Combinatorial operad actions on cochains

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Abstract

A classical E-infinity operad is formed by the bar construction of the symmetric groups. Such an operad has been introduced by M. Barratt and P. Eccles in the context of simplicial sets in order to have an analogue of the Milnor FK-construction for infinite loop-spaces. The purpose of this article is to prove that the associative algebra structure on the normalized cochain complex of a simplicial set extends to the structure of an algebra over the Barratt-Eccles operad. We also prove that differential graded algebras over the Barratt-Eccles operad form a closed model category. Similar results hold for the normalized Hochschild cochain complex of an associative algebra. More precisely, the Hochschild cochain complex is acted on by a suboperad of the Barratt-Eccles operad which is equivalent to the classical little square operad.

Introduction

This article follows the work of M. Mandell which compares the algebras over an E_∞-operad to the homotopy category of simplicial sets (cf. [18]). We consider the normalized cochain complex $N^*(X)$ associated to a simplicial set X. Our main purpose is to provide the complex $N^*(X)$ with the structure of an algebra over a suitable E_∞-operad. The results of M. Mandell imply that any such structure determines the homotopy type of the simplicial set X (under reasonable finiteness and completeness assumptions on X).

There is a classical E_∞-operad which is denoted by the letter $\mathcal{E}$ in this article and whose component $\mathcal{E}(r)$, $r = 0, 1, \ldots$, is given by the normalized bar construction of the symmetric group Σ_r. The operad structure is provided by an explicit composition product:

$$\mathcal{E}(r) \otimes \mathcal{E}(s_1) \otimes \cdots \otimes \mathcal{E}(s_r) \longrightarrow \mathcal{E}(s_1 + \cdots + s_r).$$

The degree 0 component of $\mathcal{E}(r)$ is the regular representation of the symmetric group Σ_r and is identified with the module generated by the multilinear monomials in r non-commutative variables. Hence, the degree 0 elements represent operations associated to the structure of an associative algebra. In this article, the differential graded operad $\mathcal{E}$ is called the Barratt-Eccles operad, because the associated simplicial operad $\mathcal{W}$ is introduced by M. Barratt and P. Eccles for the study of infinite loop spaces (cf. [1]).

We obtain the following result:

Theorem

For any simplicial set X, the normalized cochain complex $N^*(X)$ is equipped with the structure of an algebra over the Barratt-Eccles operad $\mathcal{E}$.

In fact, the existence of such an algebra structure is given by a Theorem of Justin Smith (cf. [28]). Our purpose is to give an explicit construction for it. This is achieved by the Theorems 1.3.2 and 2.2.7 in the article. As a corollary, we obtain a simple formula for the evaluation product

$$\mathcal{E}(r)\otimes N^*(S^n)^{\otimes r} \longrightarrow N^*(S^n)$$

on the cochain complex of the spheres S^n (cf. Theorem 3.2.4 and Proposition 3.2.5).

The Barratt-Eccles operad comes equipped with a diagonal $\mathcal{E}(r) \longrightarrow \mathcal{E}(r)\otimes\mathcal{E}(r)$ because it is the chain operad associated to a simplicial operad. This structure has interesting applications which make the Barratt-Eccles operad suitable for calculations in homotopy theory. For instance, we prove that differential graded algebras over the Barratt-Eccles operad form a closed model category. (It is not clear that this property is satisfied for algebras over a general E_∞-operad.) In fact, we have a more general result. To be precise, a classical construction replaces an operad $\mathcal{Q}$ by the operad $\mathcal{P} = \mathcal{E}\otimes\mathcal{Q}$ such that $\mathcal{P}(r) = \mathcal{E}(r)\otimes\mathcal{Q}(r)$. The operad $\mathcal{P}$ is a Σ_*-projective resolution of $\mathcal{Q}$. The Barratt-Eccles operad $\mathcal{P} = \mathcal{E}$ corresponds to the case $\mathcal{Q} = \mathcal{C}$ of the commutative operad. The following Theorem is proved in Section 3.1:

THEOREM

Let $\mathcal{Q}$ be an operad. We let $\mathcal{P} = \mathcal{E}\otimes\mathcal{Q}$ be the standard Σ_*-projective resolution of $\mathcal{Q}$. The $\mathcal{P}$-algebras are equipped with a closed model category structure for which the weak equivalences (respectively, the fibrations) are the algebra morphisms which are weak equivalences (respectively, fibrations) in the category of dg-modules.

The simplicial Barratt-Eccles operad has also a filtration

$$F_1\mathcal{W} \subseteq F_2\mathcal{W} \subseteq \cdots \subseteq F_n\mathcal{W} \subseteq \cdots \subseteq \mathcal{W}$$

by simplicial suboperads $F_n\mathcal{W}$ whose topological realization $|F_n\mathcal{W}|$ is equivalent as a topological operad to the classical operad of little n-cubes (cf. C. Berger [3], T. Kashiwabara [15], J. Smith [27]). We have an induced filtration on the associated differential graded operad

$$F_1\mathcal{E} \subseteq F_2\mathcal{E} \subseteq \cdots \subseteq F_n\mathcal{E} \subseteq \cdots \subseteq \mathcal{E}.$$

The operad $F_1\mathcal{E}$ is identified with the associative operad. The operad $F_2\mathcal{E}$ is the chain operad on a simplicial operad equivalent to the little square operad. The next result implies that this operad $F_2\mathcal{E}$ gives a solution to a conjecture of Deligne:

THEOREM

The normalized Hochschild cochain complex of an associative algebra is equipped with the structure of an algebra over the operad $F_2\mathcal{E}$.

The pivot of our constructions is formed by a quotient operad $\mathcal{X}$ of the Barratt-Eccles operad $\mathcal{E}$ which we call the surjection operad. This operad is also introduced (with different sign conventions) by J. McClure and J. Smith in their work on the Deligne conjecture (cf. [21], [22]) to which we refer in this article. Our results follow from the next Theorem. In fact, the surjection operad $\mathcal{X}$ acts naturally on the cochain complex associated to a simplicial set. In this context, the idea of the surjection operad goes back to D. Benson (cf. [2]) and to the original construction of the reduced square operations by N. Steenrod (cf. [29]). The introduction of the surjection operad in the context of the Deligne conjecture is due to J. McClure and J. Smith. In fact, the surjection operad $\mathcal{X}$ is endowed with a filtration

$$F_1\mathcal{X} \subseteq F_2\mathcal{X} \subseteq \cdots \subseteq F_n\mathcal{X} \subseteq \cdots \subseteq \mathcal{X}$$

by suboperads $F_n\mathcal{X}$ such that $F_1\mathcal{X}$ is identified with the associative operad. Furthermore, one observes that the operad $F_2\mathcal{X}$ acts naturally on the Hochschild cochain complex of an associative algebra (cf. [21], [22]).

THEOREM

The surjection operad $\mathcal{X}$ is a quotient of the Barratt-Eccles operad $\mathcal{E}$. More precisely, there is a surjective morphism of filtered operads $TR : \mathcal{E} \longrightarrow \mathcal{X}$ such that $F_1 TR : F_1\mathcal{E} \longrightarrow F_1\mathcal{X}$ is the identity isomorphism of the associative operad.

We call the quotient morphism $TR : \mathcal{E} \longrightarrow \mathcal{X}$ the table reduction morphism, by reference to its construction.

J. McClure and J. Smith have proved that the surjection operad $F_n\mathcal{X}$ is equivalent to the chain-operad of the little n-cubes. (As a consequence, the operad $F_2\mathcal{X}$ is also a solution to the Deligne conjecture.) We prove more precisely that the table reduction morphism induces a weak equivalence $F_n TR : F_n\mathcal{E} \longrightarrow F_n\mathcal{X}$. But, since the operads $F_n\mathcal{X}$ are not directly associated to a simplicial operad, it is not clear that some of our constructions are available in the context of algebras over the surjection operad. This motivates the introduction of the Barratt-Eccles operad and of the table reduction morphism.

We refer to Section 0 for the conventions on operads which are adopted throughout the article. The detailed definition of the Barratt-Eccles operad is given in Section 1.1. The surjection operad is introduced in Section 1.2. Section 1.3 is devoted to the construction of the table reduction morphism. The properties of the morphism are proved in Sections 1.4 and 1.5. (These two Sections may be skipped in a first reading.) The little cube filtration on the operads and the applications to the Deligne conjecture are explained in Section 1.6. The construction of the algebra structure on the normalized cochain complex of a simplicial set is achieved in Section 2. The applications of our constructions to homotopical algebra are explained in Section 3.

Contents

Acknowledgements:

The morphisms from the Barratt-Eccles operad to the surjection operad and to the Eilenberg-Zilber operad have been constructed (modulo the signs) by the second author and lectured on at the fall-school "Lola" in Malaga in November 2000. The second author would like to thank the organizers and the participants of the school who provided the first motivation for this work. We thank Jim McClure and Jeff Smith for their interest on our work which, at some places, should complement their results on the Deligne conjecture. In fact, the first author realized that the Barratt-Eccles operad provides a solution to the Deligne conjecture after reading the announcement of the article [22].

§0. Conventions

0.1. *On permutations*

We let Σ_r, $r \in \mathbb{N}$, denote the permutation groups. The element $1_r \in \Sigma_r$ is the identical permutation $1_r = (1, 2, \dots, r)$. In general, a permutation $\sigma \in \Sigma_r$ is specified by the sequence $(\sigma(1), \sigma(2), \dots, \sigma(r))$ formed by its values.

The permutation $\sigma \in \Sigma_r$ determines a block permutation $\sigma_*(s_1, \dots, s_r) \in \Sigma_{s_1+\dots+s_r}$ (for $s_1, \dots, s_r \in \mathbb{N}$). In the sequence $(\sigma(1), \sigma(2), \dots, \sigma(r))$, we replace the occurence of $k = 1, \dots, r$ by the sequence $1, 2, \dots, s_k$ together with a shift of $s_1 + s_2 + \dots + s_{k-1}$. For instance, if σ is the transposition $\sigma = (2, 1)$, then $\sigma_*(p, q)$ is the block transposition $\sigma_*(p, q) = (p + 1, \dots, p + q, 1, \dots, p)$.

0.2. *On the symmetric monoidal category of dg-modules*

Let us fix a ground ring $\mathbb{F}$. The category of $\mathbb{F}$-modules is denoted by $\mathcal{M}od_{\mathbb{F}}$. The category of differential graded modules is denoted by $dg\,\mathcal{M}od_{\mathbb{F}}$.

If V is a differential graded module (or, for short, a dg-module), then V has either an upper $V = V^*$ or a lower $V = V_*$ graduation. The equivalence between lower and upper graduations is given by the classical rule. To be explicit, the component of upper degree d is equivalent to the component of lower degree $-d$ of the dg-module. Thus: $V^* = V_{-*}$. Let us mention that differential degrees are assumed to range over the integers without any restriction. In general, the differential of a dg-module is denoted by the letter $\delta : V^* \longrightarrow V^{*+1}$.

The category of differential graded modules is equipped with its classical tensor product

$$\otimes : dg\,\mathcal{M}od_{\mathbb{F}} \times dg\,\mathcal{M}od_{\mathbb{F}} \longrightarrow dg\,\mathcal{M}od_{\mathbb{F}}\,.$$

If $U, V \in dg\,\mathcal{M}od_{\mathbb{F}}$, then $U \otimes V \in dg\,\mathcal{M}od_{\mathbb{F}}$ is the dg-module which has the module

$$(U \otimes V)^n = \bigoplus_{d\in\mathbb{Z}} U^d \otimes V^{n-d}$$

in degree n. The differential of $U \otimes V$ is given by the derivation rule:

$$\delta(u \otimes v) = \delta(u) \otimes v + (-1)^d u \otimes \delta(v),$$

for all homogeneous elements $u \in U^d$ and $v \in V^e$. The symmetry operator $\tau : U \otimes V \longrightarrow V \otimes U$ follows the rule of signs. If $u \in U^d$ and $v \in V^e$, then we have:

$$\tau(u \otimes v) = (-1)^{de} v \otimes u.$$

In general, a permutation of homogeneous symbols, which have d and e as degrees, produces the sign $(-1)^{de}$. For instance, in the derivation rule, the permutation of the differential δ, which has degree 1, with the element u, homogeneous of degree d, produces the sign $(-1)^d$.

A morphism $f : U \longrightarrow V$ is homogeneous of upper degree d if it raises upper degrees by d and lowers lower degrees by d. The differential of this morphism $\delta(f) : U \longrightarrow V$ is given by the commutator of f with the differentials of U and V:

$$\delta(f) = \delta \cdot f - (-1)^d f \cdot \delta.$$

The dg-module formed by the homogeneous morphisms $f : U \longrightarrow V$ is denoted by $\mathrm{Hom}^*_{\mathbb{F}}(U,V)$. Thus, we have explicitly:

$$\mathrm{Hom}^d_{\mathbb{F}}(U,V) = \prod_{*\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{F}}(U^*, V^{*+d}) = \prod_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{F}}(U_{*+d}, V_*).$$

This dg-module $\mathrm{Hom}^*_{\mathbb{F}}(U,V)$ is an internal hom in the category of dg-modules.

0.3. *The symmetric monoidal categories*

In fact, we are concerned with the following classical symmetric monoidal categories.

1) The category of $\mathbb{F}$-modules $\mathcal{M}od_{\mathbb{F}}$ together with the classical tensor product:

$$\otimes : \mathcal{M}od_{\mathbb{F}} \times \mathcal{M}od_{\mathbb{F}} \longrightarrow \mathcal{M}od_{\mathbb{F}}.$$

2) The category of dg-modules $dg\,\mathcal{M}od_{\mathbb{F}}$ together with the tensor product of dg-modules:

$$\otimes : dg\,\mathcal{M}od_{\mathbb{F}} \times dg\,\mathcal{M}od_{\mathbb{F}} \longrightarrow dg\,\mathcal{M}od_{\mathbb{F}},$$

whose definition is recalled in the Paragraph above.

3) The category of sets $\mathcal{S}et$ together with the cartesian product:

$$\times : \mathcal{S}et \times \mathcal{S}et \longrightarrow \mathcal{S}et.$$

4) The category of simplicial sets $\mathcal{S}$ together with the cartesian product of simplicial sets:

$$\times : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}.$$

Let us also introduce the symmetric monoidal category of (differential graded) coalgebras. Classically, a coalgebra is a (differential graded) module $X \in \mathcal{M}od_{\mathbb{F}}$ together with an associative diagonal $\Delta : X \longrightarrow X \otimes X$. If X and Y are coalgebras, then the tensor product $X \otimes Y \in \mathcal{M}od_{\mathbb{F}}$ is equipped with a canonical diagonal given by the composite

$$X \otimes Y \xrightarrow{\Delta\otimes\Delta} X \otimes X \otimes Y \otimes Y \xrightarrow{(2\ 3)} X \otimes Y \otimes X \otimes Y.$$

It follows that the coalgebras form a symmetric monoidal category.

0.4. *On normalized chains and cochains*

If X is a simplicial set, then $C_*(X)$ denotes the dg-module whose degree d component $C_d(X)$ is the free $\mathbb{F}$-module generated by the set X_d of the d-dimensional simplices in X. The differential of $x \in X_d$ in $C_*(X)$ is given by the classical formula: $\delta(x) = \sum_{i=0}^d (-1)^i d_i(x)$. The normalized chain complex $N_*(X)$ is the quotient of the dg-module $C_*(X)$ by the degeneracies:

$$N_d(X) = C_d(X)/s_0 C_{d-1}(X) + \cdots + s_{d-1} C_{d-1}(X).$$

We consider also the dual cochain complexes $C^*(X) = \mathrm{Hom}_{\mathbb{F}}(C_*(X), \mathbb{F})$ and $N^*(X) = \mathrm{Hom}_{\mathbb{F}}(N_*(X), \mathbb{F})$.

Let us recall that the normalized chain complex $X \mapsto N_*(X)$ is a (symmetric) monoidal functor from the category of simplicial sets to the category of dg-modules. More precisely, the Eilenberg-Zilber morphism provides a natural equivalence

$$EZ : N_*(X) \otimes N_*(Y) \xrightarrow{\sim} N_*(X \times Y)$$

which is associative and commutes with the symmetry isomorphism.

The dg-module $N_*(X)$ is also equipped with a natural diagonal. This diagonal is given by the classical Alexander-Whitney formula. Since, furthermore, the diagonal is compatible with the Eilenberg-Zilber morphism, we conclude that the normalized chain complex $X \mapsto N_*(X)$ is a (symmetric) monoidal functor from the category of simplicial sets to the category of dg-coalgebras.

We refer to the classical textbook of S. Mac Lane (cf. [17, Chapter VIII]) for the properties of the Eilenberg-Zilber equivalence.

0.5. *On the notion of an operad*

The main structures involved in our article are operads in the symmetric monoidal category of dg-modules. But, for simplicity, we recall the definition of an operad in the category of $\mathbb{F}$-modules. In fact, the notion of an operad makes sense in any symmetric monoidal category. For an introduction to the subject, we refer to the works of E. Getzler and J. Jones (cf. [9]), V. Ginzburg and M. Kapranov (cf. [10]) and to the works of P. May (cf. [20]).

A Σ_*-module M is a sequence $M = M(r)$, $r \in \mathbb{N}$, where $M(r)$ is a representation of the permutation group Σ_r. A morphism of Σ_*-modules $f : M \longrightarrow N$ is a sequence of morphisms of representations $f : M(r) \longrightarrow N(r)$.

An operad $\mathcal{P}$ is a Σ_*-module equipped with a composition product

$$\mathcal{P}(r) \otimes \mathcal{P}(s_1) \otimes \cdots \otimes \mathcal{P}(s_r) \longrightarrow \mathcal{P}(s_1 + \cdots + s_r),$$

defined for $r \geq 1$, $s_1, \ldots, s_r \geq 0$, and which statisfies some natural equivariance and associativity properties. A morphism of operads $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is a morphism of Σ_*-modules which commutes with the operad composition product.

In general, an element $p \in \mathcal{P}(r)$ represents a (multilinear) operation in r variables $x_1, \ldots, x_r$. Thus, we may write $p = p(x_1, \ldots, x_r)$. The action of a permutation $\sigma \in \Sigma_r$ on $p \in \mathcal{P}(r)$ is given by the permutation of the variables. Explicitly, we have $\sigma \cdot p = p(x_{\sigma(1)}, \ldots, x_{\sigma(r)})$. The composition product of $p \in \mathcal{P}(r)$ and $q_1 \in \mathcal{P}(s_1), \ldots, q_r \in \mathcal{P}(s_r)$ is denoted by $p(q_1, \ldots, q_r) \in \mathcal{P}(s_1 + \cdots + s_r)$. In fact, this operation is given by the substitution of the variables $x_1, \ldots, x_r$ by the operations $q_1, \ldots, q_r$.

We assume that the composition product of an operad has a unit $1 \in \mathcal{P}(1)$. This unit represents the identical operation $1(x_1) = x_1$. In this context, there are also partial composition products:

$$\circ_k : \mathcal{P}(r) \otimes \mathcal{P}(s) \longrightarrow \mathcal{P}(r + s - 1),$$

for $k = 1, \ldots, r$. If $p \in \mathcal{P}(r)$ and $q \in \mathcal{P}(s)$, then $p \circ_k q = p(1, \ldots, 1, q, 1, \ldots, 1)$. (The operation q is set at the kth entry of p.)

0.6. *On algebras over an operad*

The operad $\mathcal{P}$ has an associated category of algebras denoted by $\mathcal{A}lg_{\mathcal{P}}$. The set of the $\mathcal{P}$-algebra morphisms from $A \in \mathcal{A}lg_{\mathcal{P}}$ to $B \in \mathcal{A}lg_{\mathcal{P}}$ is denoted by $\operatorname{Hom}_{\mathcal{P}}(A, B)$. To fix a definition, a $\mathcal{P}$-algebra is an $\mathbb{F}$-module A equipped with an evaluation product

$$\mathcal{P}(r) \otimes A^{\otimes r} \longrightarrow A,$$

equivariant and defined for any $r \leq 0$. This evaluation product has to be associative with respect to the operad composition product and unital with respect to the operad unit. The image of the elements $p \in \mathcal{P}(r)$ and $a_1, \ldots, a_r \in A$ under the evaluation product is denoted by $p(a_1, \ldots, a_r) \in A$. The equivariance relation reads $(\sigma \cdot p)(a_1, \ldots, a_r) = p(a_{\sigma(1)}, \ldots, a_{\sigma(r)})$. A morphism of $\mathcal{P}$-algebras $f : A \longrightarrow B$ is a morphism of $\mathbb{F}$-modules which commutes with the evaluation product. The free $\mathcal{P}$-algebra generated by an $\mathbb{F}$-module V is denoted by $\mathcal{P}(V)$.

0.7. *The morphism operad*

If V is an $\mathbb{F}$-module, then the morphism operad associated to V is the operad such that $\operatorname{Hom}_V(r) = \operatorname{Hom}_{\mathbb{F}}(V^{\otimes r}, V)$. The composition product of the morphism operad is just given by the composition of multilinear maps. If $f : V^{\otimes r} \longrightarrow V$, then $\sigma \cdot f : V^{\otimes r} \longrightarrow V$ is defined by $f(v_1, \ldots, v_r) = f(v_{\sigma(1)}, \ldots, v_{\sigma(r)})$. Observe that V is an algebra over Hom_V. This algebra structure is universal. If $\mathcal{P}$ is an operad, then a structure of a $\mathcal{P}$-algebra on V is equivalent to an operad morphism $\mathcal{P} \longrightarrow \operatorname{Hom}_V$ (cf. [10]).

0.8. *On operads in symmetric monoidal categories*

We have mentioned that the notion of an operad makes sense in any symmetric monoidal category. One has just to replace the classical tensor product by any other tensor product in the definitions. For instance, there are operads in the category of sets and in the category of simplicial sets. In this case, the tensor product is replaced by the cartesian product. The symmetric group is always assumed to operate on the left.

0.9. *On the normalized chain complex associated to a simplicial operad*

If $\mathcal{O}$ is a simplicial operad, then the associated normalized chain complexes $\mathcal{P}(r) = N_*(\mathcal{O}(r))$ form an operad in the category of dg-coalgebras, because the normalized chain complex functor $N_*(-)$ is symmetric monoidal as reminded in the Paragraph 0.4. Explicitly, the dg-module $N_*(\mathcal{O}(r))$ is equipped with the Alexander-Whitney diagonal

$$N_*(\mathcal{O}(r)) \longrightarrow N_*(\mathcal{O}(r)) \otimes N_*(\mathcal{O}(r)).$$

The composition product is defined as the composite of the composition product of $\mathcal{O}$ with the Eilenberg-Zilber equivalence:

$$\begin{aligned} N_*(\mathcal{O}(r)) \otimes N_*(\mathcal{O}(s_1)) \otimes \cdots \otimes N_*(\mathcal{O}(s_r)) &\longrightarrow N_*(\mathcal{O}(r) \times \mathcal{O}(s_1) \times \cdots \times \mathcal{O}(s_r)) \\ &\longrightarrow N_*(\mathcal{O}(s_1 + \cdots + s_r)). \end{aligned}$$

0.10. *On Hopf operads*

Operads in the monoidal category of coalgebras are also known as Hopf operads (cf. [9]). Explicitly, a Hopf operad is an operad in the category of dg-modules $\mathcal{P}$ together with a diagonal on each component

$$\mathcal{P}(r) \longrightarrow \mathcal{P}(r) \otimes \mathcal{P}(r),\ r \in \mathbb{N}.$$

The composition product

$$\mathcal{P}(r) \otimes \mathcal{P}(s_1) \otimes \cdots \otimes \mathcal{P}(s_r) \longrightarrow \mathcal{P}(s_1 + \cdots + s_r)$$

has to commute with the diagonal.

If $\mathcal{P}$ is an Hopf operad, then a $\mathcal{P}$-algebra denotes a (differential graded) module together with a $\mathcal{P}$-algebra structure. In fact, a $\mathcal{P}$-algebra in the symmetric monoidal category of coalgebras is called a Hopf-$\mathcal{P}$-algebras. (We should not consider such structures.) Here is the property of Hopf operads which interests us. If $A, B \in \mathcal{A}lg_{\mathcal{P}}$ are $\mathcal{P}$-algebras, then the tensor product $A \otimes B \in \mathcal{M}od_{\mathbb{F}}$ is equipped with a natural $\mathcal{P}$-algebra structure. The evaluation product of $A \otimes B$ is given by the composite of the evaluation products of A and B together with the diagonal of the operad:

$$\begin{aligned}\mathcal{P}(r) \otimes (A \otimes B)^{\otimes r} &\longrightarrow (\mathcal{P}(r) \otimes \mathcal{P}(r)) \otimes (A \otimes B)^{\otimes r} \\ &\longrightarrow (\mathcal{P}(r) \otimes A^{\otimes r}) \otimes (\mathcal{P}(r) \otimes B^{\otimes r}) \\ &\longrightarrow A \otimes B.\end{aligned}$$

Explicitly, let $\sum_i p^i_{(1)} \otimes p^i_{(2)} \in \mathcal{P}(r) \otimes \mathcal{P}(r)$ be the diagonal of an operation $p \in \mathcal{P}(r)$. Then, in $A \otimes B$, we have $p(a_1 \otimes b_1, \ldots, a_r \otimes b_r) = \sum_i p^i_{(1)}(a_1, \ldots, a_r) \otimes p^i_{(2)}(b_1, \ldots, b_r)$.

0.11. *The permutation operad*

The permutation groups Σ_r, $r \in \mathbb{N}$, form an operad in the category of sets. This operad is called the permutation operad. The composition product

$$\Sigma_r \times \Sigma_{s_1} \times \cdots \times \Sigma_{s_r} \longrightarrow \Sigma_{s_1 + \cdots + s_r}$$

is characterized by the relation

$$1_r(1_{s_1}, \ldots, 1_{s_r}) = 1_{s_1 + \cdots + s_r}.$$

Thus, if $u \in \Sigma_r$ and $v_1 \in \Sigma_{s_1}, \ldots, v_r \in \Sigma_{s_r}$, then, in $\Sigma_{s_1 + \cdots + s_r}$, we have the relation

$$\begin{aligned}u(v_1, \ldots, v_r) &= v_1 \oplus \cdots \oplus v_r \cdot u_*(s_1, \ldots, s_r) \\ &= u_*(s_1, \ldots, s_r) \cdot v_{u(1)} \oplus \cdots \oplus v_{u(r)}.\end{aligned}$$

In fact, the composed permutation $u(v_1, \ldots, v_r) \in \Sigma_{s_1 + \cdots + s_r}$ can be obtained by the following explicit substitution process. In the sequence $(u(1), u(2), \ldots, u(r))$, we replace the occurence of $k = 1, \ldots, r$ by the sequence $(v_k(1), v_k(2), \ldots, v_k(s_k))$ together with the appropriate shift on the indices. Precisely, we increase the index $v_k(j)$ by $s_1 + s_2 + \cdots + s_{k-1}$. The process is the same for a partial composition product

$$u \circ_k v = u(1, \ldots, 1, v, 1, \ldots, 1) \in \Sigma_{r+s-1}.$$

In this case, we have just to replace the occurence of k by the sequence $(v(1) + k - 1, v(2) + k - 1, \ldots, v(s) + k - 1)$ and to increase by s the elements $u(i)$ such that $u(i) > k$. For instance, if $u = (3, 2, 1)$ and $v = (1, 3, 2)$, then we obtain $u \circ_2 v = (5, \mathbf{2}, \mathbf{4}, \mathbf{3}, 1)$.

0.12. *The associative and the commutative operads*

The algebras over the permutation operad are just the associative monoids. There is also an operad in the category of sets, whose algebras are the associative and commutative monoids. We consider the associated operads in the category of $\mathbb{F}$-modules.

The associative operad $\mathcal{A}$ denotes the operad in the category of $\mathbb{F}$-modules whose algebras are the associative algebras. The module $\mathcal{A}(r)$ is generated by the multilinear monomials in r non-commutative variables. Explicitly, we have:

$$\mathcal{A}(r) = \bigoplus_{(i_1,\ldots,i_r)} \mathbb{F} \cdot x_{i_1} \cdots x_{i_r}$$

where $(i_1, \ldots, i_r)$ ranges over the permutations of $(1, \ldots, r)$.

The commutative operad $\mathcal{C}$ is the operad in the category of $\mathbb{F}$-modules whose algebras are the associative and commutative algebras. The module $\mathcal{C}(r)$ is generated by the multilinear monomials in r commutative variables. Thus, we have

$$\mathcal{C}(r) = \mathbb{F} \cdot x_1 \cdots x_r$$

and $\mathcal{C}(r)$ is the trivial representation of the symmetric group Σ_r.

§1. The operads

1.1. The Barratt-Eccles operad

1.1.1. *Summary*

In general, an E_∞-operad $\mathcal{E}$ is a Σ_*-projective resolution of the commutative operad $\mathcal{C}$. By definition, an operad is Σ_*-projective if it is projective as a Σ_*-module. (In fact, a Σ_*-module M is projective if each component $M(r)$ is a projective representation of the symmetric group.) Furthermore, an operad $\mathcal{E}$ is a resolution of the commutative operad $\mathcal{C}$, if it is endowed with an augmentation morphism $\mathcal{E} \longrightarrow \mathcal{C}$ which induces an isomorphism from the homology of $\mathcal{E}$ to the the commutative operad $\mathcal{C}$.

There is a canonical E_∞-operad $\mathcal{E}$ given by the bar construction on the symmetric groups. The aim of this section is to recall the definition and to make explicit the structures of this operad. Let us mention that the degree 0 component of $\mathcal{E}$ is equal to the associative operad. In fact, we have a factorization

$$\mathcal{A} \longrightarrow \mathcal{E} \xrightarrow{\sim} \mathcal{C}$$

of the operad morphism from the associative to the commutative operad. In addition, the operad $\mathcal{E}$ is equipped with a diagonal

$$\mathcal{E}(r) \longrightarrow \mathcal{E}(r) \otimes \mathcal{E}(r)$$

and forms a Hopf operad.

In the article, the dg-operad $\mathcal{E}$ is referred to as the *Barratt-Eccles operad*. In fact, the dg-operad $\mathcal{E}$ is the normalized chain complex on a simplicial operad $\mathcal{W}$ introduced by M. Barratt and P. Eccles for the study of infinite loop spaces (cf. [1]).

1.1.2. *The dg-module*

The dg-module $\mathcal{E}(r)$ is the normalized homogeneous bar complex associated to the symmetric group Σ_r. To be more explicit, the module $\mathcal{E}(r)_d$ is the $\mathbb{F}$-module generated by the $d+1$-tuples $(w_0, \ldots, w_d)$, where $w_i \in \Sigma_r$, $i = 0, 1, \ldots, d$. If some consecutive permutations w_i and w_{i+1} are equal, then, in $\mathcal{E}(r)_d$, we have $(w_0, \ldots, w_d) = 0$. The differential of $\mathcal{E}(r)$ is given by the classical formula:

$$\delta(w_0, \ldots, w_r) = \sum_{i=0}^{d} (-1)^i (w_0, \ldots, \widehat{w_i}, \ldots, w_d).$$

The permutations $\sigma \in \Sigma_r$ act diagonaly on $\mathcal{E}(r)_*$:

$$\sigma \cdot (w_0, \ldots, w_d) = (\sigma \cdot w_0, \ldots, \sigma \cdot w_d).$$

The quasi-isomorphism $\mathcal{E}(r) \xrightarrow{\sim} \mathcal{C}(r)$ is the classical augmentation $\mathcal{E}(r)_0 \longrightarrow \mathbb{F}$ from the homogeneous bar construction to the trivial representation. The degree 0 component of $\mathcal{E}(r)$ is clearly the regular representation of the symmetric group Σ_r. Therefore, we have a canonical morphism of dg-modules $\mathcal{A}(r) \longrightarrow \mathcal{E}(r)$.

It is possible to specify an element $(w_0, \dots, w_d) \in \mathcal{E}(r)$ by a table with $d+1$ rows indexed by $0, \dots, d$. The row i consists of the sequence $(w_i(1), \dots, w_i(r))$ and determines the permutation w_i. For instance, the table:

$$\left| \begin{array}{l} 1,2 \\ 2,1 \\ 1,2 \\ 2,1 \end{array} \right.$$

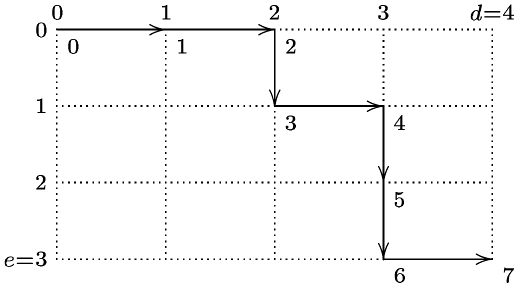
represents the element $(w_0, w_1, w_2, w_3) \in \mathcal{E}(2)_3$ such that $w_1 = w_3 = (2,1)$ is the transposition of Σ_2 and $w_0 = w_2 = (1,2)$ is the identity permutation of Σ_2.

1.1.3. *The composition product*

The composition product of the operad $\mathcal{E}$ is induced by the composition product of permutations. More precisely, if $u = (u_0, \dots, u_d) \in \mathcal{E}(r)_d$ and $v = (v_0, \dots, v_e) \in \mathcal{E}(s)_e$, then, in $\mathcal{E}(r+s-1)_{d+e}$, we have the identity

$$u \circ_k v = \sum\nolimits_{(x_*, y_*)} \pm (u_{x_0} \circ_k v_{y_0}, \dots, u_{x_{d+e}} \circ_k v_{y_{d+e}}).$$

The sum ranges over the set of paths in an $d \times e$-diagram such as:



The indices involved in the sum are given by the coordinates of the vertices. In the example above, we have:

$$(x_*, y_*) \;=\; (0,0),\ (1,0),\ (2,0),\ (2,1),\ (3,1),\ (3,2),\ (3,3),\ (4,3).$$

The sign associated to a term is determined by a permutation of the horizontal and vertical segments of the path. More precisely, one considers the shuffle permutation which takes the horizontal segments to the first places and the vertical segments to the last places. The sign is just defined as the signature of the shuffle. In the example, the segments

$$(0,1), (1,2), (2,3), (3,4), (4,5), (5,6), (6,7)$$

are permuted to

$$(0,1), (1,2), (3,4), (6,7), (2,3), (4,5), (5,6)$$

and, as a result, the associated sign is -1.

1.1.4. *The diagonal*

There is also a diagonal $\Delta : \mathcal{E}(r) \longrightarrow \mathcal{E}(r) \otimes \mathcal{E}(r)$ which represents the cup-product on the cohomology of the symmetric groups. This diagonal is given by the classical formula:

$$\Delta(w_0, \dots, w_n) = \sum_{d=0}^{n} (w_0, \dots, w_d) \otimes (w_d, \dots, w_n),$$

for any $(w_0, \dots, w_n) \in \mathcal{E}(r)_d$.

1.1.5. *On the Barratt-Eccles operad*

As mentioned in the introduction of the section, the operad $\mathcal{E}$ is associated to a simplicial operad $\mathcal{W}$ (the classical Barratt-Eccles operad). Let us recall the definition of the operad $\mathcal{W}$.

If X is a (discrete) set, then $\mathcal{W}(X)$ denotes the classical contractible simplicial set associated to X, which has

$$\mathcal{W}(X)_n = \underbrace{X \times \cdots \times X}_{n+1}$$

as a set of n-dimensional simplices. The faces and degeneracies of $\mathcal{W}(X)$ are given by the omission and repetition of components. Explicitly, if $(x_0, \dots, x_n) \in \mathcal{W}(X)_n$, then we have

$$\begin{aligned} & d_i(x_0, \dots, x_n) = (x_0, \dots, \widehat{x_i}, \dots, x_n), && \text{for } i = 0, 1, \dots, n, \\ \text{and} \quad & s_j(x_0, \dots, x_n) = (x_0, \dots, x_j, x_j, \dots, x_n), && \text{for } j = 0, 1, \dots, n. \end{aligned}$$

Observe that $X \mapsto \mathcal{W}(X)$ defines a (symmetric) monoidal functor from the category of sets to the category of simplicial sets. We have $\mathcal{W}(X \times Y) = \mathcal{W}(X) \times \mathcal{W}(Y)$. As a consequence, if $\mathcal{O}$ is an operad in the category of sets, then $\mathcal{W}(\mathcal{O})$ is an operad in the category of simplicial sets.

The Barratt-Eccles operad is formed by the contractible simplicial sets $\mathcal{W}(r) = \mathcal{W}(\Sigma_r)$, $r \in \mathbb{N}$, associated to the permutation operad. To be explicit, the simplices $(w_0, \dots, w_n) \in \mathcal{W}(\Sigma_r)_n$ support the diagonal action of the symmetric group. We have:

$$\sigma \cdot (w_0, \dots, w_n) = (\sigma \cdot w_0, \dots, \sigma \cdot w_n),$$

for any $\sigma \in \Sigma_r$. Furthermore, given $u = (u_0, \dots, u_n) \in \mathcal{W}(\Sigma_r)_n$ and $v = (v_0, \dots, v_n) \in \mathcal{W}(\Sigma_s)_n$, the composite operation $u \circ_k v \in \mathcal{W}(\Sigma_{r+s-1})_n$ is given by the formula:

$$u \circ_k v = (u_0 \circ_k v_0, \dots, u_n \circ_k v_n).$$

The augmentation maps

$$\mathcal{W}(\Sigma_r) \xrightarrow{\sim} pt$$

define a morphism of simplicial operads to the (set) commutative operad.

The identity $\mathcal{E}(r) = N_*(\mathcal{W}(\Sigma_r))$ is clear. Furthermore, the diagonal $\mathcal{E}(r) \longrightarrow \mathcal{E}(r) \otimes \mathcal{E}(r)$ defined in Paragraph 1.1.4 is clearly the Alexander-Whitney diagonal on the normalized chain complex $N_*(\mathcal{W}(\Sigma_r))$. Our definition of the composition product in the operad $\mathcal{E}$ follows from a classical explicit representation of the Eilenberg-Zilber equivalence:

$$N_d(\mathcal{W}(\Sigma_r)) \otimes N_e(\mathcal{W}(\Sigma_s)) \xrightarrow{\sim} N_{d+e}(\mathcal{W}(\Sigma_r) \times \mathcal{W}(\Sigma_s))$$

(cf. [7]).

1.2. The surjection operad

The *surjection operad* $\mathcal{X}$ is a quotient of the Barratt-Eccles operad $\mathcal{E}$. The structure of the surjection operad $\mathcal{X}$ is made explicit in this Section. The quotient morphism is introduced in the next section.

1.2.1. *The module of surjections*

The module $\mathcal{X}(r)_d$ is generated by the *non-degenerate* surjections $u : \{1,\dots,r+d\} \longrightarrow \{1,\dots,r\}$. The degenerate surjections, whose definition is given below, are equivalent to zero in $\mathcal{X}(r)_d$. The non-surjective maps represent also the zero element $0 \in \mathcal{X}(r)_d$. In the context of the surjection operad, a map $u : \{1,\dots,r+d\} \longrightarrow \{1,\dots,r\}$ is represented by a sequence $(u(1),\dots,u(r+d))$. By definition, a surjection $u : \{1,\dots,r+d\} \longrightarrow \{1,\dots,r\}$ is degenerate if the associated sequence contains a repetition (more explicitly, if we have $u(i) = u(i+1)$ for some $0 < i < r+d$). The module $\mathcal{X}(r)_d$ has a canonical Σ_r-module structure. In fact, if $\sigma \in \Sigma_r$, then the surjection $\sigma \cdot u \in \mathcal{X}(r)_d$ is represented by the sequence $(\sigma(u(1)),\dots,\sigma(u(r+d)))$.

In the sequel, we say that $v = (v(1),\dots,v(s))$ is a *subsequence* of $u = (u(1),\dots,u(r+d))$ if $v(1) = u(i_1), v(2) = u(i_2),\dots,v(s) = u(i_s)$, for some $1 \leq i_1 < i_2 < \cdots < i_s \leq r+d$.

1.2.2. *The table arrangement of a surjection*

We define the *table arrangement* of a surjection $u \in \mathcal{X}(r)_d$. In fact, the table arrangement is specified by the decomposition of u in certain subsequences

$$u_i = (\underbrace{u(r_0+\cdots+r_{i-1}+1)}_{u_i(1)}, \underbrace{u(r_0+\cdots+r_{i-1}+2)}_{u_i(2)}, \dots, \underbrace{u(r_0+\cdots+r_{i-1}+r_i)}_{u_i(r_i)})$$

where $r_0, r_1, \dots, r_d \geq 1$ and $r_0 + r_1 + \cdots + r_d = r + d$. But, in general, the values of the surjection u are just arranged on a table with $d+1$ rows indexed by $0,\dots,d$. The subsequence u_i consists of the terms on the row i:

$$u(1),\dots,u(r+d) \;=\; \left|\begin{array}{llll} u_0(1), & \cdots & u_0(r_0-1), & u_0(r_0), \\ u_1(1), & \cdots & u_1(r_1-1), & u_1(r_1), \\ \vdots & & \vdots & \vdots \\ u_{d-1}(1), & \cdots & u_{d-1}(r_{d-1}-1), & u_{d-1}(r_{d-1}), \\ u_d(1), & \cdots & u_d(r_d). & \end{array}\right.$$

Therefore, the subsequence u_i is referred to as the line i of the table arrangement of the surjection u. The elements

$$u_0(s_0), u_1(s_1), \dots, u_{d-1}(s_{d-1})$$

which are at the end of a row (except for the last one) are the *caesuras* of the surjection. The caesuras are fixed as the elements of the sequence $(u(1),\dots,u(r+d))$ which do not represent the last occurence of a value $k = 1,\dots,r$. As an example, the sequence $u = (1,3,2,4,2,1)$ has the following table arrangement:

$$1,3,2,1,4,2,1 \;=\; \left|\begin{array}{l} 1, \\ 3,2, \\ 1, \\ 4,2,1. \end{array}\right.$$

In general, if $u(i_1),\dots,u(i_{n-1}),u(i_n))$ are the occurences of $k=1,\dots,r$, then the element $u(i_n)$ is the *final occurence* of k. The elements $u(i_1),\dots,u(i_{n-1})$ are also referred to as the *inner occurences* of k. Thus, by definition, the inner occurences are caesuras of the surjection. The final occurences are inside a row of the table arrangement. These latter elements

$$u_0(1),\dots,u_0(s_0-1),\dots,u_{d-1}(1),\dots,u_{d-1}(s_{d-1}-1),u_d(1),\dots,u_d(s_d),$$

form a permutation.

1.2.3. *The complex of surjections*

In this Paragraph, we equip the module of surjections with a differential $\delta:\mathcal{X}(r)_*\to\mathcal{X}(r)_{*-1}$. The identity $\delta^2=0$ follows from results which we prove in the next section.

By definition, if the surjection $u\in\mathcal{X}(r)_d$ is specified by the sequence $(u(1),\dots,u(r+d))$, then the differential $\delta(u)\in\mathcal{X}(r)_{d-1}$ is the sum:

$$\delta(u(1),\dots,u(r+d))=\sum_{i=1}^{r+d}\pm(u(1),\dots,\widehat{u(i)},\dots,u(r+d)).$$

The signs are determined by the table arrangement. Precisely, the caesuras of the surjection are marked with alterned signs. Otherwise, if $u(i)$ is inner a line, then $u(i)$ is by definition the last occurence of $k=u(i)$ in the sequence. If k occurs only once, then the sequence $(u(1),\dots,\widehat{u(i)},\dots,u(r+d))$ does not represent a surjection and, therefore, vanishes in $\mathcal{X}(r)_{d-1}$. Otherwise, we mark $u(i)$ with the opposite of the sign associated to the previous occurence of k.

As an example, for the surjection $u=(1,3,2,4,2,1)\in\mathcal{X}(4)_2$, we obtain the signs:

$$\left|\begin{array}{l}1^+,\\3,2^-,\\1^+,\\4,2^+,1^-.\end{array}\right.$$

Therefore, we have $\delta(1,3,2,4,2,1)=(3,2,4,2,1)-(1,3,4,2,1)+(1,3,2,4,1)-(1,3,2,4,2)$.

1.2.4. *The operadic composition of surjections*

We equip the module of surjections with a partial composition product $\mathcal{X}(r)_d\otimes\mathcal{X}(s)_e\longrightarrow\mathcal{X}(r+s-1)_{d+e}$. In fact, if $u=(u(1),\dots,u(r+d))\in\mathcal{X}(r)_d$ and $v=(v(1),\dots,v(s+e))\in\mathcal{X}(s)_e$, then the product $u\circ_k v\in\mathcal{X}(r+s-1)_{d+e}$ is given by the substitution of the occurences of k in $(u(1),\dots,u(r+d))$ by elements of $(v(1),\dots,v(s+e))$.

Precisely, assume that k has n occurences in $(u(1),\dots,u(r+d))$ which are the terms $u(i_1),\dots,u(i_n)$. In this case, we decompose the sequence $(v(1),\dots,v(s+e))$ in n components:

$$(v(j_0),\dots,v(j_1))\quad(v(j_1),\dots,v(j_2))\quad\cdots\quad(v(j_{n-1}),\dots,v(j_n))$$

where $1=j_0\leq j_1\leq j_2\leq\cdots\leq j_{r-1}\leq j_n=s+e$. Then, in $(u(1),\dots,u(r+d))$, we replace the term $u(i_m)$ by the sequence $(v(j_{m-1}),\dots,v(j_m))$. Furthermore, we increase the indices $v(j)$ by $k-1$. The indices $u(i)$ such that $u(i) < k$ are fixed. The indices $u(i)$ such that $u(i) > k$ are increased by $s-1$. The sequence which arise from this process represents a *composed surjection.*

The composition product $u \circ_k v$ is the sum of the composed surjections together with a sign which we specify in the next Paragraph. The sum ranges over all the decompositions of $(v(1), \ldots, v(s+e))$.

As an example, we have:

$$(1,2,1,3) \circ_1 (1,2,1) = \pm(1,3,1,2,1,4) \pm (1,2,3,2,1,4) \pm (1,2,1,3,1,4).$$

In fact, there are 2 occurences of $k=1$ in $(1,2,1,3)$. Therefore, we cut the sequence $(1,2,1)$ in 2 pieces. There are 3 possibilities $(1)(1,2,1)$, $(1,2)(2,1)$ and $(1,2,1)(1)$. Hence, the substitution gives the composed surjections $(\mathbf{1},3,\mathbf{1},\mathbf{2},\mathbf{1},4)$, $(\mathbf{1},\mathbf{2},3,\mathbf{2},\mathbf{1},4)$ and $(\mathbf{1},\mathbf{2},\mathbf{1},3,\mathbf{1},4)$.

1.2.5. *The signs in the composition of surjections*

In fact, our substitution process requires some sequence-permutations and these permutations determine a sign which has to appear in the expansion of the composition product.

In more details, the subsitution process works as follows. The sequence $(u(1), \ldots, u(r+d))$ is cut on the occurences of k. Thus, with the notation above, we obtain explicitly:

$$(u(1), \ldots, u(i_1)) \quad (u(i_1), \ldots, u(i_2)) \quad \cdots \quad (u(i_{n-1}), \ldots, u(i_n)) \quad (u(i_n), \ldots, u(r+d)).$$

Let us define also $i_0 = 0$ and $i_{n+1} = r+d$. The sequence $(v(1), \ldots, v(s+e))$ is decomposed in

$$(v(j_0), \ldots, v(j_1)) \quad (v(j_1), \ldots, v(j_2)) \quad \cdots \quad (v(j_{n-1}), \ldots, v(j_n)).$$

The sequence $(v(j_{m-1}), \ldots, v(j_m))$ is inserted on the position of $u(i_m)$. Therefore, if we move the sequence from the right to the left, then we have to permute this sequence with the components

$$(u(i_n), \ldots, u(i_{n+1})) \quad (u(i_{n-1}), \ldots, u(i_n)) \quad (u(i_m), \ldots, u(i_{m+1})).$$

The sign is precisely produced by these transpositions according to the permutation rule in differential graded calculus. Then, we just concatenate the sequences in order to achieve the subsitution.

The sign associated to the permutation depends on a degree which equips the components of a surjection. By definition, the sequence $(u(i_{m-1}), \ldots, u(i_m))$ has degree d' if it intersects $d'+1$ lines in the table arrangement of the surjection $(u(1), \ldots, u(r+d))$. In fact, if d_m, $m = 1, \ldots, n+1$, denotes the degree of the sequence $(u(i_{m-1}), \ldots, u(i_m))$, then the definition implies that $(u(i_{m-1}), \ldots, u(i_m))$ intersects the lines:

$$d_1 + \cdots + d_{m-1}, d_1 + \cdots + d_{m-1} + 1, \ldots, d_1 + \cdots + d_{m-1} + d_m.$$

The degree of the components $(v(j_{m-1}), \ldots, v(j_m))$ of $(v(1), \ldots, v(s+e))$ are fixed by the same rule.

1.2.6. *Example*

As an example, let us determine the signs which occur in the composition product

$$(1,2,1,3)\circ_1(1,2,1)=\pm(1,3,1,2,1,4)\pm(1,2,3,2,1,4)\pm(1,2,1,3,1,4)$$

considered above. The sequence $(1,2,1,3)$ has the following table arrangement:

$$1,2,1,3\;=\;\left|\begin{array}{l}1,\\ 2,1,3.\end{array}\right.$$

This sequence is decomposed in $(1)(1,2,1)(1,3)$. The component (1) which is contained in the first line has degree 0. The component $(1,2,1)$ intersects the lines 1 and 2 and has degree 1. The component $(1,3)$ contained in the last line has degree 0. The degrees of the components of $(1,2,1)$ are determined similarly. Now, we perform the following permutations and concatenations (the subscripts denote the degree of the components):

$$\begin{array}{l}
(1)_0(1,3,1)_1(1,4)_0(\mathbf{1})_0(\mathbf{1},\mathbf{2},\mathbf{1})_1\mapsto+(1)_0(\mathbf{1})_0(1,3,1)_1(\mathbf{1},\mathbf{2},\mathbf{1})_1(1,4)_0\mapsto+(\mathbf{1},3,\mathbf{1},\mathbf{2},\mathbf{1},4),\\
(1)_0(1,3,1)_1(1,4)_0(\mathbf{1},\mathbf{2})_1(\mathbf{2},\mathbf{1})_0\mapsto-(1)_0(\mathbf{1},\mathbf{2})_1(1,3,1)_1(\mathbf{2},\mathbf{1})_0(1,4)_0\mapsto-(\mathbf{1},\mathbf{2},3,\mathbf{2},\mathbf{1},4),\\
(1)_0(1,3,1)_1(1,4)_0(\mathbf{1},\mathbf{2},\mathbf{1})_1(\mathbf{1})_0\mapsto-(1)_0(\mathbf{1},\mathbf{2},\mathbf{1})_1(1,3,1)_1(\mathbf{1})_0(1,4)_0\mapsto-(\mathbf{1},\mathbf{2},\mathbf{1},3,\mathbf{1},4),
\end{array}$$

from which we deduce the sign associated to each term of the product $(1,2,1,3)\circ_1(1,2,1)$. Explicitly, we obtain finally: $(1,2,1,3)\circ_1(1,2,1)=(1,3,1,2,1,4)-(1,2,3,2,1,4)-(1,2,1,3,1,4)$.

We claim that our definitions make sense:

1.2.7. PROPOSITION

The constructions above provide the graded module of surjections with the structure of a dg-operad.

Explicitly, we prove that the composition product of surjections is compatible with the differential and satistifies the unit, associativity and equivariance properties of an operad composition product. In fact, we deduce the proposition from results which we prove in the next sections. The surjection operad is also a Σ_*-projective resolution of the commutative operad (cf. [22]). But, we do not use this result in the article.

1.3. The table reduction morphism

1.3.1. *The table reduction morphism*

We define a morphism $TR:\mathcal{E}\longrightarrow\mathcal{X}$ (the *table reduction morphism*). Let $w=(w_0,\dots,w_d)\in\mathcal{E}(r)_d$. Given indices $r_0,\dots,r_d\geq 1$ such that $r_0+\dots+r_d=r+d$, we form a surjection $w'=(w'(1),\dots,w'(r+d))\in\mathcal{X}(r)_d$ from the values of the permutations $w_0,\dots,w_d$. In the sequel, we say that the surjection $w'\in\mathcal{X}(r)_d$ arise from the simplex $w\in\mathcal{E}(r)_d$ by a table reduction process. The image of $w=(w_0,\dots,w_d)$ in $\mathcal{X}(r)$ is the sum of these elements

$$TR(w_0,\dots,w_d)=\sum\nolimits_{(r_0,\dots,r_d)}(w'(1),\dots,w'(r+d))$$

for all choices of $r_0, \ldots, r_d \geq 1$. Let us mention that there are only positive signs in the sum.

Fix $r_0, \ldots, r_d \geq 1$ as above. The associated surjection has a table arrangement

$$w'(1), \ldots, w'(r+d) = \left| \begin{array}{llll} w_0'(1), & \cdots & w_0'(r_0-1), & w_0'(r_0), \\ w_1'(1), & \cdots & w_1'(r_1-1), & w_1'(r_1), \\ \vdots & & \vdots & \vdots \\ w_{d-1}'(1), & \cdots & w_{d-1}'(r_{d-1}-1), & w_{d-1}'(r_{d-1}), \\ w_d'(1), & \cdots & w_d'(r_d), & \end{array} \right.$$

whose rows are formed by the values of the permutations $w_0, \ldots, w_d$. Precisely, the table is constructed as follows. The elements $(w_0'(1), \ldots, w_0'(r_0))$ are the first terms of the permutation $(w_0(1), \ldots, w_0(r))$. The elements $(w_i'(1), \ldots, w_i'(r_i))$ are the first terms of the permutation $(w_i(1), \ldots, w_i(r))$ which do not occur inside a line above in the table. Equivalently, we omit the values represented in the sequence $w'(1), \ldots, w_0'(r_0-1), w_1'(1), \ldots, w_1'(r_1-1), \ldots, w_{i-1}'(1), \ldots, w_{i-1}'(r_{i-1}-1)$ from the permutation $(w_i(1), \ldots, w_i(r))$.

For instance, we have:

$$TR \left| \begin{array}{l} 1,2,3,4 \\ 1,4,3,2 \\ 1,2,4,3 \end{array} \right. = \left| \begin{array}{l} 1,2, \\ 4, \\ 2,4,3 \end{array} \right. + \left| \begin{array}{l} 1,2, \\ 4,3, \\ 2,3. \end{array} \right.$$

(We drop the degenerate terms, which are equivalent to 0, from the result.)

We claim:

1.3.2. THEOREM

The table reduction morphism $TR : \mathcal{E} \longrightarrow \mathcal{X}$ defined in the Paragraph above is a surjective morphism of differential graded operads. The surjection operad $\mathcal{X}$ is a quotient of the operad $\mathcal{E}$.

The demonstration is achieved in the next Sections. Our purpose is to prove both the Proposition 1.2.7 and the Theorem 1.3.2. In fact, as mentioned in the Theorem, we prove that $\mathcal{X}$ is a quotient operad of the classical operad $\mathcal{E}$. The table reduction morphism $TR : \mathcal{E} \longrightarrow \mathcal{X}$ is identified with the quotient arrow. Therefore, we prove the following Assertions first:

1.3.3. ASSERTION: *The table reduction morphism $TR : \mathcal{E} \longrightarrow \mathcal{X}$ is well defined.*

1.3.4. LEMMA: *The table reduction morphism $TR : \mathcal{E} \longrightarrow \mathcal{X}$ is surjective.*

Then, the next Assertion implies that the differential on $\mathcal{X}(r)$ verifies the identity $\delta^2 = 0$. Equivalently, we prove that $\mathcal{X}(r)$ is a quotient dg-module of $\mathcal{E}(r)$.

1.3.5. ASSERTION: *The table reduction morphism $TR : \mathcal{E} \longrightarrow \mathcal{X}$ maps the differential in the Barratt-Eccles operad $\mathcal{E}(r)$ to the differential in the surjection operad $\mathcal{X}(r)$ (specified in Paragraph 1.2.3).*

The next Assertion is obvious from the definition of te table reduction morphism:

1.3.6. ASSERTION: *The table reduction morphism* $TR : \mathcal{E} \longrightarrow \mathcal{X}$ *is equivariarant.*

The next Assertion implies that the composition product of the surjection operad satisfies the properties of an operad composition product and is compatible with the differential.

1.3.7. ASSERTION: *The table reduction morphism* $TR : \mathcal{E} \longrightarrow \mathcal{X}$ *maps the composition product in the Barratt-Eccles operad to the composition product in the surjection operad (specified in Paragraph 1.2.4).*

The announced conclusion follows: the surjection operad $\mathcal{X}$ is a quotient operad of the Barratt-Eccles operad $\mathcal{E}$.

The Assertions 1.3.3, 1.3.4 and 1.3.5 are proved in Section 1.4. The Section 1.5 is devoted to the demonstration of the last Assertion 1.3.7.

1.4. The operad morphism properties

1.4.1. *Proof of the Assertion 1.3.3:*

If $w \in \mathcal{E}(r)_d$ is a degenerate simplex in the Barratt-Eccles operad, then the associated surjections $w' \in \mathcal{X}(r)_d$ are also degenerate. Therefore, the element $TR(w)$ vanishes in the normalized surjection operad. This proves that the table reduction morphism $TR : \mathcal{E} \longrightarrow \mathcal{X}$ is well defined. To be more explicit, suppose that $w = s_j(w_0, \ldots, w_d) = (w_0, \ldots, w_j, w_j, \ldots, w_d)$. Consider a surjection w' which arise from the table reduction of $(w_0, \ldots, w_j, w_j, \ldots, w_d)$. In the associated table arrangement, the line j consists of the terms

$$w'_j(1), \ldots, w'_j(s_j - 1), w'_j(s_j)$$

of the permutation w_j. The values $w'_j(1), \ldots, w'_j(s_j - 1)$ are ommitted in the next row (by definition of the table reduction process). Therefore, this row (which consists also of terms of the permutation w_j) starts with the term $w'_j(s_j)$. We conlude that the last element of the line j coincides with the first element of the line $j+1$. The Assertion follows.

1.4.2. *Proof of the Lemma 1.3.4:*

Let $u \in \mathcal{X}(r)_d$ be a surjection. We specify a simplex $w = (w_0, w_1, \ldots, w_d) \in \mathcal{E}(r)_d$ such that $TR(w) = u$. The permutations $(w_0, w_1, \ldots, w_d)$ are subsequences of $(u(1), \ldots, u(r+d))$ and are determined inductively from the table arrangement of the surjection.

To be explicit, the table arrangement of u has the form:

$$u(1), \ldots, u(r+d) = \left| \begin{array}{llll} s(1), & \cdots & s(i_1 - 1), & x(1), \\ s(i_1), & \cdots & s(i_2 - 1), & x(2), \\ \vdots & & \vdots & \vdots \\ s(i_{d-1}), & \cdots & s(i_d - 1), & x(d), \\ s(i_d), & \cdots & s(r), & \end{array} \right.$$

where $(s(1), \ldots, s(r))$ is a permutation. We fix $w_d = (s(1), \ldots, s(r))$. To obtain the permutation w_i from w_{i+1}, we just move the occurence of $x = x(i)$ in the given subsequence

$$(w_{i+1}(1), \ldots, w_{i+1}(r)) \subset (u(1), \ldots, u(r+d))$$

to the position of the caesura $x(i)$. Explicitly, we have:

$$\begin{aligned}
w_d &= s(1), \ldots, s(r), \\
w_{d-1} &= s(1), \ldots, s(i_d - 1), x(d), s(i_d), \ldots, \widehat{x(d)}, \ldots, s(r), \\
w_{d-2} &= s(1), \ldots, s(i_{d-1} - 1), x(d-1), s(i_{d-1}), \ldots, \widehat{x(d-1)}, \ldots \\
&\qquad\qquad \ldots, s(i_d - 1), x(d), s(i_d), \ldots, \widehat{x(d)}, \ldots, s(r)
\end{aligned}$$

and so on. The Lemma follows from the following Claim:

1.4.3. CLAIM: *If* $(w_0, \ldots, w_d)$ *are defined as above, then we have* $TR(w_0, \ldots, w_d) = (u(1), \ldots, u(r+d))$.

The permutations w_i and w_{i+1} coincide up to the position of $x(i)$. To be more explicit:

$$\begin{aligned}
w_i &= w_i(1), \ldots, w_i(p-1), x(i), w_i(p+1), \ldots, w_i(q), w_i(q+1), \ldots, w_i(r), \\
w_{i+1} &= w_i(1), \ldots, w_i(p-1), w_i(p+1), \ldots, w_i(q), x(i), w_i(q+1), \ldots, w_i(r).
\end{aligned}$$

Now, consider a term

$$\left|\begin{array}{lll}
w'_0(1), & \cdots\ w'_0(r_0 - 1), & w'_0(r_0), \\
w'_1(1), & \cdots\ w'_1(r_1 - 1), & w'_1(r_1), \\
\ \vdots & \qquad \vdots & \quad \vdots \\
w'_{d-1}(1), & \cdots\ w'_{d-1}(r_{d-1} - 1), & w'_{d-1}(r_{d-1}), \\
w'_d(1), & \cdots\ w'_d(r_d) &
\end{array}\right.$$

in $TR(w_0, \ldots, w_d)$. The caesura in the permutation w_i is necessarily put on $x(i)$. Otherwise, in the table, there is a repetition between the lines i and $i+1$. Precisely, one observes that the last term of the line i would coincide with the first term of the line $i+1$. This property holds for any permutation w_i. The Claim follows.

1.4.4. *Proof of the Assertion 1.3.5:*

Fix $w = (w_0, \ldots, w_d) \in \mathcal{E}(r)_d$. Let $w' \in \mathcal{X}(r)_d$ be a surjection associated to w by the table reduction process. If we assume $r_i = 1$, then we have:

$$w'(1), \ldots, w'(r+d) = \left|\begin{array}{l}
w'_0(1), \ldots, w'_0(r_0), \\
\ \vdots \\
w'_{i-1}(1), \ldots, w'_{i-1}(r_{i-1}), \\
w'_i(1), \\
w'_{i+1}(1), \ldots, w'_{i+1}(r_{i+1}), \\
\ \vdots \\
w'_d(1), \ldots, w'_d(r_d).
\end{array}\right.$$

Clearly, in the differential $\delta(w') \in \mathcal{X}(r)_{d-1}$, the term, which drops $w'_i(1)$ from the sequence, is equivalent to a surjection associated to $(w_0, \ldots, \widehat{w_i}, \ldots, w_d) \in \mathcal{E}(r)_{d-1}$ by the table reduction process. Furthermore, in both cases, the sign in the differential is $(-1)^i$. We observe that the other terms in $\delta(TR(w_0,\ldots,w_d))$ vanish. Therefore, the differential $\delta(TR(w_0,\ldots,w_d)) \in \mathcal{X}(r)_{d-1}$ reduces to $TR(\delta(w)) = \sum_i \pm TR(w_0,\ldots,\widehat{w_i},\ldots,w_d)$.

In fact, the ommission of an inner value in a surjection $w' \in \mathcal{X}(r)_d$ cancels the ommission of a caesura in another surjection $w'' \in \mathcal{X}(r)_d$ associated to w. Explicitly, assume that we drop the inner element $w'_i(p)$ from the surjection $w' \in \mathcal{X}(r)_d$ specified by the indices $r_0,\ldots,r_d$. As in the definition of the differential, we consider the previous occurence of $k = w'_i(p)$, which is at a caesura $w'_j(r_j)$ in the table arrangement of the surjection. Then, let $w'' \in \mathcal{X}(r)_d$ be the surjection determined by $r_0,\ldots,r_j+1,\ldots,r_i-1,\ldots,r_d$. This surjection w'' differs from w' by the lines i and j of the associated table arrangement:

$$w'' = \left| \begin{array}{l} w'_0(1),\ldots,w'_0(r_0), \\ \quad\vdots \\ w'_j(1),\ldots,w'_j(r_j-1),k,w'_j(r_j+1), \\ \quad\vdots \\ w'_i(1),\ldots,w'_i(p-1),\widehat{k},w'_i(p+1),\ldots,w'_i(r_i), \\ \quad\vdots \\ w'_d(1),\ldots,w'_d(r_d). \end{array} \right.$$

We observe that the ommission of $w'_j(r_j+1)$ in w'' yields the same result as the ommission of $w'_i(p)$ in w'. Furthermore, the signs associated to these differentials are opposite. (This assertion is an immediate consequence of the definition.) The conclusion follows.

1.5. On the composition products

The purpose of this section is to prove the Assertion 1.3.7 and to complete the demonstration of the Proposition 1.2.7 and of the Theorem 1.3.2. Thus, we compare the composition product in the surjection operad to the composition product in the Barratt-Eccles operad.

1.5.1. *On the expansion of the composition products*

Fix elements $u \in \mathcal{E}(r)_d$ and $v \in \mathcal{E}(s)_d$ in the Barratt-Eccles operad. We prove that the composition products $TR(u) \circ_k TR(v)$ and $TR(u \circ_k v)$ agree term by term. Explicitly, we have a map from the expansion of $TR(u) \circ_k TR(v)$ to the expansion of $TR(u \circ_k v)$. This map gives a one-to-one correspondence on the non-degenerate terms of the expansions. The Assertion follows.

Precisely, on one hand, in the expansion of $TR(u \circ_k v)$, the terms are indexed by the following elements:
– a path (x_*, y_*), which fixes a composition $(u_{x_*} \circ_k v_{y_*})$ in the Barratt-Eccles operad;
– indices $(t_0,\ldots,t_{d+e})$, which fixes a surjection $(u \circ_k v)'$ associated to the former composition.
By definition, these indices $(t_0,\ldots,t_{d+e})$ are characterized by the dimensions of the table arrangement of the surjection $(u \circ_k v)'$.

On the other hand, in the expansion of $TR(u) \circ_k TR(v)$, the terms are indexed by:
– indices $(r_0,\ldots,r_d)$, which fixes a surjection u' associated to u;
– indices $(s_0,\ldots,s_e)$, which fixes a surjection v' associated to v;

– indices $1 = j_0 \leq j_1 \leq \cdots \leq j_n = s + e$, which fixes a decomposition of v'.
The associated composed surjection w', which occurs in the expansion, is given by the insertion of the components of v' in the fixed surjection u'.

We analyze the table arrangement of the composed surjection w' in the next Paragraphs. We observe that the insertion process determines a path (x_*, y_*) and a composite operation $(u_{x_*} \circ_k v_{y_*})$ in the Barratt-Eccles operad. The composed surjection w' is associated to the surjection $(u \circ_k v)'$ which results from the table reduction of this element $(u_{x_*} \circ_k v_{y_*})$ and whose table arrangement has the same dimensions.

1.5.2. *On the table arrangement of a composed surjection*
Thus, we need insights on the substitution process for surjections. We fix $u' \in \mathcal{X}(r)_d$ and $v' \in \mathcal{X}(s)_e$ as above. As in the Paragraphs 1.2.4 and 1.2.5, we let $u'(i_1), \dots, u'(i_n)$ denote the occurences of k in the sequence $u'(1), \dots, u'(r + d)$. By convention, we have also $i_0 = 1$ and $i_{n+1} = r + d$. The surjection u' is decomposed in

$$(u'(i_0), \underline{\quad}, u'(i_1)) \quad (u'(i_1), \underline{\quad}, u'(i_2)) \quad \cdots \quad (u'(i_n), \underline{\quad}, u'(i_{n+1})).$$

Furthermore, we fix a decomposition

$$(v'(j_0), \underline{\quad}, v'(j_1)) \quad (v'(j_1), \underline{\quad}, v'(j_2)) \quad \cdots \quad (v'(j_{n-1}), \underline{\quad}, v'(j_n))$$

of the surjection v'. These components are inserted in the decomposition of u'

$$\begin{aligned}(u'(i_0), \underline{\quad}, u'(i_1)) \quad (v'(j_0), \underline{\quad}, v'(j_1)) \quad (u'(i_1), \underline{\quad}, u'(i_2)) \quad \cdots\\ \cdots \quad (v'(j_{n-1}), \underline{\quad}, v'(j_n)) \quad (u'(i_n), \underline{\quad}, u'(i_{n+1}))\end{aligned}$$

The composed surjection w' is represented by the sequence obtained by concatenation. Explicitly:

$$\begin{aligned}u'(i_0), \underline{\quad}, u'(i_1 - 1), v'(j_0), \underline{\quad}, v'(j_1), u'(i_1 + 1), \dots\\ \dots, u'(i_{n-1} - 1), v'(j_{n-1}), \underline{\quad}, v'(j_n), u'(i_n + 1), \underline{\quad}, u'(i_{n+1}).\end{aligned}$$

We omit to mark the substitution-shift for simplicity (precisely, we omit to increase the elements $v'(j)$ by $k - 1$ and to increase the elements $u'(i) > k$ by $s - 1$).

It is easy to determine the caesuras of the composed surjection:

1.5.3. Observation: *The caesuras of the composed surjection above w' are classified as follows. The caesuras $u'_x(r_x)$ of the surjection u' such that $u'_x(r_x) \neq k$ give a caesura in the composed surjection. The caesuras $u'_x(r_x)$ of the surjection u' such that $u'_x(r_x) = k$ are the terms $u'(i_1), \dots, u'(i_{n-1})$. The insertion process move such a caesura $u'(i_m)$ to the last element $v'(j_m)$ of the component $(v'(j_{m-1}), \underline{\quad}, v'(j_m))$ which is inserted on $u'(i_m)$. The other elements $(v'(j_{m-1}), \underline{\quad}, v'(j_m - 1), \widehat{v'(j_m)})$, which are a caesura for v', give another caesura in the composed surjection.*

Our next purpose is to relate the table arrangement of the composed surjection to the composition procedure in the Barratt-Eccles operad.

1.5.4. *The path determined by the insertion of a surjection*

We define the path (x_*, y_*) associated to a composed surjection. We let d_m and e_m denote respectively the degree of $(u'(i_{m-1}), \underline{\quad}, u'(i_m))$ and $(v'(j_{m-1}), \underline{\quad}, v'(j_m))$. The definition implies that the sequence $(u'(i_{m-1}), \underline{\quad}, u'(i_m))$ intersects the lines

$$d_1 + \cdots + d_{m-1}, d_1 + \cdots + d_{m-1} + 1, \ldots, d_1 + \cdots + d_{m-1} + d_m$$

in the table arrangement of u'. Similarly, the sequence $(v'(i_{m-1}), \underline{\quad}, v'(i_m))$ intersects the lines

$$e_1 + \cdots + e_{m-1}, e_1 + \cdots + e_{m-1} + 1, \ldots, e_1 + \cdots + e_{m-1} + d_m$$

in the table arrangement of v'.

By definition, the path (x_*, y_*) associated to the composed surjection is given by the concatenation of horizontal components of length d_m, $m = 1, \ldots, n+1$, and of vertical components of length e_m, $m = 1, \ldots, n$, as in the insertion process. The indices (x_c, y_c) are given by the following formulas.

$$\text{If } c = d_1 + e_1 + d_2 + e_2 + \cdots + d_{m-1} + e_{m-1} + x', \text{ where } 0 \leq x' \leq d_m,$$

$$\text{then we have } \begin{cases} x_c = d_1 + d_2 + \cdots + d_{m-1} + x', \\ y_c = e_1 + e_2 + \cdots + e_{m-1}. \end{cases}$$

$$\text{If } c = d_1 + e_1 + d_2 + e_2 + \cdots + d_{m-1} + e_{m-1} + d_m + y', \text{ where } 0 \leq y' \leq e_m,$$

$$\text{then we have } \begin{cases} x_c = d_1 + d_2 + \cdots + d_{m-1} + d_m, \\ y_c = e_1 + e_2 + \cdots + e_{m-1} + y'. \end{cases}$$

This definition is motivated by the table arrangement of the composed surjection.

1.5.5. OBSERVATION:

In the component $(u'(i_{m-1}), \underline{\quad}, u'(i_m))$*, the elements* $u'(i) \neq k$*, which come from the line* $x = d_1 + \cdots + d_{m-1} + x'$ *of the table arrangement of* u'*, are inserted in the line* $c = d_1 + e_1 + \cdots + d_{m-1} + e_{m-1} + x'$ *of the table arrangement of the composed surjection* w'*.*

In the component $(v'(j_{m-1}), \underline{\quad}, v'(j_m))$*, the elements* $v'(j)$ *which come from the line* $y = e_1 + \cdots + e_{m-1} + y'$ *of the table arrangement of* v'*, are inserted on the line* $c = d_1 + e_1 + \cdots + d_{m-1} + e_{m-1} + d_m + y'$ *of the table arrangement of the composed surjection* w'*.*

(These assertions follow immediately from the classification of the caesuras in a composed surjection.)

Now, we have:

1.5.6. CLAIM: *The path associated to a composed surjection* w' *as above fixes a composition* $(u_{x_*} \circ_k v_{y_*}) \in \mathcal{E}(r+s-1)_*$ *in the Barratt-Eccles operad. We claim that* w' *agrees with the surjection* $(u \circ_k v)'$ *which results from the table reduction of* $(u_{x_*} \circ_k v_{y_*}) \in \mathcal{E}(r+s-1)_*$ *and whose table arrangement has the same dimensions.*

As mentioned in Paragraph 1.5.1, we associate the composed surjection w' in the expansion of $TR(u) \circ_k TR(v)$ to the surjection $(u \circ_k v)'$ in the expansion of $TR(u \circ_k v)$.

We prove the Claim above by induction. More precisely, we assume that the surjections agree on the lines of the table arrangement such that $x_c < x$, where $0 \leq x \leq d$. Then:

1.5.7. CLAIM: *The surjection $(u \circ_k v)'$ agrees with the composed surjection w' on the next lines of the table arrangement (on the lines such that $x_c = x$).*

Proof:

We assume that $c', c'+1, \ldots, c''$ are the indices of the lines such that $x_c = x$. We have:

$$(x_{c'}, y_{c'}) = (x, y'),\ (x_{c'+1}, y_{c'+1}) = (x, y'+1),\ \ldots,\ (x_{c''}, y_{c''}) = (x, y'').$$

In the table arrangement of the composed surjection w', the lines $c', c'+1, \ldots, c''$ either do not contain or contain an occurence of the surjection v'. In the first case, we have $c' = c''$ and the line considered reads:

$$|\, u'_x(1), \ldots, u'_x(r_x)$$

In the second case, the lines $c', c'+1, \ldots, c''$ of the table arrangement of the composed surjection have the following form:

$$\left|\begin{array}{l} u'_x(1), \ldots, u'_x(i'-1), v'_y(j'), \ldots, v'_{y'}(s_{y'}), \\ v'_{y'+1}(1), \ldots, v'_{y'+1}(s_{y'+1}), \\ \qquad \vdots \\ v'_{y''-1}(1), \ldots, v'_{y''-1}(s_{y''-1}), \\ v'_{y''}(1), \ldots, v'_{y''}(s_{y''}), u'_x(i'+1), \ldots, u'_x(r_x), \end{array}\right.$$

where $1 \leq i' \leq r_x$. It is also possible to have $c' = c''$. Then, the diagram above collapses to one line:

$$|\, u'_x(1), \ldots, u'_x(i'-1), v'_y(j'), \ldots, v'_{y'}(s_{y'}), u'_x(i'+1), \ldots, u'_x(r_x),$$

In the table arrangement of the surjection $(u \circ_k v)'$, the lines $c', c'+1, \ldots, c''$ arise from the table reduction of the permutations $u_x \circ_k v_y$, where $y = y', y'+1, \ldots, y''$. The permutation $u_x \circ_k v_y$ is represented by the sequence:

$$u_x(1), \ldots, u_x(p-1), v_y(1), \ldots, v_y(s), u_x(p+1), \ldots, u_x(r),$$

where $u_x(p) = k$. By definition, in this sequence, we have just to omit the elements, which already occur inside a line of the surjection. By the induction hypothesis, these elements occupy the same position in the table arrangement of the composed surjection w'. Therefore, we conclude: the elements, which we have to drop from the permutations $u_x \circ_k v_y$, are the same as the elements which are ommitted in the next lines of the composed surjection w'. This observation allow us to prove that the next lines of the surjection $(u \circ_k v)'$ agree with the composed surjection w'. We follow the construction.

Let us assume that we have not to omit all elements $v_{y'}(1),\dots,v_{y'}(s)$ in the construction. As observed, the elements $u_x(q)$ which we have to drop from $u_x(1),\dots,u_x(p-1)$ are also ommited in the sequence $u'_x(1),\dots,u'_x(r_x)$. It follows that the first terms inserted on the line c' of the surjection $(u\circ_k v)'$ and of the composed surjection w' agree. If the line is not completed, then the elements which we insert from $v_{y'}(1),\dots,v_{y'}(s)$ and from $u_x(p+1),\dots,u_x(r)$ are same for the same reason. If $c'<c''$, then on the lines $c=c'+1,\dots,c''$ of the table arrangement of $(u\circ_k v)'$, we omit $u_x(1),\dots,u_x(p-1)$ because these elements occur either inside the line c' or still inside a line above in the table. Therefore, the next elements that we have to insert come from $v_y(1),\dots,v_y(s)$, $y=y'+1,\dots,y''$. These elements correspond again to the next elements $v'_y(1),\dots,v'_y(s_{y'})$ of the composed surjection w'.

If we have to omit all elements $v_{y'}(1),\dots,v_{y'}(s)$, then, for the table arrangement of the composed surjection, we obtain:

$$| \, u'_x(1),\dots,u'_x(r_x).$$

Furthermore, the elements $u'_x(1),\dots,u'_x(r_x)$ belong necessarily to the last component $(u'(i_n),\dots,u'(i_{n+1}))$ of the surjection u'. Therefore, the value k has to be omitted from $u_x(1),\dots,u_x(p-1),k,u_x(p+1),\dots,u_x(r)$ in the construction of $u'_x(1),\dots,u'_x(r_x)$. The omission of the block $v_y(1),\dots,v_y(s)$ in the permutation $u_x\circ_k v_y$ gives the same result. The conclusion follows in this case.

The demonstration of the Claim is complete.

The next Claim implies that the non-degenerate terms $(u\circ_k v)'$ in the expansion of $TR(u\circ_k v)$ are associated to a composed surjection:

1.5.8. CLAIM: *Fix a surjection $(u\circ_k v)'$ in the expansion of $TR(u\circ_k v)$. Fix $x=0,\dots,d$. We assume $x_c=x$, for $c=c',c'+1,\dots,c''$. Consider the sequence formed by the lines $c=c',c'+1,\dots,c''$ of the surjection $(u\circ_k v)'$:*

$$\left|\begin{array}{l}(u_x\circ_k v_{y'})'(1),\dots,(u_x\circ_k v_{y'})'(t_{c'}),\\ (u_x\circ_k v_{y'+1})'(1),\dots,(u_x\circ_k v_{y'+1})'(t_{c'+1}),\\ \qquad\vdots\\ (u_x\circ_k v_{y''})'(1),\dots,(u_x\circ_k v_{y''})'(t_{c''}).\end{array}\right.$$

This sequence is equivalent to a concatenation:

$$(u'_x(1),\dots,u'_x(i'-1),k)\quad(v'(j'),\dots,v'(j''))\quad(k,u'_x(i'+1),\dots,u'_x(r_x)),$$

otherwise the surjection $(u\circ_k v)'$ is degenerate.

Proof:

The property is immediate if $c'=c''$. We assume $c'<c''$. In the table, all lines except the last one have to end with a term $v'_y(j)$ and all lines except the first one have to start with a term $v'_y(1)$, otherwise the table contains a repetition.

In fact, if the line y ends with $u'_x(i')$, then all values $u_x(i)$, which come before $u'_x(i')$ in the sequence $u_x(1),\dots,\widehat{k},\dots,u_x(r)$, are ommitted in the next lines of the surjection. If

k comes before $u'_x(i')$, then all values $v_y(1), \ldots, v_y(s)$ already occur as a final element in the table. Therefore, the elements $v_y(1), \ldots, v_y(s)$ are also ommitted in the next lines. As a conclusion, the line $y+1$ has to start with the element $u'_x(i')$ and a repetition occurs in the table.

Now, if we assume that the line $y-1$ ends with $v'_{y-1}(j)$, then the elements $u_x(i)$ which come before k already occur as a final element in the table. Therefore, the line y starts with an element $v'_y(1)$.

The Claim follows.

1.5.9. CLAIM: *A composed surjection w' associated to u' and v' is non-degenerate if and only if the surjections u' and v' are non-degenerate. In this case, the composed surjection w' uniquely determines the surjections u' and v' together with the decomposition of v' which gives the composition.*

Proof:

The first assertion is immediate from the insertion process for surjections. The degeneracies $u'(i) = u'(i+1)$ such that $u'(i) \neq k$ are not removed by the insertion process. So do the degeneracies $v'(j) = v'(j+1)$. If the surjection u' has a degeneracy $u'(i) = u'(i+1)$, such that $u'(i) = k$, then, according to our notation, we have $i = i_m$ and $i+1 = i_{m+1}$, for some $m = 0, 1, \ldots, n$. In the composed surjection, this degeneracy is equivalent to the repetition of the element $v'(j_m)$.

The sequence $u'_x(1), \ldots, u'_x(r_x)$, where $x = 0, \ldots, d$, is uniquely determined by the terms of the surjection w' which are on the lines such that $x_c = x$. Explicitly, the sequence $u'_x(1), \ldots, u'_x(r_x)$ is recovered by the following operations. The elements such that $k \leq w'_c(i) \leq k+t-1$ are replaced by k. The elements such that $k+t \leq w'_c(i)$ are decreased by $t-1$. The consecutive occurences of k are collapsed to one term and this achieve the process. (If the surjection u' is non-degenate, then the repetition of k follows from the insertion process.) Similarly, the sequences $u'_y(1), \ldots, u'_y(s_y)$, $y = 0, \ldots, e$, are determined by the terms of the surjection w' which are on the lines such that $y_c = y$. In this case, we withdraw the elements such that $w'_c(i) \leq k-1$ or $k+t \leq w'_c(i)$ and we decrease by $k-1$ the elements such that $k \leq w'_c(i) \leq k+t-1$. Consecutive occurences of a same value are also collapsed to one term.

This achieve the demonstration of the Claim.

The Claims above identify the non-degenerate terms in the expansion of $TR(u) \circ_k TR(v)$ to the non-degenerate terms in the expansion of $TR(u \circ_k v)$. To complete the demonstration, it remains to compare the associated signs. By definition, in $TR(u \circ_k v)$, the sign is determined by a shuffle of the horizontal and vertical components of the path (x_*, y_*). In $TR(u) \circ_k TR(v)$, the sign is produced by the insertions of the components $(v'(j_{m-1}), \ldots, v'(j_m))$ of the surjection v' in the surjection u'. It is immediate from the definition that these shuffles are same. Therefore:

1.5.10. OBSERVATION: *The sign produced by the insertion of a surjection in the composition process agrees with the sign determined by the path (x_*, y_*) associated to the composed surjection.*

The demonstration of the Assertion 1.3.7 is now complete.

1.6. The little cube filtrations on operads

The Barratt-Eccles operad has a filtration

$$F_1\mathcal{W}(r) \subseteq F_2\mathcal{W}(r) \subseteq \cdots \subseteq F_n\mathcal{W}(r) \subseteq \cdots \subseteq \mathcal{W}(r)$$

by simplicial suboperads $F_n\mathcal{W}(r)$ whose topological realization $|F_n\mathcal{W}|$ is (as a topological operad) homotopy equivalent to the classical operad of the little n-cubes $F_d\mathcal{D}$ (cf. M. Boardmann, R Vogt [4] and P. May [20]). Precisely, the topological operads $|F_n\mathcal{W}|$ and $F_n\mathcal{D}$ are connected by operad morphisms

$$|F_n\mathcal{W}| \xleftarrow{\sim} \cdot \xrightarrow{\sim} F_n\mathcal{D}$$

which are isomorphisms in the homotopy category of spaces.

The filtration on the Barratt-Eccles operad is introduced by J. Smith in the article [27] in order to generalize the Milnor FK-construction and to provide a simplicial model for iterated loop spaces. The work of T. Kashiwabara (cf. [15]) proves that the simplicial set $F_n\mathcal{W}(r)$ has the same homology as the space of the little n-cubes $F_n\mathcal{D}(r)$. The equivalence as a topological operad follows from the work of C. Berger (cf. [3]). The idea is to relate the filtration to a particular cellular structure on these operads.

We have an induced filtration on the associated dg-operad

$$F_1\mathcal{E}(r) \subseteq F_2\mathcal{E}(r) \subseteq \cdots \subseteq F_n\mathcal{E}(r) \subseteq \cdots \subseteq \mathcal{E}(r)$$

such that $F_n\mathcal{E}(r)$ is equivalent to the dg-operad formed by the chain complexes on the spaces of the little n-cubes. J. McClure and J. Smith have introduced a similar filtration

$$F_1\mathcal{X}(r) \subseteq F_2\mathcal{X}(r) \subseteq \cdots \subseteq F_n\mathcal{X}(r) \subseteq \cdots \subseteq \mathcal{X}(r)$$

on the surjection operad (cf. [22]). Furthermore, J. McClure and J. Smith have proved that the dg-operad $F_n\mathcal{X}$ is also equivalent to the little cube operad. In fact, it is possible to adapt the methods of C. Berger to the context of the surjection operad. This result implies that we have quasi-isomorphisms:

$$F_n\mathcal{E} \xleftarrow{\sim} \cdot \xrightarrow{\sim} F_n\mathcal{X}.$$

Our purpose is to prove the following Lemma:

1.6.1. LEMMA

The table reduction morphism $TR : \mathcal{E} \longrightarrow \mathcal{X}$ preserves the filtrations. Furthermore, the induced morphism $F_n TR : F_n\mathcal{E} \longrightarrow F_n\mathcal{X}$ is a quasi-isomorphism of differential graded operads.

We state a more precise result below.

1.6.2. *The cellular structures*

We recall the definition of the cellular structure which occurs on the operads $\mathcal{P} = \mathcal{E}$ and $\mathcal{P} = \mathcal{X}$. A cell is associated to a pair (μ, σ) such that $\sigma \in \Sigma_r$ is a permutation and μ is a collection of non-negative integers $\chi_{x,y}(\mu) \in \mathbb{N}$ (where $\{x, y\} \subseteq \{1, \ldots, r\}$). The permutation $\sigma \in \Sigma_r$ determines also a collection $\chi_{x,y}(\sigma)$. Explicitly, the element $\chi_{x,y}(\sigma)$ is the permutation of (x, y) formed by the occurences of x and y in the sequence $(\sigma(1), \ldots, \sigma(r))$ which represents the permutation σ. Equivalently, we assume $x < y$. The permutation $\chi_{x,y}(\sigma)$ is the identity permutation (x, y) if $\sigma(x) < \sigma(y)$ and $\chi_{x,y}(\sigma)$ is the transposition (y, x) if $\sigma(x) > \sigma(y)$.

The cellular structure is specified by sub-Σ_r-modules $F_{(\mu,\sigma)}\mathcal{P}(r) \subseteq \mathcal{P}(r)$. The filtration is related to the cellular structure by the relation:

$$F_n\mathcal{P}(r) = \sum_{\mu_{x,y} < n} F_{(\mu,\sigma)}\mathcal{P}(r).$$

The sum ranges over the elements (μ, σ) such that $\chi_{x,y}(\mu) < n$ for all pairs $x < y$.

The cellular structure is in some sense compatible with the operad composition product (cf. C. Berger [3]). Furthermore, the cells $F_{(\mu,\sigma)}\mathcal{P}(r)$ are acyclic in both cases $\mathcal{P} = \mathcal{E}$ (cf. C. Berger [3]) and $\mathcal{P} = \mathcal{X}$ (cf. J. McClure and J. Smith [22]). We just recall the definition of the cells $F_{(\mu,\sigma)}\mathcal{P}(r) \subseteq \mathcal{P}(r)$ in the next Paragraphs.

1.6.3. *The cellular structure on the Barratt-Eccles operad*

Let $w = (w_0, \ldots, w_d) \in \mathcal{E}(r)_d$. Given a pair $x < y$, we consider the sequence $\chi_{x,y}(w) = (\chi_{x,y}(w_0), \ldots, \chi_{x,y}(w_d))$ formed by the permutations of x and y associated to $(w_0, \ldots, w_d)$. The simplex $w = (w_0, \ldots, w_d)$ belongs to $F_{(\mu,\sigma)}\mathcal{E}(r)_d \subset \mathcal{E}(r)_d$ if, for all pairs $x < y$, either the sequence $\chi_{x,y}(w)$ has no more than $\chi_{x,y}(\mu) - 1$ variations, or the sequence $\chi_{x,y}(w)$ has exactly $\chi_{x,y}(\mu)$ variations and we have $\chi_{x,y}(w_d) = \chi_{x,y}(\sigma)$. The simplex $w = (w_0, \ldots, w_d)$ belongs to $F_n\mathcal{E}(r)_d \subset \mathcal{E}(r)_d$ if and only if, for all pairs $x < y$, the sequence $\chi_{x,y}(w)$ has no more than n variations. For instance, we have $((1, 2), (2, 1), (1, 2)) \in F_2\mathcal{W}(2)_3$.

1.6.4. *The cellular structure on the surjection operad*

Fix a surjection $u \in \mathcal{X}(r)_d$. Given a pair $x < y$, we let $\chi_{x,y}(u)$ denote the subsequence of $(u(1), \ldots, u(r + d))$ formed by the occurences of x and y in the surjection. For instance, we have $(2, 1, 3, 4, 3, 1)_{1,2} = (2, 1, 1)$, $(2, 1, 3, 4, 3, 1)_{1,3} = (1, 3, 3, 1)$ and $(2, 1, 3, 4, 3, 1)_{2,3} = (2, 3, 3)$. The surjection $u \in \mathcal{X}(r)_n$ belongs to $F_{(\mu,\sigma)}\mathcal{X}(r)_d \subset \mathcal{X}(r)_d$ if, for all pairs $x < y$, either the sequence $\chi_{x,y}(u)$ has no more than $\chi_{x,y}(\mu)$ variations, or the sequence $\chi_{x,y}(u)$ has exactly $\chi_{x,y}(\mu) + 1$ variations and the permutation formed by the final occurences of x and y in $\chi_{x,y}(u)$ is equal to $\chi_{x,y}(\sigma)$. The surjection $u \in \mathcal{X}(r)_n$ belongs to $F_n\mathcal{X}(r)_d \subset \mathcal{X}(r)_d$ if and only if, for all pairs $x < y$, the sequence $\chi_{x,y}(u)$ has no more than $n + 1$ variations. For instance, we have $(1, 2, 1, 2) \in F_2\mathcal{X}(2)_3$. Similarly, for the surjection above, we have $(2, 1, 3, 4, 3, 2) \in F_2\mathcal{X}(4)_3$.

The Lemma 1.6.1 follows from the following result:

1.6.5. LEMMA

The table reduction morphism $TR : \mathcal{E} \longrightarrow \mathcal{X}$ preserves the cellular structures. Explicitly, if $w \in F_{(\mu,\sigma)}\mathcal{E}$, then we have $TR(w) \in F_{(\mu,\sigma)}\mathcal{X}$.

The result above implies that the induced morphism $F_n TR : F_n\mathcal{E} \longrightarrow F_n\mathcal{X}$ is a weak equivalence, because we have the commutative diagram:

$$\begin{tikzcd}
\operatorname*{hocolim}_{(\mu,\sigma)} F_{(\mu,\sigma)}\mathcal{E} \arrow[r, "F_{(\mu,\sigma)}\, TR"] \arrow[d, "\sim"] & \operatorname*{hocolim}_{(\mu,\sigma)} F_{(\mu,\sigma)}\mathcal{X} \arrow[d, "\sim"] \\
F_n\mathcal{E} \arrow[r, "F_n\, TR"] & F_n\mathcal{X}
\end{tikzcd}$$

(cf. [3], [22]). The upper horizontal arrow is a weak equivalence because the cells $F_{(\mu,\sigma)}\mathcal{E}$ and $F_{(\mu,\sigma)}\mathcal{X}$ are acyclic.

The proof of the Lemma above is postponed to the end of this Section.

1.6.6. *The relationship to operations on the Hochschild cochains*

According to J. McClure and J. Smith, the operad $F_2\mathcal{X}$ is generated by the operations $(1,2) \in \mathcal{X}(2)$ and $(1,2,1,3,1,\ldots,1,r,1) \in \mathcal{X}(r)$. Furthermore, if A is an associative algebra, then the normalized Hochschild cochain complex $N^*(A,A)$ is equipped with the structure of an algebra over $F_2\mathcal{X}$ (cf. [21], [22]). The generator $(1,2) \in \mathcal{X}(2)$ is mapped to the cup product operation

$$x_1 \smile x_2 : N^*(A,A) \otimes N^*(A,A) \longrightarrow N^*(A,A).$$

The generator $(1,2,1,3,1,\ldots,1,r,1) \in \mathcal{X}(r)$ is mapped to the brace-operations

$$x_1\{x_2,\ldots,x_r\}_{r-1} : N^*(A,A)^{\otimes r} \longrightarrow N^*(A,A)$$

which are introduced by E. Getzler (cf. [8]) and by T. Kadeishvili (cf. [14]). J. McClure and J. Smith conclude that the surjection operad is an instance of a dg-operad which is equivalent to the little square operad and which operates on the normalized Hochschild cochain complexes. This result answers a conjecture of P. Deligne. The Lemma 1.6.1 above implies the same result about our simplicial model of the little square operad (regardless of the homotopy type of the operad $F_2\mathcal{X}$):

1.6.7. THEOREM

The normalized Hochschild cochain complex associated to an associative algebra is equipped with the structure of an algebra over the operad $F_2\mathcal{E}$.

1.6.8. *Proof of the Lemma 1.6.5*

Let $w = (w_0, w_1, \ldots, w_d) \in \mathcal{E}(r)_d$. Let w' be a surjection associated to w by the table reduction process. Fix a pair $x < y$. If $\chi_{x,y}(w)$ has no more than n variations, then we claim that $\chi_{x,y}(w')$ has no more than $n+1$ variations.

We assume that the final occurence of x comes before the final occurence of y in the surjection w' (the arguments remain same in the inverse case). We assume that the final occurence of x in w' belongs to the line m of the table arrangement of w'. Consider the subsequence of $\chi_{x,y}(w')$ which is formed by all the occurences of x in the surjection w' and by the occurences of y which are on the lines $0,1,\ldots,m-1$ of the table arrangement of w'. Equivalently, the sequence $\chi_{x,y}(w')$ is truncated on the final occurence of x. In $\chi_{x,y}(w')$, there is at least an occurence of y which comes after the final occurence of x. Therefore, the sequence $\chi_{x,y}(w')$ has one more variation than the truncated subsequence.

If x occurs on the line i of the table arrangement, then x comes necessarily before y in the sequence $w_i(1),\dots,w_i(r)$, because we assume that the final occurence of y lies below in the table. Hence, in this case, we have $\chi_{(x,y)}(w_i) = (x,y)$. Similarly, if y occurs on a line $w'_i(1),\dots,w'_i(r_i)$, such that $i < m$, then y comes necessarily before x in the sequence $w_i(1),\dots,w_i(r)$. Hence, in this case, we have $\chi_{(x,y)}(w_i) = (y,x)$. These observations imply that the truncated subsequence introduced above has no more variations than $\chi_{(x,y)}(w_0),\chi_{(x,y)}(w_1),\dots,\chi_{(x,y)}(w_m)$. The conclusion follows.

If $\chi_{x,y}(w')$ has exactly one more variation than $\chi_{x,y}(w)$, then the permutation formed by the final occurence of x and y in w' agree with $\chi_{x,y}(w_d)$. In fact, under this assumption, the sequence $\chi_{x,y}(w) = \chi_{(x,y)}(w_0),\chi_{(x,y)}(w_1),\dots,\chi_{(x,y)}(w_d)$ has as much variation as the truncated sequence $\chi_{(x,y)}(w_0),\chi_{(x,y)}(w_1),\dots,\chi_{(x,y)}(w_m)$. Therefore, we have $\chi_{(x,y)}(w_m) = \chi_{(x,y)}(w_{m+1}) = \cdots = \chi_{(x,y)}(w_d)$. The conclusion follows from the observations above (we have $\chi_{(x,y)}(w_m) = (x,y)$ in the case considered).

The Lemma is proved.

1.6.9. *Remark*

It is also straighforward to prove that the morphism $F_n TR : F_n\mathcal{E} \longrightarrow F_n\mathcal{X}$ is surjective. In fact, if $u \in \mathcal{X}(r)_d$, then we consider the simplex $w \in \mathcal{E}(r)_d$ introduced in Paragraph 1.4.2 and such that $TR(w) = u$. From the definition, it follows easily that $w \in F_n\mathcal{E}$ if $u \in F_n\mathcal{X}$.

We can improve the construction of the simplex $w \in \mathcal{E}(r)_d$. More precisely, we have a morphism of dg-modules $EZ : \mathcal{X} \longrightarrow \mathcal{E}$ which preserves the cellular structures on the operads and which is right inverse to the table reduction morphism $TR : \mathcal{E} \longrightarrow \mathcal{X}$. This construction is postponed to a next article.

§2. The interval-cut operations on chains

2.0.1. *Conventions for simplicial sets*

We refer to D. Curtis (cf. [5]), P. Gabriel and M. Zisman (cf. [7]), P. Goerss and J. Jardine (cf. [11]), P. May (cf. [19]) for our conventions on simplicial sets. We adopt the classical notation $[n]$, $n \in \mathbb{N}$, for the objects of the simplicial category Δ. A morphism $u : [m] \longrightarrow [n]$ is equivalent to a sequence $0 \leq u(0) \leq u(1) \leq \cdots \leq u(m) \leq n$. If X is a simplicial set and $x \in X_n$ is an n-dimensional simplex in X, then the expression $x(u(0), u(1), \ldots, u(m)) \in X_m$ denotes the image of $x \in X_n$ under the map $u^* : X_n \longrightarrow X_m$ associated to $u : [m] \longrightarrow [n]$. For instance, we have the identity $x = x(0, 1, \ldots, n)$. By this convention, we have also:

$$\begin{aligned} & d_i(x) = x(0, \ldots, i-1, i+1, \ldots, n), \quad \text{for } i = 0, 1, \ldots, n, \\ \text{and} \quad & s_j(x) = x(0, \ldots, j, j, \ldots, n), \quad \text{for } j = 0, 1, \ldots, n. \end{aligned}$$

The standard generators of the category of simplicial sets are denoted by Δ^n, $n \in \mathbb{N}$. The element $\Delta(0, 1, \ldots, n) \in (\Delta^n)_n$ is the universal simplex in Δ^n. The m-dimensional simplices of Δ^n are the elements

$$\Delta(u(0), \ldots, u(m)) \in (\Delta^n)_m,$$

where $u : [m] \longrightarrow [n]$. The simplex $\Delta(u(0), \ldots, u(m))$ is non-degenerate if and only if the application $u : [m] \longrightarrow [n]$ is injective.

If $x \in X_n$, then $\tilde{x} : \Delta^n \longrightarrow X$ is the unique morphism such that $\tilde{x}(\Delta(0, 1, \ldots, n)) = x$. In fact, $\tilde{x}(\Delta(u(0), \ldots, u(m))) = x(u(0), \ldots, u(m))$. If $u : [m] \longrightarrow [n]$, then $u_* : \Delta^m \longrightarrow \Delta^n$ is the unique morphism which maps the simplex $\Delta(0, 1, \ldots, m) \in (\Delta^m)_m$ to $\Delta(u(0), \ldots, u(m)) \in (\Delta^n)_m$.

2.1. On the Eilenberg-Zilber operad

The purpose of the section is to prove the following Theorem:

2.1.1. THEOREM

Let X be a simplicial set. There is a natural evaluation coproduct $\mathcal{E}(r) \otimes N_*(X) \longrightarrow N_*(X)^{\otimes r}$ which equips the normalized chain complex $N_*(X)$ with the structure of a coalgebra over the operad $\mathcal{E}$. The dual cochain complex $N^*(X)$ is equipped with the structure of an $\mathcal{E}$-algebra.

2.1.2. *On coalgebras over an operad*

An algebra over an operad $\mathcal{P}$ is defined as a dg-module A equipped an evaluation product $\mathcal{P}(r) \otimes A^{\otimes r} \longrightarrow A$. Dually, a coalgebra over an operad $\mathcal{P}$ is a dg-module C equipped an equivariant evaluation coproduct $\mathcal{P}(r) \otimes C \longrightarrow C^{\otimes r}$ which is coassociative with respect to the operad product and unital with respect to the operad unit. (Then, the dual dg-module C' is equipped with the structure of a $\mathcal{P}$-algebra.) In the context of coalgebras, an element $p \in \mathcal{P}(r)$ determines a cooperation $p^* : C \longrightarrow C^{\otimes r}$. In fact, the dg-modules $\mathrm{Hom}^c_C(r) = \mathrm{Hom}_{\mathbb{F}}(C, C^{\otimes r})$ form a dg-operad and the structure of a $\mathcal{P}$-coalgebra is equivalent to a morphism of dg-operads $\mathcal{P} \longrightarrow \mathrm{Hom}^c_C$.

To be more explicit, let V be a dg-module. The morphism operad associated to V is the dg-operad such that $\mathrm{Hom}_V(r) = \mathrm{Hom}_{\mathbb{F}}(V^{\otimes r}, V)$. Dually, the *comorphism operad* associated to V is the dg-operad such that $\mathrm{Hom}^c_V(r) = \mathrm{Hom}_{\mathbb{F}}(V, V^{\otimes r})$. Thus, if $g_1 : V \longrightarrow V^{\otimes s_1}, \dots, g_r : V \longrightarrow V^{\otimes s_r}$ and $f : V \longrightarrow V^{\otimes r}$, then $f(g_1, \dots, g_r) : V \longrightarrow V^{\otimes s_1 + \dots + s_r}$ is the composite morphism $f(g_1, \dots, g_r) = \pm g_1 \otimes \dots \otimes g_r \cdot f$. The sign follows from the permutation of $g_1, \dots, g_r$ with f. The dg-module V is equipped with the structure of a coalgebra over the comorphism operad Hom^c_V and the canonical morphism $\mathrm{Hom}^c_V(r) \otimes V \longrightarrow V^{\otimes r}$ is a universal evaluation coproduct.

2.1.3. *The Eilenberg-Zilber operad*

The Eilenberg-Zilber operad $\mathcal{Z}$ is the universal dg-operad together with a natural evaluation coproduct

$$\mathcal{Z}(r) \otimes N_*(X) \longrightarrow N_*(X)^{\otimes r}$$

defined for $X \in \mathcal{S}$. We refer to V. Hinich and V. Schechtman (cf. [13]) and V. Smirnov (cf. [24]). The idea of the construction goes back to the work of A. Dold (cf. [6]) on the Steenrod operations. The dg-module $\mathcal{Z}(r)$ is formed by the morphisms

$$f_X : N_*(X) \longrightarrow N_*(X)^{\otimes r}$$

in the product of the comorphism operads $\prod_X \mathrm{Hom}_{\mathbb{F}}(N_*(X), N_*(X)^{\otimes r})$ which define a natural transformation in $X \in \mathcal{S}$. Therefore, for a fixed $X \in \mathcal{S}$, we have a canonical operad morphism $\mathcal{Z}(r) \longrightarrow \mathrm{Hom}_{\mathbb{F}}(N_*(X), N_*(X)^{\otimes r})$ which is equivalent to an evaluation coproduct as above.

Classically, a natural transformation $f_X : C_*(X) \longrightarrow N_*(X)^{\otimes r}$ is determined by the image of the generators $\Delta(0,1,\dots,n) \in (\Delta^n)_n$, $n \in \mathbb{N}$. Furthermore, such a natural morphism induces a natural transformation on the normalized complex $f_X : N_*(X) \longrightarrow N_*(X)^{\otimes r}$ if the tensors $f_X(\Delta(0,1,\dots,n)) \in N_*(\Delta^n)^{\otimes r}$, $n \in \mathbb{N}$, are cancelled by the degeneracies $s^j : \Delta^n \longrightarrow \Delta^{n+1}$. To conclude, we have the identity:

$$\mathcal{Z}(r) = \prod_n \Big\{ \bigcap_{j=0}^{n} \ker(s^j : N_*(\Delta^n)^{\otimes r} \to N_*(\Delta^{n+1})^{\otimes r}) \Big\}.$$

To be more precise, a tensor of degree d in $N_*(\Delta^n)^{\otimes r}$ is associated to an operation of degree $d - n$ in the Eilenberg-Zilber operad.

2.2. The interval-cut operations in the Eilenberg-Zilber operad

To be precise, our aim is to construct an operad morphism $AW : \mathcal{X} \longrightarrow \mathcal{Z}$. We have then a sequence of operad morphisms

$$\mathcal{E} \xrightarrow{TR} \mathcal{X} \xrightarrow{AW} \mathcal{Z},$$

which, according to the discussion above, provide the chain complexes $N_*(X)$, $X \in \mathcal{S}$, with the structure of a coalgebra over $\mathcal{E}$, as claimed by the Theorem. The cooperation $AW(u) : N_*(X) \longrightarrow N_*(X)^{\otimes r}$, which is associated to a surjection $u \in \mathcal{X}(r)_d$, generalizes the Alexander-Whitney diagonal. This justifies the notation $AW : \mathcal{X} \longrightarrow \mathcal{Z}$ for the morphism.

We have in fact

$$AW(u)(\Delta(0,1,\dots,n)) = \sum \pm \Delta(C_{(1)}) \otimes \cdots \otimes \Delta(C_{(r)}) \in N_*(\Delta^n)^{\otimes r}.$$

The element $\Delta(C_{(1)}) \otimes \cdots \otimes \Delta(C_{(r)})$ arises from a decomposition of the interval $\{0,1,\dots,n\}$. The details of the construction are fixed in the next Paragraph. The sum ranges over all decompositions. The sign which is associated to a term is specified in Paragraph 2.2.4. By reference to our construction, we call *interval cuts* the multi-simplices $\Delta(C_{(1)}) \otimes \cdots \otimes \Delta(C_{(r)})$ which we associate to the surjection u. The element $AW(u) \in \mathcal{Z}(r)_d$ is also denoted as an *interval cut operation.*

This generalization of the Alexander-Whitney diagonal goes back to the original construction of the reduced square operations by N. Steenrod (cf. [29]). Such a morphism $AW : \mathcal{X} \longrightarrow \mathcal{Z}$ is also defined by D. Benson (cf. [2]) and J. McClure and J. Smith (cf. [22]). Our construction differs simply by the sign conventions.

2.2.1. *On the interval-cut associated to a surjection*

We define the simplices $\Delta(C_{(k)}) \in N_*(\Delta^n)$, $k = 1,\dots,r$, which are associated to a surjection $u \in \mathcal{X}(r)_d$ for a fixed decomposition of the interval $\{0,1,\dots,n\}$. In fact, we have $\Delta(C_{(k)}) = c_{(k)}^* \Delta(0,\dots,n)$ for certain morphisms $c_{(k)} : [m_k] \longrightarrow [n]$, $k = 1,\dots,r$, in the simplicial category. Thus, the letter $C_{(k)}$ is a notation for the sequence $0 \leq c_{(k)}(1) \leq \cdots \leq c_{(k)}(m_k) \leq n$, according to our Convention 2.1 on simplicial sets.

Explicitly, the interval $\{0,\dots,n\}$ is cut on positions fixed by indices

$$0 = n_0 \leq n_1 \leq \cdots \leq n_{r+d-1} \leq n_{r+d} = n$$

as represented in the following diagram

The sequence $C_{(k)}$ is the concatenation of the intervals $[n_{i-1}, n_i]$ such that $u(i) = k$. Hence, if $u(i_1),\dots,u(i_e)$ denote the occurences of k in the sequence $(u(1),\dots,u(r+d))$, then the element $\Delta(C_{(k)})$ is represented by the simplex:

$$\Delta(n_{i_1-1} \text{——} n_{i_1}, n_{i_2-1} \text{——} n_{i_2}, \dots, n_{i_e-1} \text{——} n_{i_e}).$$

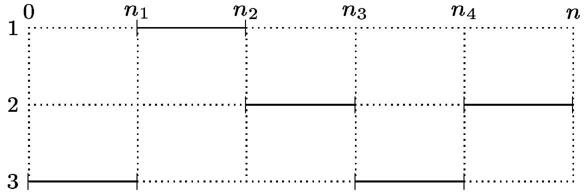
This simplex is non-degenerate if $n_{i_{x-1}} < n_{i_x-1}$, for $x = 1,\dots,e-1$. Next, we identify the dimension of the simplex to the length of the intervals $[n_{i_x-1}, n_{i_x}]$:

$$\begin{aligned} m_k &= \dim \Delta(n_{i_1-1} \text{——} n_{i_1}, n_{i_2-1} \text{——} n_{i_2}, \dots, n_{i_e-1} \text{——} n_{i_e}) \\ &= \operatorname{length}_u(n_{i_1-1} \text{——} n_{i_1}) + \operatorname{length}_u(n_{i_2-1} \text{——} n_{i_2}) + \cdots + \operatorname{length}_u(n_{i_e-1} \text{——} n_{i_e}). \end{aligned}$$

As mentioned in the introduction, we refer to the simplices $\Delta(C_{(1)}),\dots,\Delta(C_{(r)})$ as interval-cut elements.

2.2.2. *Example*

Here is a convenient graphical representation which we adopt in the article. The intervals $[n_{i-1}, n_i]$ are arranged on r lines in a diagram which has $r+d$ columns delimited by the variables n_i. The interval $[n_{i-1}, n_i]$ is put on the line indexed by $u(i)$ in the ith column of the diagram. As an example, the surjection, which is specified by the sequence $(3,1,2,3,2)$, gives the interval-cut diagram



and is associated to the tensor

$$\Delta(n_1 \text{\_\_\_} n_2) \otimes \Delta(n_2 \text{\_\_\_} n_3, n_4 \text{\_\_\_} n) \otimes \Delta(0 \text{\_\_\_} n_1, n_3 \text{\_\_\_} n_4).$$

The dimensions of these simplices are respectively $m_1 = (n_2 - n_1)$, $m_2 = (n_3 - n_2 + 1) + (n - n_4)$ and $m_3 = (n_1 + 1) + (n_4 - n_3)$. Equivalently, the diagram is a representation of the morphisms

$$\begin{array}{lllll} & 0 \text{\_\_\_} m_1 & \xrightarrow{\simeq} & n_1 \text{\_\_\_} n_2 & \hookrightarrow & 0 \text{\_\_\_} n, \\ & 0 \text{\_\_\_} m_2 & \xrightarrow{\simeq} & n_2 \text{\_\_\_} n_3, n_4 \text{\_\_\_} n & \hookrightarrow & 0 \text{\_\_\_} n \\ \text{and} & 0 \text{\_\_\_} m_3 & \xrightarrow{\simeq} & 0 \text{\_\_\_} n_1, n_3 \text{\_\_\_} n_4 & \hookrightarrow & 0 \text{\_\_\_} n \end{array}$$

which are denoted by $c_{(k)} : [m_k] \longrightarrow [n]$ in the definition.

2.2.3. *On the interval length determined by an interval-cut*

In the interval cut associated to a surjection, there are *inner* and *final* intervals. The interval $[n_{i-1}, n_i]$ is final, if $u(i)$ is a final value of the surjection. The interval $[n_{i-1}, n_i]$ is inner, if $u(i)$ is a caesura of the surjection. Consider again the simplex:

$$\Delta(n_{i_1-1} \text{\_\_\_} n_{i_1}, n_{i_2-1} \text{\_\_\_} n_{i_2}, \dots, n_{i_e-1} \text{\_\_\_} n_{i_e})$$

which is associated to the occurences of k in the surjection u. The interval $[n_{i_x-1}, n_{i_x}]$ is inner for $x = 1, \dots, e-1$ and final for $x = e$.

The length of an interval $[n_{i-1}, n_i]$ is defined by $\operatorname{length}_u[n_{i-1}, n_i] = n_i - n_{i-1}$ for a final interval and by $\operatorname{length}_u[n_{i-1}, n_i] = n_i - n_{i-1} + 1$ for an inner interval. Hence, in the context above, we have

$$\begin{array}{lll} & \operatorname{length}_u(n_{i_x-1} \text{\_\_\_} n_{i_x}) = n_{i_x} - n_{i_x-1} + 1, & \text{for } x = 1, \dots, e-1, \\ \text{and} & \operatorname{length}_u(n_{i_x-1} \text{\_\_\_} n_{i_x}) = n_{i_x} - n_{i_x-1}, & \text{for } x = e. \end{array}$$

Therefore, we have the formula:

$$\begin{aligned} &\dim \Delta(n_{i_1-1} \text{\_\_\_} n_{i_1}, n_{i_2-1} \text{\_\_\_} n_{i_2}, \dots, n_{i_e-1} \text{\_\_\_} n_{i_e}) \\ &= \operatorname{length}_u(n_{i_1-1} \text{\_\_\_} n_{i_1}) + \operatorname{length}_u(n_{i_2-1} \text{\_\_\_} n_{i_2}) + \dots + \operatorname{length}_u(n_{i_e-1} \text{\_\_\_} n_{i_e}). \end{aligned}$$

2.2.4. *On the signs associated to an interval-cut*

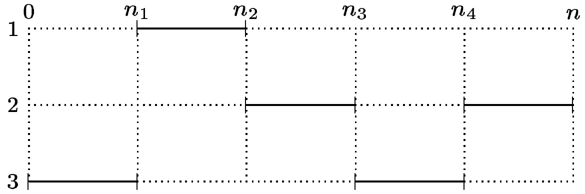
The interval-cut defined in Paragraph 2.2.1 has an associated permutation sign and an associated position sign. The permutation sign depends on the relative position of the intervals $[n_{i-1}, n_i]$. The position sign depends explicitly on the absolute position of the inner intervals $[n_{i-1}, n_i]$. The total sign associated to the interval-cut is the product of both the permutation sign and the position sign.

The permutation sign is determined as follows. Explicitly, we consider the shuffle which takes $(u(1), \dots, u(r+d))$ to the ordered sequence $(1, \dots, 1, 2, \dots, 2, \dots, r, \dots, r)$. The permutation of the associated intervals produces a sign according to the permutation rule in dg-calculus with the lengths of the intervals as degrees. This sign is the permutation sign associated to the interval-cut.

The position signs are provided by the inner intervals. Explicitly, if the interval $[n_{i-1}, n_i]$ is inner, then this interval has n_i as an associated position sign-exponent.

2.2.5. *Example*

As an example, take the interval-cut introduced in Paragraph 2.2.2, which is associated to the sequence $u = (3, 1, 2, 3, 2)$:



In this case, the final intervals are $[n_1, n_2]$, $[n_3, n_4]$ and $[n_4, n]$. These interval have length:

$$\begin{aligned} &\mathrm{length}_u(n_1 \underline{\hspace{2em}} n_2) = n_2 - n_1, \qquad \mathrm{length}_u(n_3 \underline{\hspace{2em}} n_4) = n_4 - n_3 \\ &\text{and} \qquad \mathrm{length}_u(n_4 \underline{\hspace{2em}} n) = n - n_4. \end{aligned}$$

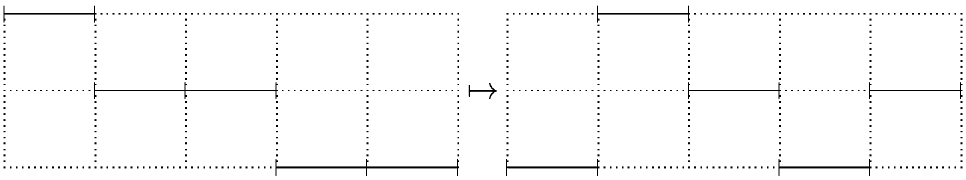
The inner intervals are $[0, n_1]$ and $[n_2, n_3]$ and have length:

$$\mathrm{length}_u(0 \underline{\hspace{2em}} n_1) = n_1 + 1 \qquad \text{and} \qquad \mathrm{length}_u(n_2 \underline{\hspace{2em}} n_3) = n_3 - n_2 + 1.$$

The associated position sign is given by the exponent $\tau = n_1 + n_3$. The permutation sign is determined by the shuffle:

$$[n_1, n_2], [n_2, n_3], [n_4, n], [0, n_1], [n_3, n_4] \mapsto [0, n_1], [n_1, n_2], [n_2, n_3], [n_3, n_4], [n_4, n]$$

In diagrams:



Hence, the permutation sign-exponent associated to the interval cut is the sum $\sigma = (n_1 + 1)(n_2 - n_1) + (n_1 + 1)(n_3 - n_2 + 1) + (n_1 + 1)(n - n_4) + (n_4 - n_3)(n - n_4)$.

2.2.6. *The interval cut operation associated to a surjection*

The interval cut operation $AW(u) \in \mathcal{Z}(r)$ associated to $u \in \mathcal{X}(r)_d$ maps the simplex $\Delta(0,\dots,n) \in \Delta^n$ to the sum

$$AW(u)(\Delta(0,\dots,n)) = \sum \pm\Delta(C_{(1)}) \otimes \cdots \otimes \Delta(C_{(r)})$$

over the indices $0 = n_0 \leq n_1 \leq \cdots \leq n_{r+d-1} \leq n_{r+d} = n$, where $\Delta(C_{(1)}),\dots,\Delta(C_{(r)})$ are the interval cut simplices defined in Paragraph 2.2.1. The sign associated to a term is the product of the permutation and position signs defined in Paragraph 2.2.4.

In general, the image of a simplex $x \in X_n$ is determined by the formula:

$$AW(u)(x) = \sum \pm x(C_{(1)}) \otimes \cdots \otimes x(C_{(r)}).$$

As an example, if $u = (3,1,2,3,2)$, then we have

$$AW(u)(x) = \sum \pm x(n_1 \rule{1.5em}{0.4pt} n_2) \otimes x(n_2 \rule{1.5em}{0.4pt} n_3, n_4 \rule{1.5em}{0.4pt} n) \otimes x(0 \rule{1.5em}{0.4pt} n_1, n_3 \rule{1.5em}{0.4pt} n_4)$$

for any n-dimensional simplex $x \in X_n$. Equivalently, we have:

$$AW(u)(x) = \sum \pm c_{(1)}^*(x) \otimes \cdots \otimes c_{(r)}^*(x),$$

where $c_{(1)},\dots,c_{(r)}$ are the morphisms in the simplicial category which are represented by the sequences $C_{(1)},\dots,C_{(r)}$.

2.2.7. THEOREM

The map $u \mapsto AW(u)$ defines an operad morphism $AW : \mathcal{X} \longrightarrow \mathcal{Z}$.

We refer to J. McClure and J. Smith (cf. [22]) for a proof of the Theorem above. Our demonstration is very similar. In fact, it is also straighforward to verify that our signs are coherent using the invariance properties of the lengths in an interval cut diagram.

2.2.8. *On the Alexander-Whitney diagonal*

The operation $AW(1,2) : N_*(X) \longrightarrow N_*(X)^{\otimes 2}$ is identified with the Alexander-Whitney diagonal. We have clearly:

$$AW(1,2)(x) = \sum_{0 \leq i \leq n} x(0 \rule{1.5em}{0.4pt} i) \otimes x(i \rule{1.5em}{0.4pt} n).$$

Similarly, the operations

$$AW(\theta_d) : N_*(X) \longrightarrow N_*(X)^{\otimes 2}$$

where $\theta_d = (1,2,1,2,\dots) \in \mathcal{X}(2)_d$. identify to the higher Alexander-Whitney diagonals, which induce the cup-d products on cochains (cf. [29]). We have explicitly:

$$\begin{aligned} AW(1,2,1)(x) \quad &= \sum\nolimits_{i<j} \pm x(0 \rule{1.5em}{0.4pt} i, j \rule{1.5em}{0.4pt} n) \otimes x(i \rule{1.5em}{0.4pt} j), \\ AW(1,2,1,2)(x) \quad &= \sum\nolimits_{i<j<k} \pm x(0 \rule{1.5em}{0.4pt} i, j \rule{1.5em}{0.4pt} k) \otimes x(i \rule{1.5em}{0.4pt} j, k \rule{1.5em}{0.4pt} n), \end{aligned}$$

and so on. The sign-exponent is $(n-j)(j-i)+i$ for $AW(1,2,1)$ and $(k-j)(j-i+1)+i+j$ for $AW(1,2,1,2)$.

§3. On closed model structures

3.1. The closed model structure

The purpose of this Section is to state the following Theorem:

3.1.1. THEOREM

The $\mathcal{E}$-algebras are equipped with a closed model category structure for which the weak equivalences (respectively, the fibrations) are the algebra morphisms which are weak equivalences (respectively, fibrations) in the category of dg-modules.

In fact, our arguments extend to a wide class of operads. In general, an operad $\mathcal{P}$ has a canonical Σ_*-projective resolution which is provided by the tensor product with the Barratt-Eccles operad. More explicitly, we consider the operad such that $(\mathcal{E} \otimes \mathcal{P})(r) = \mathcal{E}(r) \otimes \mathcal{P}(r)$. We have a canonical augmentation $\mathcal{E} \otimes \mathcal{P} \longrightarrow \mathcal{P}$ which is induced by the augmentation $\mathcal{E} \longrightarrow \mathcal{C}$ on the Barratt-Eccles operad. Our arguments extend to the following situation:

3.1.2. ASSUMPTION: *The augmentation morphism $\mathcal{E} \otimes \mathcal{P} \longrightarrow \mathcal{P}$ has a section $\mathcal{P} \longrightarrow \mathcal{E} \otimes \mathcal{P}$ (which is supposed to be an operad morphism).*

To be explicit:

3.1.3. THEOREM

If the operad $\mathcal{P}$ satisfies the Assumption 3.1.2 above, then the $\mathcal{P}$-algebras are equipped with a closed model category structure for which the weak equivalences (respectively, the fibrations) are the algebra morphisms which are weak equivalences (respectively, fibrations) in the category of dg-modules.

The assumption is satisfied by the Barratt-Eccles operad $\mathcal{E}$. In this case, a section $\mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{E}$ is given by the Alexander-Whitney diagonal (cf. Paragraph 1.1.4). More generally, we have:

3.1.4. FACT: *The Assumption 3.1.2 is satisfied by the Σ_*-projective resolution $\mathcal{P} = \mathcal{E} \otimes \mathcal{Q}$ associated to an operad $\mathcal{Q}$. The section $\mathcal{P} \longrightarrow \mathcal{E} \otimes \mathcal{P}$ is provided by the diagonal on the Barratt-Eccles operad $\mathcal{E} \otimes \mathcal{Q} \longrightarrow \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{Q}$.*

The assumption is also satisfied in the following situation:

3.1.5. FACT: *The Assumption 3.1.2 is satisfied by the cofibrant operads $\mathcal{P}$. In this case, the existence of the section $\mathcal{P} \longrightarrow \mathcal{E} \otimes \mathcal{P}$ follows from the left lifting property of operad cofibrations (cf. V. Hinich, [12]).*

By definition, a morphism of dg-modules is a fibration if it is surjective (in all degrees) and a weak-equivalence if it induces an isomorphism in homology. The cofibrations are the morphisms which have the left lifting property with respect to acyclic fibrations. The Theorem does not hold for any operad. The Property which allow to achieve the proof of the Theorem is stated in the next Lemma. In fact, in the context of algebras over a dg-operad, it is known that this Lemma implies precisely the Theorem (cf. V. Hinich, [12]). But, it is also possible to adapt classical methods of D. Quillen (cf. D. Quillen, [23]) to our situation (cf. M. Livernet, [16]) and, in the sequel, we follow this latter approach.

3.1.6. LEMMA

If the Assumption 3.1.2 is satisfied, then the $\mathcal{P}$-algebra morphisms which have the left-lifting property with respect to fibrations are weak-equivalences.

The Lemma has the following consequence. The morphism, which we consider, is related to the notion of a cell-extension in the context of algebras over an operad. We consider the free $\mathcal{P}$-algebras $\mathcal{P}(V)$ generated by certain dg-modules V. Consider the dg-module $C(n) = \mathbb{F} \cdot \mathbf{c}^{n-1} \oplus \mathbb{F} \cdot \mathbf{e}^n$ generated by the elements $\mathbf{c}^{n-1}$ in degree $n-1$ and $\mathbf{e}^n$ in degree n together with the differential $\delta(\mathbf{c}^{n-1}) = \mathbf{e}^n$. The morphism $0 \longrightarrow C(n)$ is an acyclic cofibration in the category of dg-modules. Consequently, the associated morphism of $\mathcal{P}$-algebras $\mathcal{P}(0) \longrightarrow \mathcal{P}(C(n))$ has the left-lifting property with respect to fibrations. Consider the cobase extension

$$\begin{tikzcd}
\mathcal{P}(0) \arrow[r] \arrow[d] & \mathcal{P}(C(n)) \arrow[d] \\
A \vee \mathcal{P}(0) \arrow[r] & A \vee \mathcal{P}(C(n))
\end{tikzcd}$$

where A is any $\mathcal{E}$-algebra. Since $\mathcal{P}(0)$ is the initial $\mathcal{P}$-algebra, we have also $A = A \vee \mathcal{P}(0)$. Finally, the morphism $A \longrightarrow A \vee \mathcal{P}(C(n))$ has also the left-lifting property with respect to fibrations. Let us conclude:

3.1.7. PROPERTY: *If the Assumption 3.1.2 is satisfied, then the canonical morphism $A \longrightarrow A \vee \mathcal{P}(C(n))$ is a weak equivalence for any $\mathcal{P}$-algebra A.*

There are examples of Σ_*-projective operads $\mathcal{P}$ for which this Property fails. In general, we have just the following weaker property:

3.1.8. FACT: *The canonical morphism $A \longrightarrow A \vee \mathcal{P}(C(n))$ is a weak equivalence if A is a cell-algebra.*

These issues are discussed by H. Hinich (cf. [12]) and M. Mandell (cf. [18]) to whom we refer for the notion of a cell-algebra. We just observe that, in the situation of the Assumption 3.1.2, the $\mathcal{P}$-algebras are equipped with canonical path objects. The existence of a path object provides an easy argument for the Lemma 3.1.6, as in the context of simplicial algebras (cf. D. Quillen [23]).

3.1.9. *On path objects*

Let A be a $\mathcal{P}$-algebra. Recall that a path object for A is a $\mathcal{P}$-algebra $\tilde{A}$ together with morphisms

$$\begin{tikzcd}
A \arrow[r, "\sim", "s_0"'] & \tilde{A} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & A
\end{tikzcd}$$

such that s_0 is a weak equivalence and $d_0 s_0 = d_1 s_0 = 1_A$. If the operad $\mathcal{P}$ satisfies the Assumption 3.1.2, then such a diagram is given by the tensor products:

$$\begin{tikzcd}
N^*(\Delta^0) \otimes A \arrow[r, "\sim", "s_0"'] & N^*(\Delta^1) \otimes A \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & N^*(\Delta^0) \otimes A.
\end{tikzcd}$$

In fact, the tensor product $K\otimes A$ of the $\mathcal{P}$-algebra A with an $\mathcal{E}$-algebra K forms a $\mathcal{E}\otimes\mathcal{P}$-algebra. If there is an operad morphism $\mathcal{P} \longrightarrow \mathcal{E}\otimes\mathcal{P}$, then the tensor product $K\otimes A$ is a $\mathcal{P}$-algebra by restriction of structure. Furthermore, the $\mathcal{E}$-algebra $N^*(\Delta^0)$ is identified with the ground ring $\mathbb{F}$, which is equipped with the $\mathcal{E}$-algebra structure obtained by restriction through the operad augmentation $\mathcal{E} \longrightarrow \mathcal{C}$. Consequently, the tensor product $N^*(\Delta^0)\otimes A$ is identified with the dg-module A which is equipped with the $\mathcal{E}\otimes\mathcal{P}$-algebra structure obtained by restriction through the augmentation $\mathcal{E}\otimes\mathcal{P} \longrightarrow \mathcal{P}$. Therefore, if the morphism $\mathcal{P} \longrightarrow \mathcal{E}\otimes\mathcal{P}$ is a section of the augmentation $\mathcal{E}\otimes\mathcal{P} \longrightarrow \mathcal{P}$, then the dg-module $N^*(\Delta^0)\otimes A$ is identified with A as a $\mathcal{P}$-algebra. We conclude that the tensor product $N^*(\Delta^1)\otimes A$ forms a path object for A.

Let us mention that the tensor product $N^*(\Delta^1)\otimes A$ is identified with the classical cylinder construction in the category of dg-modules. Therefore, we prefer to dualize the terminology and call $N^*(\Delta^1)\otimes A$ the *cylinder algebra* on A. To recapitulate:

3.1.10. PROPOSITION

If the Assumption 3.1.2 is satisfied, then a $\mathcal{P}$-algebra A has a canonical path object which is represented by the tensor product $N^*(\Delta^1)\otimes A$.

The Lemma 3.1.6 is a consequence of the following classical observation (cf. [16], [23]):

3.1.11. OBSERVATION: *The morphisms which have the left-lifting property with respect to fibrations are strong deformations retracts.*

Just recall that a morphism $f : A \longrightarrow F$ is a strong deformation retract if there is a morphism $r : F \longrightarrow A$ such that $rf = 1_F$ and a morphism $h : F \longrightarrow N^*(\Delta^1)\otimes F$ such that $d_0 h = fr$, $d_1 h = 1_F$ and $hf = s_0 f$. Clearly, these identities imply that $f : A \longrightarrow F$ is a weak-equivalence.

3.2. On spheres, cones and suspensions

3.2.1. *On spheres*

We let S^n denote the standard simplicial model of the n-dimensional sphere:

$$S^n = \Delta^n / \cup_{i=0}^n \Delta^n(0,\dots,i-1,i+1,\dots,n).$$

This simplicial set has just a non-degenerate simplex $\Delta(0,1,\dots,n)\in S^n$ in dimension n and a base point $*\in S^n$ in dimension 0. We let $S(n)$ denote the reduced normalized cochain complex $S(n) = \tilde{N}^*(S^n)$ associated to S^n. In fact, the reduced chain complex $\tilde{N}_*(S^n)$ is concentrated in degree n and is generated by $\Delta(0,1,\dots,n)$. We let $\mathbf{e}^n \in \tilde{N}^*(S^n)$ denote the dual element. Hence, we have:

$$S(n) = \mathbb{F}\cdot\mathbf{e}^n.$$

If $n = 1$, then, for simplicity, we write $\mathbf{e} = \mathbf{e}^1$.

The $\mathcal{E}$-algebra product

$$\mathcal{E}(r)_d \otimes (S(n)^{\otimes r})^* \longrightarrow S(n)^{*-d}$$

has a non-trivial component in degree $* = nr$ and $d = n(r-1)$. One purpose of this Section is to make this component explicit.

3.2.2. *On the pointed interval*

Similarly, we let $C(1)$ denote the reduced normalized cochain complex on the standard simplicial interval Δ^1 equipped with the base point $* = \Delta(0)$. In fact, the reduced chain complex $\tilde{N}_*(\Delta^1)$ is generated by $\Delta(0,1) \in \tilde{N}_1(C^n)$ and $\Delta(1) \in \tilde{N}_0(C^n)$. Therefore, if $\mathbf{e} \in \bar{N}^1(\Delta^1)$ and $\mathbf{c} \in \bar{N}^0(\Delta^1)$ denote the dual elements, then we have:

$$C(1) = \mathbb{F} \cdot \mathbf{e} \oplus \mathbb{F} \cdot \mathbf{c},$$

together with the differential $\delta(\mathbf{c}) = \mathbf{e}$. We make also the $\mathcal{E}$-algebra products in $C(1)$ explicit.

To conclude this paragraph, we have the canonical exact sequence:

$$0 \longrightarrow S(1) \longrightarrow C(1) \longrightarrow S(0) \longrightarrow 0.$$

More explicitly, the circle $S(1)$ is isomorphic the sub dg-module $\mathbb{F} \cdot \mathbf{e} \hookrightarrow C(1)$ and the zero-sphere $S(0)$ is isomorphic to the quotient dg-module $C(1) \to \mathbb{F} \cdot \mathbf{c}$.

3.2.3. *Certain fundamental cochains*

We introduce a cochain $\epsilon_s : \mathcal{E}(r)_* \longrightarrow \mathbb{F}$ in order to make the algebra structure of $S(1)$ and $C(1)$ explicit. Consider a d-dimensional simplex $(w_0, \ldots, w_d) \in \mathcal{E}(r)_d$. If the sequence $(w_0(1), \ldots, w_d(1))$ does not form a permutation of $(1, \ldots, s)$, then we set $\epsilon_s(w_0, \ldots, w_d) = 0$. In particular, the cochain $\epsilon_s : \mathcal{E}(r)_* \longrightarrow \mathbb{F}$ has to vanish in degree $d \neq s-1$. Otherwise, we define $\epsilon_s(w_0, \ldots, w_{s-1})$ as the signature of the permutation:

$$\epsilon_s(w_0, \ldots, w_{s-1}) = \operatorname{sgn}(w_0(1), \ldots, w_{s-1}(1)).$$

By convention, the 0-cochain $\epsilon_0 : \mathcal{E}(r)_* \longrightarrow \mathbb{F}$ is the classical augmentation on the bar complex.

3.2.4. THEOREM

1) On the circle $S(1)$, the $\mathcal{E}$-algebra evaluation product is given by the following formula:

i) $$w(\mathbf{e}_{(1)}, \ldots, \mathbf{e}_{(r)}) = (-1)^\sigma \cdot \epsilon_r(w) \cdot \mathbf{e},$$

where $\sigma = r(r-1)/2$.

2) On the cone $C(1)$, the evaluation product is characterized by the following equations:

$$\begin{aligned} &i) && w(\mathbf{e}_{(1)}, \ldots, \mathbf{e}_{(r)}) = (-1)^\sigma \cdot \epsilon_r(w) \cdot \mathbf{e}, \\ &ii) && w(\mathbf{e}_{(1)}, \ldots, \mathbf{e}_{(s)}, \mathbf{c}_{(s+1)}, \ldots, \mathbf{c}_{(r)}) = (-1)^\sigma \cdot \epsilon_s(w) \cdot \mathbf{e} \\ &iii) && \text{and} \quad w(\mathbf{c}_{(1)}, \ldots, \mathbf{c}_{(r)}) = (-1)^\sigma \cdot \epsilon_0(w) \cdot \mathbf{c} \end{aligned}$$

(where, whenever it makes sense, $0 < s < r$). The subscripts in these formulas specify the places of the copies of $\mathbf{e}$ and $\mathbf{c}$. In fact, the equation i) is the instance $s = r$ of the equation ii). In all cases, the sign-exponent is given by $\sigma = s(s-1)/2$ where $s = r$ for equation i) and $s = 0$ for equation iii).

We have morphisms of dg-modules

$$\begin{array}{lcccl} & \tilde{N}^*(S^1 \wedge X) & \to & \tilde{N}^*(S^1) \otimes \tilde{N}^*(X) & = & \Sigma^* \tilde{N}^*(X) \\ \text{and} & \tilde{N}^*(\Delta^1 \wedge X) & \to & \tilde{N}^*(\Delta^1) \otimes \tilde{N}^*(X) & = & C^* \tilde{N}^*(X) \end{array}$$

given by the classical Eilenberg-Zilber equivalence. But, these morphisms are not an compatible with the $\mathcal{E}$-algebra structure in general.

Nevertheless, it is possible to generalize the formula above to the n-sphere $S(n)$. In fact, we obtain the following result, which we mention as a remark:

3.2.5. PROPOSITION

The canonical isomorphim $S(n) \longrightarrow S(1)^{\otimes n}$ *which identifies the element* $\mathbf{e}^n \in S(n)$ *to the tensor* $\mathbf{e}^{\otimes n} \in S(1)^{\otimes n}$ *is an isomorphism of* $\mathcal{E}$*-algebras.*

3.2.6. *On cones and suspensions*

If $V \in dg\,\mathcal{M}od_{\mathbb{F}}$ is an (upper-graded) dg-module, then $C^*V \in dg\,\mathcal{M}od_{\mathbb{F}}$ is the classical cone associated to V in the category of dg-modules. In fact, $C^*V = C(1) \otimes V$. Therefore, we have

$$(C^*V)^d = \mathbf{c} \otimes V^d \oplus \mathbf{e} \otimes V^{d-1}$$

and the differential of C^*V is given by

$$\delta(\mathbf{c} \otimes x + \mathbf{e} \otimes y) = \mathbf{c} \otimes \delta(x) + \mathbf{e} \otimes (x - \delta(y)).$$

Similarly, the suspension of V in the category of dg-modules, denoted by $\Sigma^*V \in dg\,\mathcal{M}od_{\mathbb{F}}$, is identified with $S(1) \otimes V$. Equivalently:

$$(\Sigma^*V)^d = \mathbf{e} \otimes V^{d-1}.$$

There is also a short exact sequence:

$$0 \longrightarrow \Sigma^*V \longrightarrow C^*V \longrightarrow V \longrightarrow 0.$$

If A is an $\mathcal{E}$-algebra, then $C^*A = C(1) \otimes A$ is a tensor product of $\mathcal{E}$-algebras. Therefore, the cone C^*A has the structure of an $\mathcal{E}$-algebra. Similarly, the suspension $\Sigma^*A = S(1) \otimes A$ has a natural $\mathcal{E}$-algebra structure. We obtain immediately:

3.2.7. PROPOSITION

Let A *be an* $\mathcal{E}$*-algebra. The cone* $C^*A = \mathbf{e} \otimes A \oplus \mathbf{c} \otimes A$ *is equipped with a natural* $\mathcal{E}$*-algebra structure. Furthermore, the suspension* $\Sigma^*A = \mathbf{e} \otimes A \hookrightarrow C^*A$ *is a subalgebra of this cone* C^*A *and the canonical surjection* $C^*A \to \mathbf{c} \otimes A = A$ *is a morphism of* $\mathcal{E}$*-algebras. In* C^*A*, the products are given by the following formulas:*

$$\begin{aligned} & w(\mathbf{e} \otimes a_{(1)}, \ldots, \mathbf{e} \otimes a_{(r)}) = \pm(-1)^{\sigma} \cdot \mathbf{e} \otimes (\epsilon_r \cap w)(a_{(1)}, \ldots, a_{(r)}), \\ & w(\mathbf{e} \otimes a_{(1)}, \ldots, \mathbf{e} \otimes a_{(s)}, \mathbf{c} \otimes a_{(s+1)}, \ldots, \mathbf{c} \otimes a_{(r)}) = \pm(-1)^{\sigma} \cdot \mathbf{e} \otimes (\epsilon_s \cap w)(a_{(1)}, \ldots, a_{(r)}), \\ & \textit{and} \qquad w(\mathbf{c} \otimes a_{(1)}, \ldots, \mathbf{c} \otimes a_{(r)}) = \pm(-1)^{\sigma} \cdot \mathbf{c} \otimes w(a_{(1)}, \ldots, a_{(r)}), \end{aligned}$$

where $0 < s < r$. *In fact, the first formula is the instance* $s = r$ *of the second formula. But, the formula for* $s = 0$ *differs from the general case.*

In all cases, we have $\sigma = s(s-1)/2$ *(as in Theorem 3.2.4). The unspecified signs are determined by the commutation of the elements* $\mathbf{e}$ *with the factors* $w, a_{(1)}, \dots, a_{(r)}$ *and, therefore, have* $\deg(w) \cdot s + \deg(a_1) \cdot (s-1) + \cdots + \deg(a_{s-1}) \cdot 1$ *as a sign-exponent.*

In our context, the cap product of the cochain $\phi : \mathcal{E}(r)_d \longrightarrow \mathbb{F}$ with the chain $w = (w_0, \dots, w_n) \in \mathcal{E}(r)_n$ is the element

$$\phi \cap w = \phi(w_0, \dots, w_d) \cdot (w_d, \dots, w_n)$$

in $\mathcal{E}(r)_{n-d}$.

3.2.8. *On the operad suspension*

There is an operad $\Lambda^* \mathcal{E}$, associated to $\mathcal{E}$, whose algebras are the suspensions $\Sigma^* A$ of an $\mathcal{E}$-algebra A. We have in fact:

$$\Lambda^* \mathcal{E}(r)_d = \operatorname{sgn}(r) \otimes \mathcal{E}(r)_{d-r+1}$$

where $\operatorname{sgn}(r)$ denotes the signature representation (cf. [9]). If the letter w denotes an element of $\mathcal{E}(r)_d$, then $\Lambda^* w$ is the associated operation in $\Lambda^* \mathcal{E}(r)_{d+r-1}$. The suspension $\Sigma^* A$ of an $\mathcal{E}$-algebra A is equipped with the evaluation product

$$\Lambda^* \mathcal{E}(r) \otimes (\Sigma^* A)^{\otimes r} \longrightarrow (\Sigma^* A)$$

such that

$$(\Lambda^* w)(\mathbf{e} \otimes a_1, \dots, \mathbf{e} \otimes a_r) = \pm \mathbf{e} \otimes w(a_1, \dots, a_r).$$

The sign is produced by the commutation of the elements $\mathbf{e}$ with the factors $w, a_1, \dots, a_r$. Thus, the sign-exponent is identified with $\deg(w) \cdot r + \deg(a_1) \cdot (r-1) + \cdots + \deg(a_{r-1}) \cdot 1 + \deg(a_r) \cdot 0$.

The previous Proposition has the following consequence:

3.2.9. Proposition

The cap products $\epsilon_r \cap - : \mathcal{E}(r)_* \longrightarrow \mathcal{E}(r)_{*-r+1}$, $r > 0$, *define an operad morphism* $\epsilon_* \cap - : \mathcal{E} \longrightarrow \Lambda^* \mathcal{E}$.

3.2.10. *Remark*

The operads $\mathcal{X}$ and $\mathcal{Z}$ are also equipped with such morphisms $\mathcal{X} \longrightarrow \Lambda^* \mathcal{X}$ and $\mathcal{Z} \longrightarrow \Lambda^* \mathcal{Z}$. Furthermore, we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "TR"] \arrow[d] & \mathcal{X} \arrow[r, "AW"] \arrow[d, dashed] & \mathcal{Z} \arrow[d, dashed] \\
\Lambda^* \mathcal{E} \arrow[r, two heads, "\Lambda^* \alpha"'] & \Lambda^* \mathcal{X} \arrow[r, "\Lambda^* \rho"'] & \Lambda^* \mathcal{Z}
\end{tikzcd}$$

The morphism $\mathcal{X}(r)_* \longrightarrow \mathcal{X}(r)_{*-r+1}$ is very similar to $\epsilon_r \cap - : \mathcal{E}(r)_* \longrightarrow \mathcal{E}(r)_{*-r+1}$. Given $u \in \mathcal{X}(r)_d$, we let $\epsilon_r \cap u \in \mathcal{X}(r)_{d-r+1}$ be the sequence such that

$$\epsilon_r \cap u = \operatorname{sgn}(u(1), \dots, u(r)) \cdot (u(r), u(r+1), \dots, u(r+d)),$$

if $(u(1), \dots, u(r))$ is a permutation of $(1, \dots, r)$, and $\epsilon_r \cap u = 0$, otherwise. This cap product is an operad morphism $\epsilon_* \cap - : \mathcal{X} \longrightarrow \Lambda^* \mathcal{X}$ and makes the left-hand square commute as stated.

The morphism $\mathcal{Z}(r)_* \longrightarrow \mathcal{Z}(r)_{*-r+1}$ is due to V. Smirnov (cf. [25], [26]). Let $\sigma : N_*(\Delta^n) \longrightarrow N_{*-1}(\Delta^{n-1})$ be the dg-morphism such that:

$$\sigma(\Delta(i_0, i_1, \dots, i_n)) = \begin{cases} \Delta(i_1 - 1, \dots, i_n - 1), & \text{if } i_0 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The tensor powers

$$\sigma^{\otimes r} : (N_*(\Delta^n)^{\otimes r})_d \longrightarrow (N_*(\Delta^{n-1})^{\otimes r})_{d-r}$$

are equivalent to a morphism

$$\mathcal{Z}(r)_{d-n} \longrightarrow \mathcal{Z}(r)_{d-r+1-n}.$$

In fact, this morphism of dg-modules is a morphism of operads $\mathcal{Z} \longrightarrow \Lambda^* \mathcal{Z}$. It is also straightforward to prove that it makes the right-hand square commute.

3.3. Some proofs

In this section, we determine the structure of the chain complexes

$$\tilde{N}_*(X) \;=\; \tilde{N}_*(S^0),\ \tilde{N}_*(\Delta^1) \text{ and } \tilde{N}_*(S^1)$$

as coalgebras over the surjection operad $\mathcal{X}$. The algebra structure of $\tilde{N}^*(X)$, as stated in the Theorem 3.2.4, follows from these calculations and from our definitions.

To begin with, let us recall the following fact:

3.3.1. FACT: *There are morphisms of pointed simplicial sets $S^0 \longrightarrow \Delta^1 \longrightarrow S^1$ which induce the morphisms of coalgebras:*

$$\tilde{N}_*(S^0) \longrightarrow \tilde{N}_*(\Delta^1) \longrightarrow \tilde{N}_*(S^1).$$

We have explicitly:

$$\tilde{N}_*(S^0) = \mathbb{F} \cdot \Delta(0), \qquad \tilde{N}_*(\Delta^1) = \mathbb{F} \cdot \Delta(1) \oplus \mathbb{F} \cdot \Delta(0,1) \qquad \textit{and} \qquad \tilde{N}_*(S^1) = \mathbb{F} \cdot \Delta(0,1).$$

The morphism $\tilde{N}_*(S^0) \longrightarrow \tilde{N}_*(\Delta^1)$ takes $\Delta(0) \in \tilde{N}_0(S^0)$ to $\Delta(1) \in \tilde{N}_0(\Delta^1)$. The morphism $\tilde{N}_*(\Delta^1) \longrightarrow \tilde{N}_*(S^1)$ cancels $\Delta(1) \in \tilde{N}_0(\Delta^1)$ and maps $\Delta(0,1) \in \tilde{N}_1(\Delta^1)$ to $\Delta(0,1) \in \tilde{N}_1(S^1)$.

We make explicit the cooperations $AW(u) : \tilde{N}_*(X) \longrightarrow \tilde{N}_*(X)^{\otimes r}$ associated to a fixed surjection $u \in \mathcal{X}(r)_{s-1}$. This surjection is represented by the sequence

$$u = (u(1), \dots, u(s), \dots, u(s+r-1)).$$

The next assertion is immediate:

3.3.2. Fact: *Fix a surjection* $u \in \mathcal{X}(r)_0$. *In* $\tilde{N}_*(S^0)^{\otimes r}$, *we have* $AW(u)(\Delta(0)) = \Delta(0)^{\otimes r}$. *As a consequence, in* $\tilde{N}_*(\Delta^1)^{\otimes r}$, *we have* $AW(u)(\Delta(1)) = \Delta(1)^{\otimes r}$.

Our next purpose is to determine the components of the coproduct

$$AW(u)(\Delta(0,1)) = \sum \Delta(C_{(1)}) \otimes \cdots \otimes \Delta(C_{(r)})$$

in $\tilde{N}_*(\Delta^1)^{\otimes r}$. In fact, the module $\tilde{N}_*(\Delta^1)^{\otimes r}$ is generated in degree s by the permutations of the tensor $\Delta(0,1)^{\otimes s} \otimes \Delta(1)^{\otimes r-s}$ (for $0 \leq s \leq r$). If u has degree $s-1$, then $AW(u)(\Delta(0,1))$ has degree s. The next Lemma gives the component $\Delta(0,1)^{\otimes s} \otimes \Delta(1)^{\otimes r-s}$ of $AW(u)(\Delta(0,1))$, for $1 \leq s \leq r$. This result suffices to determine all components of the coproduct $AW(u)(\Delta(0,1))$ by Σ_r-equivariance.

3.3.3. Lemma

Fix a surjection $u \in \mathcal{X}(r)_{s-1}$, *where* $1 \leq s \leq r$. *The coproduct* $AW(u)(\Delta(0,1)) \in \tilde{N}_*(\Delta^1)^{\otimes r}$ *has no component* $\Delta(0,1)^{\otimes s} \otimes \Delta(1)^{\otimes r-s}$ *unless the sequences*

$$(u(1), u(2), \ldots, u(s)) \qquad \textit{and} \qquad (u(s), \ldots, u(s+r-1))$$

are permutations of $(1, \ldots, s)$ *and* $(1, \ldots, r)$. *In this case, we have:*

$$AW(u)(\Delta(0,1)) = \operatorname{sgn}(u(1), \ldots, u(s)) \cdot \Delta(0,1)^{\otimes s} \otimes \Delta(1)^{\otimes r-s}$$

in $\tilde{N}_*(\Delta^1)^{\otimes r}$.

As a consequence:

3.3.4. Fact:

Fix a surjection $u \in \mathcal{X}(r)_{r-1}$. *The coproduct* $AW(u)(\Delta(0,1))$ *vanishes in* $\tilde{N}_*(S^1)^{\otimes r}$ *unless the sequences* $(u(1), \ldots, u(r))$ *and* $(u(r), \ldots, u(r+r-1))$ *are both permutations of* $(1, \ldots, r)$. *In this case, we have* $AW(u)(\Delta(0,1)) = \operatorname{sgn}(u(1), \ldots, u(r)) \cdot \Delta(0,1)^{\otimes r}$.

The proof of the Lemma is postponed to the end of the section. We now determine the coproducts $w^* : \tilde{N}_*(X) \longrightarrow \tilde{N}_*(X)^{\otimes r}$ associated to an operation $w \in \mathcal{E}(r)_d$. First, let us record the following fact:

3.3.5. Observation: *Fix* $1 \leq s \leq r$. *If the surjection* $u \in \mathcal{X}(r)_{s-1}$ *verifies the condition of the Lemma, then it has the following table arrangement:*

$$\left| \begin{array}{l} u(1), \\ u(2), \\ \vdots \\ u(s-1), \\ u(s), \ldots, u(s+r-1). \end{array} \right.$$

The next assertion follows from this observation.

3.3.6. CLAIM: *Fix* $w = (w_0, \dots, w_d) \in \mathcal{E}(r)_d$. *Assume* $d = s-1$. *Let* $w' \in \mathcal{X}(r)_d$ *be given by the sequence*

$$w' = (w_0(1), w_1(1), \dots, w_{d-1}(1), w_d(1), \dots, w_d(r)).$$

This surjection w' *arises by table reduction of* w *for* $r_0 = r_1 = \cdots = r_{d-1} = 1$ *and* $r_d = r$. *The coproduct* $AW(TR(w)) : \tilde{N}_*(X) \longrightarrow \tilde{N}_*(X)^{\otimes r}$, *where* $X = \Delta^1$ *or* $X = S^1$, *reduces to the operations* $AW(w') : \tilde{N}_*(X) \longrightarrow \tilde{N}_*(X)^{\otimes r}$.

The Theorem 3.2.4 follows from this Claim, from the Lemma 3.3.3 and the Facts 3.3.2 and 3.3.4 above. It is straighforward to complete our calculations and to obtain the formulas stated. In fact, the tensor

$$\mathbf{e}_{(1)} \otimes \cdots \otimes \mathbf{e}_{(s)} \otimes \mathbf{c}_{(s+1)} \otimes \cdots \otimes \mathbf{c}_{(r)} \in C(1)^{\otimes r},$$

where $0 \leq s \leq r$, is dual to $\Delta(0,1)^{\otimes s} \otimes \Delta(1)^{\otimes r-s}$. To be more precise, because of the commutation rules, the duality pairing

$$\left\langle \mathbf{e}_{(1)} \otimes \cdots \otimes \mathbf{e}_{(s)} \otimes \mathbf{c}_{(s+1)} \otimes \cdots \otimes \mathbf{c}_{(r)} \middle| \Delta(0,1)^{\otimes s} \otimes \Delta(1)^{\otimes r-s} \right\rangle$$

is equal to $(-1)^{\sigma}$, where $\sigma = s(s-1)/2$.

3.3.7. *Proof of the Lemma 3.3.3:*

The surjection is represented by the sequence $(u(1), \dots, u(s), \dots, u(s+r-1))$. We let $\Delta(C_{(1)}) \otimes \cdots \otimes \Delta(C_{(r)})$ be the multi-simplex which is determined by the sequence $0 = n_0 \leq n_1 \leq \cdots \leq n_{s+r-1} \leq n_{s+r} = 1$. To be explicit, we assume $n_i = 0$, for $i = 0, \dots, j-1$, and $n_i = 1$, for $i = j, \dots, s+r$. By definition, the vertex $\Delta(0)$ is a face of $\Delta(C_{(i)})$ if $i \in \{ u(1), \dots, u(j) \}$. Similarly, the vertex $\Delta(1)$ is a face of $\Delta(C_{(i)})$ if $i \in \{ u(j), \dots, u(s+r-1) \}$. Furthermore, the simplex $\Delta(C_{(i)})$ is degenerate if the index s occurs twice in $u(1), \dots, u(j)$ or in $u(j), \dots, u(s+r-1)$. Therefore, if the tensor $\Delta(C_{(1)}) \otimes \cdots \otimes \Delta(C_{(r)})$ is equal to $\Delta(0,1)^{\otimes s} \otimes \Delta(1)^{\otimes r-s}$ in $\tilde{N}_*(C^1)^{\otimes r}$, then $(u(1), \dots, u(j))$ is a permutation of $(1, \dots, s)$ and $(u(j), \dots, u(s+r-1))$ is a permutation of $(1, \dots, r)$. This property may occur for $j = s$ only and supposes that the surjection has the following table arrangement:

$$\left|\begin{array}{l} u(1), \\ \vdots \\ u(s-1), \\ u(s), \dots, u(s+r-1). \end{array}\right.$$

Let us now determine the sign which occurs in the definition of $AW(u)(\Delta(0,1))$. The intervals $[n_{i-1}, n_i]$ ($i = 1, \dots, s-1$), which are reduced to the point $\{0\}$, have length 1 and have 0 as a position sign-exponent. The other intervals are associated to a final-value of the surjection. Thus, the interval $[n_{s-1}, n_s]$, which is equal to $[0,1]$, has length 1. The last intervals $[n_{i-1}, n_i]$ ($i = s+1, \dots, s+r-1$), which are reduced to $\{1\}$, have length 0. As a consequence, the permutation sign associated to u is identified with the signature $\mathrm{sgn}(u(1), \dots, u(s))$. The formula of the Lemma follows.

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