

# ON REALIZING DIAGRAMS OF $\Pi$ -ALGEBRAS

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ABSTRACT. Given a diagram of  $\Pi$ -algebras (graded groups equipped with an action of the primary homotopy operations), we ask whether it can be realized as the homotopy groups of a diagram of spaces. The answer given here is in the form of an obstruction theory, of somewhat wider application, formulated in terms of *generalized  $\Pi$ -algebras*. This extends a program begun in [DKS1, BDG] to study the realization of a single  $\Pi$ -algebra. In particular, we explicitly analyze the simple case of a single map, and provide a detailed example, illustrating the connections to higher homotopy operations.

## 0. INTRODUCTION

A recurring problem in algebraic topology is the rectification of homotopy-commutative diagrams: given a diagram  $F : \mathbb{D} \rightarrow \text{ho } \mathcal{T}_*$  (i.e., a functor from a small category to the homotopy category of topological spaces), we ask whether  $F$  lifts to  $\hat{F} : \mathbb{D} \rightarrow \mathcal{T}_*$ , and if so, in how many ways.

Such questions arise naturally in determining if a given  $H$ -space is a loop space; in defining Steenrod operations; in analyzing structured ring spectra; and so on. Our goal here is to present an obstruction-theoretic approach to an algebraic version of this question.

**0.1. Diagrams of  $\Pi$ -algebras.** Recall that a  $\Pi$ -algebra is a graded group equipped with an action of the primary homotopy operations (Whitehead products and compositions), modeled on the homotopy groups of a space (see §1 below). In [DKS1, DKS2], Dwyer, Kan, and Stover set out to construct an obstruction theory for realizing a given  $\Pi$ -algebra  $\Lambda$  as  $\Lambda \cong \pi_* X$ , for some space  $X$ . The program was completed in [BDG], using methods developed by Dwyer and Kan in a series of papers which established a general obstruction theory for rectifying homotopy-commutative diagrams (see [DK1, DK2, DK3, DK4, DKSm]). Our goal here is to extend this program to address the following:

**0.2. Diagram realization question.** Can a given diagram of  $\Pi$ -algebras  $\Lambda : \mathbb{D} \rightarrow \Pi\text{-Alg}$  be *realized* – that is, lifted to a diagram of spaces  $\hat{\Lambda} : \mathbb{D} \rightarrow \mathcal{T}_*$  with  $\pi_* \circ \hat{\Lambda} = \Lambda$ ?

The answer we provide is in the form an obstruction theory: we inductively define a sequence of cohomology classes  $k_n \in H^{n+2}(\Lambda; \Omega^n \Lambda)$ , and show that  $\Lambda$  is realizable precisely when  $k_n = 0$  for all  $n$ . The case of a single  $\Pi$ -algebra was treated in [BDG], and the extension to our context is straightforward. However, the description there was in terms of moduli spaces, and it seems worthwhile making obstruction theory explicit. A further generalization of this theory appears in [Bl6], but it is not easy to extract from it the simpler version needed here.

**0.3. Generalized  $\Pi$ -algebras.** In fact, it turns out that this approach may be carried out somewhat more generally, for any  $E^2$ -model category  $s\mathcal{C}$  (see Section 3), once we

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have chosen a set  $\mathcal{A}$  of homotopy cogroup objects in  $\mathcal{C}$  to play the role of the spheres  $\{\mathbf{S}^n\}_{n=1}^\infty$  in  $\mathcal{T}_*$ :

Note that a  $\Pi$ -algebra can be thought of as a product-preserving functor  $T : \Pi^{\text{op}} \rightarrow \text{Set}_*$ , where  $\Pi$  is the subcategory of finite wedges of spheres in  $\text{ho } \mathcal{T}_*$ . Similarly defining  $\Pi_{\mathcal{A}} \subseteq \text{ho } \mathcal{C}$  for any  $\mathcal{A}$  as above, we define a  $\Pi_{\mathcal{A}}$ -algebra to be a product-preserving functor  $\Pi_{\mathcal{A}}^{\text{op}} \rightarrow \text{Set}_*$ .

For example, a map  $\phi : \Gamma \rightarrow \Lambda$  of ordinary  $\Pi$ -algebras corresponds to a diagram in  $(\Pi_{\mathcal{A}}\text{-Alg})^{\mathbb{D}}$ , where  $\mathbb{D}$  has two objects and a single non-identity map  $0 \rightarrow 1$ . Setting

$$\mathcal{A} := \{\mathbf{S}^n \xrightarrow{\text{Id}} \mathbf{S}^n, * \rightarrow \mathbf{S}^n\}_{n \in \mathbb{N}},$$

we can think of  $\phi$  as a generalized  $\Pi_{\mathcal{A}}$ -algebra. The realization question for diagrams of  $\Pi$ -algebras is thus a special case of the the following:

**0.4. General Realization Question.** Given a model category  $\mathcal{C}$  with set of models  $\mathcal{A}$ , when is a  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$  realizable in  $\mathcal{C}$ ? That is, is there an  $X \in \mathcal{C}$  such that  $\pi_{\mathcal{A}}X \cong \Lambda$  (where  $\pi_{\mathcal{A}}X$  is defined by  $A \mapsto [A, X]_{\mathcal{C}}$ ?)

Again, this is not meant to be a gratuitous exercise in generalization: it allows us to answer in a systematic way the same question for (diagrams of) localized or  $n$ -connected spaces, spectra,  $n$ -types, and so on.

**0.5. Notation and conventions.**  $\mathcal{T}$  will denote the category of topological spaces, and  $\mathcal{T}_*$  that of pointed connected topological spaces. By a *space* we shall always mean an object in  $\mathcal{T}_*$ .

The category of groups is denoted by  $\mathcal{Gp}$ , and that of pointed sets by  $\text{Set}_*$ . For any category  $\mathcal{C}$ ,  $\text{gr}_{\mathcal{A}}\mathcal{C}$  denotes the category of  $\mathcal{A}$ -graded objects over  $\mathcal{C}$  (i.e., the category  $\mathcal{C}^{\mathcal{A}}$  of diagrams indexed by the discrete category  $\mathcal{A}$ ), and  $s\mathcal{C}$  that of simplicial objects over  $\mathcal{C}$ . The category of simplicial sets will be denoted by  $\mathcal{S}$ , that of pointed connected simplicial sets by  $\mathcal{S}_*$ , and that of simplicial groups by  $\mathcal{G}$ . For any  $Z \in \mathcal{C}$ , write  $c(Z)$  for the constant simplicial object determined by  $Z$ .

The suspension in a model category  $\mathcal{C}$  will denote the usual pushout of the inclusions in two cones (i.e. factorizations of the final map as a cofibration followed by an acyclic fibration), following Quillen [Q1, I, §2]. This operation will be indicated by  $\Sigma_{\mathcal{C}}$  henceforth.

**0.6. Definition.** The category of simplicial objects  $X_0, \dots, X_n$  truncated at the  $n$ -th dimension will be denoted by  $s_n\mathcal{C}$ . If  $\mathcal{C}$  has enough colimits, the obvious truncation functor  $\text{tr}_n : s\mathcal{C} \rightarrow s_n\mathcal{C}$  has a left adjoint  $\rho_n : s_n\mathcal{C} \rightarrow s\mathcal{C}$ , and the composite  $\text{sk}_n := \rho_n \circ \text{tr}_n : s\mathcal{C} \rightarrow s\mathcal{C}$  is called the  *$n$ -skeleton* functor. Thus  $\text{sk}_n X_{\bullet}$  is “freely generated” as a simplicial object by  $X_0, \dots, X_n$ .

**0.7. Definition.** Let  $\Delta[n]$  denote the standard  $n$ -simplex in  $\mathcal{S}$ , generated by  $\sigma_n \in \Delta[n]_n$ , with boundary  $\partial\Delta[n]$  (the sub-object generated by  $d_i\sigma_n$  for  $0 \leq i \leq n$ ). Similarly, the  $k$ th-horn  $\Lambda^k[n]$  is the sub-object generated by  $d_i\sigma_n$  for  $i \neq k$ . The simplicial  $n$ -sphere is  $\mathbf{S}^n := \Delta[n]/\partial\Delta[n]$ .

If  $\mathcal{C}$  has enough colimits, for  $M \in \mathcal{S}_*$  and  $X \in \mathcal{C}$ , we define  $X \hat{\otimes} M \in s\mathcal{C}$  by  $(X \hat{\otimes} M)_n := \coprod_{m \in M_n} X$ , with face and degeneracy maps induced from those of  $M$ . For  $Y \in s\mathcal{C}$ , define  $Y \otimes M \in s\mathcal{C}$  by  $(Y \otimes M)_n := \coprod_{m \in M_n} Y_m$ . The simplicial suspension functor  $- \otimes \mathbf{S}^n$  (on  $s\mathcal{C}$ ) is defined by  $Y \otimes \mathbf{S}^n := Y \otimes (\Delta[n]/\partial\Delta[n])$ .

The main result of this paper is an obstruction theory for dealing with the general realization question, expressed in the following:

**0.8. Theorem.** [Theorems 5.6 & 5.7 below] A  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$  can be realized in  $\mathcal{C}$  if and only if an inductively-defined sequence of cohomology classes in  $H_{\Lambda}^{n+3}(\Lambda; \Omega^{n+1}\Lambda)$  all vanish. The different realizations (if any) are classified (up to homotopy) by elements of  $H_{\Lambda}^{n+2}(\Lambda; \Omega^{n+1}\Lambda)$ .

**0.9. Higher homotopy operations.**

Higher order homotopy operations appear as obstructions to rectifying homotopy commutative diagrams, so, as one might expect, they tie in with our approach (in more than one way). One of the original motivations for this paper was to try to understand the intriguing relationship between the diagram realization question, framed in the algebraic language of  $\Pi$ -algebras and cohomology, and the motivating topological problem of rectifying homotopy commutative diagrams. A general answer is still beyond us (but see Remark 0.16 below). We shall, however, show how this connection appears in a specific example, which we will be using as a leitmotif to illustrate various constructions throughout this paper.

**0.10. Definition.** Given a homotopy commutative diagram:

$$(0.11) \quad \begin{array}{ccccccc} & & * & & & & \\ & \curvearrowright & & \curvearrowleft & & & \\ \mathbf{W} & \xrightarrow{f} & \mathbf{X} & \xrightarrow{g} & \mathbf{Y} & \xrightarrow{h} & \mathbf{Z} \\ & & & & & & \\ & & & & & & * \end{array}$$

the *Toda bracket*  $\langle f, g, h \rangle \subseteq [\Sigma\mathbf{W}, \mathbf{Z}]$  is the set of all homotopy classes which are pushout maps  $k$  in the following diagram:

$$(0.12) \quad \begin{array}{ccccc} \mathbf{W} & \xrightarrow{i_1} & C\mathbf{W} & & \\ \downarrow i_2 & & \downarrow & \searrow G \circ Cf & \\ C\mathbf{W} & \xrightarrow{\text{PO}} & \Sigma\mathbf{W} & & \\ & \searrow h \circ F & \dashrightarrow k & \searrow & \\ & & & & \mathbf{Z} \end{array}$$

where  $G : h \circ g \sim *$  and  $F : g \circ f \sim *$  are any nullhomotopies.

Note that  $\langle f, g, h \rangle$  is the obstruction to rectifying the homotopy commutative diagram (0.11), in the sense that it vanishes (i.e., contains the null class) if and only if (0.11) can be rectified (that is, realized by a strictly commutative diagram, with the null maps represented by actual zero maps).

**0.13. Example.** Recall that in the stable range:

$$(0.14) \quad \pi_i \mathbf{S}^k \cong \begin{cases} \mathbb{Z}\langle \iota \rangle & \text{for } i = k \\ (\mathbb{Z}/2)\langle \eta \rangle & \text{for } i = k + 1 \\ (\mathbb{Z}/4)\langle \eta^2 \rangle & \text{for } i = k + 2 \\ (\mathbb{Z}/24)\langle \nu \rangle & \text{for } i = k + 3 \\ 0 & \text{for } i = k + 4, k + 5 \end{cases}$$

where  $\eta^3 = 12\nu$  (cf. [T, 14.1]). Thus, for  $k \geq 3$ , the sequence:

$$\mathbf{S}^{k+2} \xrightarrow{\eta} \mathbf{S}^{k+1} \xrightarrow{2} \mathbf{S}^{k+1} \xrightarrow{\eta} \mathbf{S}^k$$

is an instance of (0.11), with the corresponding Toda bracket:

$$(0.15) \quad \langle \eta, 2, \eta \rangle = \{ \nu, \nu + \eta^3 \} = \{ \pm \nu \} \subseteq \pi_{k+3} \mathbf{S}^k$$

(See [T, (5.4)]).

0.16. *Remark.* Given a homotopy-commutative diagram  $F : \mathbb{D} \rightarrow \text{ho } \mathcal{T}_*$  of topological spaces (for most reasonable indexing diagrams  $\mathbb{D}$ ), a suitable higher homotopy operation appears as the obstruction to rectifying  $F$  (that is, lifting it to  $\mathcal{T}_*$ ). However, in many applications all spaces in the diagram (except perhaps  $F(*)$ , where  $*$  is terminal in  $\mathbb{D}$ ) are (wedges of) spheres – as in Example 0.13.

In this case we can replace  $F$  by the corresponding diagram of  $\Pi$ -algebras  $\pi_* \circ F : \mathbb{D} \rightarrow \Pi\text{-Alg}$  with no loss of generality (beyond the choice of realization for  $\pi_* F(*)$ ), and any obstruction to realizing  $\pi_* \circ F$  is in particular an obstruction to rectifying  $F$ . Thus Theorem 0.8 provides a way to describe many higher homotopy operations algebraically, in terms of suitable cohomology classes. We hope to pursue this point further in a future paper.

### 0.17. Organization.

In Section 1 we define our objects of study,  $\Pi_{\mathcal{A}}$ -algebras and some related algebraic concepts. Section 2 begins a detailed analysis of resolution model category structures on  $s\mathcal{C}$ , and their basic properties, giving several important examples. Section 3 defines  $E^2$ -model categories, which are a special kind of resolution model category provided with additional structures, such as Eilenberg-Mac Lane objects and Postnikov towers. The motivating examples of diagram categories of spaces, as well as the main algebraic categories, are all instances of this. In fact, we show that any diagram category on an  $E^2$ -model category is another, which provides a broad class of examples.

In Section 4, we define the cohomology theory associated to an  $E^2$ -model category structure and describe some of its basic properties. We illustrate this for the simplest example of a diagram category, namely an arrow category, and show how the cohomology of an arrow relates to that of the source and target objects.

The technical heart of the paper is the obstruction theory for dealing with the general realization question, which appears in Section 5. As expected, we induct up the construction of the Postnikov tower of our (putative) simplicial object expected to yield a realization of  $\Lambda$ . Section 6 provides a more explicit description of the single map case, illustrating it with a detailed example.

## 1. $\Pi_{\mathcal{A}}$ -ALGEBRAS

The functor  $X \mapsto \pi_* X$  is corepresented by spheres in the homotopy category of spaces. If we want to include the group structures, Whitehead products, and  $\pi_1$ -actions as well, we expand the domain category (choices of the argument  $?$  for  $[?, X]$ ) to finite wedges of spheres, and require that wedges be sent to products. This definition extends to other model categories, using the relevant properties of spheres:

**1.1. Definition.** Let  $\mathcal{C}$  be a cofibrantly generated pointed model category which is *right proper* – that is, the pullback of a weak equivalence along a fibration is a weak equivalence. A collection of *models* for  $\mathcal{C}$  is a set  $\mathcal{A}$  of cofibrant homotopy cogroup objects in  $\mathcal{C}$ , closed under suspension in  $\mathcal{C}$  (denoted by  $\Sigma_{\mathcal{C}}$ ).

**1.2. Definition.** Given a model category  $\mathcal{C}$  as above and a set  $\mathcal{A}$  of models for  $\mathcal{C}$ , let  $\Pi_{\mathcal{A}}$  denote the full subcategory of  $\text{ho } \mathcal{C}$  consisting of fibrant and cofibrant objects weakly

equivalent to finite coproducts of objects from  $\mathcal{A}$  (which become products in  $\Pi_{\mathcal{A}}^{\text{op}}$ ). A  $\Pi_{\mathcal{A}}$ -algebra is defined to be a product-preserving functor  $\Pi_{\mathcal{A}}^{\text{op}} \rightarrow \text{Set}_*$ , and the category of  $\Pi_{\mathcal{A}}$ -algebras (and natural transformations) will be denoted by  $\Pi_{\mathcal{A}}\text{-Alg}$ .

Since the suspension operator in  $\mathcal{C}$  preserves the class of cofibrant homotopy cogroup objects, in many of our examples  $\mathcal{A}$  is generated under  $\Sigma_{\mathcal{C}}$  by a much smaller set. For example, the set of spheres used to define ordinary  $\Pi$ -algebras is generated by the circle  $\mathbf{S}^1$ .

**1.3. Example.** The canonical example of a  $\Pi_{\mathcal{A}}$ -algebra is a *realizable*  $\Pi_{\mathcal{A}}$ -algebra – that is, one given by  $[?, X]_{\mathcal{C}}$  for some  $X \in \mathcal{C}$ . This will be referred to as the *homotopy  $\Pi_{\mathcal{A}}$ -algebra* of  $X$ ; it defines a functor  $\pi_{\mathcal{A}} : \text{ho}\mathcal{C} \rightarrow \Pi_{\mathcal{A}}\text{-Alg}$ .

**1.4. Remark.** When  $\mathcal{C} = \mathcal{G}p$  is the category of groups, and  $\mathcal{A} = \{\mathbb{Z}\}$ , the category of  $\Pi_{\mathcal{A}}$ -algebras is equivalent to  $\mathcal{G}p$  itself. In §2.8(f), we allow for a range of universal algebras as examples for  $\mathcal{C}$ . As noted in [Q1, §II.], there is an (unique) object  $D \in \mathcal{C}$  such that, for  $\mathcal{A} = \{D\}$ , the category  $\Pi_{\mathcal{A}}\text{-Alg}$  is equivalent to  $\mathcal{C}$ .

On the other hand, in the resulting resolution model category  $\mathcal{G} = s\mathcal{C}$  with  $\mathcal{A} = \{\mathbb{Z}\}$ , (under the constant embedding of  $\mathcal{C}$  in  $s\mathcal{C}$ ), the category  $\Pi_{\mathcal{A}}$ , consisting of all suspensions of  $\mathbb{Z}$  and coproducts thereof, is just the  $\mathcal{G}$ -version of the collection of all wedges of spheres (in  $T_*$ ), so  $\Pi_{\mathcal{A}}\text{-Alg}$  is the original category of  $\Pi$ -algebras (cf. [St, §2]). See §0.13 and §1.21 for examples of such  $\Pi$ -algebras.

### 1.5. The Free functor.

There is a forgetful functor  $\mathcal{O} : \Pi_{\mathcal{A}}\text{-Alg} \rightarrow \text{gr}_{\mathcal{A}}\text{Set}_*$  to the category of  $\mathcal{A}$ -graded pointed sets, with left adjoint  $F : \text{gr}_{\mathcal{A}}\text{Set}_* \rightarrow \Pi_{\mathcal{A}}\text{-Alg}$ . We call  $F(W)$  the *free  $\Pi_{\mathcal{A}}$ -algebra generated by  $W \in \text{gr}_{\mathcal{A}}\text{Set}_*$* . Thus  $\Pi_{\mathcal{A}}\text{-Alg}$  is an FP-sketchable variety of universal algebras as in §2.8(f), sketched by the  $\mathfrak{G}$ -theory  $\Theta := \Pi_{\mathcal{A}}$ . In particular,  $\Pi_{\mathcal{A}}\text{-Alg}$  is complete and cocomplete (see [AR, §1]).

### 1.6. Products and coproducts.

We now describe a variety of constructions which will be used at various points later. Given two  $\Pi_{\mathcal{A}}$ -algebras  $\Lambda$  and  $\Gamma$  over a fixed  $\Pi_{\mathcal{A}}$ -algebra  $B$ , we define their *fibered product*  $\Lambda \times_B \Gamma$  in  $\Pi_{\mathcal{A}}\text{-Alg}/B$  by declaring its value on an object  $U \in \Pi_{\mathcal{A}}$  to be the set-theoretic pullback:

$$(1.7) \quad \begin{array}{ccc} (\Lambda \times_B \Gamma)(U) & \longrightarrow & \prod_{\beta}(\Lambda(U) \times_{\prod_{\gamma} B(U_{\gamma})} \Gamma(U_{\beta})) \\ \downarrow & & \downarrow \sim \\ \prod_{\alpha}(\Lambda(U_{\alpha}) \times_{\prod_{\gamma} B(U_{\gamma})} \Gamma(U)) & \xrightarrow{\sim} & \prod_{\alpha} \prod_{\beta}(\Lambda(U_{\alpha}) \times_{\prod_{\gamma} B(U_{\gamma})} \Gamma(U_{\beta})) \end{array}$$

whenever  $U = \coprod_{\alpha} U_{\alpha}$  for  $U_{\alpha} \in \Pi_{\mathcal{A}}$ .

Similarly, the *coproduct*  $\Lambda_0 \amalg \Lambda_1$  of two  $\Pi_{\mathcal{A}}$ -algebras  $\Lambda_0$  and  $\Lambda_1$  may be characterized explicitly by first setting  $\Lambda_0 \amalg \Lambda_1 := F(W_0 \vee W_1)$ , if  $\Lambda_0 = F(W_0)$  and  $\Lambda_1 = F(W_1)$  are free; and, more generally, as the natural group quotient

$$(F\mathcal{O}\Lambda_0 \amalg F\mathcal{O}\Lambda_1)/I$$

where  $I$  is the smallest ideal containing the kernels  $K_i$  of  $F\mathcal{O}\Lambda_i \rightarrow \Lambda_i$  for  $i = 0, 1$ . Note there is also a coequalizer in  $\Pi_{\mathcal{A}}\text{-Alg}$ :

$$(F\mathcal{O})^2\Lambda_0 \amalg (F\mathcal{O})^2\Lambda_1 \rightrightarrows (F\mathcal{O})\Lambda_0 \amalg (F\mathcal{O})\Lambda_1 \rightarrow \Lambda_0 \amalg \Lambda_1$$

induced by the two adjunction maps  $F\mathcal{O} \rightarrow \text{Id}$  and  $\text{Id} \rightarrow \mathcal{O}F$ .

**1.8. Definition.** An *ideal* in a  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$  is a sub- $\Pi_{\mathcal{A}}$ -algebra  $I \subset \Lambda$ , such that for any  $U \in \Pi_{\mathcal{A}}$ , the top arrow in the commuting diagram:

$$(1.9) \quad \begin{array}{ccc} \Lambda(U) \times I(U) & \longrightarrow & I(U) \\ \downarrow & & \downarrow \\ \Lambda(U) \times \Lambda(U) & \longrightarrow & \Lambda(U) \end{array}$$

exists. (Uniqueness follows from injectivity of  $I(U) \rightarrow \Lambda(U)$ ). For example, the kernel  $\text{Ker}(f) := * \times_{\Gamma} \Lambda$  of a map of  $\Pi_{\mathcal{A}}$ -algebras  $f : \Lambda \rightarrow \Gamma$  is an ideal.

**1.10. Definition.** For a fixed  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$ , a  $\Lambda$ - $\Pi_{\mathcal{A}}$ -algebra is a map of  $\Pi_{\mathcal{A}}$ -algebras  $i : \Lambda \rightarrow \Gamma$ . In particular, given  $W \in \text{gr}_{\mathcal{A}} \text{Set}_*$ , the *free*  $\Lambda$ - $\Pi_{\mathcal{A}}$ -algebra on  $W$  is defined by  $F_{\Lambda}(W) := F(W) \amalg \Lambda$ . Similarly, we can define the  $\Lambda$ -coproduct  $\Gamma_1 \amalg_{\Lambda} \Gamma_2$  of two  $\Lambda$ - $\Pi_{\mathcal{A}}$ -algebras  $\Gamma_1$  and  $\Gamma_2$  as a coequalizer in  $\Pi_{\mathcal{A}}\text{-Alg}$ :

$$\Lambda \rightrightarrows \Gamma_1 \amalg \Gamma_2 \rightarrow \Gamma_1 \amalg_{\Lambda} \Gamma_2$$

where the left pair of maps is defined using the maps to left/right factors  $\Lambda \rightrightarrows \Lambda \amalg \Lambda$  together with the coproduct of the  $\Lambda$ -algebra structure maps for  $\Gamma_i$ ,  $i = 1, 2$ .

Given an ideal  $I \subseteq \Lambda$ , the *quotient*  $\Pi_{\mathcal{A}}$ -algebra of  $\Lambda$  by  $I$  is then defined:  $\Lambda/I := * \amalg_I \Lambda$ .

**1.11. Definition.** If  $\Lambda$  is a  $\Pi_{\mathcal{A}}$ -algebra, define the *loop*  $\Pi_{\mathcal{A}}$ -algebra  $\Omega\Lambda$  by  $\Omega\Lambda(U) := \Lambda(\Sigma_{\mathcal{C}}U)$ , where  $\Sigma_{\mathcal{C}}U$  is the suspension of  $U$  in  $\mathcal{C}$ .

### 1.12. Abelian $\Pi_{\mathcal{A}}$ -algebras.

An abelian group object  $M$  in  $\Pi_{\mathcal{A}}\text{-Alg}$  is called an *abelian  $\Pi_{\mathcal{A}}$ -algebra* – i.e., if  $\text{Hom}_{\Pi_{\mathcal{A}}\text{-Alg}}(B, M)$  has a natural abelian group structure for any  $B$ . Note that the structure is induced by the underlying  $\mathcal{A}$ -graded group structure in  $\Pi_{\mathcal{A}}\text{-Alg}$ , so in particular  $\mathcal{O}M$  is an  $\mathcal{A}$ -graded *abelian group*.

Denote by  $\text{Ab}(\Pi_{\mathcal{A}}\text{-Alg})$  the subcategory of abelian  $\Pi_{\mathcal{A}}$ -algebras. The inclusion functor  $\text{Ab}(\Pi_{\mathcal{A}}\text{-Alg}) \rightarrow \Pi_{\mathcal{A}}\text{-Alg}$  has a left adjoint  $\text{Ab}$ , called the *abelianization functor*, defined for  $\Lambda = F(W)$  by:

$$(\text{Ab}(F(W)))(A) := \oplus_{W_{\mathcal{A}}} \text{Ab}(\pi_{\mathcal{A}}(A)) .$$

For general  $\Lambda$ , define  $\text{Ab}(\Lambda)$  to be the coequalizer in  $\Pi_{\mathcal{A}}\text{-Alg}$ :

$$\text{Ab}((F\mathcal{O})^2\Lambda) \rightrightarrows \text{Ab}((F\mathcal{O})\Lambda) \rightarrow \text{Ab}(\Lambda).$$

Note that the composite  $\text{Ab} \circ F : \text{gr}_{\mathcal{A}} \text{Set}_* \rightarrow \text{Ab}(\Pi_{\mathcal{A}}\text{-Alg})$  is left adjoint to the forgetful functor, so it is the *free abelian  $\Pi_{\mathcal{A}}$ -algebra* functor. From the adjointness we get a natural abelianization map  $\rho : \Lambda \rightarrow \text{Ab}(\Lambda)$  and we define the ideal  $W(\Lambda) \subseteq \Lambda$  as  $\text{Ker}(\rho)$ .

Then  $W(\Lambda)$  may be viewed as the *ideal of primary operations acting on elements of  $\Lambda$* , and we have:  $\Lambda/W(\Lambda) \cong \text{Ab}(\Lambda)$ .

### 1.13. Modules.

For a fixed  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$ , a *module over  $\Lambda$*  is an abelian group object  $p : M \rightarrow \Lambda$  in the over-category  $\Pi_{\mathcal{A}}\text{-Alg}/\Lambda$ . This means that it is endowed with maps

$$m : M \times_{\Lambda} M \rightarrow M \quad \text{and} \quad i : M \rightarrow M$$

in  $\Pi_{\mathcal{A}}\text{-Alg}/\Lambda$ , as well as a section  $s : \Lambda \rightarrow M$  for  $p$  (which represents the unit element in the abelian group  $\text{Hom}_{\Lambda}(\Lambda, M)$ ). The category of modules over  $\Lambda$  is denoted by  $\Lambda\text{-Mod}$ .

Moreover, given a map of  $\Pi_{\mathcal{A}}$ -algebras  $\Lambda \rightarrow \Gamma$ , the associated restriction functor  $\Gamma\text{-Mod} \rightarrow \Lambda\text{-Mod}$  has a left adjoint, which we denote by  $(-)*_{\Lambda} \Gamma$ .

Note that  $K := \text{Ker}(p)$  is itself an abelian  $\Pi_{\mathcal{A}}$ -algebra, as we can see by mapping  $0 : X \rightarrow \Lambda$  to  $p : M \rightarrow \Lambda$  in  $\Pi_{\mathcal{A}}\text{-Alg}/\Lambda$  for any  $\Pi_{\mathcal{A}}$ -algebra  $X$ , so we have a split exact sequence of  $\Pi_{\mathcal{A}}$ -algebras

$$(1.14) \quad 0 \longrightarrow K \longrightarrow M \overset{\curvearrowright}{\longrightarrow} \Lambda \longrightarrow 0 ,$$

and in particular  $\mathcal{O}M = \mathcal{O}\Lambda \times \mathcal{O}K$  is a semi-direct product of groups.

However,  $K$  is not just an abelian  $\Pi_{\mathcal{A}}$ -algebra; it also has an action of  $\Lambda$  on it, determined by an *action map*

$$\phi_f : \Lambda(U) \times K(U) \rightarrow K(V)$$

for each  $f : V \rightarrow U$  in  $\Pi_{\mathcal{A}}$ , subject to the requirements that:

- (1) The composite  $K(U) \rightarrow \Lambda(U) \times K(U) \xrightarrow{\phi_f} K(V)$  is equal to  $K(f)$ ;
- (2) For  $g : W \rightarrow V$  in  $\Pi_{\mathcal{A}}$ , the action map  $\phi_{f \circ g}$  equals the composite

$$\Lambda(U) \times K(U) \xrightarrow{\Delta \times \text{Id}} \Lambda(U) \times (\Lambda(U) \times K(U)) \xrightarrow{\Lambda(f) \times \phi_f} \Lambda(V) \times K(V) \xrightarrow{\phi_g} K(W).$$

We sometimes say that  $K$  itself, endowed with this action of  $\Lambda$ , is a  $\Lambda$ -module (which corresponds to the traditional description of an  $R$ -module, for a ring  $R$ ), and write  $M = \Lambda \times K$ .

Note that  $\text{Ab} \circ F_{\Lambda} \cong \Lambda \times (\text{Ab} \circ F)$ , so  $\text{Ab} \circ F_{\Lambda} : \text{gr}_{\mathcal{A}} \text{Set}_* \rightarrow \Lambda\text{-Mod}$  can be viewed as the free  $\Lambda$ -module functor.

1.15. *Remark.* When  $\Pi_{\mathcal{A}}\text{-Alg} = \Pi\text{-Alg}$ , a  $\Lambda$ -module  $K$  is simply an abelian  $\Pi$ -algebra, equipped with mappings  $\langle\langle \cdot, \cdot \rangle\rangle : \Lambda_p \times K_q \rightarrow K_{p+q}$ , commuting with compositions, such that for each  $q > 0$ ,  $\alpha \circ x := \langle\langle \alpha, x \rangle\rangle - x$  defines an action of  $\Lambda_0$  on  $K_q$ , satisfying  $\langle\langle b, a \rangle\rangle \circ (a \circ x) = -\langle\langle a, b \rangle\rangle \circ x - \langle\langle a, x \rangle\rangle$ , while for  $p > 0$ ,  $\langle\langle \cdot, \cdot \rangle\rangle : \Lambda_p \times K_q \rightarrow K_{p+q}$  is bilinear, and satisfies:

$$\langle\langle \alpha, \langle\langle \beta, x \rangle\rangle \rangle\rangle = \langle\langle \langle \alpha, \beta \rangle, x \rangle\rangle + (-1)^{pq} \langle\langle \beta, \langle\langle \alpha, x \rangle\rangle \rangle\rangle .$$

1.16. **Example.** For a  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$ , define the  $\Pi_{\mathcal{A}}$ -algebra  $\Omega_+ \Lambda$  by  $\Omega_+ \Lambda(A) := \Lambda((\Sigma_{\mathcal{C}} A) \vee A)$ . There is then a split exact sequence:

$$(1.17) \quad * \longrightarrow \Omega \Lambda \longrightarrow \Omega_+ \Lambda \overset{\curvearrowright}{\longrightarrow} \Lambda \longrightarrow * ,$$

which gives  $\Omega_+ \Lambda$  the structure of a module over  $\Lambda$ .

1.18. **Example.** The *fold map*  $\nabla : \Lambda \amalg \Lambda \rightarrow \Lambda$  possesses two sections. Let  $K := \text{Ker}(\nabla)$ . Define the *Kähler differentials* of  $\Lambda$  by  $\Omega_{\Lambda} := \text{Ab}(K)$ . Then the split exact sequence:

$$(1.19) \quad * \longrightarrow \Omega_{\Lambda} \longrightarrow \Omega_{\Lambda} \amalg_K (\Lambda \amalg \Lambda) \overset{\curvearrowright}{\longrightarrow} \Lambda \longrightarrow *$$

gives  $\Omega_{\Lambda}$  the structure of a  $\Lambda$ -module.

We will see in §4.6 that the Kähler differentials are closely related to our cohomology theories.

Our key examples of modules come in Proposition 2.14, where we will see that for  $n > 0$ , the natural homotopy groups  $\pi_n^{\natural} Y_{\bullet}$  (see §1.3) and their loop algebras are modules over  $\pi_0^{\natural} Y_{\bullet}$ .

1.20. *Remark.* We have in view two types of categories for  $\mathcal{C}$  here: one type are “algebraic” categories, such as  $\mathcal{G}p$  and  $\Pi_{\mathcal{A}}\text{-Alg}$ , in which the model category structures are trivial (in the sense that the only weak equivalences are isomorphisms), so the associated realization question is also trivial.

The other type is “topological” – for example,  $\mathcal{G}$  or  $T_*$ . Here the associated algebraic invariants, such as homotopy groups, give rise to meaningful realization questions; and the associated simplicial categories possess nontrivial resolution model category structures, suited to addressing such questions.

However, as we shall see, in trying to construct a “topological” object realizing a given “algebraic” invariant, we will need to apply the constructions provided in this paper to objects in both types of category, which is why we set up our machinery in a form suitable for both contexts.

1.21. **A space and its  $\Pi$ -algebra.** We now give an example of a  $\Pi$ -algebra which will be used later to illustrate the general theory.

For  $k \geq n$ , let  $\Pi\text{-Alg}_n^k$  denote the category of  $k$ -truncated and  $(n-1)$ -connected  $\Pi$ -algebras  $\Lambda$ , with  $\Lambda_i = 0$  for  $i < n$  or  $i > k$ . Note that in the stable range – that is, if  $k < 2n$  – this is an abelian category. By restricting attention to  $(n-1)$ -connected spaces, and truncating higher homotopy groups, we may (and shall) assume that  $\text{tr}_k \pi_* \mathbf{X}$  takes values in  $\Pi\text{-Alg}_n^k$ . More formally, we may work in the context of §2.18(c)-(d) below.

From now on, we take  $n \geq 4$  with  $k := n + 2$ , and let  $\mathcal{S}^r := \pi_* \mathbf{S}^r$  and  $\mathcal{S}^r \langle x \rangle := \text{tr}_{n+2} \mathcal{S}^r$  denote the free monogenic algebra (in  $\Pi\text{-Alg}$  or  $\Pi\text{-Alg}_n^{n+2}$ ) on a generator  $x$  in degree  $r$ .

For  $n \geq 4$ , let  $\mathbf{X} := \mathbf{S}^n \cup_2 \mathbf{e}^{n+1} = \Sigma^{n-1} \mathbb{R}P^2$ . Then:

$$\pi_i \mathbf{X} \cong \begin{cases} (\mathbb{Z}/2)\langle \alpha \rangle & \text{for } i = n \\ (\mathbb{Z}/2)\langle \alpha \circ \eta \rangle & \text{for } i = n + 1 \\ (\mathbb{Z}/4)\langle \beta \rangle & \text{for } i = n + 2 \\ (\mathbb{Z}/2)\langle \alpha \circ \nu \rangle \oplus (\mathbb{Z}/2)\langle \beta \circ \eta \rangle & \text{for } i = n + 3 \end{cases}$$

with  $2\beta = \alpha \circ \eta^2$ . Note that the inclusion  $\varphi : \text{tr}_{n+2} \pi_* \mathbf{X} \rightarrow \mathcal{S}^{n-1}$ , defined by  $\varphi(\alpha) = \eta$  (and  $\varphi(\beta) = 6\nu$ , necessarily), is a morphism of  $(n+2)$ -truncated  $\Pi$ -algebras (in fact, even of  $(n+3)$ -truncated  $\Pi$ -algebras, if  $n \geq 5$ ).

1.22. *Remark.* There is one other non-trivial map of (truncated)  $\Pi$ -algebras  $\psi : \pi_* \mathbf{X} \rightarrow \mathcal{S}^{n-1}$ , defined by  $\psi(\alpha) = 0$  and  $\psi(\beta) = \eta^3 = 12\nu$ . This is induced by a map of spaces – namely, the composite of the pinch map  $p : \mathbf{X} = \mathbf{S}^n \cup_2 \mathbf{e}^{n+1} \rightarrow \mathbf{S}^{n+1}$  with  $\eta^2 : \mathbf{S}^{n+1} \rightarrow \mathbf{S}^{n-1}$ .

## 2. RESOLUTION MODEL CATEGORIES

In order to study the realization questions mentioned in the Introduction, we need a suitable *resolution model category* structure on the associated simplicial model category  $s\mathcal{C}$ , originally defined by Dwyer, Kan and Stover in [DKS1], and later extended by Bousfield in [Bo]. A variant, called a *spiral* model category, is defined by Baues in [Ba, Ch. D, §2]. We begin with some definitions:

2.1. **Definition.** Let  $(-) \otimes (-) : s\mathcal{C} \times s\text{Set}_* \rightarrow s\mathcal{C}$  be the action of simplicial sets on the simplicial category  $s\mathcal{C}$  (see 0.7 or [Q1, II, §1]).

For any finite simplicial set  $K$ , the *matching functor*  $M_K : s\mathcal{C} \rightarrow \mathcal{C}$  is characterized as a right adjoint by the relation:

$$\mathrm{Hom}_{s\mathcal{C}}(c(Z)_\bullet \otimes K, X_\bullet) \cong \mathrm{Hom}_{\mathcal{C}}(Z, M_K X_\bullet).$$

In particular,  $M_n X_\bullet := M_{\partial\Delta[n]} X_\bullet := \lim_{[n] \rightarrow [k]} X_k$ . Dually, the *latching functor*  $L_n : s\mathcal{C} \rightarrow \mathcal{C}$  is defined by:

$$L_n X_\bullet := \mathrm{colim}_{[k] \rightarrow [n]} X_k.$$

Similarly, we may characterize  $C_K : s\mathcal{C} \rightarrow \mathcal{C}$  by means of a right adjunction:

$$\mathrm{Hom}_{s\mathcal{C}}(c(Z)_\bullet \wedge K, X_\bullet) \cong \mathrm{Hom}_{\mathcal{C}}(Z, C_K X_\bullet),$$

where  $Y_\bullet \wedge K$  is the pushout in  $s\mathcal{C}$ :

$$(2.2) \quad \begin{array}{ccc} Y_\bullet \otimes * & \longrightarrow & * \\ \downarrow & & \downarrow \\ (Y_\bullet \otimes *) \otimes K & \longrightarrow & Y_\bullet \wedge K \end{array}.$$

In particular,  $C_n X_\bullet := C_M X_\bullet$  for  $M := \Delta[n]/\Lambda^0[n]$  and  $Z_n X_\bullet := C_{\mathbf{S}^n} X_\bullet$  (see §0.7).

**2.3. Remark.** There is a natural sequence:

$$Z_{n+1} X_\bullet \xrightarrow{i_{n+1}} C_{n+1} X_\bullet \xrightarrow{d_0} Z_n X_\bullet \xrightarrow{i_n} C_n X_\bullet,$$

where the composite  $i_n d_0$  is induced by the map  $\delta_0 : \Delta[n]/\Lambda^0[n] \rightarrow \Delta[n+1]/\Lambda^0[n+1]$ .

Recall that we assume  $\mathcal{C}$  to be a right proper cofibrantly generated pointed model category, and  $\mathcal{A}$  a set of models (i.e., cofibrant homotopy cogroup objects) in  $\mathcal{C}$ .

**2.4. Definition.** A map  $p : X \rightarrow Y$  in  $\mathrm{ho}\mathcal{C}$  is called  $\mathcal{A}$ -*epic* if  $p_* : [A, X]_{\mathcal{C}} \rightarrow [A, Y]_{\mathcal{C}}$  is surjective for each  $A \in \mathcal{A}$ . An object  $W \in \mathrm{ho}\mathcal{C}$  is called  $\mathcal{A}$ -*projective* if  $p_* : [W, X]_{\mathcal{C}} \rightarrow [W, Y]_{\mathcal{C}}$  is surjective for each  $\mathcal{A}$ -epic map  $p : X \rightarrow Y$  in  $\mathrm{ho}\mathcal{C}$ . Finally, an object (respectively, map) of  $\mathcal{C}$  is called  $\mathcal{A}$ -*projective* (resp.,  $\mathcal{A}$ -*epic*) if it is so in  $\mathrm{ho}\mathcal{C}$ .

**2.5. Definition.** (a) a map  $f : X_\bullet \rightarrow Y_\bullet$  in  $s\mathcal{C}$  is a *Reedy fibration* if the induced map  $X_n \rightarrow Y_n \times_{M_n Y_\bullet} M_n X_\bullet$  is a fibration in  $\mathcal{C}$  for all  $n \geq 0$ ;

(b) a map  $g$  in  $\mathcal{C}$  is an  $\mathcal{A}$ -*projective cofibration* if  $g$  is a cofibration in  $\mathcal{C}$ , and has the left lifting property with respect to the class of fibrations in  $\mathcal{C}$  which are, in addition,  $\mathcal{A}$ -epic.

**2.6. The resolution model category.** Given  $\mathcal{C}$  and  $\mathcal{A}$  as above, a map  $f : X_\bullet \rightarrow Y_\bullet$  in  $s\mathcal{C}$  is:

- (a) an  $\mathcal{A}$ -*weak equivalence* if  $f_* : [A, X_\bullet]_{\mathcal{C}} \rightarrow [A, Y_\bullet]_{\mathcal{C}}$  is a weak equivalence of simplicial groups for all  $A \in \mathcal{A}$ ;
- (b) an  $\mathcal{A}$ -*fibration* if  $f$  is a Reedy fibration and  $f_* : [A, X_\bullet]_{\mathcal{C}} \rightarrow [A, Y_\bullet]_{\mathcal{C}}$  is a fibration of simplicial groups for all  $A \in \mathcal{A}$ ;
- (c) an  $\mathcal{A}$ -*cofibration* if the induced map  $X_n \amalg_{L_n X_\bullet} L_n Y_\bullet \rightarrow Y_n$  (§2.1) is an  $\mathcal{A}$ -projective cofibration in  $\mathcal{C}$  for all  $n \geq 0$ .

**2.7. Theorem.** *If  $\mathcal{C}$  is a pointed right proper simplicial model category with a set of models  $\mathcal{A}$ , then  $s\mathcal{C}$ , with the  $\mathcal{A}$ -weak equivalences,  $\mathcal{A}$ -fibrations, and  $\mathcal{A}$ -cofibrations, and the external simplicial category structure (§0.7 and [Q1, II, §1]), is a right proper simplicial model category, called the  $\mathcal{A}$ -resolution model category, and denoted by  $s\mathcal{C}_{\mathcal{A}}$ .*

*Proof.* See [J, Theorem 2.2].  $\square$

**2.8. Example.** If  $\mathcal{C} = \mathcal{T}_*$  and  $\mathcal{A} := \{\mathbf{S}^n\}_{n=1}^\infty$ , (generated by  $\mathbf{S}^1$ ), the resulting  $\mathcal{A}$ -resolution model category structure on the category  $s\mathcal{T}_*$  of pointed simplicial spaces is the original “ $E^2$ -model category” of [DKS1].

In constructing cofibrant replacements for objects in an  $\mathcal{A}$ -resolution model category, we shall have occasion to use the following:

**2.9. Definition.** A *CW complex* is an object  $X_\bullet \in s\mathcal{C}_\mathcal{A}$  such that

- For each  $n \geq 0$ ,  $X_n \cong \bar{X}_n \amalg L_n X_\bullet$  for some  $\bar{X}_n \in \text{Obj } \Pi_\mathcal{A}$ ;
- $d_i|_{\bar{X}_n} = *$  for all  $i \geq 1$ .

The *attaching map*  $d_0|_{\bar{X}_n} : \bar{X}_n \rightarrow L_{n-1} X_\bullet$  is denoted by  $\bar{d}_0$ . The collection  $\{\bar{X}_n\}_{n=0}^\infty$  is called a *CW basis* for  $X_\bullet$ . It is straightforward to check that a CW complex in  $s\mathcal{C}_\mathcal{A}$  is  $\mathcal{A}$ -cofibrant.

**2.10. Definition.** The  $n$ -th *natural homotopy group* of  $X_\bullet \in s\mathcal{C}$  with coefficients in  $A \in \mathcal{A}$  is defined to be  $\pi_n^\natural(X_\bullet, A) := \pi_0 \text{map}_{s\mathcal{C}}(A \hat{\otimes} \mathbf{S}^n, Y_\bullet)$  (cf. §0.7), where  $X_\bullet \rightarrow Y_\bullet$  is a Reedy fibrant replacement of  $X_\bullet$ . It can be equivalently defined by the exact sequence:

$$[A, C_{n+1} Y_\bullet]_{\mathcal{C}} \xrightarrow{(d_0)^*} [A, Z_n Y_\bullet]_{\mathcal{C}} \rightarrow \pi_n^\natural(X_\bullet, A) \rightarrow 0.$$

(see [M, 17.3]). Denote the  $\mathcal{A}$ -graded group  $(\pi_n^\natural(X_\bullet, A))_{A \in \mathcal{A}}$  by  $\pi_n^\natural(X_\bullet, \mathcal{A})$ .

**2.11. Remark.** Since  $A \in \mathcal{C}$  is a homotopy cogroup object, whenever  $X_\bullet \in s\mathcal{C}$  is Reedy fibrant we may identify  $[A, C_n X_\bullet]_{\mathcal{C}}$  with  $C_n[A, X_\bullet]_{\mathcal{C}}$  (the  $n$ -chains group (§2.1) for the simplicial group  $[A, X_\bullet]_{\mathcal{C}}$ ).

**2.12. Definition.** By applying the functors  $[A, -]_{\mathcal{C}}$  for  $A \in \mathcal{A}$  to a simplicial object  $X_\bullet \in s\mathcal{C}$ , we obtain a simplicial group  $[A, X_\bullet]_{\mathcal{C}}$ , since our models are homotopy cogroup objects by assumption. This leads to another kind of homotopy group for  $X_\bullet$ , namely:  $\pi_n(X_\bullet, A) := \pi_n[A, X_\bullet]_{\mathcal{C}}$ . Write  $\pi_n \pi_\mathcal{A} X_\bullet$  for the  $\mathcal{A}$ -graded group  $(\pi_n(X_\bullet, A))_{A \in \mathcal{A}}$ .

As shown in [DKS2, 8.1], and more generally in [GH1, 3.4], the two types of  $\mathcal{A}$ -graded homotopy groups are related by a *spiral exact sequence*:

$$(2.13) \quad \begin{array}{ccccccc} \dots & \rightarrow & \Omega \pi_n^\natural(X_\bullet, A) & \xrightarrow{s_n} & \pi_n^\natural(X_\bullet, A) & \xrightarrow{h_n} & \pi_n \pi_\mathcal{A} X_\bullet & \xrightarrow{\partial_n} & \Omega \pi_{n-2}^\natural(X_\bullet, A) & \rightarrow \\ & & & & \dots & \rightarrow & \pi_1^\natural(X_\bullet, A) & \rightarrow & \pi_1 \pi_\mathcal{A} X_\bullet & \dots \end{array}$$

where  $\Omega \pi_n^\natural(X_\bullet, A) := \pi_n^\natural(X_\bullet, \Sigma_{\mathcal{C}} A)$ , for  $\Sigma_{\mathcal{C}} A$  the suspension of  $A$  in  $\mathcal{C}$ . In fact:

**2.14. Proposition** (Cf. [BDG, Prop. 7.13]). *For any simplicial object  $X_\bullet \in s\mathcal{C}_\mathcal{A}$ , there are natural actions of  $\pi_0^\natural(X_\bullet, \mathcal{A}) \cong \pi_0 \pi_\mathcal{A} X_\bullet$  on  $\pi_n^\natural(X_\bullet, \mathcal{A})$  and  $\Omega \pi_n^\natural(X_\bullet, \mathcal{A})$ , making the spiral exact sequence (2.13) a long exact sequence of modules over  $\pi_0^\natural(X_\bullet, \mathcal{A})$ .*

*Proof.* Because  $\mathbf{S}^n = \Delta[n]/\partial\Delta[n]$  has two non-degenerate simplices, if we set  $\widehat{A \otimes \mathbf{S}^n} := (A \hat{\otimes} \Delta[n]) / (A \hat{\otimes} \partial\Delta[n])$ , the map of simplicial sets  $\mathbf{S}^n \rightarrow \Delta[0]$  has a section, which induces:

$$\widehat{A \otimes \mathbf{S}^n} \xrightarrow{i} A \hat{\otimes} \mathbf{S}^n \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{p} \end{array} A \hat{\otimes} \Delta[0],$$

and thus a natural splitting:

$$\pi_n^\natural(X_\bullet, A) \begin{array}{c} \xleftarrow{s\#} \\ \xrightarrow{p\#} \end{array} \pi_0^\natural(X_\bullet, A)$$

for each  $X_\bullet \in s\mathcal{C}$  and  $A \in \mathcal{A}$ . Using the usual homotopy cogroup structure on  $\mathbf{S}^n$  (over  $\Delta[0]$ ), we see that  $\pi_n^h(X_\bullet, \mathcal{A})$  is actually a group object over  $\pi_0^h(X_\bullet, \mathcal{A})$ . Furthermore, it is abelian because of the underlying group structure coming from the fact that each  $A \in \mathcal{A}$  is a homotopy cogroup object itself (compare [W, III, Thm. (5.21)]).  $\square$

2.15. *Remark.*  $\text{Ker}(p_\#) \cong [\widehat{A \otimes \mathbf{S}^n}, X_\bullet]$  is actually the more traditional  $n$ -th homotopy group of  $X_\bullet$  (over the base-point component).

2.16. **Algebraic categories.** It will be helpful to include the following “algebraic” examples (cf. §1.20) among our candidates for  $\mathcal{C}$  :

- (a) Let  $\mathcal{C} = \Pi_{\mathcal{A}}\text{-Alg}$  and  $\mathcal{B} = \{\pi_{\mathcal{A}}(A)\}_{A \in \mathcal{A}}$ . Then  $\mathcal{C}$  has the *trivial model category* structure, where only isomorphisms are weak equivalences and all maps are both cofibrations and fibrations (notice this implies the suspension functor  $\Sigma_{\mathcal{C}}$  is the constant functor on  $*$ ). Recall that the objects of the form  $\mathcal{A}(A, ?)$  constitute a strong generating set for  $\text{gr}_{\mathcal{A}}\text{Set}_*$  by the Yoneda lemma, and  $F\mathcal{A}(A, ?) = \pi_{\mathcal{A}}(A)$  for the free functor  $F$  defined in §1.5. Hence, the resolution model category structure on  $s\Pi_{\mathcal{A}}\text{-Alg}$  with this  $\mathcal{B}$  is identical to the usual model category structure on  $s\mathcal{C}$  inherited from the category of simplicial ( $\mathcal{A}$ -graded) groups.
- (b) More generally, let  $\mathcal{C} = \Theta\text{-Alg}$  be any *FP-sketchable* variety of (graded) universal algebras, corepresented by an FP-theory  $\Theta$  (cf. [AR, §1] or [BP, §1]): for example, the categories of  $\Pi_{\mathcal{A}}$ -algebras (corepresented by  $\Theta = \Pi_{\mathcal{A}}^{\text{op}}$ ), Lie algebras, graded commutative algebras, and so on. We assume that  $\Theta$  is a  $\mathfrak{G}$ -theory as in [BP, §2], so that each  $\Theta$ -algebra has an underlying (graded) group structure. In this case we can endow  $\mathcal{C}$  with the trivial model category structure, take  $\mathcal{A}$  to be the set of all monogenic free  $\Theta$ -algebras, and obtain the usual model category structure on  $s\mathcal{C}$  (cf. [Q1, II, §4]).
- (c) As an application of example (b) above, if  $\mathcal{C} = \mathcal{G}p$  and  $\mathcal{A} = \{\mathbb{Z}\}$ , then  $s\mathcal{C}_{\mathcal{A}}$  (where  $s\mathcal{C} = \mathcal{G}$ ) also provides a resolution model category for the homotopy theory of pointed connected topological spaces (cf. [Q1, II, §3]).

2.17. *Remark.* For many purposes it is more convenient to work with  $\mathcal{G}$  than with  $T_*$ . When we do so, we use the simplicial group spheres  $\mathbf{S}^n = F\mathbf{S}^{n-1} \in \mathcal{G}$  for  $n \geq 1$  (and  $\mathbf{S}^0 = G\mathbf{S}^0$ ) as our models  $\mathcal{A}$ . (For definitions of the various loop group constructions on simplicial sets, see, e.g., [GJ, V.6].) Note that  $\mathbb{D}$ -diagrams of simplicial spaces are then replaced by  $\mathbb{D}$ -diagrams of bisimplicial groups, which are just (more complicated) diagrams of groups, so that many constructions may be performed entrywise in  $\mathcal{G}p$ .

2.18. **Topological categories.** It is also useful to include a number of variants of the usual category of pointed topological spaces:

- (a) If  $\mathcal{C} = \mathcal{T}_*$  in the rational model structure and  $\mathcal{A} := \{\mathbf{S}_{\mathbb{Q}}^n\}_{n=2}^{\infty}$  (generated by  $\mathbf{S}_{\mathbb{Q}}^2$ ) or  $\mathcal{C} = \mathcal{T}_*$  in the  $p$ -local model structure and  $\{\mathbf{S}_{(p)}^n\}_{n=2}^{\infty}$ , then we have resolution model structures on  $s\mathcal{T}_*$  for rational or  $p$ -local simply-connected homotopy theory.
- (b) If  $\mathcal{C} = \text{Spec}$  is an appropriate category of spectra (cf. [MMSS]), and  $\mathcal{A} := \{\mathbf{S}^n\}_{n=-\infty}^{\infty}$  are all sphere spectra, we have a resolution model category structure on  $s\text{Spec}$  for simplicial spectra (see [GH1, GH2, GH3] for the details on this and other categories of structured ring spectra).

- (c) Take  $\mathcal{C}$  to be one of the model categories for  $n$ -types, such as the  $n$ -cat groups of [L] or the crossed  $n$ -cubes of [ES] and  $\mathcal{A} := \{\mathbf{S}^k\}_{k=1}^n$ , which gives a resolution model category structure on  $s\mathcal{C}$  for  $n$ -types of spaces. An alternative is to use the (left) Bousfield localization model category structure on pointed spaces (see [H, §§2.1,3.3]) for the map  $*$   $\rightarrow$   $\mathbf{S}^{n+1}$  (see [DF, §1.E.1]).
- (d) Take  $\mathcal{C} = \mathcal{T}_*$  and  $\mathcal{A} = \{\mathbf{S}^n\}_{n=k}^\infty$  (generated by  $\mathbf{S}^k$ ); then we have the resolution model structure on  $s\mathcal{T}_*$  for the homotopy theory of “ $(k-1)$ -connected types” for spaces – that is, the right Bousfield localization model of [H, §3.3] (see [DF, §2.D.2.6]).

**2.19. Diagram categories.** The motivating type of example for this paper was the category  $\mathcal{T}_*^{\mathbb{D}}$  of  $\mathbb{D}$ -diagrams of spaces, where  $\mathbb{D}$  is a small category.

Recall that for any object  $X \in \mathcal{C}$  and  $d \in \text{Obj } \mathbb{D}$ , the free  $\mathbb{D}$ -diagram  $F(X, d)$  is defined by setting the  $e$ -entry equal to  $F(X, d)_e := \prod_{\text{Hom}_{\mathbb{D}}(d, e)} X$ , with maps induced by the identity on each factor. Then for any collection of models  $\mathcal{A}$  for  $\mathcal{C}$ , the *induced collection of models*  $\mathcal{B}$  for  $\mathcal{C}^{\mathbb{D}}$  consists of all free  $\mathbb{D}$ -diagrams  $F(A, d)$  for  $d \in \text{Obj } \mathbb{D}$  and  $A \in \mathcal{A}$ .

Note that the model category structure on  $s\mathcal{T}_*^{\mathbb{D}}$  given by Theorem 2.7 using  $\mathcal{B}$  is identical to the structure induced from that on  $s\mathcal{T}_*$  associated to  $\mathcal{A}$  (and Theorem 2.7) as in [H, §11.6]. Furthermore, the category  $\Pi_{\mathcal{A}}\text{-Alg}$  is equivalent to the category of  $\mathbb{D}$ -diagrams of (ordinary)  $\Pi$ -algebras in these cases.

**2.20. Notation.** For any  $n \in \mathbb{N}$ , let  $[n]$  denote the category with  $n+1$  objects  $0, 1, \dots, n$  and  $n$  composable maps between them. For example,  $\mathbb{D} = [1]$  has two objects and a single non-identity morphism  $0 \rightarrow 1$ .

**2.21. Examples.** (a) If  $\mathcal{C} = \mathcal{T}_*$  and  $\mathbb{D} = [1]$ , then  $\mathcal{T}_*^{\mathbb{D}}$  is the category of *maps of spaces*, and for any space  $X$ , the free object  $F(X, 0) = X \xrightarrow{\text{Id}} X$ , while  $F(X, 1) = * \rightarrow X$ . Hence in this case  $\mathcal{A} := \{*\rightarrow \mathbf{S}^n, \mathbf{S}^n \xrightarrow{\text{Id}} \mathbf{S}^n\}_{n=1}^\infty$  – that is,  $\mathcal{A}$  is generated by the pair consisting of  $* \rightarrow \mathbf{S}^1$  and  $\mathbf{S}^1 \xrightarrow{\text{Id}} \mathbf{S}^1$  – and  $\Pi_{\mathcal{A}}\text{-Alg}$  is the category of morphisms between  $\Pi$ -algebras.

(b) Suppose  $\mathcal{C} = \mathcal{T}_*$  and  $\mathbb{D} = [2]$  (with a single composable pair of nonidentity maps, denoted  $0 \rightarrow 1 \rightarrow 2$ ). Then for any space  $X$ :

$$F(X, 0) = X \xrightarrow{\text{Id}} X \xrightarrow{\text{Id}} X, \quad F(X, 1) = * \rightarrow X \xrightarrow{\text{Id}} X, \quad \text{and} \quad F(X, 2) = * \rightarrow * \rightarrow X.$$

Thus  $\mathcal{A}$  is generated by:

$$* \rightarrow * \rightarrow \mathbf{S}^1, \quad * \rightarrow \mathbf{S}^1 \xrightarrow{\text{Id}} \mathbf{S}^1, \quad \text{and} \quad \mathbf{S}^1 \xrightarrow{\text{Id}} \mathbf{S}^1 \xrightarrow{\text{Id}} \mathbf{S}^1.$$

while  $\Pi_{\mathcal{A}}\text{-Alg}$  is the category of composable pairs of maps between  $\Pi$ -algebras.

### 3. $E^2$ -MODEL CATEGORIES

There are a number of familiar constructions for topological spaces which relate to Postnikov towers and are useful to have in a resolution model category  $s\mathcal{C}_{\mathcal{A}}$ , although they need not exist in general. We shall show, however, that these constructions are available in all of the examples we wish to consider.

**3.1. Definition.** A *Postnikov tower* functor applied to an object  $X_{\bullet}$  in a resolution model category  $s\mathcal{C}_{\mathcal{A}}$  is a functorial commuting diagram:

$$(3.2) \quad \begin{array}{c} X_{\bullet} \\ \begin{array}{c} \curvearrowright \\ \searrow^{r^{(n+1)}} \\ \searrow^{r^{(n)}} \\ \searrow^{r^{(n-1)}} \end{array} \\ \dots \longrightarrow P_{n+1}X_{\bullet} \xrightarrow{p^{(n+1)}} P_nX_{\bullet} \xrightarrow{p^{(n)}} P_{n-1}X_{\bullet} \xrightarrow{p^{(n-1)}} \dots P_0X_{\bullet} \end{array}$$

of  $\mathcal{A}$ -fibrations  $p^{(n)}$  and maps  $r^{(n)}$  which induce isomorphisms:

$$\pi^{\natural}_k(P_nX_{\bullet}; \mathcal{A}) \cong \begin{cases} \pi^{\natural}_k(X_{\bullet}; \mathcal{A}) & 0 \leq k \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

**3.3. Definition.** If  $s\mathcal{C}_{\mathcal{A}}$  is a resolution model category, a *classifying object*  $B\Lambda = B_{s\mathcal{C}}\Lambda$  for a  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$  is any fibrant  $B_{\bullet} \in s\mathcal{C}$  such that  $B_{\bullet} \simeq P_0B_{\bullet}$  and  $\pi^{\natural}_0(B_{\bullet}, \mathcal{A}) \cong \Lambda$ .

**3.4. Definition.** Given an abelian  $\Pi_{\mathcal{A}}$ -algebra  $M$  and an integer  $n \geq 1$ , an  *$n$ -dimensional  $M$ -Eilenberg-Mac Lane object*  $E(M, n) = E_{s\mathcal{C}}(M, n)$  is any fibrant  $E_{\bullet} \in s\mathcal{C}$  such that  $\pi^{\natural}_n(E_{\bullet}, \mathcal{A}) \cong M$  and  $\pi^{\natural}_k(E_{\bullet}, \mathcal{A}) = 0$  for  $k \neq n$ .

**3.5. Definition.** Given a  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$ , a module  $M$  over  $\Lambda$ , and an integer  $n \geq 1$ , an  *$n$ -dimensional extended  $M$ -Eilenberg-Mac Lane object*  $E^{\Lambda}(M, n) = E_{s\mathcal{C}}^{\Lambda}(M, n)$  is any fibrant homotopy abelian group object  $E_{\bullet} \in s\mathcal{C}/B\Lambda$  satisfying:

$$(3.6) \quad \pi^{\natural}_k(E_{\bullet}, \mathcal{A}) \cong \begin{cases} \Lambda & \text{for } k = 0 \\ M \text{ (as a module over } \Lambda) & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases}$$

**3.7. Remark.** The fact that  $E_{\bullet} = E^{\Lambda}(M, n)$  is a homotopy abelian group object in  $s\mathcal{C}/B\Lambda$  implies that  $[B\Lambda, E_{\bullet}]_{s\mathcal{C}/B\Lambda}$  has a natural abelian group structure, so in particular a unit element. Thus  $E_{\bullet}$  comes equipped with a designated homotopy section  $s$  for  $r^{(0)} : E_{\bullet} \rightarrow P_0E_{\bullet} \simeq B\Lambda$ .

From the spiral exact sequence (2.13) we readily calculate:

$$(3.8) \quad \pi_k \pi_{\mathcal{A}} E^{\Lambda}(M, n) \cong \begin{cases} \Lambda & \text{for } k = 0 \\ \Omega\Lambda & \text{for } k = 2 \\ M & \text{for } k = n, \\ \Omega M & \text{for } k = n + 2, \\ 0 & \text{otherwise,} \end{cases}$$

with the obvious variant when  $n = 2$  (i.e.,  $\pi_2 \pi_{\mathcal{A}} E^{\Lambda}(M, 2) \cong \Omega\Lambda \times M$ ).

**3.9. Remark.** Note that if we apply the loop functor in the category  $s\mathcal{C}/B\Lambda$  (cf. [Q1, I, §2]) to  $E^{\Lambda}(M, n)$  – that is, take the pullback of  $B\Lambda \leftarrow E^{\Lambda}(M, n) \rightarrow B\Lambda$  – we obtain  $E^{\Lambda}(M, n - 1)$ .

**3.10. Definition.** Given a Postnikov tower functor as in §3.1, an  *$n$ -th  $k$ -invariant square* (with respect to  $\mathcal{A}$ ) is a functor that assigns to each  $X_{\bullet} \in s\mathcal{C}$  a homotopy pull-back square:

$$(3.11) \quad \begin{array}{ccc} P_{n+1}X_{\bullet} & \xrightarrow{p^{(n+1)}} & P_nX_{\bullet} \\ \downarrow \boxed{\text{hPB}} & & \downarrow k_n \\ B\Lambda & \xrightarrow{s} & E^{\Lambda}(M, n + 2) \end{array}$$

for  $\Lambda := \pi^{\natural}_0(X_{\bullet}, \mathcal{A})$  and  $M := \pi^{\natural}_{n+1}(X_{\bullet}, \mathcal{A})$ . The map  $k_n : P_n X_{\bullet} \rightarrow E^{\Lambda}(M, n+2)$  is called the  $n$ -th  $k$ -invariant for  $X_{\bullet}$ .

**3.12. Definition.** A resolution model category  $s\mathcal{C}_{\mathcal{A}}$  as in §2.6 is called an  $E^2$ -model category if:

Ax 1.  $s\mathcal{C}$  has functorial Postnikov towers.

Ax 2. For every  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$  and  $\Lambda$ -module  $M$  the classifying object  $B\Lambda$  and the  $n$ -dimensional extended  $M$ -Eilenberg-Mac Lane object  $E^{\Lambda}(M, n)$  exist, for each  $n \geq 1$ . In addition we assume the latter determines a functor

$$E^{\Lambda}(-, n) : \Lambda\text{-Mod} \rightarrow \text{Ab}(\text{ho}(s\mathcal{C})),$$

both constructions are functorial in  $\Lambda$ , and are unique up to homotopy.

Ax 3.  $s\mathcal{C}$  has  $k$ -invariant squares (with respect to  $\mathcal{A}$ ) for each  $n \geq 0$ .

Ax 4. There is a functor  $J : s\mathcal{C} \rightarrow \mathcal{C}$  such that, for  $\Lambda \in \Pi_{\mathcal{A}}\text{-Alg}$  and  $X_{\bullet} \in s\mathcal{C}$ , if  $\pi_{\mathcal{A}} X_{\bullet} \xrightarrow{\sim} B_{s\Pi_{\mathcal{A}}\text{-Alg}} \Lambda$  is a weak equivalence in  $s\Pi_{\mathcal{A}}\text{-Alg}$ , then there is an isomorphism:

$$(3.13) \quad [A, JX_{\bullet}]_{\mathcal{C}} \xrightarrow{\cong} \text{Hom}_{\Pi_{\mathcal{A}}\text{-Alg}}(\pi_{\mathcal{A}} A, \Lambda),$$

natural in  $\Lambda$  and  $A \in \mathcal{A}$ .

**3.14. Remarks.**

- Ax 1-3 imply that  $s\mathcal{C}_{\mathcal{A}}$  is a *spherical model category* in the sense of [Bl6, §2], and so in particular is *stratified* in the sense of [Sp]. These axioms are also satisfied, for example, by the category  $\mathcal{T}_*$ , which is not itself a resolution model category (but see §2.17).
- We may assume that our extended Eilenberg-Mac Lane objects are *strict* abelian group objects in  $s\mathcal{C}/B\Lambda$ , by functoriality, since the group structure morphisms for a  $\Lambda$ -module  $M$  are maps of modules.
- Not all resolution model categories have the additional structure of a spherical model category (see §3.21).
- The point of Ax 4 is that any  $X_{\bullet} \in s\mathcal{C}/B\Lambda$  with  $\pi_{\mathcal{A}} X_{\bullet} \simeq B_{s\Pi_{\mathcal{A}}\text{-Alg}} \Lambda$  in  $s\Pi_{\mathcal{A}}\text{-Alg}$  yields a realization  $JX_{\bullet}$  for  $\Lambda$  (see Theorem 5.6). See [CDI] for a way to geometrically handle cases where Ax 4 does not hold.
- The statement of Ax 4 may appear somewhat convoluted, because it is intended to apply to two rather different contexts: see Theorems 3.15 and 3.19 below. Theorem 3.15 deals with the case of universal algebras (hence the special case of  $\Pi_{\mathcal{A}}$ -algebras), while Theorem 3.16 treats the general extension to diagram categories, thereby reducing our motivating example of diagrams of spaces to a consequence of Theorem 3.19, which deals with  $s\mathcal{T}_*$  with several standard model structures on  $\mathcal{T}_*$ .

**3.15. Theorem.** *Let  $\mathcal{C} = \Theta\text{-Alg}$  be an FP-sketchable variety of (graded) universal algebras, corepresented by a  $\mathfrak{G}$ -theory  $\Theta$ , with trivial model category structure, and let  $\mathcal{A}$  consist of monogenic free  $\Theta$ -algebras, as in §2.8(f). Then  $s\mathcal{C}_{\mathcal{A}}$  is an  $E^2$ -model category.*

*Proof.* We use the constructions described in [BDG] for the case  $\mathcal{C} = \Pi\text{-Alg}$ :

**For Ax 1:** Follow [DK2, §1.2]:

Given  $Y_\bullet \in s\mathcal{C}$  and  $n \geq 0$ , first define  $Y_\bullet^{(n)} \in s\mathcal{C}$  by:

$$Y_k^{(n)} = \begin{cases} Y_k & 0 \leq k \leq n+1; \\ M_k(Y_\bullet^{(n)}) & n+2 \leq k, \end{cases}$$

with simplicial maps determined from  $\text{tr}_{n+1} Y_\bullet$  and  $\delta_k : M_k(Y_\bullet^{(n)}) \rightarrow Y_k^{(n)}$ , along with the obvious maps  $p^{(n)} : Y_\bullet^{(n)} \rightarrow Y_\bullet^{(n-1)}$  and  $r^{(n)} : Y_\bullet \rightarrow Y_\bullet^{(n)}$ .

The Postnikov tower for  $X_\bullet \in s\mathcal{C}$  is then defined by setting  $P_n X_\bullet := Y_\bullet^{(n)}$ , where  $X_\bullet \rightarrow Y_\bullet$  is a (functorial)  $\mathcal{A}$ -fibrant replacement in  $s\mathcal{C}_{\mathcal{A}}$ .

**For Ax 2:** Follow [BDG, Prop. 2.2], taking  $B\Lambda$  to be the constant simplicial object on  $\Lambda$ ,  $E(M, n)$  to be the iterated Eilenberg-Mac Lane construction  $\bar{W}$  on  $BM$  (cf. [M, §20]), and  $E^\Lambda(M, n)$  to be the semi-direct product  $B\Lambda \times E(M, n)$  (§1.13).

More explicitly, let  $W$  be a free  $\Theta$ -algebra equipped with a surjection  $\phi : W \rightarrow M$ . Define a simplicial  $\Theta$ -algebra  $B_\bullet$  by setting  $\text{sk}_{n-1} B_\bullet := \text{sk}_{n-1} B\Lambda$  and  $E_n \simeq W \amalg B\Lambda_n$ , with  $W \subseteq Z_n B_\bullet$ . A straightforward calculation shows  $C_n B\Lambda = Z_{n-1} B\Lambda = 0$ , so  $Z_n B_\bullet = C_n B_\bullet$  is the cokernel  $B\Lambda_n \times W$  of  $B\Lambda_n \rightarrow E_n = W \amalg B\Lambda_n$ . Note that  $B\Lambda_0$  embeds in  $B\Lambda_n$  as a free retract by  $s_{n-1} \cdots s_0$ , so  $B\Lambda_n \cong B\Lambda_0 \amalg L'$  for some  $\Theta$ -algebra  $L'$ , where  $L' \times W$  is a  $\Theta$ -algebra ideal in  $Z_n B_\bullet$ , with quotient  $\Theta$ -algebra  $Z_n B_\bullet / (L' \times W) \cong K_0 \times W$ . This is by definition the free  $B\Lambda_0$ -algebra generated by  $W$ , and thus  $\phi : W \rightarrow M$  extends to a map of  $B\Lambda_0$ -algebras  $\hat{\phi} : B\Lambda_0 \times W \rightarrow M$ ; precomposing with the projection  $Z_n B_\bullet \rightarrow B\Lambda_0 \times W$  defines  $\tilde{\phi} : Z_n B_\bullet \rightarrow M$ .

Let  $\bar{d}_0 : \bar{B}_{n+1} \rightarrow B_n B_\bullet := \text{Ker } \tilde{\phi}$  be any surjection from a free  $\Theta$ -algebra, let  $B\Lambda_{n+1} := \bar{B}_{n+1} \amalg L_{n+1} B_\bullet$ , and let  $B_\bullet := P_n \text{sk}_{n+1} B_\bullet$ . Then  $\pi_n B_\bullet \cong M$  (as a  $\Lambda$ -module), and  $\pi_i B_\bullet = 0$  for  $i \neq 0, n$ . The section is induced by the inclusion  $\text{sk}_{n+1} B\Lambda \hookrightarrow \text{sk}_{n+1} B_\bullet$ .

**For Ax 3:** Follow [BDG, §5-6].

Given  $X_\bullet \in s\mathcal{C}/B\Lambda$  and  $n \geq 0$ , take the pushout:

$$\begin{array}{ccc} P_{n+1} X_\bullet & \xrightarrow{p^{(n+1)}} & P_n X_\bullet \\ \downarrow & \boxed{\text{PO}} & \downarrow f \\ B\Lambda & \xrightarrow{g} & Y_\bullet, \end{array}$$

and apply the functor  $P_{n+2}$  to the resulting diagram. The connectivity argument of [BDG, Lemma 5.11] applies here, too, so the result is actually a homotopy pull-back square,  $P_{n+2} Y_\bullet$  is an extended Eilenberg-Mac Lane object (with section  $P_{n+2} g$ ), and  $P_{n+2} f$  is the  $k$ -invariant. The construction is evidently natural, since we have natural Postnikov systems.

**For Ax 4:** Use  $\pi_0 : s\mathcal{C} \rightarrow \mathcal{C}$  as the functor  $J$ . Then the trivial model category structure on  $\mathcal{C}$  gives the first identity

$$[A, J B\Lambda]_{\mathcal{C}} = \text{Hom}_{\mathcal{C}}(A, \pi_0 B\Lambda) \cong \pi_0 B\Lambda(A)$$

and the second isomorphism comes from the fact that  $A$  is monogenic free, while  $\pi_0 B\Lambda \cong \pi_0(B\Lambda) \cong \Lambda$  completes the claim.  $\square$

**3.16. Theorem.** *Let  $s\mathcal{C}_{\mathcal{A}}$  be an  $E^2$ -model category,  $\mathbb{D}$  a small category, and  $\mathcal{B}$  the induced collection of models in  $\mathcal{C}^{\mathbb{D}}$  (see §2.19); then  $(s\mathcal{C}^{\mathbb{D}})_{\mathcal{B}}$  is an  $E^2$ -model category.*

*Proof.* We use the induced collection of models  $\mathcal{B}$  (§2.19) to extend the  $E^2$ -model structure to  $s\mathcal{C}^{\mathbb{D}}$ . The underlying simplicial model category structure on  $\mathcal{C}^{\mathbb{D}}$  is that of [H, §11.6], with weak equivalences and fibrations defined objectwise; thus evaluation at  $d \in \text{Obj } \mathbb{D}$  preserves fibrations and weak equivalences and forms part of a strong Quillen pair, with left adjoint  $F(-, d)$  (the free diagram functor at  $d$ ). See [H, 11.5.26].

Hence, for  $A \in \mathcal{A}$ ,  $d \in \mathbb{D}$ , and  $X \in s\mathcal{C}^{\mathbb{D}}$ , we have a natural isomorphism:

$$(3.17) \quad [F(A, d), X]_{s\mathcal{C}^{\mathbb{D}}} \cong [A, X(d)]_{s\mathcal{C}} .$$

In particular,  $\pi_{\mathcal{B}}(-, F(A, d))$  is the same as  $\pi_{\mathcal{A}}(-, A)$  after pre-composition with evaluation at  $d$ . By the spiral exact sequence, the same holds for  $\pi_{\mathcal{B}}^{\natural}(-, \mathcal{B})$ .

The axioms of Definition 3.12 can therefore be verified by applying the various constructions of  $s\mathcal{C}$  at each  $d$  in  $\mathbb{D}$ , and checking that the requisite properties are satisfied in  $s\mathcal{C}^{\mathbb{D}}$ , once they hold objectwise:

**For Ax 1:** Since  $s\mathcal{C}$  has functorial Postnikov towers,  $s\mathcal{C}^{\mathbb{D}}$  possesses such towers, with  $(P_n X_{\bullet})(d) = P_n(X_{\bullet}(d))$ .

**For Ax 2:** Given a  $\Pi_{\mathcal{B}}$ -algebra  $\Lambda$  (that is, a functor  $\Lambda : \mathbb{D} \rightarrow \Pi_{\mathcal{A}}\text{-Alg}$ ) and a module  $M$  over  $\Lambda$ , for each  $n \geq 1$  we define the classifying object  $B\Lambda$  and extended  $M$ -Eilenberg-Mac Lane object  $E^{\Lambda}(M, n)$  objectwise, by applying the appropriate functors in  $s\mathcal{C}$  to the diagrams  $\Lambda$  and  $M$ . This is evidently functorial, unique up to homotopy, and satisfies (3.6). Note that in order for  $E^{\Lambda}(M, n)$  to be a homotopy abelian group object in  $s\mathcal{C}^{\mathbb{D}}/B\Lambda$ , we must produce structure maps:

$$(3.18) \quad \mu : E^{\Lambda}(M, n) \times_{B\Lambda} E^{\Lambda}(M, n) \rightarrow E^{\Lambda}(M, n) \quad \text{and} \quad \iota : E^{\Lambda}(M, n) \rightarrow E^{\Lambda}(M, n)$$

(over  $B\Lambda$ ), satisfying the appropriate identities. (The unit element is represented by the section  $s : B\Lambda \rightarrow E^{\Lambda}(M, n)$ .) However, since  $M$  is itself an abelian group object in  $\Pi_{\mathcal{A}}\text{-Alg}/\Lambda$ , it is equipped in turn with maps

$$m : M \times_{\Lambda} M \rightarrow M \quad \text{and} \quad i : M \rightarrow M$$

in  $\Pi_{\mathcal{A}}\text{-Alg}/\Lambda$ , which are themselves maps of  $\Lambda$ -modules, and these induce the maps of (3.18) by functoriality. Note that the functors  $E^{\Lambda}(-, n)$  in  $s\mathcal{C}$  preserve products of modules (over  $\Lambda$ ) because of the homotopy uniqueness and functoriality.

**For Ax 3:** Since Postnikov towers and extended Eilenberg-MacLane objects, as well as fibrations and weak equivalences are defined object-wise for  $d \in \text{Obj } \mathbb{D}$ , defining  $k$ -invariants in  $s\mathcal{C}^{\mathbb{D}}/B\Lambda$  objectwise will give homotopy pullback squares that are  $k$ -invariant squares.

**For Ax 4:** Suppose we are given a functor  $J : s\mathcal{C} \rightarrow \mathcal{C}$  with the requisite properties. Define  $J^{\mathbb{D}} : s\mathcal{C}^{\mathbb{D}} \rightarrow \mathcal{C}^{\mathbb{D}}$  by  $(J^{\mathbb{D}} X_{\bullet})(d) = J(X_{\bullet}(d))$ . Let  $\pi_{\mathcal{A}} X_{\bullet} \xrightarrow{\sim} B_{s(\Pi_{\mathcal{A}}\text{-Alg})^{\mathbb{D}}} \Lambda$  be a weak equivalence. Now we have two natural isomorphisms:

$$[F(A, d), J^{\mathbb{D}}(X_{\bullet})]_{\mathcal{C}^{\mathbb{D}}} \cong [A, J(X_{\bullet}(d))]_{\mathcal{C}}$$

and

$$[\pi_{\mathcal{B}} F(A, d), \Lambda]_{(\Pi_{\mathcal{A}}\text{-Alg})^{\mathbb{D}}} \cong [\pi_{\mathcal{A}} A, \Lambda(d)]_{\Pi_{\mathcal{A}}\text{-Alg}} .$$

From Ax 4, applied to  $\pi_{\mathcal{A}} X_{\bullet}(d) \xrightarrow{\sim} B_{s\Pi_{\mathcal{A}}\text{-Alg}} \Lambda(d)$  in  $s\Pi_{\mathcal{A}}\text{-Alg}$ , we have the natural isomorphism:

$$[A, J(X_{\bullet}(d))]_{\mathcal{C}} \xrightarrow{\cong} [\pi_{\mathcal{A}} A, \Lambda(d)]_{\Pi_{\mathcal{A}}\text{-Alg}} .$$

Combining all three isomorphisms gives the required natural isomorphism:

$$[F(A, d), J^{\mathbb{D}}(X_{\bullet})]_{\mathcal{C}^{\mathbb{D}}} \xrightarrow{\cong} [\pi_{\mathcal{B}} F(A, d), \Lambda]_{(\Pi_{\mathcal{A}}\text{-Alg})^{\mathbb{D}}} .$$

□

**3.19. Theorem.** *The category  $s\mathcal{T}_*$  of simplicial pointed connected topological spaces (with the spheres  $(\mathbf{S}^n)_{n=1}^\infty$  as models), and the four examples of §2.18, are all  $E^2$ -model categories.*

*Proof.* The case  $\mathcal{C} = \mathcal{T}_*$  was treated in [BDG], and all five cases may be treated similarly:

**For Ax 1:** As in the proof of Theorem 3.15.

**For Ax 2:** Follow [BDG, 7.7].

More explicitly, given  $A \in \mathcal{A}$ , for each  $n \geq 1$  recall  $\pi_n^\natural(X_\bullet, \mathcal{A}) \cong [A \hat{\otimes} \mathbf{S}^n, X_\bullet]_{s\mathcal{C}}$ , where  $A \hat{\otimes} \mathbf{S}^n$  denotes  $c(A)_\bullet \otimes \mathbf{S}^n \in s\mathcal{C}$  (see also 0.7).

For the existence of  $B\Lambda$ , let  $U, V \in \Pi_{\mathcal{A}}$  be such that  $\pi_{\mathcal{A}}U \rightarrow \Lambda$  is a free cover of  $\Lambda$ , and  $\pi_{\mathcal{A}}V \rightarrow \pi_{\mathcal{A}}U$  covers minimally the corresponding relations. For each summand  $A$  in  $V$ , attach a copy of  $A \hat{\otimes} \mathbf{S}^n$  to  $U$ . Applying  $P_0$  to the resulting object of  $s\mathcal{C}$  yields a classifying object  $B\Lambda$  as required.

For the Eilenberg-Mac Lane objects, again we follow [BDG, 7.7]:

Let  $W$  be the model for  $B\Lambda$  constructed as above. Let  $U, V \in \Pi_{\mathcal{A}}$  be such that  $\pi_{\mathcal{A}}V \rightarrow \pi_{\mathcal{A}}U \rightarrow M$  is a presentation for  $M$ . Attach a copy of  $A \hat{\otimes} \mathbf{S}^n$  for each summand  $A$  of  $U$  to form an object  $Z \in s\mathcal{C}$ , then attach a copy of  $A \hat{\otimes} \mathbf{S}^{n+1}$  to  $Z$  for each  $A$ -coproduct summand of  $V$  to form  $Z'$ . Applying  $P_n$  to  $Z'$  yields the desired  $E^\Lambda(M, n)$ . The existence of the section  $\sigma : B\Lambda \rightarrow E^\Lambda(M, n)$  follows from [BDG, Prop. 4.9].

**For Ax 3:** Again follow [BDG, §5-7], with the same construction as in the proof of Ax 3 for Theorem 3.15.

**For Ax 4:** For the standard model of  $\mathcal{C} = \mathcal{T}_*$ ,  $J$  will be the realization or diagonal functor  $\| - \| : s\mathcal{C} \rightarrow \mathcal{C}$  (left adjoint to the constant functor  $c(-)_\bullet : \mathcal{C} \rightarrow s\mathcal{C}$ ). This extends entrywise to diagrams of simplicial spaces, as does the natural spectral sequence of [Q2] (see also [BF, Thm B.5]), yielding an  $(\mathbb{N} \times \mathcal{A})$ -graded spectral sequence with:

$$(3.20) \quad E_{s, \mathcal{A}}^2 = \pi_s(X_\bullet, \mathcal{A}) \Rightarrow \pi_{\mathcal{A}} \|X_\bullet\| .$$

Then (3.13) will be the edge homomorphism of this spectral sequence, which collapses at the  $E^2$ -term if  $\pi_{\mathcal{A}}X_\bullet \simeq \pi_{\mathcal{A}}B\Lambda$ .

We can extend this spectral sequence argument to the other examples of §2.18 as follows:

- (i) For §2.18 (a): the exactness of  $- \otimes R$  for  $R \subseteq \mathbb{Q}$  allows us to obtain a localized Quillen spectral sequence to verify Ax 4 for either rational or  $p$ -local spaces.
- (ii) For §2.18 (b): the spectral sequence for the realization of a simplicial spectrum is analyzed in [GH1, §6], showing that Ax 4 is satisfied for  $sSpec$  (as well as for some structured versions of spectra). For the remaining axioms see [GH2, GH3].
- (iii) For §2.18 (c): to verify Ax 4, apply the Quillen spectral sequence to  $P_n X_\bullet$ , where  $X_\bullet$  is the usual resolution in  $s\mathcal{T}_*$ . Note that  $P_n \|X_\bullet\|$  is  $n$ -equivalent to  $\|P_n X_\bullet\|$  (as we can see from the differentials in the spectral sequence itself).
- (iv) For §2.18 (d): if  $\mathcal{A} := \{\mathbf{S}^n\}_{n=k}^\infty$ , we can use the usual Eilenberg-Mac Lane objects (noting that the connectivity assumptions are not in the simplicial direction), and again apply the Quillen spectral sequence to resolutions in which all spaces happen to be  $(k-1)$ -connected.

□

3.21. *Remark.* Note that not all resolution model categories are  $E^2$ -model categories. In particular, if we replace the spheres by Moore spaces as our models (in  $\mathcal{T}_*$ ), then we have neither Eilenberg-Mac Lane objects nor Postnikov systems for the mod  $p$  homotopy groups (see [Bl6, §3.10]). In addition, the realization of simplicial spaces does not provide the expected functor  $J$  for Ax 4, since the Bousfield-Friedlander spectral sequence for a mod  $p$  resolution does not collapse (see [Bl3, §4.6]).

3.22. **Notation.** In what follows we will often have to deal with parallel constructions of the  $E^2$ -model category structure in  $s\mathcal{C}_{\mathcal{A}}$ , as well as in the associated algebraic category  $s\Pi_{\mathcal{A}}\text{-Alg}$ . In order to distinguish between them, we shall use boldface  $\mathbf{-}$   $\mathbf{P}_n X_{\bullet}$ ,  $\mathbf{B}\Lambda := B_{s\mathcal{C}}\Lambda$ ,  $\mathbf{E}(M, n) := E_{s\mathcal{C}}(M, n)$ , and so on  $\mathbf{-}$  for the constructions in  $s\mathcal{C}$ , and tildes  $\tilde{\mathbf{-}}$   $\tilde{P}_n G_{\bullet}$ ,  $\tilde{B}\Lambda := B_{s\Pi_{\mathcal{A}}\text{-Alg}}\Lambda$ ,  $\tilde{E}(M, n) := E_{s\Pi_{\mathcal{A}}\text{-Alg}}(M, n)$ , etc.  $\tilde{\mathbf{-}}$  for the analogous constructions in  $s\Pi_{\mathcal{A}}\text{-Alg}$ .

We may still use the unadorned symbols  $P_n X_{\bullet}$ ,  $B\Lambda$ , and  $E^{\Lambda}(M, n)$ , etc., when we do not need to make this distinction.

#### 4. COHOMOLOGY THEORIES

As one might expect, the Eilenberg-Mac Lane objects in an  $E^2$ -model category can be used to define suitable cohomology theories:

4.1. **Definition.** Let  $s\mathcal{C}_{\mathcal{A}}$  be any resolution model category. A sequence of pointed contravariant functors  $(D^n : \text{ho } s\mathcal{C}_{\mathcal{A}} \rightarrow \mathbb{Z}\text{-Mod})_{n=0}^{\infty}$  is called a sequence of *cohomology functors* if they satisfy the analogues of the usual Eilenberg-Steenrod axioms:

- I.  $D^n(\coprod_{\alpha} X_{\alpha}) \cong \prod_{\alpha} D^n X_{\alpha}$  for any coproduct of cofibrant objects in  $s\mathcal{C}_{\mathcal{A}}$ .
- II.  $D^i(A \otimes \mathbf{S}^n) = 0$  for  $i \neq n$  and any  $A \in \mathcal{A}$ ;
- III. Given  $N_{\bullet} \leftarrow M_{\bullet} \xrightarrow{i} P_{\bullet}$  in  $s\mathcal{C}$ , with all objects cofibrant and  $i$  a cofibration, let  $N_{\bullet} \amalg_{M_{\bullet}} P_{\bullet}$  denote the pushout. Then there is a natural *Mayer-Vietoris* long exact sequence:

$$(4.2) \quad \begin{aligned} 0 \rightarrow D^0(N_{\bullet} \amalg_{M_{\bullet}} P_{\bullet}) \rightarrow D^0 N_{\bullet} \oplus D^0 P_{\bullet} \rightarrow D^0 M_{\bullet} \rightarrow \cdots \rightarrow D^n(N_{\bullet} \amalg_{M_{\bullet}} P_{\bullet}) \rightarrow \\ D^n N_{\bullet} \oplus D^n P_{\bullet} \rightarrow D^n M_{\bullet} \rightarrow D^{n+1}(N_{\bullet} \amalg_{M_{\bullet}} P_{\bullet}) \rightarrow \cdots \end{aligned}$$

4.3. **Definition.** Fix a  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$  and a  $\Lambda$ -module  $M$ . For  $X_{\bullet} \in s\mathcal{C}_{\mathcal{A}}/B\Lambda$  and  $n \geq 1$ , define the  $n$ -th (*andré-Quillen*) *cohomology group of  $X_{\bullet}$  over  $\Lambda$  with coefficients in  $M$* , denoted by  $H_{\Lambda}^n(X_{\bullet}; M)$ , to be:

$$H_{\Lambda}^n(X_{\bullet}; M) := [X_{\bullet}, E_{\Lambda}(M, n)]_{s\mathcal{C}_{\mathcal{A}}/B\Lambda}.$$

We would like to know that extending  $\pi_{\mathcal{A}} : s\mathcal{C}_{\mathcal{A}}/B\Lambda \rightarrow s\Pi_{\mathcal{A}}\text{-Alg}/\pi_{\mathcal{A}}B\Lambda$  to a functor  $s\mathcal{C}_{\mathcal{A}}/B\Lambda \rightarrow s\Pi_{\mathcal{A}}\text{-Alg}/\tilde{B}\Lambda$  (via  $\pi_{\mathcal{A}}B\Lambda \rightarrow \tilde{P}_0\pi_{\mathcal{A}}B\Lambda \simeq \tilde{B}\Lambda$ ) induces an isomorphism of cohomology theories over  $\Lambda$ . This holds for  $n \geq 2$  by the following generalization of [BDG, Prop. 8.7]:

4.4. **Proposition.** *There is a natural map  $\zeta : \pi_{\mathcal{A}}\mathbf{E}^{\Lambda}(M, n) \rightarrow \tilde{E}^{\Lambda}(M, n)$  such that*

$$\phi_n(X_{\bullet}) : [X_{\bullet}, \mathbf{E}^{\Lambda}(M, n)]_{s\mathcal{C}_{\mathcal{A}}/B\Lambda} \rightarrow [\pi_{\mathcal{A}}X_{\bullet}, \tilde{E}^{\Lambda}(M, n)]_{s\Pi_{\mathcal{A}}\text{-Alg}/\tilde{B}\Lambda},$$

*defined as the composite of the maps induced by  $\zeta$  and  $\pi_{\mathcal{A}} : s\mathcal{C} \rightarrow s\Pi_{\mathcal{A}}\text{-Alg}$ , is an isomorphism for  $n \geq 2$ .*

*Proof.* The section  $\sigma : B\Lambda \rightarrow \mathbf{E}^{\Lambda}(M, n)$  (§3.7) induces a section  $s : \pi_{\mathcal{A}}B\Lambda \rightarrow \tilde{P}_n\pi_{\mathcal{A}}\mathbf{E}^{\Lambda}(M, n)$  for the map  $\tilde{p}^{(n)} : \tilde{P}_n\pi_{\mathcal{A}}\mathbf{E}^{\Lambda}(M, n) \rightarrow \tilde{P}_{n-1}\pi_{\mathcal{A}}\mathbf{E}^{\Lambda}(M, n) = \pi_{\mathcal{A}}B\Lambda$  (cf.

§3.1) over  $\tilde{B}\Lambda$ . Moreover,  $\pi_{\mathcal{A}}\mathbf{E}^{\Lambda}(M, n)$  is known from (3.8). Therefore, the  $(n-1)$ -st  $k$ -invariant for  $\pi_{\mathcal{A}}\mathbf{E}^{\Lambda}(M, n)$  fits into a homotopy-commutative diagram:

$$\begin{array}{ccccc}
 \pi_{\mathcal{A}}\mathbf{B}\Lambda & & & & \\
 \searrow^s & & \xrightarrow{r} & & \\
 & \tilde{P}_n\pi_{\mathcal{A}}\mathbf{E}^{\Lambda}(M, n) & \xrightarrow{\quad} & \tilde{B}\Lambda & \\
 & \downarrow \tilde{p}^{(n)} & \boxed{\text{hPB}} & \downarrow \tau & \\
 & \pi_{\mathcal{A}}\mathbf{B}\Lambda & \xrightarrow{\tilde{k}_{n-1}} & \tilde{E}^{\Lambda}(M, n+1) & \\
 \swarrow & & & & \\
 & & & & 
 \end{array}$$

where  $\tilde{p}^{(n)}$  is induced by  $\pi_{\mathcal{A}}(\mathbf{p}^{(n)}) : \pi_{\mathcal{A}}\mathbf{E}^{\Lambda}(M, n) \rightarrow \pi_{\mathcal{A}}\mathbf{B}\Lambda$ , and  $r$  and the unlabelled arrow is the unique terminal map in  $s\Pi_{\mathcal{A}}\text{-Alg}/\tilde{B}\Lambda$ . Thus  $\tilde{k}_{n-1} = \tau \circ r$ , yielding two consecutive homotopy pullback squares:

$$\begin{array}{ccccc}
 \tilde{P}_n\pi_{\mathcal{A}}\mathbf{E}^{\Lambda}(M, n) & \xrightarrow{\zeta} & \tilde{E}^{\Lambda}(M, n) & \xrightarrow{\quad} & \tilde{B}\Lambda \\
 \downarrow \tilde{p}^{(n)} & \boxed{\text{hPB}} & \downarrow & \boxed{\text{hPB}} & \downarrow \\
 \pi_{\mathcal{A}}\mathbf{B}\Lambda & \xrightarrow{r} & \tilde{B}\Lambda & \xrightarrow{\tau} & \tilde{E}^{\Lambda}(M, n+1) \\
 & & \xrightarrow{\tilde{k}_{n-1}} & & 
 \end{array}$$

in which the required  $\zeta$  is a structure map for the left square.

Now let:

$$\Phi_n(X_{\bullet}) : \text{map}_{s\mathcal{C}_{\mathcal{A}}/\mathbf{B}\Lambda}(X_{\bullet}, \mathbf{E}^{\Lambda}(M, n)) \rightarrow \text{map}_{s\Pi_{\mathcal{A}}\text{-Alg}/\tilde{B}\Lambda}(\pi_{\mathcal{A}}X_{\bullet}, \tilde{E}^{\Lambda}(M, n))$$

be the analogously defined map, with  $\phi_n(X) = \pi_0\Phi_n(X_{\bullet})$ .

Because  $\pi_{\mathcal{A}}$  takes homotopy pushouts in  $s\mathcal{C}_{\mathcal{A}}$  to homotopy pushouts of simplicial  $\Pi_{\mathcal{A}}$ -algebras, it follows that the source and target of  $\Phi_n(-)$  take homotopy pushouts to homotopy pullbacks. Now every object of  $s\mathcal{C}_{\mathcal{A}}$  is, up to homotopy, a filtered colimit of objects constructed from copies of  $A\hat{\otimes}\mathbf{S}^m$  by finitely many homotopy pushouts. Thus, since source and target of  $\Phi_n$  take filtered colimits to homotopy limits of simplicial sets, it suffices to show that  $\Phi_n(A\hat{\otimes}\mathbf{S}^m)$  is a  $\pi_0$ -equivalence for all  $m \geq 2$  and  $A \in \mathcal{A}$ . As  $A\hat{\otimes}\mathbf{S}^m$  corepresents  $\pi_n^{\dagger}(\cdot, \mathcal{A})$  in  $\text{ho } s\mathcal{C}_{\mathcal{A}}/\mathbf{B}\Lambda$  and  $\pi_{\mathcal{A}}(A\hat{\otimes}\mathbf{S}^m)$  corepresents  $\pi_n\pi_{\mathcal{A}}(\cdot)$  in  $\text{ho } s\Pi_{\mathcal{A}}\text{-Alg}$  for  $n \geq 2$ , the Proposition follows from the naturality of  $\zeta$  and Definition 3.5.  $\square$

The restriction  $n \geq 2$  is needed because  $\pi_1\pi_{\mathcal{A}}(\cdot)$  is not known to be corepresentable (see [DKS2, §7(ii)]).

**4.5. Corollary.**  $H_{\Lambda}^*(-; M)$  are cohomology functors on  $s\mathcal{C}_{\mathcal{A}}/\mathbf{B}\Lambda$  and  $s\Pi_{\mathcal{A}}\text{-Alg}/\tilde{B}\Lambda$ .

*Proof.* This follows from [Q1, II, §5].  $\square$

**4.6. Remark.** If  $\mathcal{C}$  is the category  $\Pi_{\mathcal{A}}\text{-Alg}$ , or more generally any category of  $\Theta$ -algebras as in Theorem 3.15, we have an equivalence:

$$H_{\Lambda}^n(G_{\bullet}; M) \cong \pi_0 \text{map}_{sG_{\bullet}\text{-Mod}/\mathbf{B}\Lambda}(\mathbb{L}\Omega_{G_{\bullet}}, E^{\Lambda}(M, n)).$$

Here  $\mathbb{L}\Omega_{G_{\bullet}}$  denotes the *cotangent complex* associated to  $G_{\bullet}$ , defined by:

$$\mathbb{L}\Omega_{G_{\bullet}} := \Omega_{G'_{\bullet}} *_{G'_{\bullet}} G_{\bullet}$$

where  $G'_{\bullet}$  is a cofibrant replacement of  $G_{\bullet}$  in  $s\mathcal{C}_{\mathcal{A}}$  and the group of Kähler differentials  $\Omega_{G'_{\bullet}}$  is defined in 1.19.

4.7. *Remark.* In fact, this previous observation can be carried a little further. Given a (simplicial)  $\Pi_{\mathcal{A}}$ -algebra  $G_{\bullet}$  and a  $G_{\bullet}$ -module  $M$ , define the *group of algebraic extensions*  $\text{exal}_{\Lambda}(G_{\bullet}; M)$  to be the set of equivalence classes of the form (1.14) with  $K = M$ . This set is a functor in both variables (via pullbacks and pushouts) and forms an abelian group with unit  $M \times_{\Lambda} G_{\bullet}$  and addition induced by the diagonal  $G_{\bullet} \rightarrow G_{\bullet} \times_{\Lambda} G_{\bullet}$  and the group addition  $M \times_{\Lambda} M \rightarrow M$ .

Assume now that  $G_{\bullet}$  is cofibrant. Following [I, III.1.2.3], there is a natural isomorphism

$$(4.8) \quad \text{exal}_{\Lambda}(G_{\bullet}; E^{\Lambda}(M, n)) \xrightarrow{\cong} H_{\Lambda}^{n+1}(G_{\bullet}; M)$$

sending an algebraic extension  $(E^{\Lambda}(M, n) \rightarrow X \rightarrow G_{\bullet})$  of simplicial  $\Pi_{\mathcal{A}}$ -algebras to the induced homotopy coboundary  $(G_{\bullet} \rightarrow E^{\Lambda}(M, n+1))$ . For general  $G_{\bullet}$ , there is an isomorphism

$$(4.9) \quad H_{\Lambda}^{n+1}(G_{\bullet}; M) \cong \text{colim}_{\text{Wk}(G_{\bullet})} \text{exal}_{\Lambda}(G'_{\bullet}; E^{\Lambda}(M, n+1))$$

where  $\text{Wk}(G_{\bullet})$  is the category of cofibrant replacements  $G'_{\bullet} \rightarrow G_{\bullet}$  in simplicial  $\Pi_{\mathcal{A}}$ -algebras

#### 4.10. The cohomology of a diagram.

Let  $\mathbb{D}$  be a small category. Observe that a map of  $\mathbb{D}$ -diagrams is just a natural transformation: a collection of maps on objects which commute with the maps in each diagram.

4.11. **Fact.** *Given two functors  $X, Y : \mathbb{D} \rightarrow \mathcal{C}$ , the set  $\text{Hom}_{\mathcal{C}^{\mathbb{D}}}(X, Y)$  of diagram maps between them fits into the equalizer diagram*

$$(4.12) \quad \text{Hom}_{\mathcal{C}^{\mathbb{D}}}(X, Y) \hookrightarrow \prod_{d \in \mathbb{D}} \text{Hom}_{\mathcal{C}}(X_d, Y_d) \rightrightarrows \prod_{d, e \in \mathbb{D}} \prod_{\eta \in \text{Hom}_{\mathbb{D}}(d, e)} \text{Hom}_{\mathcal{C}}(X_d, Y_e),$$

where the two parallel arrows map to each factor indexed by  $\eta : d \rightarrow e$  in  $\mathbb{D}$  by the appropriate projection, followed by  $Y(\eta)_{*} : \text{Hom}_{\mathcal{C}}(X_d, Y_d) \rightarrow \text{Hom}_{\mathcal{C}}(X_d, Y_e)$ , or  $X(\eta)^{*} : \text{Hom}_{\mathcal{C}}(X_e, Y_e) \rightarrow \text{Hom}_{\mathcal{C}}(X_d, Y_e)$ , respectively.

4.13. *Remark.* If  $\mathcal{C}$  is a simplicial model category, and  $Y_d$  is an abelian group object for each  $d \in \text{Obj } \mathbb{D}$ , we can replace the equalizer diagram (4.12) by an exact sequence of simplicial abelian mapping spaces (using the mapping space construction of [Q1, II, 3.1]):

$$(4.14) \quad 0 \rightarrow \text{map}_{\mathcal{C}^{\mathbb{D}}}(X, Y) \rightarrow \prod_{d \in \mathbb{D}} \text{map}_{\mathcal{C}}(X_d, Y_d) \xrightarrow{\xi} \prod_{d, e \in \mathbb{D}} \prod_{\eta : d \rightarrow e} \text{map}_{\mathcal{C}}(X_d, Y_e),$$

where  $\xi$  is the difference of the two parallel arrows of (4.12).

If this were a fibration sequence after the mapping spaces are restricted to appropriate over-categories, we could apply  $\pi_0$  and compute cohomology in the diagram category directly from the exact sequence. However, it is not a fibration sequence in general, so we concentrate for now on the special case of  $\mathbb{D} = [1]$ :

#### 4.15. The cohomology of a map.

For the arrow category  $\mathcal{C}(\rightarrow)$ , the exact sequence of (4.14), suitably modified, is in fact a fibration sequence. To show this, we need some technical results on model categories:

4.16. **Lemma.** *Suppose*

$$(4.17) \quad \begin{array}{ccc} X & \xrightarrow{f} & W \\ & \searrow g & \downarrow \psi \\ & & Z \end{array}$$

is a diagram in a model category  $\mathcal{C}$  which commutes up to homotopy, with  $X$  cofibrant and  $\psi$  a fibration. Then there is a homotopic map  $f \simeq f' : X \rightarrow W$  such that  $\psi \circ f' = g$ . Dually, if

$$(4.18) \quad \begin{array}{ccc} X & & \\ \downarrow \phi & \searrow f & \\ Y & \xrightarrow{g} & Z \end{array}$$

commutes up to homotopy, with  $Z$  fibrant and  $\phi$  a cofibration, then there is a homotopic map  $g \simeq g' : Y \rightarrow Z$  such that  $f = g' \circ \phi$ .

*Proof.* Assume  $\psi$  is a fibration. Cofibrancy of  $X$  implies  $i_0 : X \rightarrow \text{cyl}(X)$  is an acyclic cofibration by [H, 7.3.7]. Given a homotopy  $H : \text{cyl}(X) \rightarrow Z$  with  $H \circ i_0 = \psi \circ f$  and  $H \circ i_1 = g$ , we may use the left lifting property in

$$(4.19) \quad \begin{array}{ccc} X & \xrightarrow{f} & W \\ \text{acyc. cof} \downarrow i_0 & \hat{H} \nearrow & \downarrow \psi \text{ fib} \\ \text{cyl}(X) & \xrightarrow{H} & Z \end{array}$$

to factor  $H$  as  $\psi \circ \hat{H}$ , and set  $f' := \hat{H} \circ i_1$ . If instead  $\phi$  is a cofibration and  $Z$  is fibrant, use the dual argument.  $\square$

4.20. **Corollary.** *Suppose*

$$(4.21) \quad \begin{array}{ccc} X & \xrightarrow{f} & W \\ \downarrow \phi & & \downarrow \psi \\ Y & \xrightarrow{g} & Z \end{array}$$

is a commutative diagram in a model category  $\mathcal{C}$ . If  $\psi$  is a fibration and  $X$  is cofibrant, then to any homotopic map  $g' \simeq g$  there corresponds a homotopic map  $f' \simeq f$  such that  $\psi \circ f' = g' \circ \phi$ . Dually, if  $\phi$  is a cofibration and  $Z$  is fibrant, then to any homotopic map  $f' \simeq f$  there corresponds a homotopic map  $g' \simeq g$  such that  $\psi \circ f' = g' \circ \phi$ .

4.22. *Remark.* Since we assume that fibrations and weak equivalences in our diagram categories are defined objectwise, then if  $\phi$  is a cofibrant object in  $\mathcal{C}(\rightarrow)$  it follows that  $\phi$  is a cofibration in  $\mathcal{C}$  with cofibrant source. Thus if  $\psi$  is a fibration with fibrant target in  $\mathcal{C}$ , it makes sense to consider homotopy classes of maps  $[\phi, \psi]$  in (4.17) – in fact, the mapping space  $\text{map}_{\mathcal{C}(\rightarrow)}(\phi, \psi)$  has homotopical meaning, and  $[\phi, \psi] \cong \pi_0 \text{map}_{\mathcal{C}(\rightarrow)}(\phi, \psi)$ .

4.23. **Proposition.** *Let  $\vartheta : U \rightarrow V$  be a fixed map in a simplicial model category  $\mathcal{C}$  and let  $\phi : X \rightarrow Y$  and  $\psi : W \rightarrow Z$  be maps in  $\mathcal{C}(\rightarrow)/\vartheta$ . If  $\phi$  is a cofibration with cofibrant source and  $Z \rightarrow V$  is a fibration in  $\mathcal{C}$ , with  $W$  and  $Z$  abelian group*

objects, then the restriction of the exact sequence of simplicial abelian mapping spaces from Remark 4.13

$$(4.24) \quad \text{map}_{\mathcal{C}(\rightarrow)/\vartheta}(\phi, \psi) \rightarrow \text{map}_{\mathcal{C}/U}(X, W) \times \text{map}_{\mathcal{C}/V}(Y, Z) \xrightarrow{\xi} \text{map}_{\mathcal{C}/V}(X, Z)$$

is a fibration sequence (in  $\mathcal{S}$ ).

*Proof.* First, by [Q1, II, §3, Prop. 1], we know that  $\xi$  of (4.24) is a fibration in  $\mathcal{G}$  (and so in  $\mathcal{S}$ ) if and only if it surjects onto the basepoint component of the target space  $\text{map}_{\mathcal{C}/V}(X, Z) \in \mathcal{S}$  – or equivalently, onto any component of  $\text{map}_{\mathcal{C}/V}(X, Z)$  which it hits.

Now, if  $k : X \times \Delta[n] \rightarrow Z$  is any map in the image of  $\xi$ , then there are maps  $f : X \times \Delta[n] \rightarrow W$  in  $\mathcal{C}/U$  and  $g : Y \times \Delta[n] \rightarrow Z$  in  $\mathcal{C}/V$  such that in the (not commutative) diagram

$$(4.25) \quad \begin{array}{ccc} X \otimes \Delta[n] & \xrightarrow{f} & W \\ \phi \otimes \text{Id} \downarrow & \searrow k & \downarrow \psi \\ Y \otimes \Delta[n] & \xrightarrow{g} & Z \end{array}$$

we have  $\psi \circ f - g \circ (\phi \otimes \text{Id}) = k$  in  $\mathcal{C}/V$ .

Finally, if  $k'$  is in the same component as  $k$  in  $\text{map}_{\mathcal{C}/V}(X, Z)$ , we can write  $\psi \circ f - g \circ (\phi \otimes \text{Id}) \sim_V k'$  (since  $X$  is cofibrant and  $Z$  is fibrant in  $\mathcal{C}/V$ ) or equivalently, since  $\pm$  preserves homotopies,  $\psi \circ f - k' \sim_V g \circ (\phi \otimes \text{Id})$ , where  $\sim_V$  indicates homotopy in  $\mathcal{C}/V$ . By Lemma 4.16 applied to the diagram

$$(4.26) \quad \begin{array}{ccc} X \otimes \Delta[n] & & \\ \downarrow \phi \otimes \text{Id} & \searrow \psi \circ f - k' & \\ Y \otimes \Delta[n] & \xrightarrow{g} & Z \end{array}$$

viewed in  $\mathcal{C}/V$ , we can replace  $g$  by a homotopic map  $g'$  over  $V$  such that  $\psi \circ f - k' = g' \circ (\phi \otimes \text{Id})$ . But then  $\xi(f, g') = k'$ , so  $\xi$  indeed surjects onto the component of  $k$ .  $\square$

**4.27. Corollary.** For  $\phi : X_\bullet \rightarrow Y_\bullet$ , a morphism in  $s\mathcal{C}$  over a map  $B\lambda : B\Lambda_0 \rightarrow B\Lambda_1$ , suppose  $\psi : E^{\Lambda_0}(M_0, n) \rightarrow E^{\Lambda_1}(M_1, n)$  is the morphism of extended Eilenberg-Mac Lane objects induced by a module  $\tau : M_0 \rightarrow M_1$  over  $\lambda : \Lambda_0 \rightarrow \Lambda_1$ . Then there is a long exact sequence:

$$(4.28) \quad \begin{aligned} 0 \rightarrow H_\lambda^0(\phi, \tau) \rightarrow H_{\Lambda_0}^0(X_\bullet; M_0) \oplus H_{\Lambda_1}^0(Y_\bullet; M_1) &\xrightarrow{\psi_* - \phi^*} H_{\Lambda_1}^0(X_\bullet; M_1) \rightarrow H^1(\phi, \tau) \rightarrow \dots \\ \rightarrow H_{\Lambda_1}^{n-1}(X_\bullet; M_1) \rightarrow H_\lambda^n(\phi; \tau) \xrightarrow{\theta} H_{\Lambda_0}^n(X_\bullet; M_0) \oplus H_{\Lambda_1}^n(Y_\bullet; M_1) &\xrightarrow{\psi_* - \phi^*} H_{\Lambda_1}^n(X_\bullet; M_1) \end{aligned}$$

where  $\theta$  is induced by the obvious forgetful functors.

*Proof.* Recall from Remark 3.14 that we may assume that our extended Eilenberg-Mac Lane objects are strict abelian group objects, so that the previous discussion applies. Note also that  $H_\Gamma^{n-r}(W_\bullet, N) \cong \pi_r \text{map}_{s\mathcal{C}}(W_\bullet, E^\Gamma(N, n))$  for  $W_\bullet \in s\mathcal{C}/B\Gamma$ ,  $N$  a  $\Gamma$ -module, and  $0 \leq r \leq n$ . Similarly  $H_\lambda^{n-r}(\phi, \tau) \cong \pi_r \text{map}_{s\mathcal{C}(\rightarrow)}(\phi, E^\lambda(\tau, n))$ . Thus the fibration sequence (4.24) yields the desired long exact sequence in homotopy (though the last map in  $\pi_0$  need not be surjective).  $\square$

We can identify the image of  $\psi_* - \phi^*$  in cohomological terms as:

$$\text{Ker}(q_* : H^n(X_\bullet; M_1) \rightarrow H^n(X_\bullet; C)) \cap \text{Im}(\phi^* : H^n(Y_\bullet; M_1) \rightarrow H^n(X_\bullet; M_1)) ,$$

where  $q : M_1 \twoheadrightarrow C := \text{Coker}(\tau)$ .

#### 4.29. An example of the cohomology of a map.

Note that in the stable range any  $\Lambda$ -module is *trivial* – that is,  $\langle\langle -, - \rangle\rangle \equiv 0$  (in the notation of §1.15) (although of course it need not be trivial as an abelian  $\Pi$ -algebra – i.e., compositions may be non-zero).

In our example, for  $\Lambda := \text{tr}_{n+2} \pi_* \mathbf{X}$  (§1.21), and  $M := \Omega\Lambda$ , we have:

$$M_i = \begin{cases} (\mathbb{Z}/2)\langle\alpha\rangle & \text{for } i = n - 1 \\ (\mathbb{Z}/2)\langle\alpha \circ \eta\rangle & \text{for } i = n \\ (\mathbb{Z}/4)\langle\beta\rangle & \text{for } i = n + 1 \\ 0 & \text{for } i = n + 2 , \end{cases}$$

with  $2\beta = \alpha \circ \eta^2$ .

Since  $\Pi\text{-Alg}_n^{n+2}$  is an abelian category, by the Dold-Kan correspondence we can use chain-complex notation to describe a free simplicial resolution  $\mathcal{V}_\bullet$  of  $\Lambda$  as follows:

(4.30)

$$\begin{array}{ccccccc} \mathcal{S}^{n+2}\langle s \rangle & \xrightarrow{2} & \mathcal{S}^{n+2}\langle t \rangle & & \mathcal{S}^{n+2}\langle w \rangle & \xrightarrow{2} & \mathcal{S}^{n+2}\langle y \rangle \longrightarrow \beta \\ & & \searrow \eta & & \searrow & & \searrow -\eta^2 \\ & & \mathcal{S}^{n+1}\langle v \rangle & \xrightarrow{2} & \mathcal{S}^{n+1}\langle u \rangle & & \mathcal{S}^n\langle z \rangle \xrightarrow{2} \mathcal{S}^n\langle x \rangle \longrightarrow \alpha \\ & & & & \searrow \eta & & \searrow \\ & & & & & & \end{array}$$

$$\mathcal{V}_5 \xrightarrow{\partial_5} \mathcal{V}_4 \xrightarrow{\partial_4} \mathcal{V}_3 \xrightarrow{\partial_3} \mathcal{V}_2 \xrightarrow{\partial_2} \mathcal{V}_1 \xrightarrow{\partial_1} \mathcal{V}_0 \xrightarrow{\partial_0} \Lambda ,$$

A minimal free resolution  $\mathcal{V}_\bullet$  of  $\Lambda$

(where  $\partial_1(w) = 2y - x \circ \eta^2 \in \mathcal{V}_0$ ) – so we can calculate  $C^* := \text{Hom}_{\Lambda\text{-Mod}}(\mathcal{V}_\bullet, \Omega\Lambda)$  as follows:

$$\begin{array}{ccccccccccc} C^5 & \leftarrow & C^4 & \leftarrow & C^3 & \leftarrow & C^2 & \leftarrow & C^1 & \leftarrow & C^0 \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ 0 & \xleftarrow{0} & 0 & \xleftarrow{0} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/2 & \xleftarrow{0} & \mathbb{Z}/2 \end{array}$$

which implies that:

$$H^i(\Lambda; \Omega\Lambda) = \begin{cases} \mathbb{Z}/2 & \text{for } i = 0, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,  $\text{Hom}(\mathcal{V}_\bullet, \Omega\mathcal{S}^{n-1})$  is  $0 \leftarrow 0 \xleftarrow{0} \mathbb{Z}/24 \xleftarrow{2} \mathbb{Z}/24 \xleftarrow{12} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2$ , so that:

$$H^i(\Lambda; \Omega\mathcal{S}^{n-1}) = \begin{cases} \mathbb{Z}/2 & \text{for } i = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

with  $\varphi_* : H^0(\Lambda; \Omega \mathcal{S}^{n-1}) \rightarrow H^0(\Lambda; \Omega \Lambda)$  the identity, while  $\varphi_* : H^3(\Lambda; \Omega \mathcal{S}^{n-1}) \rightarrow H^3(\Lambda; \Omega \Lambda)$  is trivial (and similarly for  $\psi$  of §1.22).

On the other hand, since  $\mathcal{S}^{n-1}$  is a free  $\Pi$ -algebra, for any module  $M$  we have:

$$H^i(\mathcal{S}^{n-1}; M) = \begin{cases} M & \text{for } i = 0 \\ 0 & \text{otherwise .} \end{cases}$$

From the long exact sequence (4.28) we conclude that:

$$(4.31) \quad H_\varphi^i(\varphi; \Omega \varphi) = H_\psi^i(\psi; \Omega \psi) = \begin{cases} \mathbb{Z}/2 & \text{for } i = 3, 4 \\ 0 & \text{for } 0 < i < 3 \text{ or } 4 < i. \end{cases}$$

## 5. REALIZATIONS OF A $\Pi_{\mathcal{A}}$ -ALGEBRA

Our aim now is to address the general realization question described in the introduction – namely, given an  $E^2$ -model category  $s\mathcal{C}_{\mathcal{A}}$  and a  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$ , is there a *realization* of  $\Lambda$  in  $\mathcal{C}$  – that is, is there a  $Y \in \mathcal{C}$  such that  $\pi_{\mathcal{A}} Y \cong \Lambda$  as  $\Pi$ -algebras?

Before we state our main result, we need the following variation on the Postnikov system:

**5.1. Definition.** A *quasi-Postnikov tower* for an  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$  is a tower of fibrations:

$$(5.2) \quad \cdots \xrightarrow{p^{(n+1)}} X\langle n+1 \rangle_{\bullet} \xrightarrow{p^{(n)}} X\langle n \rangle_{\bullet} \xrightarrow{p^{(n-1)}} \cdots \xrightarrow{p^{(0)}} X\langle 0 \rangle_{\bullet} \simeq \mathbf{B}\Lambda$$

in  $s\mathcal{C}$  such that  $\pi_{\mathcal{A}} X\langle n \rangle_{\bullet} \simeq \tilde{E}^{\Lambda}(\Omega^{n+1}\Lambda, n+2)$  for every  $n > 0$ , with the sections  $s : \tilde{B}\Lambda \rightarrow \pi_{\mathcal{A}} X\langle n \rangle_{\bullet}$  (§3.7) induced by the maps  $p^{(n)}$ . The object  $X\langle n \rangle_{\bullet} \in s\mathcal{C}$  will be called an *n-th quasi-Postnikov section* for  $\Lambda$ .

**5.3. Remark.** Thus a tower (5.2) is a quasi-Postnikov tower for  $\Lambda$  if

$$(5.4) \quad \pi_k \pi_{\mathcal{A}} X\langle n \rangle_{\bullet} \cong \begin{cases} \Lambda & \text{for } k = 0, \\ \Omega^{n+1}\Lambda & \text{for } k = n+2, \\ 0 & \text{otherwise ,} \end{cases}$$

and it is equipped with maps  $\rho^{(n)} : \tilde{B}\Lambda \rightarrow \pi_{\mathcal{A}} X\langle n \rangle_{\bullet}$  over  $\tilde{B}\Lambda$ , for each  $n \geq 0$ , commuting with the maps  $p_{\#}^{(n)}$ .

We then deduce from the exact sequence (2.13) that:

$$(5.5) \quad \pi_k^{\natural}(X\langle n \rangle_{\bullet}, \mathcal{A}) \cong \begin{cases} \Omega^k \Lambda & \text{for } 0 \leq k \leq n, \\ 0 & \text{otherwise .} \end{cases}$$

Note that (5.5) implies in turn that the (ordinary) Postnikov sections  $\mathbf{P}_k X\langle n \rangle_{\bullet}$  of  $X\langle n \rangle_{\bullet}$  constitute quasi-Postnikov sections for  $\Lambda$ , for  $k \leq n$  (see also [BDG, Prop. 9.11]).

We are now in a position to state the two key results addressing our realization question (the proofs are deferred to §§5.15-5.16):

**5.6. Theorem.** *If  $s\mathcal{C}_{\mathcal{A}}$  is an  $E^2$ -model category and  $\Lambda \in \Pi_{\mathcal{A}}\text{-Alg}$ , the following are equivalent:*

- (1)  $\Lambda$  is realizable – that is, there is a  $Y \in \mathcal{C}$  with  $\pi_{\mathcal{A}} Y \cong \Lambda$ ;
- (2) There is an  $X_{\bullet} \in s\mathcal{C}$  with  $\pi_{\mathcal{A}} X_{\bullet} \simeq \tilde{B}\Lambda$ .
- (3) There is a quasi-Postnikov tower for  $\Lambda$ .

**5.7. Theorem.** *Let  $X\langle n-1\rangle_\bullet \in s\mathcal{C}$  be an  $(n-1)$ -st quasi-Postnikov section for a  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$ . Then:*

- (a) *Up to homotopy, there is a unique  $X\langle n\rangle_\bullet \in s\mathcal{C}$  satisfying (5.4) and (5.5), with  $\mathbf{P}_{n-1}X\langle n\rangle_\bullet = X\langle n-1\rangle_\bullet$ .*
- (b) *This  $X\langle n\rangle_\bullet$  is an  $n$ -th quasi-Postnikov section for  $\Lambda$  if and only if the  $(n+2)$ -nd  $\tilde{k}$ -invariant for  $\pi_{\mathcal{A}}X\langle n\rangle_\bullet$  vanishes in  $H_\Lambda^{n+3}(\tilde{B}\Lambda; \Omega^{n+1}\Lambda)$ .*
- (c) *In that case,  $X\langle n+1\rangle_\bullet$  exists, by (a); furthermore, the different choices for the map  $p^{(n)} : X\langle n+1\rangle_\bullet \rightarrow X\langle n\rangle_\bullet$  – or equivalently, choices of the section  $\tilde{s}_n : \tilde{B}\Lambda \rightarrow \tilde{E}^\Lambda(\Omega^{n+1}\Lambda, n+2) = \pi_{\mathcal{A}}X\langle n\rangle_\bullet$  of §3.7 – are in one-to-one correspondence with elements of  $H_\Lambda^{n+2}(\tilde{B}\Lambda; \Omega^{n+1}\Lambda)$ .*

Compare [Ba, Ch. D, (7.9)].

Our approach to constructing an  $X_\bullet$  in Theorem 5.6 (2) will be inductive, using its Postnikov system, which serves as a quasi-Postnikov tower for  $\Lambda$ . Thus at each stage we will have the obstruction of Theorem 5.7 (b) to moving up one more level. To explain why this works (and prove the two Theorems), we shall need some facts about:

### 5.8. Connections between the Postnikov systems.

Given any simplicial object  $X_\bullet \in s\mathcal{C}$ , consider its  $n$ -th Postnikov section  $\mathbf{P}_n X_\bullet$ , for some  $n > 0$ , and let  $\Lambda := \pi_0^\natural(X_\bullet, \mathcal{A}) = \pi_0 \pi_{\mathcal{A}} X_\bullet$ . We want to describe the simplicial  $\Pi_{\mathcal{A}}$ -algebra  $\pi_{\mathcal{A}} \mathbf{P}_n X_\bullet$  (up to homotopy) in terms of  $\pi_{\mathcal{A}} X_\bullet$ , and whatever other information is necessary.

First, observe that (2.13) also implies:

$$(5.9) \quad \pi_k \pi_{\mathcal{A}} \mathbf{P}_n X_\bullet \cong \begin{cases} \pi_k \pi_{\mathcal{A}} X_\bullet & \text{for } k \leq n, \\ \text{Coker}(h_{n+1}^X : \pi_{n+1}^\natural(X_\bullet, \mathcal{A}) \rightarrow \pi_{n+1} \pi_{\mathcal{A}} X_\bullet) & \text{for } k = n+1, \\ \Omega \pi_n^\natural(X_\bullet, \mathcal{A}) & \text{for } k = n+2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, when  $\pi_{\mathcal{A}} X_\bullet \simeq \tilde{B}\Lambda$ , (5.9) simplifies to:

$$(5.10) \quad \pi_k \pi_{\mathcal{A}} \mathbf{P}_n X_\bullet \cong \begin{cases} \Lambda & \text{for } k = 0, \\ \Omega^{n+1}\Lambda & \text{for } k = n+2, \\ 0 & \text{otherwise,} \end{cases}$$

**5.11. Lemma.** *For any  $X_\bullet \in s\mathcal{C}$ , we have a homotopy fibration sequence in  $s\Pi_{\mathcal{A}}\text{-Alg}/\tilde{B}\Lambda$  (that is, a homotopy pullback square over  $\tilde{B}\Lambda$ ):*

$$\pi_{\mathcal{A}} \mathbf{P}_{n+1} X_\bullet \xrightarrow{p_\#^{(n)}} \pi_{\mathcal{A}} \mathbf{P}_n X_\bullet \xrightarrow{(\mathbf{k}_n)_\#} \pi_{\mathcal{A}} \mathbf{E}^\Lambda(\pi_{n+1}^\natural(X_\bullet, \mathcal{A}), n+2).$$

*Proof.* Definition 2.6(b) implies that  $(\mathbf{k}_n)_\# : \pi_{\mathcal{A}} \mathbf{P}_n X_\bullet \rightarrow \pi_{\mathcal{A}} \mathbf{E}^\Lambda(\pi_{n+1}^\natural(X_\bullet, \mathcal{A}), n+2)$  is an  $\mathcal{A}$ -fibration over  $\pi_{\mathcal{A}} \mathbf{B}\Lambda$ . Denote its fiber by  $F_\bullet$ , with a natural map of simplicial  $\Pi_{\mathcal{A}}$ -algebras  $\varphi : \pi_{\mathcal{A}} \mathbf{P}_{n+1} X_\bullet \rightarrow F_\bullet$ .

Because the functors  $\pi_k \pi_{\mathcal{A}} : s\mathcal{C} \rightarrow \Pi_{\mathcal{A}}\text{-Alg}$  are corepresentable for  $k > 1$  (cf. [DKS2, §7.4]), applying  $\pi_{\mathcal{A}}$  to the homotopy pull-back (3.11) yields a “quasi-fibration” of simplicial  $\Pi_{\mathcal{A}}$ -algebras, and so a long exact sequence in homotopy (in dimensions  $\geq 2$ ), which implies that  $\varphi_\#$  is an isomorphism in dimensions  $\geq 2$ ; since this is trivially true in dimensions 0 and 1,  $\varphi$  is a weak equivalence.  $\square$

5.12. **Lemma.** *If we write  $E_\bullet := \mathbf{E}^\Lambda(\pi^{\natural}_{n+1}(X_\bullet, \mathcal{A}), n+2)$ , then applying  $\pi_{n+2}\pi_{\mathcal{A}}$  to the  $k$ -invariant  $\mathbf{k}_n : \mathbf{P}_n X_\bullet \rightarrow E_\bullet$  yields the homomorphism  $s_{n+1} : \Omega \pi^{\natural}_n(X_\bullet, \mathcal{A}) \rightarrow \pi^{\natural}_{n+1}(X_\bullet, \mathcal{A})$  of (2.13).*

*Proof.* First, note that, in the following commutative diagram:

$$\begin{array}{ccccccc}
& & 0 & \longrightarrow & \pi^{\natural}_{n+1}(\Omega E_\bullet, \mathcal{A}) & \xrightarrow{\cong} & \pi_{n+1}\pi_{\mathcal{A}}\Omega E_\bullet \\
& & \downarrow & & \downarrow \cong & & \downarrow \\
\pi_{n+2}\pi_{\mathcal{A}}\mathbf{P}_{n+1}X_\bullet & \xrightarrow{\partial_{n+2}} & \Omega \pi^{\natural}_n(\mathbf{P}_{n+1}X_\bullet, \mathcal{A}) & \xrightarrow{s_{n+1}} & \pi^{\natural}_{n+1}(\mathbf{P}_{n+1}X_\bullet, \mathcal{A}) & \xrightarrow{h_{n+1}} & \pi_{n+1}\pi_{\mathcal{A}}\mathbf{P}_{n+1}X_\bullet \\
& \downarrow & \downarrow \cong & & \downarrow & & \\
\pi_{n+2}\pi_{\mathcal{A}}\mathbf{P}_n X_\bullet & \xrightarrow{\cong} & \Omega \pi^{\natural}_n(\mathbf{P}_n X_\bullet, \mathcal{A}) & \longrightarrow & \pi^{\natural}_{n+1}(\mathbf{P}_n X_\bullet, \mathcal{A}) = 0 & & \\
& \downarrow (\mathbf{k}_n)_\# & & & & & \\
& & \pi_{n+2}\pi_{\mathcal{A}}E_\bullet & & & & 
\end{array}$$

the isomorphisms of  $\pi_{n+2}\pi_{\mathcal{A}}\mathbf{P}_n X_\bullet$  with  $\Omega \pi^{\natural}_n(\mathbf{P}_{n+1}X_\bullet, \mathcal{A})$ , and  $\pi^{\natural}_{n+1}(\mathbf{P}_{n+1}X_\bullet, \mathcal{A})$  with  $\pi_{n+1}\pi_{\mathcal{A}}\Omega E_\bullet$ , are natural. Also, the columns here are exact either by the long exact sequence in  $\pi^{\natural}_*$  for a fibration in  $s\mathcal{C}$ , or by Lemma 5.11.

The result now follows from the naturality of the exact sequence (2.13), applied to the fibration sequence:

$$\Omega E_\bullet \simeq \mathbf{E}^\Lambda(\pi^{\natural}_{n+1}(X_\bullet, \mathcal{A}), n+1) \rightarrow \mathbf{P}_{n+1}X_\bullet \rightarrow \mathbf{P}_n X_\bullet \xrightarrow{\mathbf{k}_n} E_\bullet.$$

□

5.13. **Lemma.** *If  $\pi_{\mathcal{A}}X_\bullet \simeq \tilde{B}\Lambda$ , then the spiral exact sequence (2.13) for  $X_\bullet$  from  $\pi_{n+3}\pi_{\mathcal{A}}X_\bullet$  down is determined by the homomorphism  $\partial_{n+3}^* : \pi_{n+3}\pi_{\mathcal{A}}X_\bullet \rightarrow \Omega \pi^{\natural}_{n+1}(X_\bullet, \mathcal{A})$ .*

*Proof.* First, observe that given  $\mathbf{P}_n X_\bullet$ , we know the exact sequence (2.13) for  $X_\bullet$  only from  $\Omega \pi^{\natural}_{n-1}(X_\bullet, \mathcal{A})$  down. However, when  $r_\#^{(n)} : \pi_*\pi_{\mathcal{A}}X_\bullet \rightarrow \pi_*\pi_{\mathcal{A}}\mathbf{P}_n X_\bullet$  is also known, and  $\pi_{\mathcal{A}}X_\bullet \simeq \tilde{B}\Lambda$ , then all we need in order to determine (2.13) for  $X_\bullet$  from  $\pi_{n+3}\pi_{\mathcal{A}}X_\bullet$  down is the homomorphism  $(r_\#^{(n+1)})_{n+3} : \pi_{n+3}\pi_{\mathcal{A}}X_\bullet \rightarrow \pi_{n+3}\pi_{\mathcal{A}}\mathbf{P}_{n+1}X_\bullet$  – which is just  $\partial_{n+3}^* : \pi_{n+3}\pi_{\mathcal{A}}X_\bullet \rightarrow \Omega \pi^{\natural}_{n+1}(X_\bullet, \mathcal{A})$ .

□

5.14. **Lemma.** *If  $\tilde{k}_{n+1}(\pi_{\mathcal{A}}X_\bullet) : \tilde{P}_{n+1}\pi_{\mathcal{A}}X_\bullet \rightarrow \tilde{E}^\Lambda(\pi_{n+2}\pi_{\mathcal{A}}X_\bullet, n+3)$  is the  $(n+1)$ -st  $\tilde{k}$ -invariant for  $\pi_{\mathcal{A}}X_\bullet$ , then the  $(n+1)$ -st  $\tilde{k}$ -invariant*

$$\tilde{k}_{n+1}(\pi_{\mathcal{A}}\mathbf{P}_n X_\bullet) : \tilde{P}_{n+1}\pi_{\mathcal{A}}\mathbf{P}_n X_\bullet \rightarrow \tilde{E}^\Lambda(\Omega \pi^{\natural}_n(X_\bullet, \mathcal{A}), n+3)$$

*satisfies:*  $(\partial_{n+2}^*)_* \circ \tilde{k}_{n+1}(\pi_{\mathcal{A}}X_\bullet) = \tilde{k}_{n+1}(\pi_{\mathcal{A}}\mathbf{P}_n X_\bullet) \circ \tilde{P}_{n+1}(r_\#^{(n)})$ .

*Proof.* This follows from the naturality of the  $\tilde{k}$ -invariants (Ax 3 of §3.12) and Lemma 5.13. □

### 5.15. Proof of Theorem 5.6.

(1)  $\iff$  (2): Given  $Y$ , let  $X_\bullet := c(Y)_\bullet$ . Conversely, if  $X_\bullet \in s\mathcal{C}/\mathbf{B}\Lambda$  satisfies  $\pi_{\mathcal{A}}X_\bullet \simeq \tilde{B}\Lambda$ , then by Ax 4 of §3.12, there is a functor  $J : s\mathcal{C}_{\mathcal{A}} \rightarrow \mathcal{C}$  equipped with an isomorphism:

$$[A, JX_\bullet]_{\mathcal{C}} \cong \text{Hom}_{\Pi_{\mathcal{A}}\text{-Alg}}(\pi_{\mathcal{A}}A, \Lambda),$$

natural in  $A \in \mathcal{A}$ . Thus  $\pi_{\mathcal{A}} JX_{\bullet} \cong \Lambda$  as  $\Pi_{\mathcal{A}}$ -algebras, by Yoneda's Lemma, so we can take  $Y := JX_{\bullet}$ .

(2)  $\iff$  (3): By [BDG, Prop. 9.11] we know that  $\pi_{\mathcal{A}} X_{\bullet} \simeq \tilde{B}\Lambda$  if and only if  $\pi_{\mathcal{A}} \mathbf{P}_n X_{\bullet} \simeq \tilde{E}^{\Lambda}(\Omega^{n+1}\Lambda, n+2)$ .

Thus given  $X_{\bullet}$  with  $\pi_{\mathcal{A}} X_{\bullet} \simeq \tilde{B}\Lambda$ , the ordinary Postnikov tower  $\mathbf{P}_k X_{\bullet}$  of  $X\langle n \rangle_{\bullet}$  constitutes a quasi-Postnikov tower for  $\Lambda$ , by (5.10).

Conversely, given a quasi-Postnikov tower (5.2) for  $\Lambda$ , let  $X_{\bullet} := \text{holim}_n X\langle n \rangle_{\bullet}$ . Since  $\tilde{P}_{n+1}\rho^{(n)} : \tilde{B}\Lambda \rightarrow \tilde{P}_{n+1}\pi_{\mathcal{A}} X\langle n \rangle_{\bullet}$  is a weak equivalence for each  $n$ , the maps  $\rho^{(n)}$  induce a weak equivalence  $r : \tilde{B}\Lambda \xrightarrow{\simeq} \pi_{\mathcal{A}} X_{\bullet}$ .  $\square$

### 5.16. Proof of Theorem 5.7.

Let  $X\langle n-1 \rangle_{\bullet}$  be an  $(n-1)$ -st quasi-Postnikov section for  $\Lambda$ . By assumption  $\pi_{\mathcal{A}} X\langle n-1 \rangle_{\bullet} \simeq \tilde{E}^{\Lambda}(\Omega^n \Lambda, n+1)$ , and the map  $\rho^{(n-1)} : \tilde{B}\Lambda \rightarrow \pi_{\mathcal{A}} X\langle n-1 \rangle_{\bullet}$  is the required section.

- (a) In order to construct  $X\langle n \rangle_{\bullet}$ , we must choose a suitable  $(n-1)$ -st  $\mathbf{k}$ -invariant  $\mathbf{k}_{n-1} \in [X\langle n-1 \rangle_{\bullet}, \mathbf{E}^{\Lambda}(\Omega^n \Lambda, n+1)]_{\mathbf{B}\Lambda}$ . Note that using the long exact sequence in  $\pi^{\sharp}$  for a fibration over  $\mathbf{B}\Lambda$ , combined with (2.13), automatically ensures that any such choice yields  $X\langle n \rangle_{\bullet}$  satisfying (5.4) and (5.5).

We can use the map  $\zeta : \pi_{\mathcal{A}} \mathbf{E}^{\Lambda}(\Omega^n \Lambda, n+1) \rightarrow \tilde{E}^{\Lambda}(\Omega^n \Lambda, n+1)$  of Proposition 4.4 to define  $\mathbf{k}_{n-1} : X\langle n-1 \rangle_{\bullet} \rightarrow \mathbf{E}^{\Lambda}(\Omega^n \Lambda, n+1)$  (uniquely up to homotopy) by specifying  $\zeta \circ (\mathbf{k}_{n-1})_{\#} : \pi_{\mathcal{A}} X\langle n-1 \rangle_{\bullet} \rightarrow \tilde{E}^{\Lambda}(\Omega^n \Lambda, n+1)$ . Since  $\pi_{\mathcal{A}} X\langle n-1 \rangle_{\bullet} \simeq \tilde{E}^{\Lambda}(\Omega^n \Lambda, n+1)$ , the functoriality of Ax 2 of §3.12 implies that such a map is uniquely determined up to homotopy by a map of  $\Lambda$ -modules  $\varphi : \Omega^n \Lambda \rightarrow \Omega^n \Lambda$ , and by Lemma 5.12 this  $\varphi$  must be the given isomorphism  $(s_{n+1}) : \Omega \pi^{\sharp}_{n-1}(X\langle n-1 \rangle_{\bullet}, \mathcal{A}) \rightarrow \Omega^n \Lambda$ , if the quasi-Postnikov tower we are constructing for  $\Lambda$  is to be a Postnikov tower in  $s\mathcal{C}$ . (Note that by Lemma 5.13, we already know the long exact sequence (2.13) for  $X\langle n \rangle_{\bullet}$  from  $s_{n+1}$  down.) Thus the candidate for  $X\langle n \rangle_{\bullet}$  over  $X\langle n-1 \rangle_{\bullet}$ , satisfying (5.4) and (5.5), is determined uniquely up to homotopy by  $X\langle n-1 \rangle_{\bullet}$ .

- (b) There is only one possible obstruction to  $X\langle n \rangle_{\bullet}$  (the homotopy fiber of  $\mathbf{k}_{n-1}$  in  $s\mathcal{C}/\mathbf{B}\Lambda$ ) being an  $n$ -th quasi-Postnikov section for  $\Lambda$ : the non-existence of the lift  $\rho^{(n)} : \tilde{B}\Lambda \rightarrow \pi_{\mathcal{A}} X\langle n \rangle_{\bullet}$ . However, since  $\tilde{P}_{n+1}\pi_{\mathcal{A}} X\langle n \rangle_{\bullet} \simeq \tilde{B}\Lambda$ , by (5.5), we may use the long exact sequence in  $\pi_{\mathcal{A}}$  for the fibration sequence:

$$(5.17) \quad \pi_{\mathcal{A}} X\langle n \rangle_{\bullet} = \tilde{P}_{n+2}\pi_{\mathcal{A}} X\langle n \rangle_{\bullet} \xrightarrow{\tilde{p}^{(n+2)}} \tilde{P}_{n+1}\pi_{\mathcal{A}} X\langle n \rangle_{\bullet} \xrightarrow{\tilde{k}_{n+1}} \tilde{E}^{\Lambda}(\Omega^{n+1}\Lambda, n+3)$$

over  $\tilde{B}\Lambda$  to deduce that  $\rho^{(n-1)}$  lifts to  $\rho^{(n)}$  if and only if  $\tilde{k}_{n+1}$  is null in  $s\Pi_{\mathcal{A}}\text{-Alg}/\tilde{B}\Lambda$ .

More precisely, we want  $\rho^{(n)}$  to map to the homotopy pullback (Ax 3 of §3.12) in:

$$(5.18) \quad \begin{array}{ccc} \tilde{P}_{n+1}\tilde{B}\Lambda & \xrightarrow{\simeq} & \tilde{B}\Lambda \\ \downarrow \rho^{(n)} & \searrow & \downarrow \tilde{k}_{n+1} \\ \pi_{\mathcal{A}} \mathbf{P}_n X_{\bullet} & \xrightarrow{\quad} & \tilde{B}\Lambda \\ \downarrow \simeq & & \downarrow \tilde{k}_{n+1} \\ \tilde{B}\Lambda & \xrightarrow{\tilde{s}} & \tilde{E}^{\Lambda}(\Omega^{n+1}\Lambda, n+3), \end{array}$$

which is possible if and only if  $\tilde{k}_{n+1}$  is homotopic to the given homotopy section  $\tilde{s} : \tilde{B}\Lambda \rightarrow \tilde{E}^\Lambda(\Omega^{n+1}\Lambda, n+3)$ .

- (c) Since the fiber (over  $\tilde{B}\Lambda$ ) of  $\tilde{p}^{(n+2)}$  in (5.17) is  $\tilde{E}^\Lambda(\Omega^{n+1}\Lambda, n+2)$ , the possible choices for such lifts are distinguished by elements of:

$$[\tilde{B}\Lambda, \tilde{E}^\Lambda(\Omega^{n+1}\Lambda, n+2)]_{\tilde{B}\Lambda} = H^{n+2}(\tilde{B}\Lambda/\Lambda, \Omega^{n+1}\Lambda),$$

which are in fact just choices for  $\partial_{n+3}^* : \pi_{n+3}\tilde{B}\Lambda \rightarrow \Omega \pi_{n+1}^\natural(X\langle n+1 \rangle_\bullet, \mathcal{A})$  (see 5.13). These determine the identification of  $\pi_{\mathcal{A}}X\langle n \rangle_\bullet$  with  $\tilde{E}^\Lambda(\Omega^{n+1}\Lambda, n+2)$ , which is the only freedom in the inductive procedure we have described.  $\square$

5.19. *Remark.* To appreciate the explicit inductive construction of these obstructions provided in the above proof, let us examine more carefully the first step in realizing a  $\Pi_{\mathcal{A}}$ -algebra  $\Lambda$ :

Note first that, from the spiral exact sequence and Postnikov sections, the homotopy groups of  $\mathbf{B}\Lambda$  fit into the algebraic extension:

$$\tilde{E}^\Lambda(\Omega\Lambda, 2) \rightarrow \pi_*\mathbf{B}\Lambda \rightarrow \tilde{B}\Lambda,$$

and so yields an element of  $\text{exal}_\Lambda(\tilde{B}\Lambda; \tilde{E}^\Lambda(\Omega\Lambda, 2))$  (see Remark 4.7). Using (4.8), we may view this extension as an element of  $H^3(\tilde{B}\Lambda/\Lambda, \Omega\Lambda)$ , which is precisely the first obstruction to realizing  $\Lambda$ . Note that by Ax 4 of 3.12, this obstruction is natural in  $\Lambda$ . See [BKS] for a similar perspective on the obstructions to realizing modules over the Tate cohomology of a group  $G$  as the group cohomology of a  $G$ -module.

5.20. *Remark.* The realization problem, as formulated in this section, and its solution in Theorem 5.6 applies to  $\Pi$ -algebras associated to any of the categories listed in §2.18 -  $n$ -connected spaces,  $p$ -local or rational spaces,  $n$ -types (and so on) - as well as any diagrams of such  $\Pi$ -algebras. Note, however, that realization is a tautology when  $\mathcal{C}$  itself had a trivial model category structure - e.g., if  $\mathcal{C} = \Theta\text{-Alg}$  is a variety of universal algebras.

## 6. REALIZING MAPS OF $\Pi$ -ALGEBRAS

We now examine the diagram realization question in more detail for the simplest non-trivial case: a single map of (ordinary)  $\Pi$ -algebras  $\varphi : \Lambda \rightarrow \Gamma$ .

### 6.1. Maps of realizable $\Pi$ -algebras.

Assume for simplicity that the  $\Pi$ -algebras  $\Lambda$  and  $\Gamma$  are realizable, and replace them by cofibrant simplicial models  $\psi : K_\bullet \rightarrow L_\bullet$  in  $s\Pi\text{-Alg}$ .

Note that if we are *given* realizations  $V_\bullet, W_\bullet$  for  $K_\bullet$  and  $L_\bullet$ , respectively (equivalently: for  $\Lambda$  and  $\Gamma$ ), we have the usual obstruction theory for lifting  $f^0 := \mathbf{B}\phi \circ \mathbf{p}^{(0)} : V_\bullet \rightarrow \mathbf{B}\Gamma = \mathbf{P}_0W_\bullet$  through the successive Postnikov stages for  $W_\bullet$ , with the existence and difference obstructions all lying in the Quillen cohomology groups  $H^*(V_\bullet/\mathbf{B}\Gamma; \Omega^n\Gamma)$ . However, in our approach we want to choose the realizations for the  $\Pi$ -algebras  $\Lambda$  and  $\Gamma$ , and for the map  $\varphi$ , simultaneously - again by induction on the quasi-Postnikov system.

At the  $n$ -th stage, we assume that we have a map of simplicial spaces  $f\langle n \rangle : X\langle n \rangle_\bullet \rightarrow Y\langle n \rangle_\bullet$ , where:

- a)  $X\langle n \rangle_\bullet \simeq \mathbf{P}_n X\langle n \rangle_\bullet$  and  $Y\langle n \rangle_\bullet \simeq \mathbf{P}_n Y\langle n \rangle_\bullet$ ; and
- b)  $\tilde{P}_n(f\langle n \rangle)_\# : \tilde{P}_n \pi_{\mathcal{A}} X\langle n \rangle_\bullet \rightarrow \tilde{P}_n \pi_{\mathcal{A}} Y\langle n \rangle_\bullet$  is  $\varphi_* : \tilde{B}\Lambda \rightarrow \tilde{B}\Gamma$ .

Our goal is to extend  $f$  to  $(n+1)$ -stage Postnikov pieces. Because the sections  $\tilde{s}_n^\Lambda : \tilde{B}\Lambda \rightarrow \tilde{E}^\Lambda(\Omega^{n+1}\Lambda, n+2)$  and  $\tilde{s}_n^\Gamma : \tilde{B}\Gamma \rightarrow \tilde{E}^\Gamma(\Omega^{n+1}\Gamma, n+2)$  will ultimately be induced by the natural Postnikov maps  $W_\bullet \rightarrow \mathbf{P}_n W_\bullet \simeq X\langle n \rangle_\bullet$  and  $V_\bullet \rightarrow \mathbf{P}_n V_\bullet \simeq Y\langle n \rangle_\bullet$ , say, we know that if  $f\langle n \rangle$  extends we will have naturality for the sections, so our object is to choose  $\tilde{s}_n^\Lambda$  and  $\tilde{s}_n^\Gamma$  so that the diagram

$$(6.2) \quad \begin{array}{ccc} \tilde{B}\Lambda & \xrightarrow{\varphi\#} & \tilde{B}\Gamma \\ \downarrow \tilde{s}_n^\Lambda & & \downarrow \tilde{s}_n^\Gamma \\ \tilde{E}^\Lambda(\Omega^{n+1}\Lambda, n+2) & \xrightarrow{f\#} & \tilde{E}^\Gamma(\Omega^{n+1}\Gamma, n+2) \end{array}$$

commutes up to homotopy. This means that  $(\tilde{s}_n^\Lambda, \tilde{s}_n^\Gamma)$  is just the obstruction class in  $H_\varphi^{n+2}(\varphi; \Omega^n \varphi)$  described by Theorem 5.6.

### 6.3. An example of the obstructions to realizability.

We now apply the above theory to the map of  $\Pi$ -algebras  $\varphi : \Lambda \rightarrow \mathcal{S}^{n-1}$  considered in §4.29. By [Bl5, Thm. 3.16], we know that the resolution (4.30), as well as the constant free resolution  $W_\bullet \rightarrow \mathcal{S}^{n-1}$ , are realizable by simplicial spaces.

The relevant part of the realization of (4.30) is described in (6.4), where the indexing is based on the Stover resolution comonad in the obvious way, with  $d_0$  on  $\mathbf{S}_{\langle \beta, 2 \rangle - \langle \alpha, \eta^2 \rangle}^{n+2}$  equal to the difference of the degree 2 map to  $\mathbf{S}_\beta^{n+2}$  and  $\eta^2$  to  $\mathbf{S}_\alpha^n$ , and all face maps  $d_1$  and  $d_2$  are inclusions.

$$(6.4) \quad \begin{array}{c} \begin{array}{c} \mathbf{S}_{\langle \beta, 2 \rangle - \langle \alpha, \eta^2 \rangle}^{n+2} \\ \searrow^{d_1} \\ \mathbf{S}_{\langle \beta 2 \rangle - \langle \alpha \eta^2 \rangle}^{n+2} \cup \mathbf{e}_H^{n+3} \\ \searrow^{d_1} \\ \mathbf{S}_\beta^{n+2} = \mathbf{S}_{\alpha 2 \eta}^{n+1} \cup \mathbf{e}_{\alpha \circ F}^{n+2} \cup \mathbf{e}_{G \circ C \eta}^{n+2} \\ \searrow^{d_1} \\ \mathbf{S}_{\alpha 2}^n \cup \mathbf{e}_G^{n+1} \\ \searrow^{d_1} \\ \mathbf{S}_\alpha^n \end{array} \\ \begin{array}{c} \mathbf{S}_{\langle \alpha, 2, \eta \rangle}^{n+1} \xrightarrow{d_2} \mathbf{S}_{\langle \alpha 2, \eta \rangle}^{n+1} \cup \mathbf{e}_{G, C \eta}^{n+2} \\ \searrow^{d_1} \\ \mathbf{S}_{\langle \alpha, 2 \eta \rangle}^{n+1} \cup \mathbf{e}_{\alpha, F}^{n+2} \\ \searrow^{d_1} \\ \mathbf{S}_{\langle \alpha, 2 \rangle}^n \end{array} \\ \begin{array}{c} \mathbf{S}_{\langle \alpha, 2 \rangle}^n \xrightarrow{d_1} \mathbf{S}_{\langle \alpha 2 \rangle}^n \\ \searrow^{d_1} \\ \mathbf{S}_\alpha^n \end{array} \\ \begin{array}{c} \mathbf{S}_{\langle \alpha, 2 \rangle}^n \xrightarrow{d_0=\eta} \mathbf{S}_{\langle \alpha 2 \rangle}^n \\ \searrow^{d_0=2} \\ \mathbf{S}_\alpha^n \end{array} \\ \begin{array}{c} \mathbf{S}_{\langle \beta 2 \rangle - \langle \alpha \eta^2 \rangle}^{n+2} \xrightarrow{d_0=2} \mathbf{S}_{\langle \beta 2 \rangle - \langle \alpha \eta^2 \rangle}^{n+2} \cup \mathbf{e}_H^{n+3} \\ \searrow^{d_0=C\eta} \\ \mathbf{S}_\beta^{n+2} \\ \searrow^{d_0=F} \\ \mathbf{S}_{\alpha 2}^n \\ \searrow^{d_0=2} \\ \mathbf{S}_\alpha^n \end{array} \\ \begin{array}{c} \mathbf{S}_{\langle \beta 2 \rangle - \langle \alpha \eta^2 \rangle}^{n+2} \xrightarrow{d_0=2} \mathbf{S}_{\langle \beta 2 \rangle - \langle \alpha \eta^2 \rangle}^{n+2} \cup \mathbf{e}_H^{n+3} \\ \searrow^{\varepsilon=\beta} \\ \mathbf{S}_\beta^{n+2} \\ \searrow^{\varepsilon=H} \\ \mathbf{S}_{\alpha 2}^n \\ \searrow^{\varepsilon=G} \\ \mathbf{S}_\alpha^n \end{array} \\ \begin{array}{c} \mathbf{S}_{\langle \beta 2 \rangle - \langle \alpha \eta^2 \rangle}^{n+2} \xrightarrow{d_0=2} \mathbf{S}_{\langle \beta 2 \rangle - \langle \alpha \eta^2 \rangle}^{n+2} \cup \mathbf{e}_H^{n+3} \\ \searrow^{\varepsilon=\alpha} \\ \mathbf{S}_\alpha^n \end{array} \\ \mathbf{X} \end{array}$$

$$\mathbf{V}_2 \rightrightarrows \mathbf{V}_1 \rightrightarrows \mathbf{V}_0 \longrightarrow \mathbf{X},$$

A minimal free resolution  $\mathbf{V}_\bullet$  in  $s\mathcal{T}$

The inductive approach to realizing  $\varphi : \Lambda \rightarrow \mathcal{S}^{n-1}$  described in §6.1 begins with  $f\langle 0 \rangle : X\langle 0 \rangle_\bullet \rightarrow Y\langle 0 \rangle_\bullet$ , which is just  $\mathbf{B}\varphi : \mathbf{B}\Lambda \rightarrow \mathbf{B}\mathcal{S}^{n-1}$ . Moreover, the proof of Theorem 5.6 shows that this always extends uniquely to  $f\langle 1 \rangle : X\langle 1 \rangle_\bullet \rightarrow Y\langle 1 \rangle_\bullet$  (although the lifting  $\rho^{(1)}$  as required in §5.3 need not exist).

The construction of Postnikov systems (Ax 1 of Theorems 3.15, 3.19) shows that the existence of  $f\langle 1 \rangle$  is equivalent to having a 2-truncated augmented simplicial space  $\mathbf{V}'_\bullet \rightarrow \mathbf{S}^{n-1}$  realizing the augmented simplicial  $\Pi$ -algebra  $\mathcal{V}_\bullet \rightarrow \mathcal{S}^{n-1}$  induced by  $\varphi : \Lambda \rightarrow \mathcal{S}^{n-1}$ .

Using Lemma 4.16, we may assume that the composite of the maps  $\mathbf{S}^n \xrightarrow{2} \mathbf{S}^n \xrightarrow{\eta} \mathbf{S}^{n-1}$  is actually null, so we can describe  $\mathbf{V}'_\bullet$  explicitly by (6.5). Moreover,  $X\langle 1 \rangle_\bullet$ , and thus  $\mathbf{V}'_\bullet$ , is unique up to homotopy (in  $s\mathcal{T}$ ).

$$(6.5) \quad \begin{array}{c} \mathbf{S}^{n+2}_{\langle 2\nu, 2 \rangle - \langle \eta, \eta^2 \rangle} \xrightarrow{d_1 = \text{incl.}} \mathbf{S}^{n+2}_{\langle 4\nu \rangle - \langle \eta^3 \rangle} \cup \mathbf{e}_H^{n+3} \\ \downarrow d_0 = 2 \quad \searrow -\eta^2 \\ \mathbf{S}^{n+1}_{\langle \eta, 2, \eta \rangle} \xrightarrow{d_2 = 0} * \quad \xrightarrow{d_1 = \text{incl.}} \mathbf{S}^{n+1}_{\langle \eta, 2 \eta \rangle} \cup \mathbf{e}_{\eta, F}^{n+2} \xrightarrow{d_1 = 0} \mathbf{S}^{n+2}_{2\nu} \\ \downarrow d_0 = \eta \quad \downarrow d_0 = F \quad \downarrow \varepsilon = 2\nu \quad \downarrow \varepsilon = H \\ \mathbf{S}^n_{\langle \eta, 2 \rangle} \xrightarrow{d_1 = 0} * \quad \xrightarrow{d_0 = 2} \mathbf{S}^n_{\eta} \xrightarrow{\varepsilon = \eta} \mathbf{S}^{n-1} \end{array}$$

$$\mathbf{V}'_2 \rightrightarrows \mathbf{V}'_1 \rightrightarrows \mathbf{V}'_0 \longrightarrow \mathbf{S}^{n-1},$$

An augmentation of  $\mathbf{V}'_\bullet$  to  $\mathbf{S}^{n-1}$

However, in constructing  $\mathbf{V}'_\bullet \rightarrow \mathbf{S}^{n-1}$  we have “distorted” the original augmented simplicial space  $\mathbf{V}_\bullet \rightarrow \mathbf{X}$  in such a way that we no longer have a strict augmentation  $\mathbf{V}'_\bullet \rightarrow \mathbf{X}$ .

We can see this geometrically, using the Toda bracket

$$(6.6) \quad \langle \eta, 2, \alpha \rangle = \{ \beta, \beta + \alpha \circ \eta^2 \} \subseteq \pi_{n+2} \mathbf{X}$$

(see, e.g., [Bl1, §6]), which we used in the decomposition  $\mathbf{S}^{n+2}_\beta = \mathbf{S}^{n+1}_{\alpha 2\eta} \cup \mathbf{e}_{\alpha \circ F}^{n+2} \cup \mathbf{e}_{G \circ C\eta}^{n+2}$  in (6.4). Because we no longer have this in (6.5), we must have  $0 \in \langle \eta, 2, \alpha \rangle$  for any augmentation  $\alpha : \mathbf{S}^n \rightarrow \mathbf{X}$  on  $\mathbf{S}^n \subseteq \mathbf{V}'_0$ .

More formally, (6.6) yields a non-vanishing second-order homotopy operation in  $[\Sigma \mathbf{V}'_2, \mathbf{X}]$  which is the obstruction to rectifying the homotopy augmentation  $\mathbf{V}'_\bullet \rightarrow \mathbf{X}$  realizing  $\mathcal{V}_\bullet \rightarrow \Lambda$ , using [Bl2, Theorem 7.13 & Lemma 5.12]. But then we may use the equivalent obstruction theory of [Bl4, BDG] to deduce that the  $\tilde{k}$ -invariant  $\tilde{k}_1 \in H^3_\Lambda(\tilde{B}\Lambda; \Omega\Lambda) \cong \mathbb{Z}/2$  does not vanish, for the choice of  $X\langle 0 \rangle_\bullet$  described in (6.5) (with  $\eta : \mathbf{S}^n \rightarrow \mathbf{S}^{n-1}$  replaced by  $\alpha : \mathbf{S}^n \rightarrow \mathbf{X}$  and  $2\nu$  replaced by  $\beta : \mathbf{S}^{n+2} \rightarrow \mathbf{X}$ ).

However, since the  $\tilde{k}$ -invariants are natural (Definition 3.10), we deduce from the long exact sequence (4.28) that the corresponding obstruction for the diagram – that is,  $\tilde{k}_1 \in H^3(\varphi; \Omega\varphi) \cong \mathbb{Z}/2$  – is also non-zero, which implies that  $\varphi$  cannot be realized by a map of spaces  $f : \mathbf{X} \rightarrow \mathbf{S}^{n-1}$  (or even of suitable Postnikov sections).

6.7. *Remark.* There is a more elementary way to see that  $\varphi$  is not realizable: if it were, from (0.15) and (6.6) we would have

$$(6.8) \quad \begin{aligned} \{6\nu, 18\nu\} &= \{6\nu, 6\nu + \eta^3\} = \varphi\{\beta, \beta + \alpha \circ \eta^2\} = f_*(\langle \eta, 2, \alpha \rangle) \\ &= \langle \eta, 2, \varphi(\alpha) \rangle = \langle \eta, 2, \eta \rangle = \{\nu, 12\nu\}, \end{aligned}$$

a contradiction. Nevertheless, we hope the cohomological approach helps to illustrate how the general theory works.

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