# RIGIDIFICATION OF ALGEBRAS OVER MULTI-SORTED THEORIES

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ABSTRACT. We define the notion of a multi-sorted algebraic theory, which is a generalization of an algebraic theory in which the objects are of different "sorts." We prove a rigidification result for simplicial algebras over these theories, showing that there is a Quillen equivalence between a model category structure on the category of strict algebras over a multi-sorted theory and an appropriate model category structure on the category of functors from a multi-sorted theory to the category of simplicial sets. In the latter model structure, the fibrant objects are homotopy algebras over that theory. Our two main examples of strict algebras are operads in the category of simplicial sets and simplicial categories with a given set of objects.

## 1. INTRODUCTION

Algebraic theories are useful in studying many standard algebraic objects, such as monoids, abelian groups, and commutative rings. An algebraic theory provides a functorial means of describing particular algebraic objects without specifying generating sets for the operations to which the objects are subject, or for the relations between these operations [12]. Given a category $\mathcal{C}$ of algebraic objects, the associated algebraic theory $\mathcal{T}_{\mathcal{C}}$ (if it exists) is a small category with products satisfying the property that specifying an object of $\mathcal{C}$ is equivalent to giving a product-preserving functor $\mathcal{T}_{\mathcal{C}} \rightarrow \mathcal{S}ets$.

Consider a category $\mathcal{C}$ with an associated algebraic theory $\mathcal{T}_{\mathcal{C}}$. If a functor from $\mathcal{T}_{\mathcal{C}}$ to the category of simplicial sets preserves products, then it is essentially a simplicial object in $\mathcal{C}$ and is thus a combinatorial model for a topological object in $\mathcal{C}$, such as a topological group when $\mathcal{C}$ is the category of groups. We call such a functor a *strict* $\mathcal{T}$*-algebra* (Definition 2.3). If the functor preserves products up to homotopy, we call it a *homotopy* $\mathcal{T}$*-algebra* (Definition 2.4). A homotopy $\mathcal{T}$-algebra can be viewed as a simplicial set with the appropriate algebraic structure "up to homotopy," in a higher-order sense. Using an appropriate notion of weak equivalence on homotopy $\mathcal{T}$-algebras [2, 5.6], the following result due to Badzioch relates strict and homotopy $\mathcal{T}$-algebras:

**Theorem 1.1.** [2, 1.4] *Let* $\mathcal{T}$ *be an algebraic theory. Any homotopy* $\mathcal{T}$*-algebra is weakly equivalent as a homotopy* $\mathcal{T}$*-algebra to a strict* $\mathcal{T}$*-algebra.*

As a motivation for the work in this paper, consider the category of monoids. There is an associated algebraic theory $\mathcal{T}_M$, and thus a simplicial monoid can be specified by a $\mathcal{T}_M$-algebra. However, the notion of simplicial monoid can be

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generalized to that of a simplicial category, by which we mean a category enriched over simplicial sets, since a simplicial monoid is a simplicial category with one object. We would like to have a generalization of Badzioch's theorem which applies to simplicial categories. From the point of view of algebraic structure, the main difference between a simplicial monoid and a simplicial category with more than one object is that in the latter case the description of the algebraic structure is more complicated, in that two morphisms can be combined by the composition operation only if they satisfy certain compatibility conditions on the domain and range. Therefore, we would like to describe a more general notion of theory which is capable of describing algebraic structures in which the elements have various sorts or types, and in which the operations which can be used to combine a collection of elements depend on these sorts.

There is in fact such a "multi-sorted" theory, $\mathcal{T}_{\mathcal{O}\mathcal{C}at}$, such that a product-preserving functor $\mathcal{T}_{\mathcal{O}\mathcal{C}at} \to \mathcal{S}ets$ is essentially a category with object set $\mathcal{O}$ (Example 3.5). A simplicial category, analogously, can be viewed as a product-preserving functor $\mathcal{T}_{\mathcal{O}\mathcal{C}at} \to \mathcal{SS}ets$.

A simpler example of an algebraic structure which requires the use of a multi-sorted theory, which we will describe in more detail in Example 3.2, is the case of a group acting on a set. There are two sorts of elements, namely, the elements of the group and the elements of the set. Two elements of the group can be combined via multiplication, or an element can be inverted. An element of the group and an element of the set can be combined via the group action. However, the elements of the set cannot be combined with one another in any nontrivial way, so the operations which we allow depend on the sort of element involved. The example of a module over a ring is constructed similarly in Example 3.3.

Another application of the notion of a multi-sorted theory gives a convenient description of an operad. In Example 3.4, we characterize the theory $\mathcal{T}_{operad}$ of operads. An operad in the category of sets is then a product-preserving functor from $\mathcal{T}_{operad}$ to the category of sets.

A multi-sorted theory $\mathcal{T}$ is a category with products, so we can define strict and homotopy $\mathcal{T}$-algebras as before (see Definitions 3.6 and 3.7). Using a definition of weak equivalence for homotopy $\mathcal{T}$-algebras (Proposition 4.12), the main result which we prove for multi-sorted theories is the following generalization of Theorem 1.1:

**Theorem 1.2.** *Let $\mathcal{T}$ be a multi-sorted algebraic theory. Any homotopy $\mathcal{T}$-algebra is weakly equivalent as a homotopy $\mathcal{T}$-algebra to a strict $\mathcal{T}$-algebra.*

As Badzioch does, we will actually prove a stronger statement in terms of a Quillen equivalence of model category structures (Theorem 5.1).

Using our example of the theory $\mathcal{T}_{operad}$ of operads, an operad in the category of simplicial sets is a strict $\mathcal{T}_{operad}$-algebra. A homotopy operad, or sequence of simplicial sets with the structure of an operad only up to homotopy, is then a homotopy $\mathcal{T}_{operad}$-algebra and can be rigidified to a strict operad using this theorem.

Returning to the example of simplicial categories, let $\mathcal{O}$ be a set and $\mathcal{SC}_{\mathcal{O}}$ the category of simplicial categories with object set $\mathcal{O}$ in which the morphisms are the identity on the objects. In [3], we use Theorem 1.2 to prove a relationship between $\mathcal{SC}_{\mathcal{O}}$ and the category of Segal categories with the same set $\mathcal{O}$ in dimension zero. In [4], we use the ideas of this proof to prove an analogous relationship between the category of all small simplicial categories and the category of all Segal categories.

Throughout this paper, we frequently work in the category of simplicial sets, $\mathcal{SS}ets$. Recall that a simplicial set is a functor $\mathbf{\Delta}^{op} \rightarrow \mathcal{S}ets$, where $\Delta$ denotes the *cosimplicial category* whose objects are the finite ordered sets $[n] = (0, \ldots, n)$ and whose morphisms are the order-preserving maps. The *simplicial category* $\mathbf{\Delta}^{op}$ is then the opposite of this category. Some examples of simplicial sets are, for each $n \geq 0$, the $n$-simplex $\Delta[n]$, its boundary $\dot{\Delta}[n]$, and, for any $0 \leq k \leq n$, the simplicial set $V[n,k]$, which is $\Delta[n]$ with the $k$th face removed. More information about simplicial sets can be found in [8, I.1].

In this paper, we begin by recalling the definition of an algebraic theory and stating some of its basic properties. Using this definition as a model, we then define a multi-sorted theory. We should note here that this notion is not a new one; similar definitions are given by Adámek and Rosický [1, 3.14] and by Boardman and Vogt [5, 2.3]. (The still more general definition of a finite limit theory is used by Johnson and Walters [11].) Because our perspective is slightly different, however, we will give a precise definition followed by some examples. Given a multi-sorted theory $\mathcal{T}$, we define strict and homotopy $\mathcal{T}$-algebras over a multi-sorted theory $\mathcal{T}$ and show that the existence of a model category structure on the category of all $\mathcal{T}$-algebras. We also show the existence of a model category structure on the category of all functors $\mathcal{T} \rightarrow \mathcal{SS}ets$ in which the fibrant objects are the homotopy $\mathcal{T}$-algebras. We then show that there is a Quillen equivalence between these two model categories.

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## 2. A Summary of Algebraic Theories

We first recall the definition of an ordinary algebraic theory. More details about algebraic theories can be found in chapter 3 of [6].

**Definition 2.1.** An *algebraic theory* $\mathcal{T}$ is a small category with finite products and which has as objects $T_n$ for $n \geq 0$ together with, for each $n$, an isomorphism $T_n \cong (T_1)^n$. In particular, $T_0$ is the terminal object in $\mathcal{T}$.

We can use theories to describe certain algebraic categories, namely those which are determined by sets with $n$-ary operations for each $n \geq 2$. Consider a category $\mathcal{C}$ such that there exists a forgetful functor

$$\Phi : \mathcal{C} \rightarrow Sets$$

taking an object of $\mathcal{C}$ to its underlying set, and its left adjoint (a free functor)

$$L : Sets \rightarrow \mathcal{C}.$$

In other words, $\mathcal{C}$ is required to have free objects. If the category $\mathcal{C}$ and the adjoint pair $(\Phi, L)$ satisfy some additional technical conditions (see [6, 3.9.1] for details), we will call $\mathcal{C}$ an *algebraic category.*

Given an object $X$ of an algebraic category $\mathcal{C}$, we have a natural map

$$\eta_X : L\Phi(X) \rightarrow X$$

and given a set $A$, we have another map

$$\varepsilon_A : A \rightarrow \Phi L(A).$$

In order to discuss a theory over the algebraic category $\mathcal{C}$, consider a set $A$ together with a map $m_A : \Phi L(A) \rightarrow A$ satisfying two conditions: the composite map

$$\begin{tikzcd} A \arrow[r, "\varepsilon_X"] & \Phi L(A) \arrow[r, "m_A"] & A \end{tikzcd}$$

is the identity map on $A$, and the diagram

$$\begin{tikzcd} (\Phi L)^2 A \arrow[r, shift left, "\Phi L(m_A)"] \arrow[r, shift right, "\Phi \eta_{LA} L"'] & \Phi L(A) \arrow[r, "m_A"] & A \end{tikzcd}$$

is a coequalizer. These maps define an algebraic structure on the set $A$, specifically the structure possessed by the objects of $\mathcal{C}$ [12].

For example, if $\mathcal{C} = \mathcal{G}$, the category of groups, $\Phi$ is the forgetful functor taking a group to its underlying set, and $L$ is the free group functor taking a set to the free group on that set, then these two conditions are precisely the ones defining a group structure on the set $A$.

We would like to discuss the algebraic theory $\mathcal{T}$ corresponding to $\mathcal{C}$ to simplify this way of talking about algebraic structure. Let $X$ be an object of $\mathcal{C}$. We consider natural transformations of functors $\mathcal{C} \rightarrow \mathcal{S}ets$

$$\underbrace{\Phi(-) \times \cdots \times \Phi(-)}_{n} \rightarrow \Phi(-).$$

Using the adjointness of $\Phi$ and $L$, we have that

$$\Phi(X) \cong \operatorname{Hom}_{\mathcal{S}ets}(\{1\}, \Phi(X)) \cong \operatorname{Hom}_{\mathcal{C}}(L\{1\}, X)$$

where $\{1\}$ denotes the set with one object, and we can think of $L\{1\}$ as the free object in $\mathcal{C}$ on one generator, since $L$ is the free functor. Hence, we have

$$\begin{aligned} \Phi(X)^n &= \operatorname{Hom}_{\mathcal{S}ets}(\{1\}, \Phi(X))^n \\ &= \operatorname{Hom}_{\mathcal{S}ets}(\coprod_n \{1\}, \Phi(X)) \\ &= \operatorname{Hom}_{\mathcal{S}ets}(\{1, \ldots, n\}, \Phi(X)) \\ &= \operatorname{Hom}_{\mathcal{C}}(L\{1, \ldots, n\}, X). \end{aligned}$$

Now, by Yoneda's Lemma we have a bijection between the set of natural maps $\Phi(X)^n \rightarrow \Phi(X)$ and the set $\operatorname{Hom}_{\mathcal{C}}(L\{1\}, L\{1, \ldots, n\})$. The objects

$$L\{\phi\} = T_0, L\{1\} = T_1, \ldots, L\{1, \ldots, n\} = T_n, \ldots$$

are the objects of the algebraic theory $\mathcal{T}$ corresponding to $\mathcal{C}$. The morphisms are the opposites of the ones in $\mathcal{C}$ between these objects. More precisely stated, $\mathcal{T}$ is the opposite of the full subcategory of representatives of isomorphism classes of finitely generated free objects of $\mathcal{C}$.

Given an object $X$ of $\mathcal{C}$, define a functor $H_X : \mathcal{T} \rightarrow \mathcal{S}ets$ such that

$$H_X(L\{1, \ldots, n\}) = \operatorname{Hom}_{\mathcal{C}}(L\{1, \ldots, n\}, X) = \Phi(X)^n.$$

Now, the algebraic category $\mathcal{C}$ is equivalent to the category of the functors $H_X$, namely, the full subcategory of the category of functors $A : \mathcal{T} \rightarrow \mathcal{S}ets$ whose objects preserve products, or those for which the canonical map $A(T_n) \rightarrow A(T_1)^n$ induced by the $n$ projection maps is an isomorphism of sets for all $n \geq 0$ [12].

**Example 2.2.** Let $\mathcal{G}$ denote the category of groups. Consider the full subcategory of $\mathcal{G}$ whose objects $T_n$ are the free groups on $n$ generators for $n \geq 0$ (where $T_0$ is the trivial group). The opposite of this category is $\mathcal{T}_{\mathcal{G}}$, the theory of groups. It can be shown that the category of product-preserving functors $\mathcal{T}_{\mathcal{G}} \rightarrow \mathcal{S}ets$ is equivalent to the category $\mathcal{G}$.

Product-preserving functors from the theory $\mathcal{T}$ to $\mathcal{S}ets$ are called *algebras* over $\mathcal{T}$. We would also like to consider functors from an algebraic theory to the category $\mathcal{SS}ets$ of simplicial sets. To do so, we must first define a simplicial algebra over a theory $\mathcal{T}$. For simplicity, we will also use the term "algebra" to refer to these simplicial algebras.

**Definition 2.3.** [2, 1.1] Given an algebraic theory $\mathcal{T}$, a *(strict simplicial) $\mathcal{T}$-algebra* $A$ is a product-preserving functor $A : \mathcal{T} \rightarrow \mathcal{SS}ets$. Namely, the canonical map

$$A(T_n) \rightarrow A(T_1)^n,$$

induced by the $n$ projection maps $T_n \rightarrow T_1$, is an isomorphism of simplicial sets. In particular, $A(T_0)$ is the one-point space $\Delta[0]$.

The category of all $\mathcal{T}$-algebras will be denoted $\mathcal{A}lg^{\mathcal{T}}$. Similarly, we have the notion of a homotopy algebra, for which we only require products to be preserved up to homotopy:

**Definition 2.4.** [2, 1.2] Given an algebraic theory $\mathcal{T}$, a *homotopy $\mathcal{T}$-algebra* is a functor $X : \mathcal{T} \rightarrow \mathcal{SS}ets$ which preserves products up to homotopy. The functor $X$ preserves products up to homotopy if for each $n$ the canonical map

$$X(T_n) \rightarrow X(T_1)^n$$

is a weak equivalence of simplicial sets. In particular, we assume that $X(T_0)$ is weakly equivalent to $\Delta[0]$.

There exists a forgetful functor, or evaluation map,

$$U_{\mathcal{T}} : \mathcal{A}lg^{\mathcal{T}} \rightarrow \mathcal{SS}ets$$

such that $U_{\mathcal{T}}(A) = A(T_1)$. This functor has a left adjoint, the free $\mathcal{T}$-algebra functor

$$F_{\mathcal{T}} : \mathcal{SS}ets \rightarrow \mathcal{A}lg^{\mathcal{T}}$$

where, if $Y$ is any simplicial set,

$$F_{\mathcal{T}}(Y)(T_1) = \coprod_{n \geq 0} \operatorname{Hom}_{\mathcal{T}}(T_n, T_1) \times Y^n / \sim$$

where the identifications come from the structure of the algebraic theory [2, 2.1].

## 3. Multi-Sorted Algebraic Theories

We now generalize the definition of an algebraic theory to that of a multi-sorted theory.

**Definition 3.1.** Given a set $S$, an *$S$-sorted algebraic theory* (or *multi-sorted theory*) $\mathcal{T}$ is a small category with objects $T_{\underline{\alpha}^n}$ where $\underline{\alpha}^n =< \alpha_1, \ldots, \alpha_n >$ for $\alpha_i \in S$ and $n \geq 0$ varying, and such that each $T_{\underline{\alpha}^n}$ is equipped with an isomorphism

$$T_{\underline{\alpha}^n} \cong \prod_{i=1}^{n} T_{\alpha_i}.$$

For a particular $\underline{\alpha}^n$, the entries $\alpha_i$ can repeat, but they are not ordered. In other words, $\underline{\alpha}^n$ is a an $n$-element subset with multiplicities. There exists a terminal object $T_0$ (corresponding to the empty subset of $S$).

**Notation.** Lower-case Greek letters (with or without subscripts), say $\alpha$ or $\alpha_i$, will be used to denote objects of $S$, whereas underlined ones, say $\underline{\alpha}^n$ or simply $\underline{\alpha}$, will denote an $n$-element subset of objects of $S$ (with multiplicities) for $n \geq 1$.

Notice that a theory with a single sort is a theory in the sense of the previous section.

We would like to speak of multi-sorted theories corresponding to categories which are analogous to the algebraic categories which we had in the ordinary case. However, because we have several objects (or "sorts") $T_\alpha$ where we only had the object $T_1$ in an ordinary theory, we have many pairs of adjoint functors, one for each sort. Let $\mathcal{C}$ be a category with coproducts such that given any element $\beta \in S$, we have a forgetful functor

$$\Phi_\beta : \mathcal{C} \rightarrow \mathcal{S}ets$$

and its left adjoint, the free functor

$$L_\beta : \mathcal{S}ets \rightarrow \mathcal{C}.$$

For each object $X$ in $\mathcal{C}$ and element $\beta \in S$, we have a map

$$\eta_{X,\beta} : L_\beta \Phi_\beta(X) \rightarrow X$$

and, for each set $A$ a map

$$\varepsilon_{A,\beta} : A \rightarrow \Phi_\beta L_\beta(A).$$

As before, in order to make sense of the notion of theory, we consider a set $A$ together with, for each $\beta \in S$, a map

$$m_{A,\beta} : \Phi_\beta L_\beta(A) \rightarrow A$$

satisfying two conditions: the composite map

$$A \xrightarrow{\varepsilon_{A,\beta}} \Phi_\beta L_\beta(A) \xrightarrow{m_{A,\beta}} A$$

is the identity map on $A$, and the diagram

$$\begin{tikzcd}(\Phi_\beta L_\beta)^2 A \arrow[r, shift left, "\Phi_\beta L_\beta(m_{A,\beta})"] \arrow[r, shift right, "\Phi_\beta \eta_{L_\beta A,\beta} L_\beta"'] & \Phi_\beta L_\beta(A) \arrow[r, "m_{A,\beta}"] & A\end{tikzcd}$$

is a coequalizer. These maps define a "multi-sorted algebraic structure" on $\mathcal{C}$. In particular, we have a notion of composition for certain elements of $\mathcal{C}$ depending on their sorts. Given this structure, we can now construct the $S$-sorted theory corresponding to the category $\mathcal{C}$.

Given $\alpha_i, \beta \in S$, we consider the natural transformations of functors $\mathcal{C} \rightarrow \mathcal{S}ets$

$$\Phi_{\alpha_1}(-) \times \cdots \times \Phi_{\alpha_n}(-) \rightarrow \Phi_\beta(-).$$

As before, we can apply these functors to an object $X$ of $\mathcal{C}$ and rewrite to obtain a map

$$\operatorname{Hom}_{\mathcal{S}ets}(\{1\}, \Phi_{\alpha_1}(X)) \times \cdots \times \operatorname{Hom}_{\mathcal{S}ets}(\{1\}, \Phi_{\alpha_n}(X)) \rightarrow \operatorname{Hom}_{\mathcal{S}ets}(\{1\}, \Phi_\beta(X))$$

which, by adjointness, is equivalent to

$$\operatorname{Hom}_{\mathcal{C}}(L_{\alpha_1}\{1\}, X) \times \cdots \times \operatorname{Hom}_{\mathcal{C}}(L_{\alpha_n}\{1\}, X) \rightarrow \operatorname{Hom}_{\mathcal{C}}(L_\beta\{1\}, X).$$

Since $\mathcal{C}$ has coproducts, we can rewrite this map as

$$\operatorname{Hom}_{\mathcal{C}}(L_{\alpha_1}\{1\} \amalg \cdots \amalg L_{\alpha_n}\{1\}, X) \to \operatorname{Hom}_{\mathcal{C}}(L_\beta\{1\}, X).$$

Then, by Yoneda's Lemma, there is a bijection between the set of natural transformations

$$\Phi_{\alpha_1}(-) \times \cdots \times \Phi_{\alpha_n}(-) \to \Phi_\beta(-)$$

and the set

$$\operatorname{Hom}_{\mathcal{C}}(L_\beta\{1\}, \coprod_k L_{\alpha_k}\{1\}).$$

The objects of the theory $\mathcal{T}$ corresponding to $\mathcal{C}$ are given by finite coproducts of "free" objects $L_{\alpha_k}\{1\}$ of $\mathcal{C}$ for all choices of $\alpha_k$, and the morphisms are the opposites of those of $\mathcal{C}$. Let $X$ be an object of $\mathcal{C}$ and $(\alpha_1, \ldots, \alpha_n, \beta) \in S^{n+1}$ an $(n+1)$-tuple of elements in $S$. We define the map $H_{X,\alpha_1,\ldots,\alpha_n,\beta} : \mathcal{T}^{op} \to \mathcal{S}ets$ such that

$$H_{X,\alpha_1,\ldots,\alpha_n,\beta}(\coprod_{k=1}^n L_{\alpha_k}\{1\}) = \operatorname{Hom}_{\mathcal{C}}(\coprod_{k=1}^n L_{\alpha_k}\{1\}, X) = \Phi_{\alpha_1}(X) \times \cdots \times \Phi_{\alpha_n}(X).$$

If the category $\mathcal{C}$ satisfies analogous conditions to those of [6, 3.9.1], then $\mathcal{C}$ is equivalent to the category of all such functors.

We now consider with some examples.

**Example 3.2.** Consider pairs $(G,X)$, where $G$ is a group and $X$ is a set. We can obtain two different 2-sorted theories from these pairs, one corresponding to the category of unstructured pairs, and the other corresponding to the category of pairs $(G,X)$ with a given action of the group $G$ on the set $X$.

In each case, we have two forgetful functors and their respective left adjoints. We begin with the category of unstructured pairs, which we denote $\mathcal{P}$. The objects are the pairs $(G,X)$ and the morphisms $(G,X) \to (H,Y)$ consist of pairs $(\varphi, f)$ where $\varphi : G \to H$ is a group homomorphism and $f : X \to Y$ is a map of sets. For each sort $i = 1, 2$ we have a forgetful map

$$\Phi_i : \mathcal{P} \to \mathcal{S}$$

and its left adjoint

$$L_i : \mathcal{S} \to \mathcal{P}.$$

When $i = 1$, we have, for any group $G$ and set $X$,

$$\Phi_1(G,X) = G$$

(where on the right hand side $G$ denotes the underlying set of the group $G$) and for any set $S$

$$L_1(S) = (F_S, \phi)$$

where $F_S$ denotes the free group on the set $S$.

Similarly, when $i = 2$, we define

$$\Phi_2(G,X) = X$$

and

$$L_2(S) = (e, S)$$

where $e$ denotes the trivial group.

In order to determine the objects of our theory, consider functors

$$F_{i,j} : \mathcal{P} \to \mathcal{S}ets$$

such that $F_{i,j}(G,X) = G^i \times X^j$. In other words,

$$F_{i,j}(G,X) = \operatorname{Hom}_{\mathcal{P}}(L_1(\mathbf{i}) \amalg L_2(\mathbf{j}), (G,X))$$

where $\mathbf{i}$ denotes the set with $i$ elements and similarly for $\mathbf{j}$. The objects of the theory will be representatives of the isomorphism classes of the $L_1(\mathbf{i}) \amalg L_2(\mathbf{j})$ for all choices of $\mathbf{i}$ and $\mathbf{j}$. This coproduct in $\mathcal{P}$ is defined to be the coproduct of each element in the pairs. Thus we have

$$(G,X) \amalg (G',X') = (G * G', X \times X')$$

where $G * G'$ denotes the free product of groups. So, our corresponding theory is the opposite of the full subcategory of $\mathcal{P}$ whose objects are of the form $L_1(\mathbf{i}) \amalg L_2(\mathbf{j})$.

When we equip each pair $(G,X)$ with an action of $G$ on $X$ to obtain another category which we denote $\mathcal{PA}$, the process is identical until we have to specify the coproduct, since in this case we need to take the group actions into account. We then have the coproduct in $\mathcal{PA}$

$$(G,X) \amalg (G',X') = (H, (H \times_G X) \amalg (H \times_{G'} X'))$$

where $H = G * G'$ and we have defined

$$H \times_G X = \{(h,x) | h \in H, x \in X\}/ \sim$$

when $(hg,x) \sim (h,gx)$ for any $g \in G$. We can now take the opposite of a full subcategory of $\mathcal{PA}$ as above to obtain the corresponding theory.

**Example 3.3.** A very similar example is the case of a commutative ring $R$ and an $R$-module $A$. Again, we have two different 2-sorted theories: one where we simply have a ring $R$ and regard $A$ merely as an abelian group, and the other where we consider the $R$-module structure on $A$.

As before, we begin with $\mathcal{PR}$, the category of pairs with no additional structure. We have the forgetful map

$$\Phi_1 : \mathcal{PR} \to \mathcal{S}ets$$

where $\Phi_1(R,A) = R$ for any ring $R$ and abelian group $A$, where on the right side $R$ is the underlying set of the ring $R$. Its left adjoint is the functor

$$L_1 : \mathcal{S}ets \to \mathcal{PR}$$

where for any set $S$, $L_1(S) = (\mathbb{Z}[S], e)$, where $\mathbb{Z}[S]$ is the free commutative ring on the set $S$ and $e$ denotes the trivial (abelian) group. Then we have the map

$$\Phi_2 : \mathcal{PR} \to \mathcal{S}ets$$

such that $\Phi_2(R,A) = A$, where again on the right hand side $A$ is the underlying set of the abelian group $A$. Its left adjoint is the map

$$L_2 : \mathcal{S}ets \to \mathcal{PR}$$

where $L_2(S) = (\mathbb{Z}, FA_S)$ where $FA_S$ denotes the free abelian group on the set $S$.

To know what the objects of this 2-sorted theory are, we need to know what the coproduct is. We have that

$$(R,A) \amalg (R',A') = (R \otimes_{\mathbb{Z}} R', A \oplus A'),$$

and from there we can obtain a theory as in the previous example.

Now consider the category $\mathcal{PM}$ whose objects are pairs $(R, A)$ where $R$ is a ring and $A$ is a module over $A$. If $A$ and $A'$ are modules over $R$ and $R'$, respectively, we have a coproduct similar to that in the group action example. So, we say that

$$(R, A) \amalg (R', A') = (R \otimes_{\mathbb{Z}} R', (R' \otimes_{\mathbb{Z}} A) \oplus (R \otimes_{\mathbb{Z}} A')).$$

**Example 3.4.** Another example of a multi-sorted theory is the $\mathbb{N}$-sorted theory of operads. Recall that an operad in the category of sets is a sequence of sets $\{P(k)\}_{k \geq 0}$, a unit map $1 \in P(1)$, and operations

$$P(k) \times P(j_1) \times \cdots \times P(j_k) \to P(j_1 + \cdots + j_k)$$

satisfying associativity, unit, and equivariance conditions [14, II.1.4].

There is a notion of a free operad on $n$ generators at levels $m_1, \ldots, m_n$ [14, §II.1.9]. Specifically, such a free operad has, for each $1 \leq i \leq n$, a generator in $P(m_i)$. Note that the values of $m_i$ can repeat. For example, one can think of the free operad on $n$ generators, each at level 1, as the free monoid on $n$ generators.

In the category of operads, consider the full subcategory of free operads. Each object in this category, then, can be described as the free operad on $n$ generators at levels $m_1, \ldots, m_n$ for some $n \geq 0$ and $m_1, \ldots, m_n$. The opposite of this category is the theory of operads. Using the notation we have set up for multi-sorted theories, we have that $T_\alpha$ for $\alpha \in \mathbb{N}$ is just the free operad on one generator at level $\alpha$ and for $\underline{\alpha}^n = < \alpha_1, \ldots, \alpha_n >$, we have that $T_{\underline{\alpha}^n}$ is the free operad on $n$ generators at levels $\alpha_1, \ldots \alpha_n$.

There is also a notion of non-$\Sigma$ operads, where we no longer have an action of the symmetric group or an equivariance condition [14, II.1.14]. We can define the theory of non-$\Sigma$ operads analogously, taking the opposite of the full subcategory of free non-$\Sigma$ operads in the category of all non-$\Sigma$ operads.

**Example 3.5.** Consider the category $\mathcal{O}\mathcal{C}at$ whose objects are the categories with a fixed object set $\mathcal{O}$ and whose morphisms are the functors which are the identity map on the objects. There is a theory $\mathcal{T}_{\mathcal{O}\mathcal{C}at}$ associated to this category. The objects of the theory are categories which are freely generated by directed graphs with vertices corresponding to the elements of the set $\mathcal{O}$. This theory will be sorted by pairs of elements in $\mathcal{O}$, corresponding to the morphisms with source the first element and target the second. In other words, this theory is $(\mathcal{O} \times \mathcal{O})$-sorted.

In particular, consider $\alpha = (x, y) \in \mathcal{O} \times \mathcal{O}$. Then, if $x \neq y$, $T_\alpha$ is the category with object set $\mathcal{O}$ and one nonidentity morphism with source $x$ and target $y$. If $x = y$, then $T_\alpha$ is the category freely generated by one morphism from $x$ to itself and no other nonidentity morphisms.

In general, if $\underline{\alpha} = < \alpha_1, \ldots, \alpha_n >$, then $T_{\underline{\alpha}}$ is the category with object set $\mathcal{O}$ and morphisms freely generated by the morphisms given for each $\alpha_k$ as in the previous case.

Consider the forgetful functor $\Phi_\alpha \colon \mathcal{O}\mathcal{C}at \to \mathcal{S}ets$ where, for any object $X$ in $\mathcal{C}$,

$$\Phi_\alpha(X) = \operatorname{Hom}_X(x, y).$$

Its left adjoint then is the free functor $L_\alpha$ defined by, for a set $A$,

$$L_\alpha(A) = \begin{cases} C \text{ with } \operatorname{Hom}_C(x, y) = A & \text{if } x \neq y \\ C \text{ with } \operatorname{Hom}_C(x, y) = F_A & \text{if } x = y \end{cases}$$

where $F_A$ is the free monoid generated by the set $A$ and where in each case there are no other nonidentity morphisms in the category $C$.

As with ordinary algebraic theories, we can define strict and homotopy $\mathcal{T}$-algebras for a multi-sorted theory $\mathcal{T}$.

**Definition 3.6.** Given an $S$-sorted theory $\mathcal{T}$, a *(strict simplicial) $\mathcal{T}$-algebra* is a product-preserving functor $A : \mathcal{T} \rightarrow \mathcal{SS}ets$. Here, product-preserving means that the canonical map

$$A(T_{\underline{\alpha}^n}) \rightarrow \prod_{i=1}^n A(T_{\alpha_i})$$

induced by the projections $T_{\underline{\alpha}^n} \rightarrow T_{\alpha_i}$ for all $1 \leq i \leq n$ is an isomorphism of simplicial sets.

As before, we will denote the category of strict $\mathcal{T}$-algebras by $\mathcal{A}lg^{\mathcal{T}}$.

**Definition 3.7.** Given an $\mathcal{S}$-sorted theory $\mathcal{T}$, a *homotopy $\mathcal{T}$-algebra* is a functor $X : \mathcal{T} \rightarrow \mathcal{SS}ets$ which preserves products up to homotopy. The functor $X$ preserves products up to homotopy if the canonical map

$$X(T_{\underline{\alpha}^n}) \rightarrow \prod_{i=1}^n X(T_{\alpha_i})$$

induced by the projection maps $T_{\underline{\alpha}^n} \rightarrow T_{\alpha_i}$ for all $1 \leq i \leq n$ is a weak equivalence of simplicial sets.

We would like to prove a rigidification result similar to Theorem 1.1 above. We begin by finding model category structures for $\mathcal{T}$-algebras and homotopy $\mathcal{T}$-algebras. We then find a Quillen equivalence between these model category structures $\mathcal{T}$-algebras for any multi-sorted theory $\mathcal{T}$.

## 4. Model Category Structures

In this section, we define, given a multi-sorted theory $\mathcal{T}$, model category structures on the category of diagrams $\mathcal{T} \rightarrow \mathcal{SS}ets$ and on the category of $\mathcal{T}$-algebras. We begin with a review of model category structures.

Recall that a model category structure on a category $\mathcal{C}$ is a choice of three distinguished classes of morphisms: fibrations, cofibrations, and weak equivalences. A (co)fibration which is also a weak equivalence will be called an *acyclic (co)fibration*. With this choice of three classes of morphisms, $\mathcal{C}$ is required to satisfy axioms MC1-MC5 [7, 3.3].

An object $X$ in $\mathcal{C}$ is *fibrant* if the unique map $X \rightarrow *$ from $X$ to the terminal object is a fibration. Dually, $X$ is *cofibrant* if the unique map $\phi \rightarrow X$ from the initial object to $X$ is a cofibration. The factorization axiom MC5 guarantees that each object $X$ has a weakly equivalent fibrant replacement $\widehat{X}$ and a weakly equivalent cofibrant replacement $\widetilde{X}$. These replacements are not necessarily unique, but they can be chosen to be functorial in the cases we will use [10, 1.1.3].

The model category structures which we will discuss are all cofibrantly generated. A *cofibrantly generated* model category $\mathcal{C}$ is a model category for which there are two sets of morphisms, one of generating cofibrations and one of generating acyclic cofibrations, such that a map is a fibration if and only if it has the right lifting property with respect to the generating acyclic cofibrations, and a map is an acyclic fibration if and only if it has the right lifting property with respect to the generating cofibrations [9, 11.1.2]. To describe such model categories, we make the following definition.

**Definition 4.1.** [9, 10.5.2] Let $\mathcal{M}$ be a category and $I$ a set of maps in $\mathcal{C}$. Then an *$I$-injective* is a map which was the right lifting property with respect to every map in $I$. An *$I$-cofibration* is a map with the left lifting property with respect to every $I$-injective.

We are now able to state the theorem that we will use to prove our model category structures in this paper.

**Theorem 4.2.** [9, 11.3.1] *Let $\mathcal{M}$ be a category which has all finite limits and colimits. Suppose that $\mathcal{M}$ has a class of weak equivalences which satisfies the "two out of three property" (model category axiom MC2) and which is closed under retracts. Let $I$ and $J$ be sets of maps in $\mathcal{M}$ which satisfy the following conditions:*

1. *Both $I$ and $J$ permit the small object argument* [9, 10.5.15].
2. *Every $J$-cofibration is an $I$-cofibration and a weak equivalence.*
3. *Every $I$-injective is a $J$-injective and a weak equivalence.*
4. *One of the following conditions holds:*
   - (i) *A map that is an $I$-cofibration and a weak equivalence is a $J$-cofibration, or*
   - (ii) *A map that is both a $J$-injective and a weak equivalence is an $I$-injective.*

*Then there is a cofibrantly generated model category structure on $\mathcal{M}$ in which $I$ is a set of generating cofibrations and $J$ is a set of generating acyclic cofibrations.*

We will refer to the standard model category structure on the category $\mathcal{SS}ets$ of simplicial sets. In this case, a weak equivalence is a map of simplicial sets $f : X \to Y$ such that the induced map $|f| : |X| \to |Y|$ is a weak homotopy equivalence of topological spaces. The cofibrations are inclusions, and the fibrations are the maps with the right lifting property with respect to the acyclic cofibrations [8, I.11.3]. This model category structure is cofibrantly generated; the generating cofibrations are the maps $\dot{\Delta}[n] \to \Delta[n]$ for $n \geq 0$, and the generating acyclic cofibrations are the maps $V[n,k] \to \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$.

We will also need the notion of a simplicial model category $\mathcal{M}$, or one for which an object $X \otimes K$ is defined for any object $X$ of $\mathcal{M}$ and simplicial set $K$. In particular, $\mathcal{M}$ is a simplicial category with a model structure which is required to satisfy several axioms [9, 9.1.6].

**Definition 4.3.** For any objects $X$ and $Y$ in a simplicial model category $\mathcal{M}$, the *function complex* is the simplicial set $\operatorname{Map}(X, Y)$.

It is important to note that a function complex is only homotopy invariant in the case that $X$ is cofibrant and $Y$ is fibrant. For the general case, we have the following definition:

**Definition 4.4.** [9, 17.3.1] A *homotopy function complex* $\operatorname{Map}^h(X, Y)$ in a simplicial model category $\mathcal{M}$ is the simplicial set $\operatorname{Map}(\widetilde{X}, \widehat{Y})$ where $\widetilde{X}$ is a cofibrant replacement of $X$ in $\mathcal{M}$ and $\widehat{Y}$ is a fibrant replacement for $Y$.

Several of the model category structures that we will use will be obtained by localizing a given model category structure with respect to a map or a set of maps. Suppose that $P = \{f : A \to B\}$ is a set of maps with respect to which we would like to localize a model category $\mathcal{M}$.

**Definition 4.5.** A *P-local* object $X$ is a fibrant object of $\mathcal{M}$ such that for any $f : A \rightarrow B$ in $P$, the induced map on homotopy function complexes

$$f^* : \operatorname{Map}^h(B, W) \rightarrow \operatorname{Map}^h(A, W)$$

is a weak equivalence of simplicial sets. A map $g : X \rightarrow Y$ in $\mathcal{M}$ is then a *P-local equivalence* if for every local object $W$, the induced map on homotopy function complexes

$$g^* : \operatorname{Map}^h(Y, W) \rightarrow \operatorname{Map}^h(X, W)$$

is a weak equivalence of simplicial sets.

Given a multi-sorted theory $\mathcal{T}$, let $\mathcal{SS}ets^{\mathcal{T}}$ denote the category of functors $\mathcal{T} \rightarrow \mathcal{SS}ets$. Note that the category $\mathcal{A}lg^{\mathcal{T}}$ of strict $\mathcal{T}$-algebras is a full subcategory of $\mathcal{SS}ets^{\mathcal{T}}$.

The category $\mathcal{SS}ets^{\mathcal{T}}$ is an example of a category of diagrams. In general, given any small category $\mathcal{D}$, there is a category $\mathcal{SS}ets^{\mathcal{D}}$ of $\mathcal{D}$-diagrams in $\mathcal{SS}ets$, or functors $\mathcal{D} \rightarrow \mathcal{SS}ets$. We can obtain two model category structures on $\mathcal{SS}ets^{\mathcal{D}}$ by the following results.

**Theorem 4.6.** [8, IX 1.4] *Given the category $\mathcal{SS}ets^{\mathcal{D}}$ of $\mathcal{C}$-diagrams of simplicial sets, there is a simplicial model category structure $\mathcal{SS}ets^{\mathcal{D}}_f$ in which the weak equivalences and fibrations are objectwise and in which the cofibrations are the maps which have the left lifting property with respect to the maps which are both fibrations and weak equivalences.*

**Theorem 4.7.** [8, VIII 2.4] *There is a simplicial model category $\mathcal{SS}ets^{\mathcal{D}}_c$ in which the weak equivalences and the cofibrations are objectwise and in which the fibrations are the maps which have the right lifting property with respect to the maps which are cofibrations and weak equivalences.*

We now return to the situation where our small category is a multi-sorted theory $\mathcal{T}$. We would like to have an evaluation map and its left adjoint as in the ordinary case (see the end of section 2 above), but here we will have one for each $\gamma \in S$. These evaluation maps look like

$$U_\gamma : \mathcal{A}lg^{\mathcal{T}} \rightarrow \mathcal{SS}ets$$

such that

$$U_\gamma(A) = A(T_\gamma)$$

for any $\mathcal{T}$-algebra $A$.

Each functor $U_\gamma$ has a left adjoint, the free functor

$$F_\gamma : \mathcal{SS}ets \rightarrow \mathcal{A}lg^{\mathcal{T}}$$

such that, given a simplicial set $Y$ and object $T_{\underline{\beta}}$ in $\mathcal{T}$,

$$F_\gamma(Y)(T_{\underline{\beta}}) = \coprod_{n \geq 0} (\operatorname{Hom}(T_{\gamma,\ldots,\gamma}, T_{\underline{\beta}}) \times Y^n)/\sim$$

where the equivalence is as in the ordinary case (see the end of section 2 above).

Given a theory $\mathcal{T}$ (regular or multi-sorted), define a weak equivalence in the category $\mathcal{A}lg^{\mathcal{T}}$ of $\mathcal{T}$-algebras to be a map which induces a weak equivalence of simplicial sets after applying the evaluation functor $U_\alpha$ for each sort $\alpha$. Similarly, define a fibration of $\mathcal{T}$-algebras to be a map $f$ such that $U_\alpha(f)$ is a fibration of simplicial sets. Then define a cofibration to be a map with the left lifting property with respect to the maps which are fibrations and weak equivalences.

The following theorem is a generalization of a result by Quillen [15, II.4].

**Theorem 4.8.** *Let* $\mathcal{T}$ *be an* $S$*-sorted theory. There is a cofibrantly generated model category structure on* $\mathcal{A}lg^{\mathcal{T}}$ *with the weak equivalences, fibrations, and cofibrations as defined above.*

We first need to define sets $I$ and $J$ which will be our candidates for generating sets of cofibrations and acyclic cofibrations, respectively. We first define $I$ to be the set of maps $U_\alpha \dot{\Delta}[n] \rightarrow U_\alpha \Delta[n]$ for each $n \geq 0$ and $\alpha \in S$. Similarly, define $J$ to be the set of all maps $U_\alpha V[n,k] \rightarrow U_\alpha \Delta[n]$ for each $n \geq 1$, $1 \leq k \leq n$, and $\alpha \in S$. We now use these sets to prove our model category structure.

*Proof.* We need to show that the conditions of 4.2 are satisfied for these sets $I$ and $J$. The existence of limits and colimits and the conditions on the weak equivalences follow just as they do in the case where $\mathcal{T}$ is an ordinary theory [15, II.4].

We now show that $I$ and $J$ satisfy the small object argument. Consider some $\mathcal{T}$-algebra $X$, which can be written as a directed colimit $\operatorname{colim}_m(X_m)$ and can therefore be computed objectwise. Thus, we can show that $\dot{\Delta}[n]$ is small:

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{A}lg^{\mathcal{T}}}(F_\alpha \dot{\Delta}[n], \operatorname{colim}_m(X_m)) &= \operatorname{Hom}_{\mathcal{SS}ets}(\dot{\Delta}[n], U_\alpha \operatorname{colim}_m(X_m)) \\
&= \operatorname{Hom}_{\mathcal{SS}ets}(\dot{\Delta}[n], \operatorname{colim}_m(U_\alpha X_m)) \\
&= \operatorname{colim}_m \operatorname{Hom}_{\mathcal{SS}ets}(\dot{\Delta}[n], U_\alpha X_m) \\
&= \operatorname{colim}_m \operatorname{Hom}_{\mathcal{A}lg^{\mathcal{T}}}(F_\alpha \dot{\Delta}[n], X_m).
\end{aligned}$$

The object $V[n,k]$ can be shown to be small analogously, so we have proved statement (1).

We first prove statements (3) and (4)(ii), namely that an $I$-injective is precisely a $J$-injective and a weak equivalence. An $I$-injective $f : X \rightarrow Y$ by definition has the right lifting property with respect to the maps in $I$, but using the adjointness of $U_\alpha$ and $F_\alpha$, this fact is equivalent to $f$'s being an acyclic fibration. But then $f$ is a weak equivalence and has the right lifting property with respect to the maps in $J$, again by adjointness.

It remains to prove statement (2). Suppose $i : A \rightarrow B$ is a $J$-cofibration. Then it has the right lifting property with respect to the fibrations. Another adjointness argument shows that $i$ therefore is an $I$-cofibration and a weak equivalence, completing the proof. □

We now need a model category structure on the category of homotopy $\mathcal{T}$-algebras. However, the category of homotopy $\mathcal{T}$-algebras does not have all finite limits and colimits (axiom MC1). Thus, we instead define a model category structure on all diagrams $\mathcal{T} \rightarrow \mathcal{SS}ets$ in such a way that the fibrant objects are homotopy $\mathcal{T}$-algebras.

The following theorem holds for model categories $\mathcal{M}$ which are left proper and cellular. We will not define these conditions here, but refer the reader to [9, 13.1.1, 12.1.1] for more details. It can be shown that $\mathcal{SS}ets^{\mathcal{T}}$ satisfies both these conditions [9, 13.1.14, 12.5.1].

**Theorem 4.9.** [9, 4.1.1] *Let* $\mathcal{M}$ *be a left proper cellular model category and* $P$ *a set of morphisms of* $\mathcal{M}$*. There is a model category structure* $\mathcal{L}_P\mathcal{M}$ *on the underlying category of* $\mathcal{M}$ *such that:*

1. *The weak equivalences are the $P$-local equivalences.*
2. *The cofibrations are precisely the cofibrations of $\mathcal{M}$.*
3. *The fibrations are the maps which have the right lifting property with respect to the maps which are both cofibrations and $P$-local equivalences.*
4. *The fibrant objects are the $P$-local objects.*

To localize the model structure $\mathcal{SS}ets^{\mathcal{T}}_f$, we first need an appropriate map. To do so for ordinary algebraic theories, Badzioch [2, 2.9] uses free diagrams which are corepresented by the objects $T_n$ of the theory $\mathcal{T}$. In particular the $n$ projection maps $T_n \rightarrow T_1$ induce maps

$$\coprod_n \mathrm{Hom}(T_1, -) \rightarrow \mathrm{Hom}(T_n, -).$$

He defines his localization with respect to these maps. We would like to define similar free diagrams in a multi-sorted theory.

For each $\underline{\alpha}^n = <\alpha_1, \ldots, \alpha_n>$ and $1 \leq i \leq n$, there exists a projection map $T_{\underline{\alpha}^n} \rightarrow T_{\alpha_i}$ inducing a map

$$\mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \rightarrow \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -).$$

Taking the coproduct of all such maps results in a map

$$\kappa_{\underline{\alpha}^n} : \coprod_{i=1}^n \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \rightarrow \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -).$$

These maps are the ones which we will use to localize $\mathcal{SS}ets^{\mathcal{T}}$. We define $P$ to be the set of all such maps $\kappa_{\underline{\alpha}^n}$ for each $\underline{\alpha}^n$ and $n \geq 0$.

**Proposition 4.10.** *There is a model category structure $\mathcal{LSS}ets^{\mathcal{T}}$ on the category $\mathcal{SS}ets^{\mathcal{T}}$ with weak equivalences the $P$-local equivalences, cofibrations as in $\mathcal{SS}ets^{\mathcal{T}}_f$, and fibrations the maps which have the right lifting property with respect to the maps which are cofibrations and weak equivalences.*

*Proof.* This proposition is a special case of Theorem 4.9. □

The following propositions are proved by Badzioch for ordinary theories, and his proofs follow for multi-sorted theories.

**Proposition 4.11.** [2, 5.5] *An object $Z$ of $\mathcal{LSS}ets^{\mathcal{T}}$ is fibrant if and only if it is a homotopy $\mathcal{T}$-algebra which is fibrant as an object of $\mathcal{SS}ets^{\mathcal{T}}_f$.*

**Proposition 4.12.** [2, 5.6] *If $X$ and $X'$ are homotopy $\mathcal{T}$-algebras in $\mathcal{SS}ets^{\mathcal{T}}$ and there is a $P$-local weak equivalence $f : Z \rightarrow X'$, then $f$ is also a weak equivalence in $\mathcal{SS}ets^{\mathcal{T}}_f$, i.e. an objectwise weak equivalence.*

**Proposition 4.13.** [2, 5.8] *A map $f : X \rightarrow X'$ is a $P$-local equivalence if and only if for any $\mathcal{T}$-algebra $Y$ which is fibrant in $\mathcal{SS}ets^{\mathcal{T}}_c$, the induced map of function complexes*

$$f^* : Map(X', Y) \rightarrow Map(X, Y)$$

*is a weak equivalence of simplicial sets.*

These results can actually be stated in more generality; they are really just statements about the fibrant objects in a localized model category structure (see chapter 3 of [9] for more details).

Hence, we can consider the category $\mathcal{LSS}ets^{\mathcal{T}}$ to be our homotopy $\mathcal{T}$-algebra model category structure.

## 5. Rigidification of Algebras over Multi-Sorted Theories

We are now able to prove the following statement, which is a stronger version of Theorem 1.2:

**Theorem 5.1.** *There is a Quillen equivalence of model categories between* $\mathcal{A}lg^{\mathcal{T}}$ *and* $\mathcal{LSS}ets^{\mathcal{T}}$.

We begin with the necessary definitions. Recall that an adjoint pair

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

(where $F$ is the left adjoint and $R$ is the right adjoint) is defined by a map

$$\varphi : \operatorname{Hom}_{\mathcal{D}}(FX, Y) \rightarrow \operatorname{Hom}_{\mathcal{C}}(X, RY)$$

and is sometimes written as the triple $(F, R, \varphi)$ [13, IV.1].

**Definition 5.2.** [10, 1.3.1] If $\mathcal{C}$ and $\mathcal{D}$ are model categories, then the adjoint pair $(F, R, \varphi)$ is a *Quillen pair* if one of the following statements is true:

1. $F$ preserves cofibrations and acyclic cofibrations.
2. $R$ preserves fibrations and acyclic fibrations.

**Definition 5.3.** [10, 1.3.12] A Quillen pair is a *Quillen equivalence* if for all cofibrant $X$ in $\mathcal{C}$ and fibrant $Y$ in $\mathcal{D}$, a map $f : FX \rightarrow Y$ is a weak equivalence in $\mathcal{D}$ if and only if the map $\varphi f : X \rightarrow RY$ is a weak equivalence in $\mathcal{C}$.

We need to find an adjoint pair of functors between $\mathcal{A}lg^{\mathcal{T}}$ and $\mathcal{LSS}ets^{\mathcal{T}}$ and prove that it is a Quillen equivalence. Let

$$J_{\mathcal{T}} : \mathcal{A}lg^{\mathcal{T}} \rightarrow \mathcal{SS}ets^{\mathcal{T}}$$

be the inclusion functor. We need to show we have an adjoint functor taking an arbitrary diagram in $\mathcal{SS}ets^{\mathcal{T}}$ to a $\mathcal{T}$-algebra. We first make the following definition.

**Definition 5.4.** Let $\mathcal{C}$ be a small category and $\mathcal{SS}ets^{\mathcal{D}}$ the category of functors $\mathcal{C} \rightarrow \mathcal{SS}ets$. Let $P$ be a set of morphisms in $\mathcal{SS}ets^{\mathcal{D}}$. An object $Y$ of $\mathcal{SS}ets^{\mathcal{D}}$ is *strictly* $P$*-local* if for every morphism $f : A \rightarrow B$ in $P$, the induced map on function complexes (Definition 4.3)

$$f^* : \operatorname{Map}(B, Y) \rightarrow \operatorname{Map}(A, Y)$$

is an isomorphism of simplicial sets. A map $g : C \rightarrow D$ in $\mathcal{SS}ets^{\mathcal{D}}$ is a *strict* $P$*-local equivalence* if for every strictly $P$-local object $Y$ in $\mathcal{SS}ets^{\mathcal{D}}$, the induced map

$$g^* : \operatorname{Map}(D, Y) \rightarrow \operatorname{Map}(C, Y)$$

is an isomorphism of simplicial sets.

Now, given a category of $\mathcal{C}$-diagrams in $\mathcal{SS}ets$ and the full subcategory of strictly $P$-local diagrams for some set $P$ of maps, we have the following result.

**Lemma 5.5.** *Consider two categories, the category of all diagrams* $X : \mathcal{C} \rightarrow \mathcal{SS}ets$ *and the category of strictly local diagrams with respect to the set of maps* $P = \{f : A \rightarrow B\}$. *Then the forgetful functor from the category of strictly local diagrams to the category of all diagrams has a left adjoint.*

*Proof.* Without loss of generality, assume that we have just one map $f$ in $P$; otherwise replace $f$ by $\coprod_\alpha f_\alpha$. Given an arbitrary diagram $X$, we would like to construct a strictly local diagram from $X$. So, suppose that $X$ is not strictly local, i.e. the map

$$f^* \colon \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$$

is not an isomorphism. First suppose that $f^*$ fails to be surjective. Then we obtain an object $X'$ as the pushout in the following diagram:

$$\begin{tikzcd}
\coprod_{A \to X} A \arrow[r] \arrow[d] & X \arrow[d] \\
\coprod_{A \to X} B \arrow[r] & X'
\end{tikzcd}$$

where each coproduct is taken over all maps $A \to X$. If $X$ fails to be injective, we obtain $X'$ by taking the pushout

$$\begin{tikzcd}
\coprod (B \coprod_A B) \arrow[r] \arrow[d] & X \arrow[d] \\
\coprod B \arrow[r] & X'
\end{tikzcd}$$

again where the coproduct is over all maps $B \coprod_A B \to X$, and where the map

$$B \coprod_A B \to B$$

is the fold map. (If $f^*$ is neither injective nor surjective, apply one of the above pushouts, then apply the other to the new object $X'$ rather than to the original $X$.)

In the first case, i.e. where $f^*$ is not surjective, for any strictly local object $Y$ we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(X', Y) \arrow[r] \arrow[d, "\cong"] & \mathrm{Map}(B, Y) \arrow[d, "\cong"] \\
\mathrm{Map}(X, Y) \arrow[r] & \mathrm{Map}(A, Y)
\end{tikzcd}$$

showing that the map $X \to X'$ is a strict local equivalence since $f \colon A \to B$ is.

In the second case, where $f^*$ is not injective, we obtain a similar diagram, but it takes more work to show that the map $X \to X'$ is a strict local equivalence. We first obtain the diagram

$$\begin{tikzcd}
\mathrm{Map}(X', Y) \arrow[r] \arrow[d] & \mathrm{Map}(B, Y) \arrow[d] \\
\mathrm{Map}(X, Y) \arrow[r] & \mathrm{Map}(B \coprod_A B, Y)
\end{tikzcd}$$

It then suffices to show that the right hand vertical arrow is an isomorphism.

Recall that the object $B\coprod_A B$ is defined as the pushout in the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
B \arrow[r] & B\coprod_A B
\end{tikzcd}$$

which enables us to look at the diagram

$$\begin{tikzcd}
\mathrm{Map}(B\coprod_A B, Y) \arrow[r] \arrow[d] & \mathrm{Map}(B,Y) \arrow[d, "\cong"] \\
\mathrm{Map}(B,Y) \arrow[r] & \mathrm{Map}(A,Y).
\end{tikzcd}$$

Hence the map

$$B \to B\coprod_A B$$

is a strict local equivalence. But, this map fits into a composite

$$\begin{tikzcd}
B \arrow[r] \arrow[rr, bend right, "id"'] & B\coprod_A B \arrow[r] & B
\end{tikzcd}$$

Since the identity map is a strict local equivalence, it follows that the map

$$B\coprod_A B \to B$$

is a strict local equivalence, since it can be shown that the strictly local equivalences satisfy the "two out of three property" (model category axiom MC2).

Therefore, in either case, the map $X \to X'$ is a strict local equivalence. However, we still do not know that the map

$$\mathrm{Map}(B,X') \to \mathrm{Map}(A,X')$$

is an isomorphism. So, we repeat this process, taking a (possibly transfinite) colimit to obtain a local object $\widetilde{X}$ such that there is a local equivalence $X \to \widetilde{X}$.

It suffices to show that the functor which takes a diagram $X$ to the local diagram $\widetilde{X}$ is left adjoint to the forgetful functor. So if $J$ is the forgetful functor from the category of strictly local diagrams to the category of all diagrams and $K$ is the functor we have just defined, we need to show that

$$\mathrm{Map}(X, JY) \cong \mathrm{Map}(KX, Y)$$

for any diagram $X$ and strictly local diagram $Y$. But, proving this statement is equivalent to showing that

$$\mathrm{Map}(X,Y) \cong \mathrm{Map}(\widetilde{X}, Y)$$

which was shown above for each step, and it still holds for the colimit. □

We consider the category $\mathcal{SS}ets^{\mathcal{T}}$ of $\mathcal{T}$-diagrams and the strict localization with respect to the set of maps

$$P = \{\coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \to \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -)\}$$

defined in the last section to obtain the model category structure of homotopy $\mathcal{T}$-algebras.

Recall that we defined the inclusion map (or forgetful functor)

$$J_{\mathcal{T}} : \mathcal{A}lg^{\mathcal{T}} \rightarrow \mathcal{SS}ets^{\mathcal{T}}.$$

Applying the above lemma to this functor of diagrams, we obtain its left adjoint functor

$$K_{\mathcal{T}} : \mathcal{SS}ets^{\mathcal{T}} \rightarrow \mathcal{A}lg^{\mathcal{T}}.$$

The following proposition holds in the more general situation of an arbitrary diagram category.

**Proposition 5.6.** *The adjoint pair of functors*

$$K_{\mathcal{T}} : \mathcal{SS}ets^{\mathcal{T}} \rightleftarrows \mathcal{A}lg^{\mathcal{T}} : J_{\mathcal{T}}.$$

*is a Quillen pair.*

*Proof.* As categories, $\mathcal{A}lg^{\mathcal{T}}$ is a subcategory of $\mathcal{SS}ets^{\mathcal{T}}$, and the map $J_{\mathcal{T}}$ is an inclusion. Since in both cases, the fibrations and weak equivalences are defined objectwise, $J_{\mathcal{T}}$ preserves fibrations and acyclic fibrations. □

**Lemma 5.7.** *Each map $K_{\mathcal{T}}(\kappa_{\underline{\alpha}})$ is an isomorphism in $\mathcal{A}lg^{\mathcal{T}}$.*

*Proof.* Let $A$ be a $\mathcal{T}$-algebra. Notice that by Yoneda's Lemma we have that

$$\operatorname{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\operatorname{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A) \simeq A(T_{\underline{\alpha}}).$$

Then we have the following weak equivalences of simplicial sets:

$$\begin{aligned}
\operatorname{Map}_{\mathcal{A}lg^{\mathcal{T}}}(K_{\mathcal{T}}(\coprod_i \operatorname{Hom}_{\mathcal{T}}(T_{\alpha_i}, -)), A) &\simeq \operatorname{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\coprod_i \operatorname{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A) \\
&\simeq \prod_i \operatorname{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\operatorname{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A) \\
&\simeq \prod_i A(T_{\alpha_i}) \\
&\simeq A(T_{\underline{\alpha}^n}) \\
&\simeq \operatorname{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\operatorname{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -), A) \\
&\simeq \operatorname{Map}_{\mathcal{A}lg^{\mathcal{T}}}(K_{\mathcal{T}}(\operatorname{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -)), A)
\end{aligned}$$

It then suffices to show that the map $K_{\mathcal{T}}(\kappa)$ actually induces this isomorphism. This fact follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{SS}ets^{\mathcal{T}} \arrow[r, "K_{\mathcal{T}}"] & \mathcal{A}lg^{\mathcal{T}} \\
\mathcal{SS}ets \arrow[u, "J_{\mathcal{T}} \circ F_{\gamma}"] \arrow[ur, "F_{\gamma}"'] &
\end{tikzcd}$$

which follows since

$$K_{\mathcal{T}}(\coprod J_{\mathcal{T}}(\operatorname{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -))) \simeq \coprod K_{\mathcal{T}} J_{\mathcal{T}}(\operatorname{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -)) \simeq \coprod \operatorname{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -).$$

□

Now, we need to show that the same adjoint pair is still a Quillen pair when we replace the model structure $\mathcal{SS}ets^{\mathcal{T}}$ with the model structure $\mathcal{LSS}ets^{\mathcal{T}}$.

**Proposition 5.8.** *The adjoint pair*

$$K_{\mathcal{T}} : \mathcal{LSS}ets^{\mathcal{T}} \rightleftarrows \mathcal{A}lg^{\mathcal{T}} : J_{\mathcal{T}}$$

*is a Quillen pair.*

*Proof.* Consider again the set of maps

$$P = \{\kappa_{\underline{\alpha}} : \coprod_i \operatorname{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \to \operatorname{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -)\}.$$

The model category structure $\mathcal{LSS}ets^{\mathcal{T}}$ is obtained by localizing with respect to these maps. Then using Lemma 5.7, we have that each map $K_{\mathcal{T}}(\kappa_{\underline{\alpha}^n})$ is an isomorphism in $\mathcal{A}lg^{\mathcal{T}}$. Hence, it follows from [9, 3.3.20] that the pair of adjoints forms a Quillen pair even after the localization on $\mathcal{SS}ets^{\mathcal{T}}_f$. □

Before stating the main theorem, that the above Quillen pair is actually a Quillen equivalence, we first need a lemma. Badzioch's proof [2, 6.5] for ordinary theories is slightly different than the one here, but it would follow for multi-sorted theories as well.

**Lemma 5.9.** *If $X$ is cofibrant in $\mathcal{LSS}ets^{\mathcal{T}}$, then the unit map $\eta : X \to K_{\mathcal{T}}X = J_{\mathcal{T}}K_{\mathcal{T}}X$ is a weak equivalence in $\mathcal{LSS}ets^{\mathcal{T}}$.*

*Proof.* Case 1: The cofibrant object $X$ is a free diagram. Then $X$ looks like

$$\coprod_{\underline{\alpha}} \operatorname{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -).$$

The proof for such an object follows from the proof of Lemma 5.7.

Case 2: Let $X$ be any cofibrant diagram. Then $X \simeq \operatorname{hocolim}_{\mathbf{\Delta}^{op}} X_i$ where each $X_i$ is a free diagram. It then suffices to show that $\operatorname{Map}(K_{\mathcal{T}}X, Y) \simeq \operatorname{Map}(X, Y)$ for any $\mathcal{T}$-algebra $Y$ which is fibrant in $\mathcal{SS}ets^{\mathcal{T}}_{cof}$. Using case 1, we have the following:

$$\begin{aligned}
\operatorname{Map}(X,Y) &\simeq \operatorname{Map}(\operatorname{hocolim}_{\mathbf{\Delta}^{op}} X_i, Y) \\
&\simeq \operatorname{holim}_{\mathbf{\Delta}} \operatorname{Map}(X_i, Y) \\
&\simeq \operatorname{holim}_{\mathbf{\Delta}} \operatorname{Map}(K_{\mathcal{T}}X_i, Y) \\
&\simeq \operatorname{Map}(\operatorname{hocolim}_{\mathbf{\Delta}^{op}} K_{\mathcal{T}}X_i, Y) \\
&\simeq \operatorname{Map}(K_{\mathcal{T}}X, Y).
\end{aligned}$$

The lemma follows. □

Now, the proof of the main theorem follows from this lemma exactly as it does for ordinary theories in [2, 6.4].

**Theorem 5.10.** *The Quillen pair of functors*

$$K_{\mathcal{T}} : \mathcal{LSS}ets^{\mathcal{T}} \rightleftarrows \mathcal{A}lg^{\mathcal{T}} : J_{\mathcal{T}}.$$

*is a Quillen equivalence.*

*Proof.* Let $X$ be a cofibrant object in $\mathcal{LSS}ets^{\mathcal{T}}$, $A$ a fibrant object in $\mathcal{A}lg^{\mathcal{T}}$, and $f : X \to A = J_{\mathcal{T}}A$ a map in $\mathcal{LSS}ets^{\mathcal{T}}$. We need to show that $f$ is a $P$-local equivalence if and only if its adjoint map $g : K_{\mathcal{T}}X \rightarrow A$ is a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}$. There is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "f"'] & K_{\mathcal{T}}X \arrow[d, "g"] \\
 & A
\end{tikzcd}$$

First assume that $f$ is a $P$-local equivalence. Then $g$ must also be a $P$-local equivalence since $\eta$ is, by the previous lemma. However, $g$ is a map in $\mathcal{A}lg^{\mathcal{T}}$, and so it is an objectwise weak equivalence, or a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}$.

Conversely, suppose that $g$ is a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}$. Then it is a $P$-local equivalence. Hence, $f = g \circ \eta$ is also a $P$-local equivalence. ☐

Hence, we have a Quillen equivalence of model categories between strict $\mathcal{T}$-algebras and homotopy $\mathcal{T}$-algebras.

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