over P^n is a nonincreasing function of n for even n in a fixed congruence class mod 8, and so must achieve a stable value, which we call the stable geometric dimension $\mathrm{sgd}(\overline{n}, e)$. Here $\overline{n} \in 2\mathbf{Z}/8$ is the residue class of n. Thus we have the following result, which was proved in the first paragraph of [12].

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Proposition 1.1. *For even* $\overline{n} \in \mathbf{Z}/8$ *and* e *a positive integer, there is an integer* $\text{sgd}(\overline{n}, e)$ *which equals the geometric dimension of all bundles of order* 2^e *in* $\widetilde{KO}(P^n)$ *for sufficiently large even* n *satisfying* $n \equiv \overline{n} \mod 8$.

Maps $P^{n+8} \to P^n$ inducing a monomorphism in $\widetilde{KO}(-)$ do not exist when n is odd, and so the situation for stable geometric dimension of bundles over odd-dimensional projective spaces is much more delicate, and will be discussed in a separate paper.

In [12] and [13], the value of $\text{sgd}(\overline{n}, e)$ was determined for $e \geq 75$.

Theorem 1.2. ([13]) *Define* $\delta(\overline{n}, e)$ *by the table*

		$e \mod 4$			
		0	1	2	3
	6, 8	0	2	2	1
$\overline{n}$	2, 4	0	0	−1	−2

Then $\text{sgd}(\overline{n}, e) \geq 2e + \delta(\overline{n}, e)$*, and if* $e \geq 75$*, then* $\text{sgd}(\overline{n}, e) = 2e + \delta(\overline{n}, e)$.

In fact, as follows from the fact that sgd is the limiting value of a nonincreasing function but also is proved directly in [12], the geometric dimension of all vector bundles of order 2^e over all P^n is $\geq 2e + \delta(\overline{n}, e)$.

Our first new result lowers the condition $e \geq 75$ to $e \geq 6$.

Theorem 1.3. *If* $e \geq 6$*, then* $\text{sgd}(\overline{n}, e) = 2e + \delta(\overline{n}, e)$.

If $e \leq 5$, the value of $\text{sgd}(\overline{n}, e)$ is sometimes greater than $2e + \delta(\overline{n}, e)$. It is given in the following theorem, which is proved in Section 3. Thus we have determined the stable geometric dimension of all stable vector bundles over all even-dimensional real projective spaces.

Theorem 1.4. *For* $e \leq 5$, $\text{sgd}(\overline{n}, e)$ *is given by the following table.*

		e				
		1	2	3	4	5
	6, 8	5	6	7	11	12
$\overline{n}$	2, 4	5	5	6	10	11

We remark that Adams ([1]) initiated the study of which bundles over RP^n have specific small values of geometric dimension, and this topic has also been considered in [22, §3] and [17, §3], and in unpublished work of Lam and Randall.(e.g., [21]).

Our new approach makes heavy use of the computation of $v_1^{-1}\pi_*(SO(m))$ obtained in [2]. We begin by indicating the relationship between this computation and sgd.

Let n be even. The maps ϕ^k which are used in defining sgd can be factored as

$$P^{n+8k} \xrightarrow{\text{col}} P_{1+8k}^{n+8k} \xrightarrow{(\phi')^k} P^n. \tag{1.5}$$

If $\nu(k) \geq n/2$,[1] then James periodicity says that $P_{1+8k}^{n+8k} \simeq \Sigma^{8k}P^n$, which, preceding $(\phi')^k$, yields a v_1-map $\Sigma^{8k}P^n \to P^n$. Bousfield ([7, p.1251]) uses this v_1-map to define

$$v_1^{-1}\pi_i(Y;P^n) = \operatorname*{colim}_d [\Sigma^{8kd+i}P^n, Y]$$

for any space Y. With $Y = BSO(m)$ and $i = 0$, this becomes $\operatorname{colim}[P_{1+8kd}^{n+8kd}, BSO(m)]$. This group contains one summand which is stable in the sense that it injects as m increases, along with some unstable summands. The stable summand corresponds to multiples of the Hopf bundle ξ, which comprise $\widetilde{KO}(P^n)$. We will denote this stable summand by $\mathbf{s}$ in various contexts of v_1-periodic homotopy groups or spectral sequence groups which approximate them. It follows that

Proposition 1.6. *If n is even, then $\operatorname{sgd}(\overline{n}, e) \leq m$ if and only if, for $n \equiv \overline{n} \mod 8$ sufficiently large, the exponent of 2 of the cyclic group $\mathbf{s}v_1^{-1}\pi_0(BSO(m);P^n)$ satisfies $\nu(\mathbf{s}v_1^{-1}\pi_0(BSO(m);P^n)) \geq e$.*

This notation of $\nu(G)$ for the exponent of 2 in a cyclic group G will be adopted throughout.

The situation when $n \equiv 6, 8 \mod 8$ is particularly simple. We will prove the following key result in Section 2.

Proposition 1.7. *If $n \equiv 6, 8 \mod 8$, then*

$$\mathbf{s}v_1^{-1}\pi_0(BSO(m);P^n) \approx \mathbf{s}v_1^{-1}\pi_{-2}(SO(m)).$$

Note the simplification here—coefficients are no longer in a projective space. The groups $v_1^{-1}\pi_*(SO(m))$ were computed in [2], where the following result was proved.

Theorem 1.8. *If $8i + d \geq 11$, then*

$$\nu(\mathbf{s}v_1^{-1}\pi_{-2}(SO(8i+d))) = 4i + \begin{cases} -1 & d = -1 \\ 0 & d = 0,1,2,3 \\ 1 & d = 4,5 \\ 2 & d = 6. \end{cases}$$

[1] Slightly smaller values work, too.

Theorem 1.3 when $\overline{n} = 6$ or 8 is an immediate consequence of 1.6, 1.7, and 1.8. Indeed, for $n \equiv 6, 8 \bmod 8$, the smallest d such that

$$\nu(\mathbf{s}v_1^{-1}\pi_0(BSO(8i+d); P^n)) \geq 4i + \langle 0, 1, 2, 3\rangle$$

is $8i + \langle 0, 4, 6, 7\rangle$.

Proof of Proposition 1.8. Because of the mammoth nature of [2], we guide the reader to the relevant results. Referring always to [2], the specific statements regarding $\nu(\mathbf{s}v_1^{-1}\pi_{-2}(SO(8i+d)))$ are in 1.2 for $d = \pm 1$, 3.10 for $d = 4 \pm 1$, and 3.13 for $d = 4 \pm 2$. Specific statements are not made for $d = 4$ or 8, but only with relation to the case $d - 1$. In 3.4(last case) (resp. 3.14(last case)), it is shown that the exponent when $d = 8$ (resp. $d = 4$) is 1 greater than when $d = 7$ (resp. $d = 3$). ■

When $\overline{n} = 2, 4$, a similar program is followed but we must define and compute a modified sort of v_1-periodic homotopy group. In Section 2, we will utilize the following definition and prove Theorem 1.10, which, with 1.6, implies Theorem 1.3 when $\overline{n} = 2$ or 4 just as in the previous case.

Definition 1.9. *Let $M^{n+1}(k) = S^n \cup_k e^{n+1}$ denote the usual Moore spectrum, and*

$$N^{n+1}(k) = M^{n+1} \cup_\eta e^{n+2} \cup_2 e^{n+3},$$

and define, for any space X and any integer i,

$$v_1^{-1}\pi_i'(X) = \operatorname*{colim}_{k,e}[N^{i+1+k2^{Le}}(2^e), X].$$

The second part of this definition, analogous to the definition of $v_1^{-1}\pi_*(X)$ first given in [15], is made using v_1-maps $\Sigma^{2^{Le}}N^i(2^e) \to N^i(2^e)$ and canonical maps $N^i(2^{e+1}) \to N^i(2^e)$, similarly to the situation for Moore spaces $M^i(2^e)$.

Theorem 1.10. (1) *If $n \equiv 2, 4 \mod 8$, then*

$$\mathbf{s}v_1^{-1}\pi_0(BSO(m); P^n) \approx \mathbf{s}v_1^{-1}\pi_{-2}'(SO(m)).$$

(2) *If* $8i+d\ge 11$, *then*

$$\nu(\mathbf{s}v_1^{-1}\pi'_{-2}(SO(8i+d)))=4i+\begin{cases}0 & d=0,1\\ 1 & d=2\\ 2 & d=3\\ 3 & d=4,5,6,7.\end{cases}$$

The requirement that $e\ge 6$ in Theorem 1.3 is due to the condition $8i+d\ge 11$ in 1.8 and 1.10(2). In Section 3, we will prove the following result, which, with 1.6, 1.7, and 1.10(1), implies Theorem 1.4.

Theorem 1.11. *For* $5\le m\le 10$,

$$\nu(\mathbf{s}v_1^{-1}\pi_{-2}(SO(m))) = \begin{cases}1 & m=5\\ 2 & m=6\\ 3 & m=7,8,9,10;\end{cases}$$

$$\nu(\mathbf{s}v_1^{-1}\pi'_{-2}(SO(m))) = \begin{cases}2 & m=5\\ 3 & m=6,7,8,9\\ 4 & m=10.\end{cases}$$

The stable summand is not in the image of $v_1^{-1}\pi_{-2}(SO(4))\to v_1^{-1}\pi_{-2}(SO(5))$ *or of* $v_1^{-1}\pi'_{-2}(SO(4))\to v_1^{-1}\pi'_{-2}(SO(5))$.

When $m=5$ or 6, a slight reinterpretation of $\nu(\mathbf{s}v_1^{-1}\pi_{-2}(SO(m)))$ is required. In this case, the stable classes do not form a direct summand, and by $\nu(\mathbf{s}v_1^{-1}\pi_{-2}(SO(m)))$ we mean $\nu(\mathrm{im}(v_1^{-1}\pi_{-2}(SO(m))\to v_1^{-1}\pi_{-2}(SO))$, a definition that works for all m. A similar interpretation is used for the primed groups.

2. Proof of main results

In this section we prove Proposition 1.7 and Theorem 1.10, which we have already seen imply Theorem 1.3.

Let Φ denote the v_1-periodic spectrum functor described in [7, 7.2]. By [7, 7.2(i)], we have, if n is even,

$$\begin{aligned}v_1^{-1}\pi_0(BSO(m);P^n) &\approx [P^n,\Phi BSO(m)]\approx[P^n,\Phi SO(m)]_{-1}\\ &\approx v_1^{-1}\pi_{-1}(SO(m);P^n),\end{aligned}\tag{2.1}$$

or similarly with P^n replaced by another space with a v_1-map. We will use the four parts of (2.1) interchangeably.

Proof of Proposition 1.7. The proof utilizes the following result, which is part of [14, 4.2]. Here and throughout, $M^n(k) = S^{n-1} \cup_k e^n$ denotes a Moore spectrum.

Theorem 2.2. ([14]) *For $\epsilon = 0$ and 1, and L sufficiently large, there is a K_*-equivalence $M^{2^L}(2^{4k-\epsilon}) \to P_{2^L+1-8k}^{2^L-2\epsilon}$.*

We also note the following elementary result.

Proposition 2.3. *A K_*-equivalence $P_{b+8}^{n+8} \xrightarrow{\phi'} P_b^n$, with n even and b odd, as used in* (1.5)*, induces an isomorphism*

$$v_1^{-1}\pi_*(Y; P_b^n) \xrightarrow{\phi'^*} v_1^{-1}\pi_*(Y; P_{b+8}^{n+8})$$

for any space Y.

Proof. A K_*-equivalence $\Sigma^{2^L} P_b^n \to P_b^n$ used in defining $v_1^{-1}\pi_*(Y; P_b^n)$ can be factored as

$$\Sigma^{2^L} P_b^n \to P_{b+8}^{n+8} \xrightarrow{\phi'} P_b^n;$$

thus ϕ'^* is injective. Similarly a K_*-equivalence $\Sigma^{2^L} P_{b+8}^{n+8} \to P_{b+8}^{n+8}$ used in defining $v_1^{-1}\pi_*(Y; P_{b+8}^{n+8})$ can be factored as

$$\Sigma^{2^L} P_{b+8}^{n+8} \xrightarrow{\Sigma^{2^L}\phi'} \Sigma^{2^L} P_b^n \to P_{b+8}^{n+8},$$

and so ϕ'^* is surjective. ■

Thus

$$v_1^{-1}\pi_*(Y; P^{8k-2\epsilon}) \approx v_1^{-1}\pi_*(Y; P_{1-8k}^{-2\epsilon}) \approx v_1^{-1}\pi_*(Y; M^0(2^{4k-\epsilon})). \tag{2.4}$$

Here we use that a K_*-equivalence induces an isomorphism in $[-, \Phi Y]$, since ΦY is K_*-local, and also use the fact ([20, 3.7]) that the maps of 2.2 asymptotically respect the v_1-maps of the two spaces.

With k sufficiently large, there is, by [15, 1.7], a split short exact sequence

$$0 \to v_1^{-1}\pi_{-1}(SO(m)) \to v_1^{-1}\pi_{-1}(SO(m); M^0(2^{4k-\epsilon})) \to v_1^{-1}\pi_{-2}(SO(m)) \to 0. \tag{2.5}$$

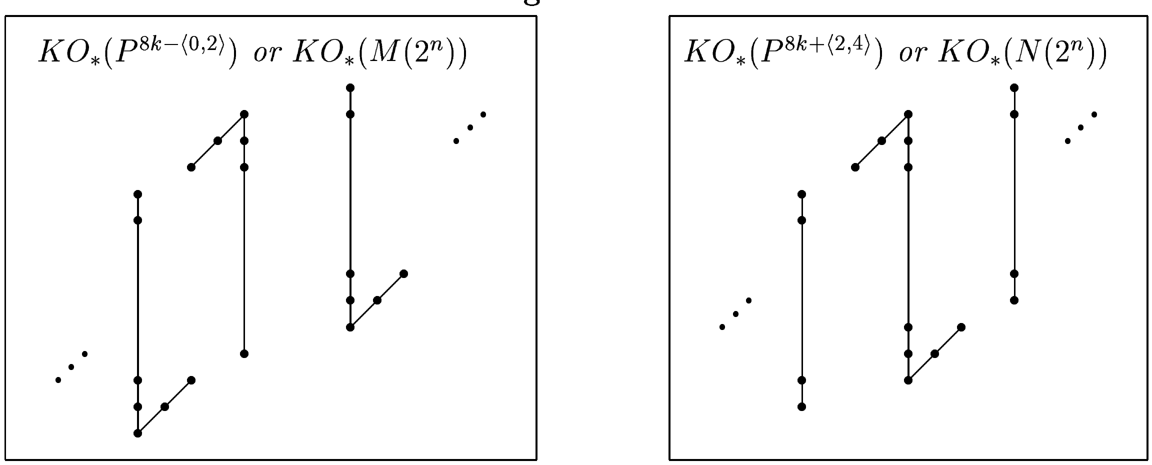
The stable summand $\mathbf{s}v_1^{-1}\pi_i(Y; SO(m))$ may be defined to be $v_1^{-1}\pi_i(Y; SO(m))$ modulo the kernel of the stabilization $v_1^{-1}\pi_i(Y; SO(m)) \to v_1^{-1}\pi_i(Y; SO)$. Similarly to [6, 1.9], $v_1^{-1}\pi_{-1}(SO) = 0$. Thus (2.5) induces an isomorphism

$$\mathbf{s}v_1^{-1}\pi_{-1}(SO(m); M^0(2^{4k-\epsilon})) \to \mathbf{s}v_1^{-1}\pi_{-2}(SO(m)). \qquad (2.6)$$

With (2.1) and (2.4), this yields the desired conclusion of Proposition 1.7. ■

Next we prove Theorem 1.10(1). One viewpoint for the relevance of $N^{n+1}(k)$ involves a comparison of charts of $KO_*(-)$ computed, for example, by the method of [12, p.41] or [14, p.133] as $v_1^{-1}ko_*(-)$. In Diagram 2.7, the left side is a chart of $KO_*(P^{8k-2\epsilon})$ with $\epsilon = 0$ or 1 and main groups of order $2^{4k-\epsilon}$, while the right side is $KO_*(P^{8k+2\delta})$ with $\delta = 1$ or 2 and larger (middle) groups of order $2^{4k+\delta+1}$. A chart for $KO_*(M(2^n))$ is given by the left side of Diagram 2.7 with main groups of order 2^n, while a chart for $KO_*(N(2^n))$ is given by the right side of the diagram with the larger groups of order 2^{n+2}. Here we have not listed a superscript for $M(-)$ or $N(-)$ since the effect of the superscript is just to translate the chart horizontally. These charts are not necessary for the proof; they merely form one way of understanding the need for resorting to $N^{n+1}(k)$. The charts for $M(2^n)$ match nicely with those of $P^{8k-\langle 0,2\rangle}$, but must be modified to those of $N(2^n)$ to match with $P^{8k+\langle 2,4\rangle}$.

Diagram 2.7.

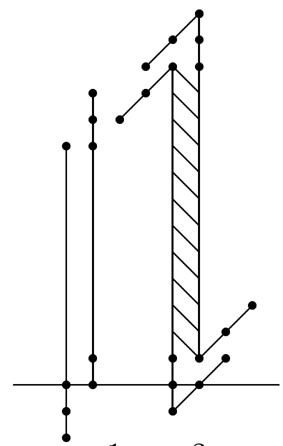


Proposition 2.8. *For sufficiently large L, there exist K_*-equivalences*

$$N^{2^{4k}L}(2^{4k}) \xrightarrow{f_1} P^2_{1-8k} \text{ and}$$
$$N^{2^{4k+1}L}(2^{4k+1}) \xrightarrow{f_2} P^4_{1-8k}.$$

Proof. In [14, 4.2], a K_*-equivalence $M^0(2^{4k}) \xrightarrow{f} P^0_{1-8k}$ was constructed. Let $\mathcal{J} = v_1^{-1}J$ denote the periodic J-spectrum. The chart for $\mathcal{J}_*(P^2_{1-8k})$ in the range $-2 \leq * \leq 5$ is given in Diagram 2.9.

Diagram 2.9.



Let $M^1 = S^0 \cup_2 e^1$. Consider the composite

$$M^1 \xrightarrow{\tilde\eta} S^{-1} \xrightarrow{j} M^0(2^{4k}) \xrightarrow{f} P^0_{1-8k} \xrightarrow{i} P^2_{1-8k} \wedge \mathcal{J}. \tag{2.10}$$

The cofiber of $j \circ \tilde\eta$ is $N^0(2^{4k})$. Note that $\mathcal{J}_*(P^{10}_{1-8k})$ has a chart like that of Diagram 2.9 with the same top and extending 4 units lower. The commutative diagram induced by the inclusion $P^2_{1-8k} \to P^{10}_{1-8k}$

$$\begin{array}{ccc} \mathcal{J}_{-1}(P^2_{1-8k}) & \xrightarrow{\tilde\eta^*} & [M^1, P^2_{1-8k} \wedge \mathcal{J}] \\ {\scriptstyle 2^4}\downarrow & & \downarrow{\scriptstyle =} \\ \mathcal{J}_{-1}(P^{10}_{1-8k}) & \xrightarrow{\tilde\eta^*} & [M^1, P^{10}_{1-8k} \wedge \mathcal{J}] \end{array}$$

implies that its top morphism is 0. Thus the composite (2.10) is trivial, and hence the extension $N^0(2^{4k}) \xrightarrow{\tilde f} P^2_{1-8k} \wedge \mathcal{J}$ of $i \circ f$ exists.

By [24], $P^2_{1-8k} \wedge \mathcal{J}$ is the telescope $v_1^{-1}P^2_{1-8k}$ over v_1-maps of $P^{2-2^{4k}L}_{1-8k-2^{4k}L}$ as $L \to \infty$. Thus the map $\tilde f$ factors through a map f_1 whose $(2^{4k}L)$-suspension is as in the statement of the proposition. This f_1 is a K_*-equivalence by the Five Lemma applied to

$$\begin{array}{ccccc} M^{2^{4k}L}(2^{4k}) & \longrightarrow & N^{2^{4k}L}(2^{4k}) & \longrightarrow & M^{2+2^{4k}L}(2) \\ \downarrow & & \downarrow & & \downarrow \\ P^0_{1-8k} & \longrightarrow & P^2_{1-8k} & \longrightarrow & M^2(2). \end{array}$$

In [14, 4.2], a K_*-equivalence $M^0(2^{4k+1}) \to P^0_{-8k-1}$ is constructed. As in the proof of the first part, this yields a K_*-equivalence $N^{2^{4k+1}L'}(2^{4k+1}) \xrightarrow{f_2'} P^2_{-8k-1}$. By [16, 3.1], there is a filtration-3 K_*-equivalence $P^2_{-8k-1} \xrightarrow{h_1} P^{-4}_{-8k-7}$. The filtration-4 K_*-equivalences used in the definition of stable geometric dimension yield a K_*-equivalence $P^{-4}_{-8k-7} \xrightarrow{h_2} P^{4-2^{4k+1}}_{1-8k-2^{4k+1}} \simeq \Sigma^{-2^{4k+1}} P^4_{1-8k}$. The 2^{4k+1}-fold suspension of $h_2 \circ h_1 \circ f_2'$ is our desired K_*-equivalence f_2. ■

Thus for $\delta = 1, 2$, we have

$$\mathbf{s}v_1^{-1}\pi_0(BSO(m); P^{8k+2\delta}) \approx \mathbf{s}v_1^{-1}\pi_0(BSO(m); N^0(2^{4k+\delta-1})). \tag{2.11}$$

Here again we use from [20] that K_*-equivalences such as those in 2.8 must asymptotically commute with v_1-maps. Similarly to (2.5), for k sufficiently large, there is a split short exact sequence

$$0 \to v_1^{-1}\pi_{-1}(SO(m)) \to v_1^{-1}\pi_{-1}(SO(m); N^0(2^{4k+\delta-1})) \to v_1^{-1}\pi'_{-2}(SO(m)) \to 0, \tag{2.12}$$

which, similarly to (2.6), induces an isomorphism

$$\mathbf{s}v_1^{-1}\pi_{-1}(SO(m); N^0(2^{4k+\delta-1})) \approx \mathbf{s}v_1^{-1}\pi'_{-2}(SO(m)). \tag{2.13}$$

We will expand slightly upon the proof of (2.12) following Definition 2.14. Theorem 1.10(1) is an immediate consequence of (2.11), (2.1), and (2.13). ■

We expand 1.9 to include another related spectrum.

Definition 2.14. *Let* $T^n = S^n \cup_\eta e^{n+2} \cup_2 e^{n+3}$.

The reason for the choice of names of these spaces is "next letter of alphabet." The space T^n has appeared in other guises as variations on a sphere. In [8, 10.7], it was called C, and its K_*-localization was shown in [8, 10.6] to be the only other K_*-local spectrum to have the same $K_*(-)$-groups as S_K. The spectrum *bsp*, which was used in many papers of the second and third authors (e.g. [12, p.41], [14, p.127], [16, p.41]) involving the J-spectrum, equals $T^0 \wedge bo$.

The split short exact sequence (2.12) is induced from cofiber sequences

$$T^{2^eL-1} \to N^{2^eL}(2^{4k+\delta-1}) \to S^{2^eL} \xrightarrow{2^{4k+\delta-1}}, \tag{2.15}$$

where k is large enough that $SO(m)$ has H-space exponent $2^{4k+\delta-1}$, and e and L are large. This induces a split short exact sequence

$$0 \to \pi_{2^eL-1}(SO(m)) \to [N^{2^eL-1}(2^{4k+\delta-1}), SO(m)] \to [T^{2^eL-2}, SO(m)] \to 0,$$

and, similarly to [15, 2.6], there is a direct system of these split short exact sequences with respect to increasing 2^eL, the direct limit of which is (2.12). See also (2.17), which suggests that (2.12) can be obtained by applying $[-, \Phi BSO(m)]$ to (2.15).

We will use the following spectral sequence to compute $v_1^{-1}\pi'_*(X)$, which was defined in 1.9.

Proposition 2.16. *If X is an odd sphere or simply-connected compact Lie group, there is a spectral sequence converging to $v_1^{-1}\pi'_*(X)$ with E_2-term*

$$\widetilde{E}_2^{s,t} \approx \operatorname{Ext}^s_{\mathcal{A}}(QK^1(X; \mathbf{Z}_2^\wedge)/\operatorname{im}(\psi^2), K^1(S^t; \mathbf{Z}_2^\wedge)).$$

Note that the E_2-term is isomorphic to that of the spectral sequence of [4] converging to $v_1^{-1}\pi_*(X)$. We will call it $\widetilde{E}_2(X)$ when it is the initial term of the spectral sequence converging to $v_1^{-1}\pi'_*(X)$.

Proof of Proposition 2.16. We begin by mimicking the proof of [7, 7.5]. With D denoting S-duality, there are isomorphisms

$$\begin{aligned} v_1^{-1}\pi'_*(X) &\approx \lim_k [N^1(2^k), \Phi(X)]_* \approx \operatorname{colim}_k \pi_*(DN^1(2^k) \wedge \Phi(X)) \\ &\approx \pi_*(D(T^0) \wedge M^0(\mathbf{Z}/2^\infty) \wedge \Phi(X)) \\ &\approx [T^0, \Phi(X)]_*. \end{aligned} \tag{2.17}$$

Here we have used that applying $\wedge M^0(\mathbf{Z}/2^\infty)$ to the torsion spectra $S^0 \cup_2 e^1$ and $\Phi(X)$ leaves them unchanged.

By [7, 10.4],[2] there is a spectral sequence converging to $[T^0, \Phi(X)]_*$ with

$$E_2^{s,t} \approx \operatorname{Ext}^s_{\mathcal{A}}(K^*(\Phi(X); \mathbf{Z}_2^\wedge), K^*(T^t; \mathbf{Z}_2^\wedge)).$$

[2] Although many results of [7] only work when p is odd, this one also works when $p = 2$.

Bousfield ([9]) has proved that for X as in this theorem, there is an isomorphism in $\mathcal{A}$

$$K^i(\Phi X) \approx \begin{cases} 0 & i = 0 \\ QK^1(X)/\operatorname{im}(\psi^2) & i = 1, \end{cases}$$

where $Q(-)$ denotes the indecomposables. This is the 2-primary analogue of [7, 9.2]. The proof of the proposition is completed by noting that there is an isomorphism in $\mathcal{A}$, $K^*(S^t; \mathbf{Z}_2^\wedge) \approx K^*(T^t; \mathbf{Z}_2^\wedge)$. Indeed, the morphism in $K^*(-; \mathbf{Z}_2^\wedge)$ induced by the inclusion $S^t \hookrightarrow T^t$ is a monomorphism onto multiples of 2. ■

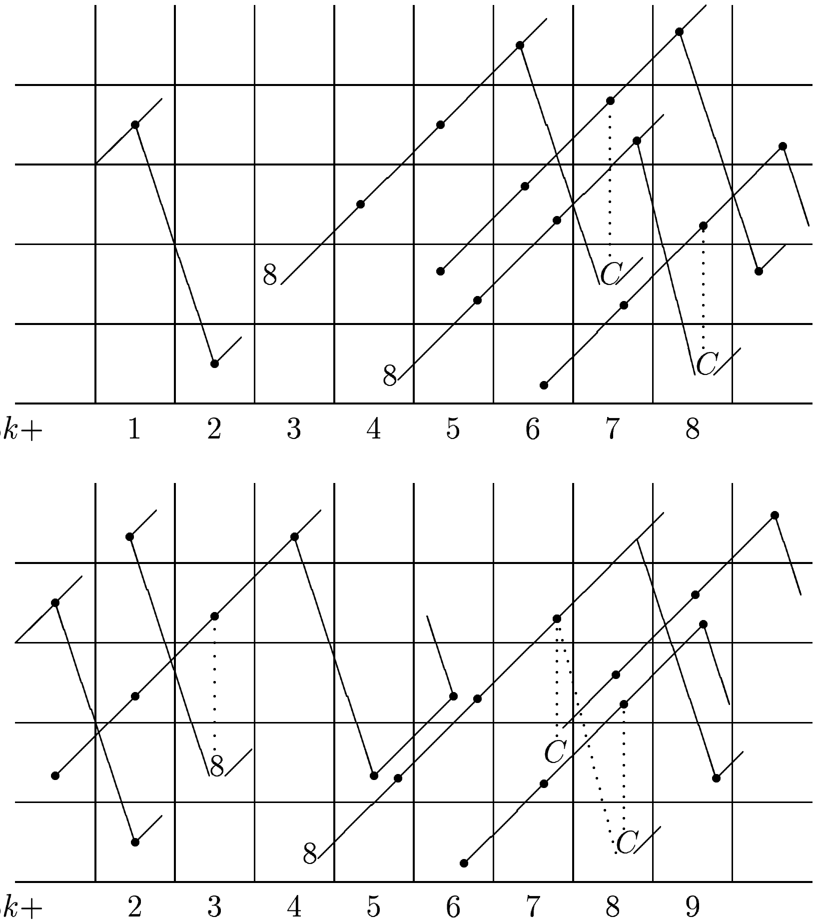
The next result gives the primed v_1-periodic homotopy groups of odd spheres. The conclusion is that the d_3-differentials between the eta-towers in the spectral sequence for $v_1^{-1}\pi'_*(S^{2n+1})$ are the opposite of the way they are in the spectral sequence for $v_1^{-1}\pi_*(S^{2n+1})$. Here n can be even or odd.

Theorem 2.18. *The spectral sequence of 2.16 converging to $v_1^{-1}\pi'_*(S^{2n+1})$ is as pictured in Diagram 2.19. Here 8 means $\mathbf{Z}/8$, while C is $\mathbf{Z}/2^{\min(n,4+\nu(k+1))}$. We do not picture many portions of eta-towers which are involved in nontrivial d_3's. The dotted differential when $n \equiv 1, 2$ is nonzero unless $\nu(k+1)+4 > n$, in which case $d_3 = 0$ and the extension in $v_1^{-1}\pi'_{2n+8k+7}(S^{2n+1})$ occurs. The action of h_1 on the generator of C in position $(2n+8k+8, 1)$ is nontrivial, but the class which it hits depends upon whether or not $\nu(k+1)+4 > n$.*

Diagram 2.19.

$v_1^{-1}\pi'_*(S^{2n+1})$
$n \equiv 0, 3 \mod 4$

$v_1^{-1}\pi'_*(S^{2n+1})$
$n \equiv 1, 2 \mod 4$



Proof. We begin by using a J-homology approach to determine $v_1^{-1}\pi'_*(S^{2n+1})$. These methods were developed in [24], and described quite thoroughly in [10, §3,§4,§5]. We assume that the reader has some familiarity with those methods. For a reader who has no such expertise, an alternate proof is given after this one.

Let $U^i = S^{i-3}\cup_2 e^{i-2}\cup_\eta e^i$. Note that T^i and U^{-i} are S-dual. The map $\Omega^{2n+1}S^{2n+1} \to QP^{2n}$ of [10, 3.3] induces an isomorphism in $v_1^{-1}\pi'_*(-)$. Thus

$$\begin{aligned} v_1^{-1}\pi'_i(S^{2n+1}) &\approx v_1^{-1}[T^i, \Sigma^\infty\Sigma^{2n+1}P^{2n}] \\ &\approx v_1^{-1}\pi_i(U^{2n+1}\wedge P^{2n}) \\ &\approx v_1^{-1}J_i(U^{2n+1}\wedge P^{2n}). \end{aligned}$$

Arguing similarly to [10, p.1011], there is a short exact sequence of A_1-modules

$$0 \to H^*U^5 \to A_1//A_0 \to \mathbf{Z}_2 \to 0,$$

and hence isomorphisms

$$\mathrm{Ext}_{A_1}^{s,t}(H^*(U^5 \wedge X), \mathbf{Z}_2) \xrightarrow{\approx} \mathrm{Ext}_{A_1}^{s+1,t}(H^*X, \mathbf{Z}_2)$$

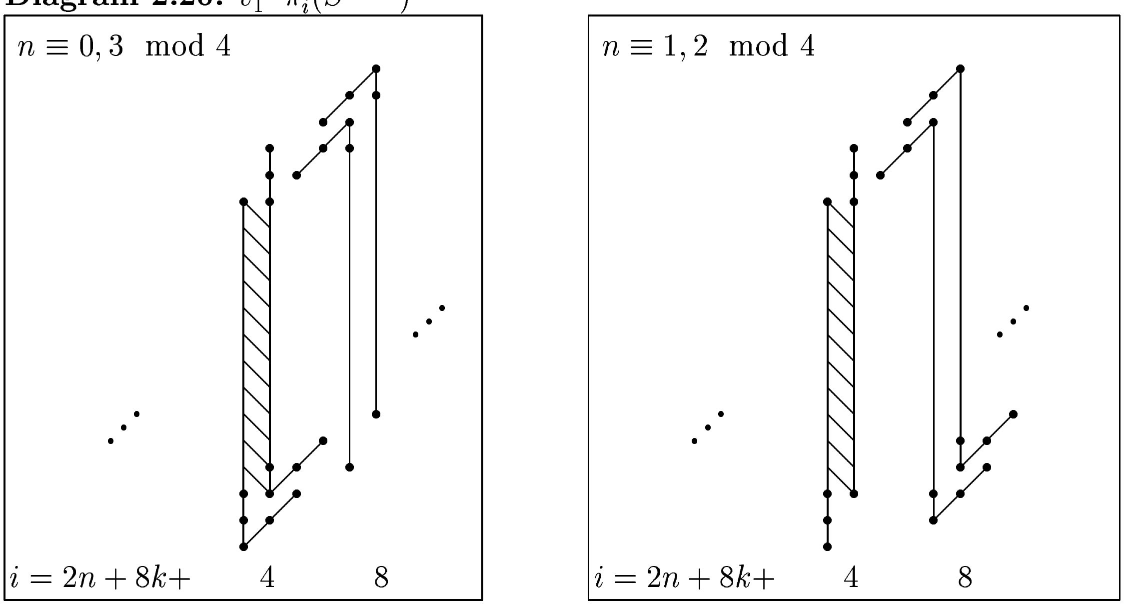
for $s > 1$ and any space X. Here A_1 is the subalgebra of the mod 2 Steenrod algebra generated by Sq^1 and Sq^2, which is relevant since the E_2-term of the Adams spectral sequence converging to $\pi_*(X \wedge bo)$ is $\mathrm{Ext}_{A_1}(H^*X, \mathbf{Z}_2)$. Inverting v_1, we conclude

$$v_1^{-1}\pi_i(U^0 \wedge X \wedge bo) \approx v_1^{-1}\pi_{i+4}(X \wedge bo).$$

Thus, since $v_1^{-1}J$ is the fiber of $\psi^3 - 1 : v_1^{-1}bo \to v_1^{-1}bo$, the chart for $v_1^{-1}J_*(U^0 \wedge X)$ is like that of $v_1^{-1}J_*(X)$ pushed back by 4, but the differentials between adjacent towers (corresponding to $\psi^3 - 1$) of $U^0 \wedge X$ are the same as those in X in the same dimension.

We obtain charts for $v_1^{-1}\pi'_*(S^{2n+1})$ as in Diagram 2.20. Here the differential between the second pair of towers in either box is $d_{\nu(4k+4)}$. The height (number of dots) of the towers in the left box is n. The height of the smaller (left) towers in the right box is $n-1$, while that of the larger towers is $n+1$.

Diagram 2.20. $v_1^{-1}\pi'_i(S^{2n+1})$



By Proposition 2.16, the E_2-term in 2.19 is the same as that for $v_1^{-1}\pi_*(S^{2n+1})$ as given, for example, in [3, p.488]. The d_3-differentials in 2.19 are the only way of inserting them to yield groups which agree with $v_1^{-1}\pi'_*(S^{2n+1})$ as given in 2.20. ■

Now we easily deduce the following key result.

Proposition 2.21. *The spectral sequence of* (2.16) *for* $v_1^{-1}\pi'_*(\mathrm{Spin}(m))$ *has* $\widetilde{E}_2$ *as given in* [2, 1.3,3.4,3.7,3.12,3.14] *but with* d_3*-differentials between eta-towers the opposite of those given there.*

Proof. As described in [2, §5], an eta-tower is a family of $\mathbf{Z}_2$ elements related by $h_1 : \widetilde{E}_2^{s,t} \to \widetilde{E}_2^{s+1,t+2}$, beginning in filtration 1, 2, or 3. If x is an eta-tower, then there is an eta-tower with the same name appearing every 4 (horizontal) dimensions, and either all those congruent mod 8 to x support d_3-differentials hitting the others, or else all those congruent mod 8 to x are hit by d_3-differentials from the others. In [2], it was shown that all these d_3's in $\mathrm{Spin}(m)$ could be determined by naturality from those in the odd spheres. Since we saw in 2.18 that the d_3's in the spectral sequence for $v_1^{-1}\pi'_*(S^{2n+1})$ are opposite of those in the spectral sequence for $v_1^{-1}\pi_*(S^{2n+1})$, we can deduce that the same happens for $\mathrm{Spin}(m)$. ■

Now we give an alternate proof of Theorem 2.18 which does not involve J-chart technology. This argument can probably be used to prove Proposition 2.21 concurrently with 2.18.

Alternate proof of Theorem 2.18. Let t be odd, and let $M_t(\eta) = S^t \cup_\eta e^{t+2}$. The obvious cofibration induces a short exact sequence in $\mathcal{A}$

$$0 \to K^*(S^{t+2}) \to K^*(M_t(\eta)) \to K^*(S^t) \to 0,$$

and hence, for any $\mathcal{A}$-object N, an exact sequence

$$\mathrm{Ext}_{\mathcal{A}}^{s,t+2}(N) \to \mathrm{Ext}_{\mathcal{A}}^{s}(N, K^*M_t(\eta)) \to \mathrm{Ext}_{\mathcal{A}}^{s,t}(N) \xrightarrow{h_1} \mathrm{Ext}_{\mathcal{A}}^{s+1,t+2}(N).$$

If, as is the case when $N = K^*(\Phi S^{2n+1})$ or $K^*(\Phi\,\mathrm{Spin}(n))$, $\mathrm{Ext}_{\mathcal{A}}^{s,t}(N) \xrightarrow{h_1} \mathrm{Ext}_{\mathcal{A}}^{s+1,t+2}(N)$ is an isomorphism for $s > 2$, then $\mathrm{Ext}_{\mathcal{A}}^{s}(N, K^*M_t(\eta)) = 0$ for $s > 2$.

Now we consider the cofiber sequence

$$S^{t+2} \xrightarrow{\alpha} M_t(\eta) \xrightarrow{i} T^t \xrightarrow{q} S^{t+3},$$

where α is a coextension of 2, i the inclusion, and q the collapse. It induces a short exact sequence in $\mathcal{A}$

$$0 \to K^*(T^t) \to K^*(M_t(\eta)) \to K^*(S^{t+2}) \to 0$$

and hence an exact sequence

$$\mathrm{Ext}^s_{\mathcal{A}}(N, K^*T^t) \to \mathrm{Ext}^s_{\mathcal{A}}(N, K^*M_t(\eta)) \to \mathrm{Ext}^{s,t+2}_{\mathcal{A}}(N) \xrightarrow{\delta} \mathrm{Ext}^{s+1}_{\mathcal{A}}(N, K^*T^t).$$

With N as above, since $\mathrm{Ext}^s_{\mathcal{A}}(N, K^*M_t(\eta)) = 0$ for $s > 2$, δ induces an isomorphism of eta-towers. Note that since $K^*T^t \approx K^*S^t$ in $\mathcal{A}$, this δ can be considered to be a morphism

$$\mathrm{Ext}^{s,t+2}_{\mathcal{A}}(N) \to \mathrm{Ext}^{s+1,t}_{\mathcal{A}}(N) \xrightarrow{h_1^{-1}} \mathrm{Ext}^{s,t-2}_{\mathcal{A}}(N),$$

since h_1 is an isomorphism. As noted in the proof of 2.21, names of eta-towers have period 4 in t. Thus this δ maps a set of eta towers to a set of eta towers with the same name. It can be shown, using the Small Complex of [2, §11], that this δ sends an eta tower to the one with the same name, at least if $N = K^*(\Phi S^{2n+1})$. Since the proof is somewhat involved and this is only an alternate proof, it is omitted here.

Finally we note that this δ commutes with d_3-differentials since it is induced by the map q. Thus the d_3-differential on eta towers in $\widetilde{E}_2^{s,t-2}$ of the spectral sequence for $v_1^{-1}\pi'_*(S^{2n+1})$ agree with d_3 on $E_2^{s,t+2}$ of the spectral sequence for $v_1^{-1}\pi_*(S^{2n+1})$. The conclusion is that E_2 is the same for the two spectral sequences, but d_3 on eta-towers is opposite. For d_3 on the 1-line, more delicate analysis is required, which will be the focus of the next proposition. ■

We close this section by proving Theorem 1.10(2), the determination of the required $\mathbf{s}v_1^{-1}\pi'_{-2}(SO(m))$. This is accomplished using the spectral sequence of 2.16, and follows from the following result.

Proposition 2.22. *In the spectral sequence of 2.16 with $X = \mathrm{Spin}(m)$,*

- *If $4a \le m \le 4a+3$, then* $\nu(\mathbf{s}\widetilde{E}_2^{1,-1}) = 2a + \begin{cases} 0 & m \equiv 0,1,2 \mod 4 \\ 1 & m \equiv 3 \mod 4. \end{cases}$
- *There is a nontrivial extension $(\cdot 2)$ from $\widetilde{E}_\infty^{1,-1}$ to $\widetilde{E}_\infty^{3,1}$ if $m \ge 7$.*
- *$d_3 : E_3^{1,-1} \to E_3^{4,1}$ is nonzero if and only if $m \equiv 0, \pm 1 \mod 8$.*

Proof. We use the observation after 2.16 that $\widetilde{E}_2$ is isomorphic to the E_2-term of the spectral sequence converging to $v_1^{-1}\pi_*(X)$. From [2, 3.1], $\nu(\mathbf{s}E_2^{1,-1}(\mathrm{Spin}(2n+1))) = n$, while from [2, 3.3]

$$\nu(\mathbf{s}E_2^{1,-1}(\mathrm{Spin}(2n))) = \begin{cases} n-1 & \text{if } n \text{ odd} \\ n & \text{if } n \text{ even.} \end{cases}$$

The extension is into the class which would be labeled 1 in diagrams such as [2, 1.3]. This is the class corresponding to the element $x_1 \in K^1(\mathrm{Spin}(m))$. See, e.g., [2, 5.9,5.19]. This class in position $(-3, 4)$ is not depicted in [2, 1.3] because its entire eta-tower supports a nonzero d_3-differential, and such eta-towers are often omitted from the diagrams. But by 2.21, in the $\widetilde{E}_r(-)$-spectral sequence, the eta-tower labeled 1 passing through $(-3, 4)$ is hit by d_3, and only in filtration ≥ 4. Thus this class x_1 lives in $\widetilde{E}_\infty(\mathrm{Spin}(m))$ as a candidate for an extension.

To see that this extension actually takes place, we can look either at the beginning $(\mathrm{Spin}(7))$ or the end (Spin) of this sequence of spaces. The extension in $\mathrm{Spin}(m)$ for $7 \leq m \leq \infty$ then follows by naturality, since the class x_1 persists throughout this range. The case of $\mathrm{Spin}(7)$ is still in the range of some anomalous behavior, and so is dealt with in the next section. As for Spin, all the unstable classes are gone. We have

$$\widetilde{E}_\infty^{s,t}(\mathrm{Spin}) \approx \begin{cases} \mathbf{Z}/2^\infty & s=1,\ t-s \equiv 2 \mod 4 \\ \mathbf{Z}_2 & 1 \leq s \leq 3,\ t-2s \equiv 3 \mod 8 \\ 0 & \text{otherwise.} \end{cases}$$

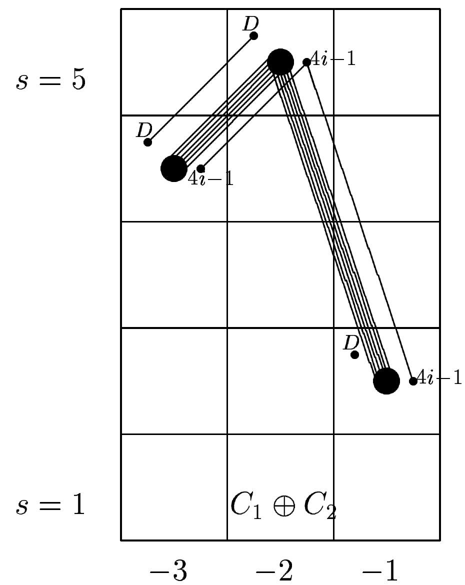
The extensions must be nontrivial by a form of Bott periodicity. A similar situation is discussed in [6, 1.19].

In [2, §7], d_3 from the 1-line of the spectral sequence converging to $v_1^{-1}\pi_*(\mathrm{Spin}(m))$ was determined by noting that $d_3(x) = y$ iff $d_3(h_1x) = h_1y$. Since d_3 from the 2-line had already been computed, it sufficed to compute h_1x. Methods for computing h_1 from the 1-line were developed in [2, 7.2,7.9]. The same methods work here in the spectral sequence converging to $v_1^{-1}\pi'_*(\mathrm{Spin}(m))$. The biggest difference is that, as shown in 2.21, the d_3's from the 2-line here are opposite of the way they were in [2].

We focus here on the cases where we must show $d_3 = 0$ on $\mathbf{s}\widetilde{E}_2^{1,-1}(\mathrm{Spin}(m))$. The nonzero d_3's when $m \equiv 0$ or ± 1 are implied by 1.2, 1.6, and 1.10, since these results imply that $\nu(\mathbf{s}v_1^{-1}\pi'_{-2}(SO(8i+d)))$ is equal to or less than the value claimed in 1.10(2) in these cases, and, given the first two parts of 2.22, the only way to make the group this small is with a nonzero d_3. The method of the proof which follows can also be used to obtain these nonzero d_3's.

We begin with the spectral sequence for $v_1^{-1}\pi'_*(\mathrm{Spin}(8i+3))$. The E_2-term equals that of [2, Diagram 3.7]. In Diagram 2.23, we present the relevant portion, with the d_3-differentials which apply to $v_1^{-1}\pi'_*(\mathrm{Spin}(8i+3))$.

Diagram 2.23. Part of the spectral sequence for $v_1^{-1}\pi'_*(\mathrm{Spin}(8i+3))$



The dual group $(\widetilde{E}_2^{2,1})^\#$ has basis $\{D, x_{4i-1}\} \cup B_C[2i,4i]$, where

$$B_C[2i,4i] = \{x_j : 2i \le j \le 4i \text{ and } j - 2^{\nu(j)+1} < 2i\}.$$

The set $B_C[2i,4i]$ has $[\log_2(16i/3)] + \delta_{\alpha(i),1}$ elements and is represented by the big $\bullet$ in Diagram 2.23. We use the same names for elements of the dual basis. By [2, 3.7] and Proposition 2.21, all basis elements of $\widetilde{E}_2^{2,1}$ except D support nonzero d_3 in the spectral sequence for $v_1^{-1}\pi'_*(\mathrm{Spin}(8i+3))$. By [2, 7.9], D is a summand of $h_1(g_1)$ in the case at hand; we will see why this is true in the next paragraph.

In order to show that $d_3(g_1) = 0$, we must show that the basis elements of $\widetilde{E}_2^{2,1}$ other than D are not summands of $h_1(g_1)$. We adopt the dual point of view as explained in the proof of [2, 7.9]. In the notation of that proof, we are in the first case considered there—$4\ell+3=8k-1$ with $\nu >> n$. Since n which we have been using in this paper to denote the dimension of a projective space is not relevant to this proposition, we are free here to use n as it was used in the proof of [2, 7.9], namely $8i+3=2n+1$ so $n=4i+1$. The four relations described there which yield $(\widetilde{E}_2^{1,8k-1})^{\#}$ are $A_1 2^n\xi_1$, $A_2 2^n\xi_1 - 2^{n+1}\Delta$, $A_3 2^n\xi_1 - 2^n\Delta$, and $u2^n\xi_1 + 2^\nu\Delta$ with u odd.[3] In fact, A_1 is even by the discussion following [2, 8.1], and A_2 is even by [2, 3.2]. Hence in the $\mathbf{Z}/2^n\oplus\mathbf{Z}/2^n$ group presented, it is only the last relation whose division by 2 lowers the order of the first (ξ_1) summand.

The fourth relation here is due to $(\psi^3-3^{4k-1})(\Delta)$, the third to $\psi^2(\Delta)$, and the first two to ψ^2 and ψ^3-3^{4k-1} acting on various x_j. It was observed in the proof of [2, 7.9] that dividing the fourth relation by 2 corresponds to modding $(\widetilde{E}_2^{1,8k-1})^{\#}$ by $h_1^{\#}(D)$. Modding $(\widetilde{E}_2^{1,8k-1})^{\#}$ by $h_1^{\#}(b)$ for other elements b in the basis of $(\widetilde{E}_2^{2,8k+1})^{\#}$ corresponds to dividing other relations $\psi^2(\Delta)$, $\psi^2(x)$, or $(\psi^3-3^{4k-1})(x)$ by 2. Since it is only dividing the fourth relation by 2 that lowers the order of the fourth summand, we deduce that the first component of $h_1^{\#}(\alpha_0 D+\sum\alpha_i x_i)$ in $(\widetilde{E}_2^{1,8k-1})^{\#}$ equals α_0 times the element of order 2, or dually that $h_1(g_1)=D$. This implies $d_3(g_1)=0$ since $d_3(D)=0$.

We prove now that $d_3=0:\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+2))\to\widetilde{E}_2^{4,1}(\mathrm{Spin}(8i+2))$. By [2, 3.3], $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+1))\to\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+2))$ is bijective. By the proof of [2, 3.11], $\widetilde{E}_2^{4,1}(\mathrm{Spin}(8i+2))\to\widetilde{E}_2^{4,1}(\mathrm{Spin}(8i+3))$ is injective.[4] By [2, 3.1], $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+1))\approx\mathbf{Z}/2^{4i}\oplus\mathbf{Z}/2^{4i}$ and $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+3))\approx\mathbf{Z}/2^{4i+1}\oplus\mathbf{Z}/2^{4i+1}$. Let $x\in\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+2))$. Then $i_*(x)=2y\in\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+3))$. Hence $i_*(d_3(x))$ is divisible by 2, and hence is 0, since it lies in a $\mathbf{Z}_2$-vector space. The injectivity of i_* on $\widetilde{E}_2^{4,1}$ implies that $d_3(x)=0$.

Next we consider $\mathrm{Spin}(8i+4)$. From [2, 6.1], we see that $\widetilde{E}_2^{4,1}(\mathrm{Spin}(8i+4))$ has basis dual to

$$\{x_{4i-1},D_+\}\cup B_C[2i,4i]\cup\{(D_+-D_-)_s,(D_+-D_-)_u\},$$

where the two classes (D_+-D_-) map nontrivially to $\widetilde{E}_2^{4,1}(S^{8i+3})$. By Diagram 2.19, d_3 acts injectively on $\widetilde{E}_2^{4,1}(S^{8i+3})$, and hence it does also on the classes (D_+-D_-).

[3] We use Δ to denote elements of $K^1(\mathrm{Spin}(8i+3))$ instead of the D that was used in [2] to avoid confusion with the element D of $(E_2^{2,8k-1})^{\#}$. Also note that k of [2, 7.9] is 0 here.

[4] The proof there deals with $E_2^{4,8k+5}$ but applies also to $E_2^{4,8k+1}$.

The element D_+ also supports a nonzero d_3 from $\widetilde{E}_3^{4,1}(\mathrm{Spin}(8i+4))$. This is true because of 2.21 and the fact that in [2, 3.7], the element D in position $(8k-3,4)$ did not support a nonzero d_3 in the spectral sequence for $v_1^{-1}\pi_*(\mathrm{Spin}(8i+3))$. Thus the only elements that $d_3(g_1)$ might hit are dual to x_{4i-1} or $B_C[2i,4i]$. By the argument used above in the case of $\mathrm{Spin}(8i+3)$, $h_1^\#$ does not send the corresponding elements of $\widetilde{E}_2^{2,1}(\mathrm{Spin}(8i+4))^\#$ to the element of order 2 in $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+4))^\#$ because dividing the corresponding relations by 2 will not lower the order of the first (ξ_1) summand. Thus $d_3(g_1)=0$ on the stable summand of $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+4))$. That the same is true in $\mathrm{Spin}(8i+5)$ and $\mathrm{Spin}(8i+6)$ follows by naturality, since

$$\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+4))\to\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+5))\to\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+6))$$

send the first summand bijectively. ■

3. Proof of results for $SO(m)$ when $m\le 10$

In this section, we prove Theorem 1.11, which we showed in Section 1 implies Theorem 1.4. Regarding the comment after 1.11 about the meaning of the so-called stable summand **s**, we will make the following distinction: as described there, **s** will refer to stable classes, those which map nontrivially to SO, while **f** will refer to the "first" summand, the summand generated by the stable classes, which may contain multiples of the stable class which are not stable, inasmuch as they become 0 upon stabilization. It is the case that **s** and **f** are equal in $\mathrm{Spin}(m)$ for $m\ge 7$, as we shall see.

By [2, 3.19], for $m=5$, 6, 7, 8, 9, and 10, $\mathbf{f}E_2^{1,-1}(\mathrm{Spin}(m))\approx\mathbf{f}\widetilde{E}_2^{1,-1}(\mathrm{Spin}(m))$ is given by

$$\mathbf{Z}/16\xrightarrow{\approx}\mathbf{Z}/16\xrightarrow{2}\mathbf{Z}/8\xrightarrow{\approx}\mathbf{Z}/8\xrightarrow{\approx}\mathbf{Z}/8\xrightarrow{\approx}\mathbf{Z}/8. \tag{3.1}$$

In the spectral sequence converging to $v_1^{-1}\pi_*(\mathrm{Spin}(m))$, when $m=7$ the extension $(\cdot 2)$ from $\mathbf{f}E_\infty^{1,-1}$ to $E_\infty^{3,1}$ is trivial by [2, 1.4], as is $d_3:\mathbf{f}E_3^{1,-1}\to E_4^{4,1}$. The same is true for $m=8$, 9, and 10 by naturality.

In the spectral sequence converging to $v_1^{-1}\pi_*(\mathrm{Spin}(6))$, differentials and extensions from $E_r^{1,-1}$ are trivial by [2, 3.11]. Indeed, there is nothing for d_3 to hit, and extensions are ruled out by $2\eta=0$. In the spectral sequence converging to $v_1^{-1}\pi_*(\mathrm{Spin}(5))$, there is a nontrivial d_3-differential and a nontrivial extension from $E_r^{-1,1}$, as can be seen by comparison with [19, 1.7], using that $\mathrm{Spin}(5) = Sp(2)$. Thus $\mathbf{f}v_1^{-1}\pi_{-2}(\mathrm{Spin}(m))$ is, for $m = 5, 6, 7, 8, 9$, and 10, given by

$$\mathbf{Z}/16 \xrightarrow{2} \mathbf{Z}/16 \xrightarrow{2} \mathbf{Z}/8 \xrightarrow{\approx} \mathbf{Z}/8 \xrightarrow{\approx} \mathbf{Z}/8 \xrightarrow{\approx} \mathbf{Z}/8.$$

Since the multiples of 4 in $v_1^{-1}\pi_{-2}(\mathrm{Spin}(6))$ and multiples of 2 in $v_1^{-1}\pi_{-2}(\mathrm{Spin}(5))$ stabilize to 0, they are not included in $\mathbf{s}$, as discussed at the beginning of this section, and so we obtain the first part of Theorem 1.11.

Since $\mathrm{Spin}(4) \approx S^3 \times S^3$, we deduce $4v_1^{-1}\pi_*(\mathrm{Spin}(4)) = 0$, and so $v_1^{-1}\pi_{-2}(\mathrm{Spin}(4)) \to v_1^{-1}\pi_{-2}(\mathrm{Spin}(5))$ cannot hit an element which stabilizes nontrivially, since 8 times such an element is nonzero in $v_1^{-1}\pi_{-2}(\mathrm{Spin}(5))$.

We will show that in the spectral sequence $\widetilde{E}_r(\mathrm{Spin}(m))$, whose $\widetilde{E}_2^{1,-1}$ is given in (3.1), there is a nontrivial extension from $\widetilde{E}_\infty^{1,-1}$ for $m = 7, 8, 9$, and 10 (but not when $m = 5$ or 6), and a nonzero d_3-differential from $\widetilde{E}_3^{1,-1}$ when $m = 5, 7, 8$, and 9 (but not when $m = 6$ and 10). From this, it is immediate that $\mathbf{f}v_1^{-1}\pi'_{-2}(\mathrm{Spin}(m))$ is, for $m = 5, 6, 7, 8, 9$, and 10, given by

$$\mathbf{Z}/8 \xrightarrow{2} \mathbf{Z}/16 \xrightarrow{1} \mathbf{Z}/8 \xrightarrow{\approx} \mathbf{Z}/8 \xrightarrow{\approx} \mathbf{Z}/8 \xrightarrow{2} \mathbf{Z}/16.$$

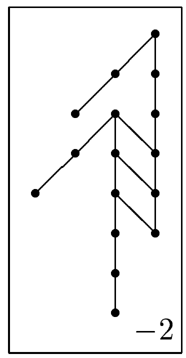
Since the multiples of 8 in $v_1^{-1}\pi_{-2}(\mathrm{Spin}(6))$ and multiples of 4 in $v_1^{-1}\pi_{-2}(\mathrm{Spin}(5))$ stabilize to 0, they are not included in $\mathbf{s}$, and so we obtain the second part of Theorem 1.11.

First we show that $d_3 \neq 0$ on $\mathbf{s}E_3^{1,-1}(\mathrm{Spin}(m))$ when $m = 9$. This implies $d_3 \neq 0$ when $m = 7$ and 8, too. By the argument after [2, 7.2], $h_1 : \widetilde{E}_2^{1,-1}(\mathrm{Spin}(m)) \to \widetilde{E}_2^{2,1}(\mathrm{Spin}(m))$ is injective. Since d_3 on eta-towers of $\widetilde{E}_3(\mathrm{Spin}(m))$ is opposite to that on $E_3(\mathrm{Spin}(m))$, we deduce from [2, 1.3] that d_3 acts injectively on $\widetilde{E}^{2,1}(\mathrm{Spin}(9))$. Naturality of h_1 now implies that d_3 acts injectively on generators of $\widetilde{E}_3^{1,-1}(\mathrm{Spin}(9))$.

Next we deduce the nonzero extension (and give another proof of the nonzero d_3) on $\widetilde{E}_r^{1,-1}(\mathrm{Spin}(7))$ by a comparison of charts for $v_1^{-1}\pi'_*(G_2)$ deduced from [18, p.666] and [5, Fig.2,p.1276]. The relevance of G_2 is the 2-primary decomposition $\mathrm{Spin}(7) \simeq G_2 \times S^7$. This extension result was already alluded to in the previous section, where it was noted that this is the beginning of a range of nontrivial extensions, and Spin is at the end, and either one of them can be used to deduce the whole batch, so that this proof is not really essential to our proof.

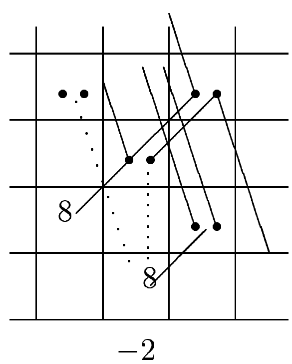
The aforementioned charts are for $v_1^{-1}\pi_*(G_2)$, and so must be modified as in the proof of 2.18 to give $v_1^{-1}\pi'_*(G_2)$. The J-chart approach of [18] must be shifted by 4 dimensions with higher differentials staying the same in a fixed dimension. Thus $v_1^{-1}\pi'_*(G_2)$ near $* = -2$ is as in Diagram 3.2.

Diagram 3.2. $v_1^{-1}\pi'_*(G_2)$



From the point of view of 2.16, we have the same E_2 as in [5], but d_3 on eta-towers is reversed. The chart near $t - s = -2$ is given in Diagram 3.3.

Diagram 3.3. $v_1^{-1}\pi'_*(G_2)$



By [2, after 7.2], $h_1 : \widetilde{E}_2^{1,-1} \to \widetilde{E}_2^{2,1}$ is injective, and since d_3 acts injectively on $\widetilde{E}_3^{2,1}$, it must also act injectively on $\widetilde{E}_3^{1,-1}$. The extension from $\widetilde{E}_\infty^{1,-1}$ must be nontrivial to give the $\mathbf{Z}/8$ group deduced from the first approach. The extensions from $\widetilde{E}_\infty^{1,-1}(\mathrm{Spin}(m))$ for $8 \le m \le 10$ are then deduced by naturality as explained in the previous section.

Similarly to the proof in the previous section for $\mathrm{Spin}(8i+2)$ with $i > 1$, we deduce that $d_3 = 0$ from $\mathbf{s}\widetilde{E}_3^{1,-1}(\mathrm{Spin}(10))$. Indeed, $\widetilde{E}_2^{4,1}(\mathrm{Spin}(10)) \to \widetilde{E}_3^{4,1}(\mathrm{Spin}(11))$

is injective, but $\mathbf{s}\widetilde{E}_2^{1,-1}(\mathrm{Spin}(10)) \to \widetilde{E}_2^{1,-1}(\mathrm{Spin}(11))$ maps onto elements divisible by 8.

The groups $v_1^{-1}\pi'_*(\mathrm{Spin}(5)) = v_1^{-1}\pi'_*(Sp(2))$ can be obtained similarly to the J-chart determination of $v_1^{-1}\pi_*(Sp(2))$ in [19]. To obtain $v_1^{-1}\pi'_*(Sp(2))$, [19, Fig.2.1] should be shifted by 4 dimensions, and d_1-differentials inserted from the new $8k+2$ to $8k+1$. But these differentials are not needed for our purposes. Since $v_1^{-1}\pi'_{-2}(S^3) = 0$ and $v_1^{-1}\pi'_{-2}(S^7) = \mathbf{Z}/8$, the exact sequence of the fibration $S^3 \to Sp(2) \to S^7$ implies that $v_1^{-1}\pi'_{-2}(Sp(2))$ is at most $\mathbf{Z}/8$. Thus the $\mathbf{Z}/16$ in $\mathbf{f}\widetilde{E}_2^{1,-1}(\mathrm{Spin}(5))$ must support a nonzero d_3 and cannot extend.

There can be no extension from $\widetilde{E}_\infty^{1,-1}(\mathrm{Spin}(6))$ by naturality. Finally d_3 is 0 on $\mathbf{f}\widetilde{E}_3^{1,-1}(\mathrm{Spin}(6))$ since its image in Spin(7) consists of multiples of 2, but the target classes $\widetilde{E}_3^{4,1}$ map injectively from Spin(6) to Spin(7) by [2, 6.1].

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