A MODEL CATEGORY STRUCTURE ON THE CATEGORY OF SIMPLICIAL CATEGORIES

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ABSTRACT. In this paper we put a cofibrantly generated model category structure on the category of small simplicial categories. The weak equivalences are a simplicial analogue of the notion of equivalence of categories.

1. Introduction

Simplicial categories, which in this paper we will take to mean categories enriched over simplicial sets, arise in the study of homotopy theories. Given any model category $\mathcal{M}$, the simplicial localization of $\mathcal{M}$ as given in [3] is a simplicial category which possesses the homotopy-theoretic information contained in $\mathcal{M}$. Finding a model category structure on the category of simplicial categories is then the first step in studying the homotopy theory of homotopy theories.

In an early version of the preprint [1], Dwyer, Hirschhorn, and Kan present a cofibrantly generated model category structure on the category of simplicial categories, but as Toën and Vezzosi point out in their paper [9], this model category structure is incorrect, in that some of the proposed generating acyclic cofibrations are not actually weak equivalences. Here we complete the work of [1] by describing a different set of generating acyclic cofibrations which are in fact weak equivalences and which, along with the generating cofibrations given in [1], enable us to prove that the desired model category structure exists.

Note that the term "simplicial category" is potentially confusing. As we have already stated, by a simplicial category we mean a category enriched over simplicial sets. If a and b are objects in a simplicial category $\mathcal{C}$, then we denote by $\operatorname{Hom}_{\mathcal{C}}(a,b)$ the function complex, or simplicial set of maps $a \rightarrow b$ in $\mathcal{C}$. This notion is more restrictive than that of a simplicial object in the category of categories. Using our definition, a simplicial category is essentially a simplicial object in the category of categories which satisfies the additional condition that all the simplicial operators induce the identity map on the objects of the categories involved [2, 2.1].

We will assume that our simplicial categories are small, namely, that they have a set of objects. A functor between two simplicial categories $f : \mathcal{C} \rightarrow \mathcal{D}$ consists of a map of sets $f : Ob(\mathcal{C}) \rightarrow Ob(\mathcal{D})$ on the objects of the two simplicial categories, and function complex maps $f : \operatorname{Hom}_{\mathcal{C}}(a,b) \rightarrow \operatorname{Hom}_{\mathcal{D}}(fa, fb)$ which are compatible with composition. Let $\mathcal{SC}$ denote the category whose objects are the small simplicial categories and whose morphisms are the functors between them. This category $\mathcal{SC}$ is the underlying category of our model category structure.

In a similar way, we can consider categories enriched over the category of topological spaces. Making slight modifications to the ideas from this paper, it is possible

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to put an analogous model category structure on the category of small topological categories.

Recall that a model category structure on a category $\mathcal{C}$ is a choice of three distinguished classes of morphisms, namely, fibrations, cofibrations, and weak equivalences. We use the term *acyclic (co)fibration* to denote a map which is both a (co)fibration and a weak equivalence. This structure is required to satisfy five axioms [4, 3.3]:

- MC1: $\mathcal{C}$ is complete and cocomplete. In other words, $\mathcal{C}$ has all small limits and colimits.
- MC2: If f and g are maps in $\mathcal{C}$ and their composite gf is defined, and if two of the three maps f, g, and gf are weak equivalences, then so is the third.
- MC3: If a map f is a retract of g and g is a fibration, cofibration, or weak equivalence, then so is f.
- MC4: Given a solid arrow commutative diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

  a dotted arrow lift exists if either
  (i) i is a cofibration and p is an acyclic fibration, or
  (ii) i is an acyclic cofibration and p is a fibration.
- MC5: Any map f can be factored in two ways:
  (i) $f = pi$ where i is a cofibration and p is an acyclic fibration, and
  (ii) $f = qj$ where j is an acyclic cofibration and q is a fibration.

In axiom MC4, we say that i has the *left lifting property* with respect to p, or equivalently that p has the *right lifting property* with respect to i.

Before defining these three classes of morphisms in $\mathcal{SC}$, we need some notation. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are two simplicial categories. Let $\pi_0\mathcal{C}$ denote the category of components of $\mathcal{C}$, namely, the category in which the objects are the same as those of $\mathcal{C}$ and the morphisms are the path components of the simplicial sets of morphisms in $\mathcal{SC}$. Explicitly, if a and b are objects of $\mathcal{C}$, then

$$\mathrm{Hom}_{\pi_0\mathcal{C}}(a,b) = \pi_0 \mathrm{Hom}_{\mathcal{C}}(a,b).$$

If $f : \mathcal{C} \rightarrow \mathcal{D}$ is a map of simplicial categories, then $\pi_0 f : \pi_0\mathcal{C} \rightarrow \pi_0\mathcal{D}$ denotes the induced map on the categories of components of $\mathcal{C}$ and $\mathcal{D}$.

If $\mathcal{C}$ is a simplicial category, say that a morphism $e \in \mathrm{Hom}_{\mathcal{C}}(a,b)_0$ is a *homotopy equivalence* if there is a map $e' \in \mathrm{Hom}_{\mathcal{C}}(b,a)_0$ such that the composite $e'e \in \mathrm{Hom}_{\mathcal{C}}(a,a)_0$ is in the same path component as the identity map on a and the composite $ee' \in \mathrm{Hom}_{\mathcal{C}}(b,b)_0$ is in the same path component as the identity map on b. Alternatively stated, e is a homotopy equivalence if e becomes an isomorphism in $\pi_0\mathcal{C}$.

Now, given these definitions, our three classes of morphisms are defined as follows.

(1) The weak equivalences are the maps $f : \mathcal{C} \rightarrow \mathcal{D}$ satisfying the following two conditions:

- (W1) For any objects a_1 and a_2 in $\mathcal{C}$, the map
$$\operatorname{Hom}_{\mathcal{C}}(a_1, a_2) \to \operatorname{Hom}_{\mathcal{D}}(fa_1, fa_2)$$
is a weak equivalence of simplicial sets.
- (W2) The induced functor $\pi_0 f : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an equivalence of categories.

(2) The fibrations are the maps $f : \mathcal{C} \to \mathcal{D}$ satisfying the following two conditions:
- (F1) For any objects a_1 and a_2 in $\mathcal{C}$, the map
$$\operatorname{Hom}_{\mathcal{C}}(a_1, a_2) \to \operatorname{Hom}_{\mathcal{D}}(fa_1, fa_2)$$
is a fibration of simplicial sets.
- (F2) For any object a_1 in $\mathcal{C}$, b in $\mathcal{D}$, and homotopy equivalence $e : fa_1 \to b$ in $\mathcal{D}$, there is an object a_2 in $\mathcal{C}$ and homotopy equivalence $d : a_1 \to a_2$ in $\mathcal{C}$ such that $fd = e$.

(3) The cofibrations are the maps which have the left lifting property with respect to the maps which are both fibrations and weak equivalences.

The weak equivalences are often called *DK-equivalences*, as they were first described by Dwyer and Kan in [2]. They are a generalization of the notion of equivalence of categories to the simplicial setting.

We now state our main theorem.

Theorem 1.1. *There is a model category structure on the category of all small simplicial categories with the above weak equivalences, fibrations, and cofibrations.*

We will actually prove the stronger statement that the above model category structure is cofibrantly generated. Recall that a cofibrantly generated model category is one for which there are two specified sets of morphisms, one of generating cofibrations and one of generating acyclic cofibrations, such that a map is a fibration if and only if it has the right lifting property with respect to the generating acyclic cofibrations, and a map is an acyclic fibration if and only if it has the right lifting property with respect to the generating cofibrations. For more details about cofibrantly generated model category structures, see [6, Ch. 11]. To prove the theorem, we will use the following proposition, which is stated in more general form by Hirschhorn [6, 11.3.1].

Proposition 1.2. *Let $\mathcal{M}$ be a category with specified classes of weak equivalences and fibrations. Define a map to be a cofibration if it has the left lifting property with respect to the acyclic fibrations, and suppose that, with these three classes of morphisms, $\mathcal{M}$ satisfies model category axioms MC1, MC2, and MC3. Suppose further that there exist sets C and A of maps in $\mathcal{M}$ satisfying the following properties:*

(1) Both C and A permit the small object argument [6, 10.5.15].
(2) A map is a fibration if and only if it has the right lifting property with respect to the maps in A.
(3) A map is an acyclic fibration if and only if it has the right lifting property with the maps in C.
(4) A map is an acyclic cofibration if and only if it has the left lifting property with respect to the fibrations.

Then there is a cofibrantly generated model category structure on $\mathcal{M}$ in which C is a set of generating cofibrations and A is a set of generating acyclic cofibrations.

Let $\mathcal{SS}ets$ denote the category of simplicial sets. Recall in $\mathcal{SS}ets$ we have for any $n \geq 0$ the n-simplex $\Delta[n]$, its boundary $\dot{\Delta}[n]$, and, for any $0 \leq k \leq n$, $V[n,k]$, which is $\dot{\Delta}[n]$ with the kth face removed. Given a simplicial set X, we denote by $|X|$ its geometric realization. The standard model category structure on $\mathcal{SS}ets$ is cofibrantly generated; the generating cofibrations are the maps $\dot{\Delta}[n] \rightarrow \Delta[n]$ for $n \geq 0$, and the generating acyclic cofibrations are the maps $V[n,k] \rightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$. More details on simplicial sets and the model category structure on them can be found in [5].

There is a functor

$$U : \mathcal{SS}ets \rightarrow \mathcal{SC} \tag{1}$$

which takes a simplicial set X to the category with objects x and y and with $\operatorname{Hom}(x,y) = X$ but no other nonidentity morphisms.

We will say that a simplicial set K is *weakly contractible* if all the homotopy groups of $|K|$ are trivial.

We will refer to the model category structure on the category of simplicial categories with a fixed set $\mathcal{O}$ of objects, denoted $\mathcal{SC}_{\mathcal{O}}$, such that all the morphisms induce the identity map on the objects, as defined by Dwyer and Kan in [3]. The weak equivalences are the maps satisfying condition W1 and the fibrations are the maps satisfying condition F1.

We can then define our generating cofibrations and acyclic cofibrations as follows. The generating cofibrations are the maps

- (C1) $U\dot{\Delta}[n] \rightarrow U\Delta[n]$ for $n \geq 0$, and
- (C2) $\phi \rightarrow \{x\}$, where ϕ is the simplicial category with no objects and $\{x\}$ denotes the simplicial category with one object x and no nonidentity morphisms.

The generating acyclic cofibrations are

- (A1) the maps $UV[n,k] \rightarrow U\Delta[n]$ for $n \geq 1$, and
- (A2) inclusion maps $\{x\} \rightarrow \mathcal{H}$, where $\{x\}$ is as in C2 and $\{\mathcal{H}\}$ is a set of representatives for the isomorphism classes of simplicial categories with two objects x and y, weakly contractible function complexes, and only countably many simplices in each function complex. Furthermore, we require that the inclusion map $\{x\} \amalg \{y\} \rightarrow \mathcal{H}$ be a cofibration in $\mathcal{SC}_{\{x,y\}}$.

The idea behind the set A2 of generating acyclic cofibrations is the fact that two simplicial categories can have a weak equivalence between them which is not a bijection on objects, much as two categories can be equivalent even if they do not have the same objects. We only require that our weak equivalences are surjective on equivalence classes of objects. Thus, we must consider acyclic cofibrations for which the object sets are not isomorphic.

In section 2, we show that these proposed generating acyclic cofibrations satisfy the necessary conditions to be a generating set. In section 3, we complete the proof of Theorem 1. In section 4, we prove a technical lemma that we needed in section 2.

Acknowledgments. The author would like to thank Bill Dwyer for many helpful conversations about this paper.

2. The Generating Acyclic Cofibrations

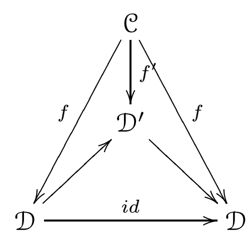
In this section, we will show that a map in $\mathcal{SC}$ satisfies properties F1 and F2 if and only if it has the right lifting property with respect to the maps in A1 and A2.

We begin by stating some facts about the Dwyer-Kan model category structure $\mathcal{SC}_{\mathcal{O}}$ on the category of simplicial categories with a fixed set $\mathcal{O}$ of objects. The weak equivalences are the maps which satisfy property W1, and the fibrations are the maps which satisfy property F1. The cofibrations are the maps which have the left lifting property with respect to the acyclic fibrations. However, we would like a more explicit description of the cofibrations in this category, for which we need some definitions. If $\mathcal{C}$ is a simplicial category, then let $\mathcal{C}_k$ denote the (discrete) category whose morphisms are the k-simplices of the morphisms of $\mathcal{C}$.

Definition 2.1. [3, 7.4] A map $f : \mathcal{C} \to \mathcal{D}$ in $\mathcal{SC}_{\mathcal{O}}$ is *free* if

(1) f is a monomorphism,
(2) if $*$ denotes the free product, then in each simplicial dimension k, the category $\mathcal{D}_k$ admits a unique free factorization $\mathcal{D}_k = f(\mathcal{C}_k) * \mathcal{F}_k$, where $\mathcal{F}_k$ is a free category, and
(3) for each $k \geq 0$, all degeneracies of generators of $\mathcal{F}_k$ are generators of $\mathcal{F}_{k+1}$.

Definition 2.2. [3, 7.5] A map $f : \mathcal{C} \to \mathcal{D}$ of simplicial categories is a *strong retract* of a map $f' : \mathcal{C} \to \mathcal{D}'$ if there exists a commutative diagram



Then, we have [3, 7.6] that the cofibrations of $\mathcal{SC}_{\mathcal{O}}$ are precisely the strong retracts of free maps. In particular, a cofibrant simplicial category is a retract of a free category.

Given these facts, we now continue with our discussion of the generating acyclic cofibrations.

Recall (1) the map $U : \mathcal{SS}ets \to \mathcal{SC}$. We first consider the set A1 of maps $UV[n,k] \to U\Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$. Using the model category structure on simplicial sets, we can see that a map of simplicial categories has the right lifting property with respect to the maps in A1 if and only if it satisfies the property F1.

We then consider the maps in A2 which we would also like to be generating acyclic cofibrations and show that maps with the right lifting property with respect to the maps in A1 and A2 are precisely the maps which satisfy conditions F1 and F2. The proof of this statement will take up the remainder of this section, and we will treat each implication separately.

Proposition 2.3. *Suppose that a map $f : \mathcal{C} \to \mathcal{D}$ of simplicial categories has the right lifting property with respect to the maps in A1 and A2. Then f satisfies condition F2.*

Before proving this proposition, we state a lemma.

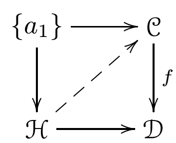
Lemma 2.4. *Let $\mathcal{F}$ be a (discrete) simplicial category with object set $\{x, y\}$ and one nonidentity morphism $g : x \to y$. Let $\mathcal{E}'$ be a simplicial category also with object set $\{x, y\}$. Let $i : \mathcal{F} \to \mathcal{E}'$ send g to a homotopy equivalence in $Hom_{\mathcal{E}'}(x, y)$. This map i can be factored as a composite $\mathcal{F} \to \mathcal{H} \to \mathcal{E}'$ in such a way that the composite map $\{x\} \to \mathcal{F} \to \mathcal{H}$ is isomorphic to a map in A2.*

We will prove this lemma in section 4. We now prove Proposition 2.3 assuming Lemma 2.4.

Proof of Proposition 2.3. Given objects a_1 in $\mathcal{C}$ and b in $\mathcal{D}$, we need to show that a homotopy equivalence $e : fa_1 \to b$ in $\mathcal{D}$ lifts to a homotopy equivalence $d : a_1 \to a_2$ for some a_2 in $\mathcal{C}$ such that $fa_2 = b$ and $fd = e$. So, we begin by considering the objects $a = fa_1$ and b in $\mathcal{D}$.

We first consider the case where $a \neq b$. Define $\mathcal{E}'$ to be the full simplicial subcategory of $\mathcal{D}$ with objects $a = fa_1$ and b, and let $\mathcal{F}$ be a simplicial category with objects a and b and a single nonidentity morphism $g : a \to b$. Let $i : \mathcal{F} \to \mathcal{E}'$ send g to a homotopy equivalence $e : a \to b$. By Lemma 2.4, we can factor this map as $\mathcal{F} \to \mathcal{H} \to \mathcal{E}'$ in such a way that the composite $\{a\} \to \mathcal{F} \to \mathcal{H}$ is isomorphic to a map in A2.

It follows that the composite $\{a_1\} \to \{a\} \to \mathcal{H}$ is also isomorphic to a map in A2. Then consider the composite $\mathcal{H} \to \mathcal{E}' \to \mathcal{D}$ where the map $\mathcal{E}' \to \mathcal{D}$ is the inclusion map. These maps fit into a diagram



The lift exists because we assume that the map $f : \mathcal{C} \to \mathcal{D}$ has the right lifting property with respect to all maps in A2. Now, composing the map $\mathcal{F} \to \mathcal{H}$ with the lift sends the map g in $\mathcal{F}$ to a map d in $\mathcal{C}$ such that $fd = e$. The map d is a homotopy equivalence since all the morphisms of $\mathcal{H}$ are homotopy equivalences and therefore map to homotopy equivalences in $\mathcal{C}$.

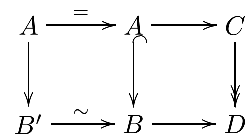
Now suppose that $a = b$. Define $\mathcal{E}'$ to be the simplicial category with two objects a and a' such that each function complex of $\mathcal{E}'$ is the simplicial set $\mathrm{Hom}_{\mathcal{D}}(a, a)$ and compositions are defined as they are in $\mathcal{D}$. We then define the map $\mathcal{E}' \to \mathcal{D}$ which sends both objects of $\mathcal{E}'$ to a in $\mathcal{D}$ and is the identity map on all the function complexes. Given this simplicial category $\mathcal{E}'$, the argument proceeds as above. □

We now prove the converse.

Proposition 2.5. *Suppose $f : \mathcal{C} \to \mathcal{D}$ is a map of simplicial categories which satisfies properties F1 and F2. Then f has the right lifting property with respect to the maps in A2.*

Again, we state a lemma before proceeding with the proof of this proposition.

Lemma 2.6. *Suppose that $A \to B$ is a cofibration, $C \to D$ is a fibration, and $B' \to B$ is a weak equivalence in a model category $\mathcal{M}$. Then in the following* *commutative diagram*



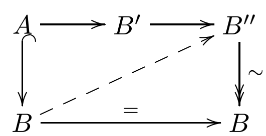
a lift $B \to C$ exists if and only if a lift $B' \to C$ exists.

Proof. If the lift $B \to C$ exists, it follows that the lift $B' \to C$ exists via composition with the map $B' \to B$.

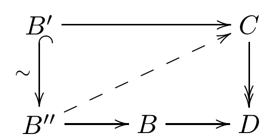
To prove the converse, we first note that the map $B' \to B$ can be factored as the composite

$$B' \hookrightarrow B'' \twoheadrightarrow B$$

of a cofibration and a fibration, where each is a weak equivalence because the map $B' \to B$ is. Therefore by model category axiom MC4, there is a lift in the diagram



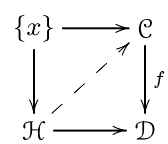
It now suffices to show that there is a lift in the diagram



However, this fact again follows from axiom MC4. □

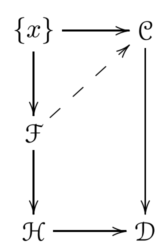
We are now able to prove the proposition.

Proof of Proposition 2.5. We need to show that there exists a lift in any diagram of the form



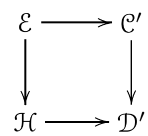
where $\{x\} \to \mathcal{H}$ is a map in A2. Since the map f satisfies property F2, then given an object a_1 in $\mathcal{C}$ and a homotopy equivalence $e : a = fa_1 \to b$, there exists an object a_2 in $\mathcal{C}$ and a homotopy equivalence $d : a_1 \to a_2$ such that $fd = e$.

Let $g : x \to y$ be a homotopy equivalence in $\mathcal{H}$. Let $\mathcal{F}$ denote the subcategory of $\mathcal{H}$ consisting of the objects x and y and g its only nonidentity morphism. Consider the composite map $\{x\} \to \mathcal{F} \to \mathcal{H}$ and the resulting diagram



Because the map $\mathcal{F} \to \mathcal{D}$ factors through $\mathcal{H}$, which consists of homotopy equivalences, the image of g in $\mathcal{D}$ is a homotopy equivalence. Thus, the existence of the lift in the above diagram follows from the fact that the map f satisfies F2.

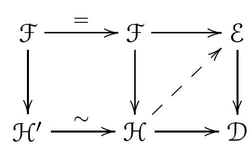
Now, we need to show that the rest of $\mathcal{H}$ lifts to $\mathcal{C}$. We begin by assuming that $a \neq b$ and therefore $a_1 \neq a_2$. Consider the full simplicial subcategory of $\mathcal{C}$ with objects a_1 and a_2, and denote by $\mathcal{C}'$ the isomorphic simplicial category with objects x and y. Define $\mathcal{D}'$ analogously where we take objects x and y rather than a and b. Now, we can work in the category $\mathcal{SC}_{\{x,y\}}$ of simplicial categories with fixed object set $\{x, y\}$. Note that the map $\mathcal{C}' \to \mathcal{D}'$ is still a fibration in $\mathcal{SC}$, and in fact it is a fibration in $\mathcal{SC}_{\{x,y\}}$. Now define $\mathcal{E}$ to be the pullback in the diagram



Then the map $\mathcal{E} \to \mathcal{H}$ is also a fibration in $\mathcal{SC}_{\{x,y\}}$ [4, 3.14(iii)].

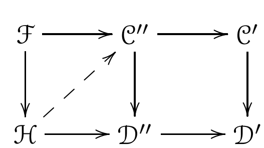
By Lemma 2.4, we can factor the map $\mathcal{F} \to \mathcal{E}$ as the composite $\mathcal{F} \to \mathcal{H}' \to \mathcal{E}$ for some simplicial category $\mathcal{H}'$ such that the composite $\{x\} \to \mathcal{F} \to \mathcal{H}'$ is isomorphic to a map in A2. Then, note that the composite map $\mathcal{H}' \to \mathcal{E} \to \mathcal{H}$ is a weak equivalence in $\mathcal{SC}_{\{x,y\}}$ since all the function complexes of $\mathcal{H}$ and $\mathcal{H}'$ are weakly contractible.

Now, we have a diagram



in which the dotted arrow lift exists by Lemma 2.6.

If $a = b$, then $\mathcal{D}'$ (and possibly $\mathcal{C}'$) as defined above will have only one object x. If this is the case, then define the simplicial category $\mathcal{D}''$ with two objects x and y such that each function complex is the simplicial set $\operatorname{Hom}_{\mathcal{D}'}(x, x)$ (as in the proof of Proposition 2.3). We can then factor the map $\mathcal{H} \to \mathcal{D}'$ through the object $\mathcal{D}''$, where the map $\mathcal{D}'' \to \mathcal{D}$ sends both objects of $\mathcal{D}''$ to a in $\mathcal{D}$ and is the identity map on each function complex. If $\mathcal{C}'$ also has one object, then we obtain a simplicial category $\mathcal{C}''$ in the same way. Then, we can repeat the argument above in the left-hand square of the diagram



to obtain a lift $\mathcal{H} \to \mathcal{C}''$, and hence a lift $\mathcal{H} \to \mathcal{C}'$ via composition. □

3. The Model Category Structure

In order to show that our proposed model category structure exists, we need to show that our definitions are compatible with one another. In particular, we need to prove that the maps with the left lifting property with respect to the fibrations are exactly the acyclic cofibrations, and that the maps with the right lifting property with respect to the generating cofibrations are exactly the maps which are fibrations and weak equivalences. Before proving these statements, however, we prove that first three model category axioms hold in $\mathcal{SC}$.

Proposition 3.1. *With the given weak equivalences, fibrations, and cofibrations,* $\mathcal{SC}$ *satisfies model category axioms MC1, MC2, and MC3.*

Proof. It can be shown that the category of all simplicial categories has all coproducts and all coequalizers, and therefore all colimits, and all products and equalizers, and therefore all limits. Thus, $\mathcal{SC}$ satisfies MC1. MC2 and MC3 follow as usual, for example, as in [4, 8.10]. □

We first consider the sets C1 and C2. Suppose we have a map $f : \mathcal{C} \rightarrow \mathcal{D}$ which is a fibration and a weak equivalence. Using simplicial set arguments, we can see that a map satisfies conditions F1 and W1 if and only if it has the right lifting property with respect to the maps $U\dot{\Delta}[n] \rightarrow U\Delta[n]$ for $n \geq 0$, where U is the map (1) from simplicial sets to simplicial categories defined in the first section.

However, the maps $U\dot{\Delta}[n] \rightarrow U\Delta[n]$ only generate those cofibrations between simplicial categories with the same number of objects, a condition that we do not require on our cofibrations of simplicial categories. Therefore, we include as a generating cofibration the map $\phi \rightarrow \{x\}$ from the simplicial category with no objects to the single-object simplicial category with no nonidentity morphisms. In other words, we are including the addition of an object as a cofibration.

Proposition 3.2. *A map in* $\mathcal{SC}$ *is a fibration and a weak equivalence if and only if it has the right lifting property with respect to the maps in C1 and C2.*

Proof. First suppose that $f : \mathcal{C} \rightarrow \mathcal{D}$ is both a fibration and a weak equivalence. By conditions F1 and W1, the map $\operatorname{Hom}_{\mathcal{C}}(a, b) \rightarrow \operatorname{Hom}_{\mathcal{D}}(fa, fb)$ is an acyclic fibration of simplicial sets for any choice of objects a and b in $\mathcal{C}$. In other words, there is a lift in any diagram of the form

$$\begin{array}{ccc} \dot{\Delta}[n] & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(a,b) \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & \operatorname{Hom}_{\mathcal{D}}(fa,fb) \end{array}$$

However, having this lift is equivalent to having a lift in the diagram

$$\begin{array}{ccc} U\dot{\Delta}[n] & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow \\ U\Delta[n] & \longrightarrow & \mathcal{D} \end{array}$$

where the objects x and y of $U\dot{\Delta}[n]$ map to a and b in $\mathcal{C}$, and analogously for $U\Delta[n]$ and $\mathcal{D}$. Hence, f has the right lifting property with respect to the maps in C1.

It remains only to show that f has the right lifting property with respect to the map $\phi \rightarrow \{x\}$. However, this property is equivalent to f being onto on objects. Being onto on homotopy equivalence classes of objects follows from condition W2. Then suppose that $e : a \rightarrow b$ is an isomorphism in $\mathcal{D}$ and there is an object a_1 in $\mathcal{C}$ such that $fa_1 = a$. Since e is a homotopy equivalence, by F2 there is a homotopy equivalence in $\mathcal{C}$ with domain a_1 and which maps to e under f. In particular, there is an object a_2 in $\mathcal{C}$ mapping to b.

Conversely, suppose that f has the right lifting property with respect to the maps in C1 and C2. Again, using the model category structure on simplicial sets, we have that the map

$$\operatorname{Hom}_{\mathcal{C}}(a,b) \rightarrow \operatorname{Hom}_{\mathcal{D}}(fa,fb)$$

is both a fibration and a weak equivalence, satisfying both F1 and W1. It follows that $\operatorname{Hom}_{\pi_0\mathcal{C}}(a,b) \rightarrow \operatorname{Hom}_{\pi_0\mathcal{D}}(fa,fb)$ is an isomorphism. As above, having the right lifting property with respect to the map $\phi \rightarrow \{x\}$ is equivalent to being onto on objects. These two facts show then that $\pi_0\mathcal{C} \rightarrow \pi_0\mathcal{D}$ is an equivalence of categories, proving condition W2.

It remains to show that f satisfies property F2. By Proposition 2.3 and the fact that satisfying F1 is equivalent to having the right lifting property with respect to maps in A1, it suffices to show that f has the right lifting property with respect to the maps in A2. But, a map $\{x\} \rightarrow \mathcal{H}$ in A2 can be written as a (possibly infinite) composition of a pushout along $\phi \rightarrow \{x\}$ followed by pushouts along maps of the form $U\dot{\Delta}[n] \rightarrow U\Delta[n]$, and f has the right lifting property with respect to all such maps since these are just the maps in C1 and C2. □

Proposition 3.3. *A map in $\mathcal{SC}$ is an acyclic cofibration if and only if it has the left lifting property with respect to the fibrations.*

The proof will require the use of the following lemma:

Lemma 3.4. *Let $\mathcal{A} \rightarrow \mathcal{B}$ be a map in A1 or A2 and $i : \mathcal{A} \rightarrow \mathcal{C}$ a map in $\mathcal{SC}$. Then in the pushout diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{D} \end{array}$$

the map $\mathcal{C} \rightarrow \mathcal{D}$ is a weak equivalence.

Proof. First suppose that the map $\mathcal{A} \rightarrow \mathcal{B}$ is in A2. Let $\mathcal{O}$ be the set of objects of $\mathcal{C}$ and define $\mathcal{O}'$ to be the set $\mathcal{O}\backslash\{x\}$. (For simplicity of notation, we assume that $ix = x$.) Assume as before that x and y are the objects of $\mathcal{H}$. We denote also by $\mathcal{O}'$ the (simplicial) category with object set $\mathcal{O}'$ and no nonidentity morphisms. Consider the diagram

$$\begin{array}{ccc} \mathcal{X} = \mathcal{O} \amalg \{y\} & \longrightarrow & \mathcal{C} \amalg \{y\} = \mathcal{C}' \\ \downarrow & & \downarrow \\ \mathcal{H}' = \mathcal{O}' \amalg \mathcal{H} & \longrightarrow & \mathcal{D} \end{array}$$

and notice that $\mathcal{D}$ is also the pushout of this diagram. Since $\mathcal{X}$ (regarded as a set) is the object set of any of these categories, note that the left hand vertical arrow is a cofibration in $\mathcal{SC}_{\mathcal{X}}$.

We factor the map $\mathcal{X} \rightarrow \mathcal{C}'$ as the composite of a cofibration and an acyclic fibration in $\mathcal{SC}_{\mathcal{X}}$

$$\mathcal{X} \hookrightarrow \mathcal{C}'' \xrightarrow{\sim} \mathcal{C}'.$$

Since $\mathcal{SC}_{\mathcal{X}}$ is proper [3, 7.3], it follows from [6, 13.5.4] that the pushouts of each row in the diagram

$$\begin{array}{ccccc}
\mathcal{H}' & \hookleftarrow & \mathcal{X} & \longrightarrow & \mathcal{C}' \\
\uparrow{=} & & \uparrow{=} & & \uparrow{\sim} \\
\mathcal{H}' & \hookleftarrow & \mathcal{X} & \hookrightarrow & \mathcal{C}'' \\
\downarrow{\sim} & & \downarrow{=} & & \downarrow{=} \\
\pi_0\mathcal{H}' & \longleftarrow & \mathcal{X} & \hookrightarrow & \mathcal{C}''
\end{array}$$

are weakly equivalent to one another. In particular, the pushout of the bottom row is weakly equivalent to $\mathcal{D}$. It remains to show that there is a weak equivalence of pushouts of the rows of the diagram

$$\begin{array}{ccccc}
\pi_0\mathcal{H}' & \longleftarrow & \mathcal{X} & \hookrightarrow & \mathcal{C}'' \\
\downarrow & & \downarrow & & \downarrow \\
\pi_0\mathcal{H}' & \longleftarrow & \mathcal{X} & \longrightarrow & \mathcal{C}'.
\end{array}$$

However, a calculation shows that the pushout of this bottom row is weakly equivalent in $\mathcal{SC}$ to the pushout of the diagram

$$\pi_0\mathcal{H} \longleftarrow \{x\} \longrightarrow \mathcal{C}$$

and therefore that the pushout of the top row is weakly equivalent to the pushout of the bottom row. It follows that the map $\mathcal{C} \to \mathcal{D}$ is a weak equivalence in $\mathcal{SC}$.

For the maps in A1, we have pushout diagrams

$$\begin{array}{ccc}
UV[n,k] & \xrightarrow{j} & \mathcal{C} \\
\downarrow & & \downarrow \\
U\Delta[n] & \longrightarrow & \mathcal{D}
\end{array} .$$

As before, define $\mathcal{O}$ to be the object set of $\mathcal{C}$. Let $\mathcal{O}'' = \mathcal{O} \backslash \{x, y\}$. (Again, for notational simplicity we will assume that $jx = x$ and $jy = y$.) Now we consider the diagram

$$\begin{array}{ccc}
\mathcal{O}'' \amalg UV[n,k] & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{O}'' \amalg U\Delta[n] & \longrightarrow & \mathcal{D}
\end{array}$$

in $\mathcal{SC}_{\mathcal{O}}$. However, since the left vertical map is a weak equivalence and assuming that the top map is a cofibration (factoring if necessary as above), we can again use the fact that $\mathcal{SC}_{\mathcal{O}}$ is proper to show that $\mathcal{C} \to \mathcal{D}$ is a weak equivalence in $\mathcal{SC}_{\mathcal{O}}$ and thus also in $\mathcal{SC}$. □

Proof of Proposition 3.3. First suppose that a map $\mathcal{C} \to \mathcal{D}$ is an acyclic cofibration. By the small object argument ([4, Sec. 7] or [6, Ch. 11]), we have a factorization of the map $\mathcal{C} \to \mathcal{D}$ as the composite $\mathcal{C} \to \mathcal{C}' \to \mathcal{D}$ where $\mathcal{C}'$ is obtained from $\mathcal{C}$ by a directed colimit of iterated pushouts along the maps in A1 and A2. Thus, by Lemma 3.4 above and the fact that a directed colimit of such maps is a weak equivalence, this map $\mathcal{C} \to \mathcal{C}'$ is a weak equivalence. Furthermore, the map $\mathcal{C}' \to \mathcal{D}$

has the right lifting property with respect to the maps in A1 and A2. Thus, by Proposition 2.3, it is a fibration. It is also a weak equivalence since the maps $\mathcal{C} \to \mathcal{D}$ and $\mathcal{C} \to \mathcal{C}'$ are, by axiom MC2. In particular, by the definition of cofibration, it has the right lifting property with respect to the cofibrations. Therefore, there is a lift in the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\simeq} & \mathcal{C}' \\ \downarrow & \nearrow & \downarrow \\ \mathcal{D} & \xrightarrow{=} & \mathcal{D} \end{array}$$

Hence the map $\mathcal{C} \to \mathcal{D}$ is a retract of the map $\mathcal{C} \to \mathcal{C}'$ and therefore also has the left lifting property with respect to fibrations.

Conversely, suppose that the map $\mathcal{C} \to \mathcal{D}$ has the left lifting property with respect to fibrations. In particular, it has the left lifting property with respect to the acyclic fibrations, so it is a cofibration by definition. We again obtain a factorization of this map as the composite $\mathcal{C} \to \mathcal{C}' \to \mathcal{D}$ where $\mathcal{C}'$ is obtained from $\mathcal{C}$ by iterated pushouts of the maps in A1 and A2. Once again, the map $\mathcal{C}' \to \mathcal{D}$ has the right lifting property with respect to the maps in A1 and A2 and thus is a fibration by Proposition 2.3. Therefore there is a lift in the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\simeq} & \mathcal{C}' \\ \downarrow & \nearrow & \downarrow \\ \mathcal{D} & \xrightarrow{=} & \mathcal{D} \end{array}$$

Again using Lemma 3.4, the map $\mathcal{C} \to \mathcal{D}$ is a weak equivalence because it is a retract of the map $\mathcal{C} \to \mathcal{C}'$. □

We have now proved everything we need for the existence of the model category structure on $\mathcal{SC}$.

Proof of 1.1. It remains to show that the four conditions of Proposition 1.2 are satisfied. It can be shown that both ϕ and $\{x\}$ are small, and using the smallness of $V[n,k]$ and $\Delta[n]$ in $\mathcal{SS}ets$ [7, 3.1.1], it can be shown that each $U\dot{\Delta}[n]$ is small relative to the set C1 and each $UV[n,k]$ is small relative to the set A1 [6, 10.5.12]. Therefore, condition 1 holds. Condition 2 follows from Propositions 2.3 and 2.5. Condition 3 is proved in Proposition 3.2, and condition 4 is proved in Proposition 3.3. □

4. Proof of 2.4

Recall that we have a (simplicial) category $\mathcal{F}$ with objects x and y and a single nonidentity morphism $g : x \to y$, and a simplicial category $\mathcal{E}'$ also with objects x and y such that there is a map $i : \mathcal{F} \to \mathcal{E}'$ which sends g to a homotopy equivalence $x \to y$ in $\mathcal{E}'$. We first replace $\mathcal{E}'$ by its subcategory of weak equivalences which we denote by $\mathcal{E}$. In order to make our constructions homotopy invariant, we take functorial cofibrant replacements $\widetilde{\mathcal{F}} \to \mathcal{F}$ and $\widetilde{\mathcal{E}} \to \mathcal{E}$ in the model category $\mathcal{SC}_{\{x,y\}}$ as given in [3, 2.5], and in this construction $\widetilde{\mathcal{F}}$ is actually isomorphic to $\mathcal{F}$.

Now, take the localization $\mathcal{F}^{-1}\mathcal{F}$ (respectively $\widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}$) obtained by formally inverting all the morphisms in each simplicial degree of $\mathcal{F}$ (respectively $\widetilde{\mathcal{E}}$). These localizations are the groupoid completions of $\mathcal{F}$ and $\widetilde{\mathcal{E}}$, respectively. (In taking a functorial cofibrant replacement and then the groupoid completion, we have taken the simplicial localizations of $\mathcal{F}$ and $\mathcal{E}$ with respect to all the morphisms in each as defined in [3].) We now have a diagram

$$\begin{array}{ccccc} \mathcal{F} & \stackrel{=}{\longleftarrow} & \mathcal{F} & \longrightarrow & \mathcal{F}^{-1}\mathcal{F} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E} & \longleftarrow & \widetilde{\mathcal{E}} & \longrightarrow & \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}. \end{array}$$

To assure that our next step is homotopy invariant, we factor the map $\mathcal{F}^{-1}\mathcal{F} \to \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}$ as the composite

$$\mathcal{F}^{-1}\mathcal{F} \xrightarrow{i} \mathcal{Z} \xrightarrow{p} \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}$$

where i is an acyclic cofibration and p is a fibration in $\mathcal{SC}_{\{x,y\}}$. However, we will continue to write $\mathcal{F}^{-1}\mathcal{F}$ rather than $\mathcal{Z}$ to avoid more notation than necessary.

We take the pullback of the bottom right hand corner of the above diagram and denote it $\mathcal{G}$:

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{F}^{-1}\mathcal{F} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{E}} & \longrightarrow & \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}} \end{array}$$

Lemma 4.1. *The composite map* $\{x\} \to \mathcal{F} \to \mathcal{G}$ *is a weak equivalence in* $\mathcal{SC}$.

Proof. Since the simplicial categories $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}$, and $\mathcal{F}^{-1}\mathcal{F}$ all consist of homotopy equivalences, so must $\mathcal{G}$. Therefore, all the morphisms of $\pi_0\mathcal{G}$ are isomorphisms.

It then suffices to show that $\mathcal{G}$ has weakly contractible function complexes. Because all of the morphisms of $\mathcal{E}$, and hence also of $\widetilde{\mathcal{E}}$, are homotopy equivalences, the map $\widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}$ is a weak equivalence in $\mathcal{SC}_{\{x,y\}}$ [3, 9.5].

Note that $\mathcal{F}^{-1}\mathcal{F}$ is the simplicial category in $\mathcal{SC}_{\{x,y\}}$ with exactly one morphism between any two objects. In particular, $\mathcal{F}^{-1}\mathcal{F}$ has weakly contractible function complexes.

Now, because all the categories have as objects x and y and all the maps involved are the identity on these objects, we can consider the above pullback diagram in $\mathcal{SC}_{\{x,y\}}$. Since this model category structure is proper [3, 7.3], every pullback of a weak equivalence along a fibration is a weak equivalence. The map $\widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}$ is a weak equivalence and the map $\mathcal{F}^{-1}\mathcal{F} \to \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}$ is a fibration, so it follows that the map $\mathcal{G} \to \mathcal{F}^{-1}\mathcal{F}$ is a weak equivalence in $\mathcal{SC}_{\{x,y\}}$, and therefore $\mathcal{G}$ has weakly contractible function complexes. Thus, the map $\{x\} \to \mathcal{G}$ satisfies the conditions to be a weak equivalence in $\mathcal{SC}$. □

However, not all the maps $\{x\} \to \mathcal{G}$ are isomorphic to maps in A2 because the simplicial categories $\mathcal{G}$ could have uncountable simplices in their function complexes. Furthermore, there is no reason to assume that the inclusion map $\{x\} \amalg \{y\} \to \mathcal{G}$ is a cofibration in $\mathcal{SC}_{\{x,y\}}$. To complete the proof, we need to show that any acyclic cofibration $\{x\} \to \mathcal{G}$ as above factors as a composite $\{x\} \to \mathcal{H} \to \mathcal{G}$ where the inclusion map $\{x\} \to \mathcal{H}$ is in A2.

Let $\mathcal{H}_0$ be the simplicial category $\mathcal{F}$. Let $i : \mathcal{H}_0 \to \mathcal{G}$ be the inclusion map. We will construct a simplicial category $\mathcal{H}$ from $\mathcal{H}_0$ satisfying the necessary properties specified in A2. We first state the following lemma:

Lemma 4.2. *Let $f : A \to B$ be a map of simplicial sets where B is weakly contractible, and let $u : S^n \to |A|$ be a map of CW-complexes for some $n \geq 0$. Then f can be factored as a composite $A \to A' \to B$ where A' is obtained from A by attaching a finite number of nondegenerate simplices and the composite map of spaces $S^n \to |A| \to |A'|$ is null homotopic.*

Proof. We first assume that the map f is a cofibration; if not, we factor it as the composite

$$A \xrightarrow{i} B' \xrightarrow{p} B$$

where in the model category structure on simplicial sets i is a cofibration and p is an acyclic fibration. Thus, we can assume that f is an inclusion map, replacing B by B' if needed.

Now consider the composite map of spaces $S^n \to |A| \to |B|$, which is necessarily null homotopic since B is weakly contractible. The composite map then factors through CS^n, the cone on S^n, and we have a diagram

$$\begin{array}{ccccc} S^n & \longrightarrow & |A| & \longrightarrow & |B| \\ \downarrow & & & \nearrow & \\ CS^n & & & & \end{array}$$

Now, since CS^n is compact, its image will intersect only a finite number of cells of $|B|$ nontrivially. Then define A' to be a simplicial set such that $|A'|$ contains $|A|$ as well as all the cells in this image. □

Now, consider the categories $\mathcal{H}_0$ and $\mathcal{G}$ as described above and the inclusion map $i : \mathcal{H}_0 \to \mathcal{G}$. Each of these categories has four function complexes to consider. For the category $\mathcal{H}$ we call them H_j, and for $\mathcal{G}$ we call them G_j for $1 \leq j \leq 4$. (The numbering is arbitrary but must match up between the two categories. So if $H_1 = \operatorname{Hom}_{\mathcal{H}}(x, y)$, then we must have $G_1 = \operatorname{Hom}_{\mathcal{G}}(x, y)$.)

Identify $n \geq 0$ such that all maps $S^m \to |H_j|$ are null homotopic for all $0 \leq m < n$ and all $1 \leq j \leq 4$, but there is a map $S^n \to |H_j|$ which is not null homotopic for some j. We then apply Lemma 4.2 to the map $H_j \to G_j$ and the map $S^n \to |H_j|$.

Replace the function complex H_j with the simplicial set A' obtained from Lemma 4.2. This process may result in more maps $S^m \to |A'|$ which are not null homotopic than for the original $|H_j|$, but only for $m > n$. Also, it will not have more than countably many more such maps than $|H_j|$ did. Now that we have added simplices to our function complex, we include all necessary compositions of these morphisms with the original morphisms of $\mathcal{H}_0$ to obtain a new simplicial category which we denote $\mathcal{H}_1$. There will be at most countably many new simplices added from these compositions. Repeat the above process with another map from S^n to a function complex of $\mathcal{H}_1$, again, where n is minimal, to obtain another category $\mathcal{H}_2$. Continue, perhaps countably many times, to obtain a category $\mathcal{H}$ such that for any n and any function complex H' of $\mathcal{H}$, any map $S^n \to |H'|$ is nullhomotopic. To show that it is possible to obtain such an $\mathcal{H}$ in this way, we need only show that there are at most countably many homotopy classes of maps from spheres to each function complex that need to be killed off. However, this fact follows from the following lemma:

Lemma 4.3. *Let A be a simplicial set with countably many simplices. Then for all $n \geq 0$ there are at most countably many distinct homotopy classes of maps $S^n \to |A|$.*

Proof. It suffices to show that there are at most countably many homotopy classes of maps from S^n into any finite CW complex X. For a simply connected CW complex X, an argument using Serre mod $\mathcal{C}$ theory [8] shows that all the homotopy groups of X are countable if and only if the homology groups of X are countable, which they are when X is finite. The case of a general CW complex X follows from this one using a universal cover argument. □

By construction, this simplicial category $\mathcal{H}$ is free, and therefore the map

$$\{x\} \amalg \{y\} \rightarrow \mathcal{H}$$

is a cofibration in $\mathcal{SC}_{\{x,y\}}$. Thus, we have obtained a factorization $\{x\} \rightarrow \mathcal{H} \rightarrow \mathcal{G}$. We are now able to complete the proof of 2.4.

Proof of 2.4. Using the simplicial category $\mathcal{H}$ from above and the map $\mathcal{H} \rightarrow \mathcal{G}$, we obtain a composite map

$$\{x\} \rightarrow \mathcal{F} \rightarrow \mathcal{H} \rightarrow \mathcal{G} \rightarrow \widetilde{\mathcal{E}} \rightarrow \mathcal{E} \rightarrow \mathcal{E}'.$$

In particular, we have a factorization $\mathcal{F} \rightarrow \mathcal{H} \rightarrow \mathcal{E}'$. As we have shown above, the composite $\{x\} \rightarrow \mathcal{F} \rightarrow \mathcal{H}$ is isomorphic to a map in A2. □

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