

ON THE 2-ADIC K -LOCALIZATIONS OF H -SPACES

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ABSTRACT. We determine the 2-adic K -localizations for a large class of H -spaces and related spaces. As in the odd primary case, these localizations are expressed as fibers of maps between specified infinite loop spaces, allowing us to approach the 2-primary v_1 -periodic homotopy groups of our spaces. The present v_1 -periodic results have been applied very successfully to simply-connected compact Lie groups by Davis, using knowledge of the complex, real, and quaternionic representations of the groups. We also functorially determine the united 2-adic K -cohomology algebras (including the 2-adic KO -cohomology algebras) for all simply-connected compact Lie groups in terms of their representation theories, and we show the existence of spaces realizing a wide class of united 2-adic K -cohomology algebras with specified operations.

1. INTRODUCTION

In [21], Mahowald and Thompson determined the p -adic K -localizations of the odd spheres at an arbitrary prime p , expressing these localizations as homotopy fibers of maps between specified infinite loop spaces. Then working at an odd prime p in [8], we generalized this result to give the p -adic K -localizations for a large class of H -spaces and related spaces. In the present paper, we obtain similar results for 2-adic K -localizations of such spaces, using our preparatory work in [10] and [11]. By a *2-adic K -localization*, we mean a $K/2_*$ -localization (see [2], [3]), which is the same as a $K^*(-; \hat{\mathbb{Z}}_2)$ -localization since the $K/2_*$ -equivalences of spaces or spectra are the same as the $K^*(-; \hat{\mathbb{Z}}_2)$ -equivalences. Our localization results in this paper will apply to many (but not all) simply-connected finite H -spaces and to related spaces such as the spheres S^{4k-1} for $k \geq 1$. We show that these results allow computations of the v_1 -periodic homotopy groups (see [13], [15]) of our spaces from their united 2-adic K -cohomologies, and thus allow computations of the v_1 -periodic homotopy groups for a large class of simply-connected compact Lie groups from their complex, real, and quaternionic representation theories. The present results will be extended in a subsequent paper to cover the remaining simply-connected compact Lie groups and various spaces related to the remaining odd spheres. This work has been applied very successfully by Davis [14] to complete

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his 13-year program (with Bendersky) of calculating the v_1 -periodic homotopy groups of all simply-connected compact Lie groups, and has also been applied by Bendersky, Davis, and Mahowald [1].

Throughout this paper, we work at the prime 2 and rely on the *united 2-adic K-cohomology*

$$K_{CR}^*(X; \hat{\mathbb{Z}}_2) = \{K^*(X; \hat{\mathbb{Z}}_2), KO^*(X; \hat{\mathbb{Z}}_2)\}$$

of a space or spectrum X as in [10]. This combines the usual periodic cohomologies with the operations between them such as complexification and realification. For our H -spaces and related spaces X , the cohomology $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ is essentially determined by the *2-adic Adams Δ -module*

$$\tilde{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2) = \{\tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2), \widetilde{KO}^{-1}(X; \hat{\mathbb{Z}}_2), \widetilde{KO}^{-5}(X; \hat{\mathbb{Z}}_2)\}$$

which combines the specified cohomologies with the additive operations among them (see 6.1). In fact, for most simply-connected finite H -spaces X , we expect to have an isomorphism $K_{CR}^*(X; \hat{\mathbb{Z}}_2) \cong \hat{L}(M)$ where $M = \{M_C, M_R, M_H\}$ is the submodule of primitives in $\tilde{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ and where \hat{L} is a functor that we introduce in 4.5 extending the 2-adic exterior algebra functor on complex components. For a simply-connected compact Lie group G , the required 2-adic Adams Δ -module may be obtained as the indecomposables $\hat{Q}R_{\Delta}G = \{\hat{Q}RG, \hat{Q}R_RG, \hat{Q}R_HG\}$ of the complex, real, and quaternionic representation ring $R_{\Delta}G = \{RG, R_RG, R_HG\}$ (see 10.1), and we have:

Theorem 1.1. *For a simply-connected compact Lie group G , there is a natural isomorphism $K_{CR}^*(G; \hat{\mathbb{Z}}_2) \cong \hat{L}(\hat{Q}R_{\Delta}G)$ of algebras.*

This will follow by Theorem 10.3. It extends results of Hodgkin [18], Seymour [24], Minami [22], and others on $K^*(G; \hat{\mathbb{Z}}_2)$ and $KO^*(G; \hat{\mathbb{Z}}_2)$. Our main result on $K/2_*$ -localizations will apply to a space X with $K_{CR}^*(X; \hat{\mathbb{Z}}_2) \cong \hat{L}M$ for a 2-adic Adams Δ -module M that is *strong* (7.11). This technical algebraic condition seems relatively mild and holds for $\hat{Q}R_{\Delta}G$ when G is a simply-connected compact simple Lie group other than E_6 or $Spin(4k+2)$ with k not a 2-power by work of Davis (see Lemma 10.5). For a strong 2-adic Adams Δ -module M , we obtain two *stable 2-adic Adams Δ -modules* (5.3) $\bar{M} = \{\bar{M}_C, \bar{M}_R, \bar{M}_H\}$ and $\bar{\rho}\bar{M} = \{\bar{M}_C, \bar{M}_R + \bar{M}_H, \bar{M}_R \cap \bar{M}_H\}$ where $\bar{M}_C = M_C$, $\bar{M}_R = \text{im}(M_R \rightarrow M_C)$, and $\bar{M}_H = \text{im}(M_H \rightarrow M_C)$; and we obtain two corresponding $K/2_*$ -local spectra $\mathcal{E}\bar{M}$ and $\mathcal{E}\bar{\rho}\bar{M}$ such that $K_{\Delta}^{-1}(\mathcal{E}\bar{M}; \hat{\mathbb{Z}}_2) = \bar{M}$, $K^0(\mathcal{E}\bar{M}; \hat{\mathbb{Z}}_2) = 0$, $K_{\Delta}^{-1}(\mathcal{E}\bar{\rho}\bar{M}; \hat{\mathbb{Z}}_2) = \bar{\rho}\bar{M}$, and $K^0(\mathcal{E}\bar{\rho}\bar{M}; \hat{\mathbb{Z}}_2) = 0$ (see 8.1). Stated briefly, our main localization result is:

Theorem 1.2. *If X is a connected space with $K_{CR}^*(X; \hat{\mathbb{Z}}_2) \cong \hat{L}M$ for a strong 2-adic Adams Δ -module M , then its $K/2_*$ -localization $X_{K/2}$ is the homotopy fiber of a map from $\Omega^{\infty}\mathcal{E}\bar{M}$ to $\Omega^{\infty}\mathcal{E}\bar{\rho}\bar{M}$ with low dimensional modifications.*

This will follow from Theorem 8.6. It will apply to simply-connected compact simple Lie groups with the above-mentioned exceptions, and it should apply to many other simply-connected finite H -spaces and related spaces; in fact, there must exist a great diversity of spaces with the required united 2-adic K -cohomology algebras by:

Theorem 1.3. *For each strong 2-adic Adams Δ -module M , there exists a simply-connected space X with $K_{CR}^*(X; \hat{\mathbb{Z}}_2) \cong \hat{L}M$.*

This will follow from Theorem 8.5. For our spaces X , we also obtain results on the 2-primary v_1 -periodic homotopy groups $v_1^{-1}\pi_*X$, which are naturally isomorphic to stable homotopy groups $\pi_*\tau_2\Phi_1X$, where $\tau_2\Phi_1X$ is the 2-torsion part of the spectrum Φ_1X obtained using the v_1 -stabilization functor Φ_1 constructed in [4], [9], [16], and [19]. From this standpoint, the homotopy $v_1^{-1}\pi_*X$ is essentially determined by the cohomology $KO^*(\Phi_1X; \hat{\mathbb{Z}}_2)$ since there is an exact sequence

$$\begin{aligned} \dots \longrightarrow KO^{n-3}(\Phi_1X; \hat{\mathbb{Z}}_2) &\xrightarrow{\psi^3-9} KO^{n-3}(\Phi_1X; \hat{\mathbb{Z}}_2) \longrightarrow (v_1^{-1}\pi_nX)^\# \\ &\longrightarrow KO^{n-2}(\Phi_1X; \hat{\mathbb{Z}}_2) \xrightarrow{\psi^3-9} KO^{n-2}(\Phi_1X; \hat{\mathbb{Z}}_2) \longrightarrow \dots \end{aligned}$$

where $(-)^{\#}$ gives the Pontrjagin dual (see Theorem 9.2). A space X is called $K/2_*$ -durable (9.3) when the $K/2_*$ -localization induces an isomorphism $v_1^{-1}\pi_*X \cong v_1^{-1}\pi_*X_{K/2}$ or equivalently $\Phi_1X \cong \Phi_1X_{K/2}$. This condition holds for all connected H -spaces (and many other spaces), and our $K/2_*$ -localization result implies:

Theorem 1.4. *If X is a connected $K/2_*$ -durable space (e.g. H -space) with $K_{CR}^*(X; \hat{\mathbb{Z}}_2) \cong \hat{L}M$ for a strong 2-adic Adams Δ -module M , then there is a (co)fiber sequence of spectra $\Phi_1X \rightarrow \mathcal{E}\bar{M} \rightarrow \mathcal{E}\bar{\rho}\bar{M}$ with a $KO^*(-; \hat{\mathbb{Z}}_2)$ cohomology exact sequence*

$$\begin{aligned} 0 \longrightarrow KO^{-8}(\Phi_1X; \hat{\mathbb{Z}}_2) &\longrightarrow \bar{M}_C/(\bar{M}_R + \bar{M}_H) \xrightarrow{\lambda^2} \bar{M}_C/\bar{M}_R \longrightarrow KO^{-7}(\Phi_1X; \hat{\mathbb{Z}}_2) \longrightarrow 0 \\ \longrightarrow \bar{M}_H/(\bar{M}_R \cap \bar{M}_H) &\longrightarrow KO^{-6}(\Phi_1X; \hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_R \cap \bar{M}_H \xrightarrow{\lambda^2} \bar{M}_H \longrightarrow KO^{-5}(\Phi_1X; \hat{\mathbb{Z}}_2) \\ &\longrightarrow 0 \longrightarrow 0 \longrightarrow KO^{-4}(\Phi_1X; \hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_C/(\bar{M}_R \cap \bar{M}_H) \xrightarrow{\lambda^2} \bar{M}_C/\bar{M}_H \\ &\longrightarrow KO^{-3}(\Phi_1X; \hat{\mathbb{Z}}_2) \longrightarrow (\bar{M}_R + \bar{M}_H)/(\bar{M}_R \cap \bar{M}_H) \xrightarrow{\lambda^2} \bar{M}_R/(\bar{M}_R \cap \bar{M}_H) \\ &\longrightarrow KO^{-2}(\Phi_1X; \hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_R + \bar{M}_H \xrightarrow{\lambda^2} \bar{M}_R \longrightarrow KO^{-1}(\Phi_1X; \hat{\mathbb{Z}}_2) \longrightarrow 0 \end{aligned}$$

This will follow from Theorem 9.5. It allows effective computations of 2-primary v_1 -periodic homotopy groups as shown by Davis [14], and its complex analogue implies that our spaces X are usually $\hat{K}\Phi_1$ -good (9.6), which means that $\hat{Q}K^n(X; \hat{\mathbb{Z}}_2)/\lambda^2 \cong K^n(\Phi_1X; \hat{\mathbb{Z}}_2)$ for $n = -1, 0$.

Theorem 1.5. *If X is as in Theorem 1.4 with $\lambda^2: M_C \rightarrow M_C$ monic, then X is $\widehat{K}\Phi_1$ -good.*

This will be used in a subsequent paper to show that all simply-connected compact Lie groups (and many other spaces) are $\widehat{K}\Phi_1$ -good, which is useful because the v_1 -periodic homotopy groups of $\widehat{K}\Phi_1$ -good spaces are often accessible by [10], even when our $K/2_*$ -localization theorems do not apply.

Throughout the paper, spaces and spectra will belong to the usual pointed simplicial or CW homotopy categories. To provide a suitably precise setting for our main theorems and proofs, we must devote considerable attention to developing the algebraic infrastructure of united 2-adic K -cohomology theory. The paper is divided into the following sections:

1. Introduction
2. The united 2-adic K -cohomologies of spectra and spaces
3. The 2-adic ϕCR -algebras
4. The universal 2-adic ϕCR -algebra functor \hat{L}
5. Stable 2-adic Adams operations and $K/2_*$ -local spectra
6. On the $K/2_*$ -localizations of infinite loop spaces
7. Strong 2-adic Adams Δ -modules
8. On the $K/2_*$ -localizations of our spaces
9. On the v_1 -periodic homotopy groups of our spaces
10. Applications to simply-connected compact Lie groups
11. Proofs of basic lemmas for \hat{L}
12. Proof of the Bott exactness lemma for \hat{L}
13. Proofs for regular modules
14. Proof of the realizability theorem for $\hat{L}M$

Although we have long been interested in the K -localizations and v_1 -periodic homotopy groups of spaces, we were prompted to develop the present results by Martin Bendersky and Don Davis. We thank them for their questions and comments.

2. THE UNITED 2-ADIC K -COHOMOLOGIES OF SPECTRA AND SPACES

We now consider the united 2-adic K -cohomologies

$$K_{CR}^*(X; \hat{\mathbb{Z}}_2) = \{K^*(X; \hat{\mathbb{Z}}_2), KO^*(X; \hat{\mathbb{Z}}_2)\}$$

of spectra and spaces X , focusing on their basic structures as 2-adic CR -modules or CR -algebras. We first recall:

2.1. The 2-adic CR -modules. By a *2-adic CR -module*, we mean a CR -module over the category of 2-profinite abelian groups (see [10, 4.1]). Thus, a 2-adic CR -module $M = \{M_C, M_R\}$ consists of \mathbb{Z} -graded 2-profinite abelian groups M_C and M_R with continuous additive operations

$$\begin{aligned} B: M_C^* &\cong M_C^{*-2}, & t: M_C^* &\cong M_C^*, & B_R: M_R^* &\cong M_R^{*-8}, \\ \eta: M_R^* &\rightarrow M_R^{*-1}, & c: M_R^* &\rightarrow M_C^*, & r: M_C^* &\rightarrow M_R^*, \end{aligned}$$

satisfying the relations

$$\begin{aligned} 2\eta &= 0, & \eta^3 &= 0, & \eta B_R &= B_R \eta, & \eta r &= 0, & c\eta &= 0, \\ t^2 &= 1, & tB &= -Bt, & rt &= r, & tc &= c, & cB_R &= B^4 c, \\ rB^4 &= B_R r, & cr &= 1 + t, & rc &= 2, & rBc &= \eta^2, & rB^{-1}c &= 0. \end{aligned}$$

For $z \in M_C^*$ and $x \in M_R^*$, the elements $tz \in M_C^*$ and $rB^2cx \in M_R^*$ are sometimes written as z^* (or $\psi^{-1}z$) and ξx . For a spectrum or space X , the united 2-adic K -cohomology

$$K_{CR}^*(X; \hat{\mathbb{Z}}_2) = \{K^*(X; \hat{\mathbb{Z}}_2), KO^*(X; \hat{\mathbb{Z}}_2)\}$$

has a natural 2-adic CR -module structure with the usual periodicities $B: K^*(X; \hat{\mathbb{Z}}_2) \cong K^{*-2}(X; \hat{\mathbb{Z}}_2)$, and $B_R: KO^*(X; \hat{\mathbb{Z}}_2) \cong KO^{*-8}(X; \hat{\mathbb{Z}}_2)$, conjugation $t: K^*(X; \hat{\mathbb{Z}}_2) \cong K^*(X; \hat{\mathbb{Z}}_2)$, Hopf operation $\eta: KO^*(X; \hat{\mathbb{Z}}_2) \rightarrow KO^{*-1}(X; \hat{\mathbb{Z}}_2)$, complexification $c: KO^*(X; \hat{\mathbb{Z}}_2) \rightarrow K^*(X; \hat{\mathbb{Z}}_2)$, and realification $r: K^*(X; \hat{\mathbb{Z}}_2) \rightarrow KO^*(X; \hat{\mathbb{Z}}_2)$.

2.2. Bott exactness. As in [10, 4.1], we say that a 2-adic CR -module M is *Bott exact* when the Bott sequence

$$\dots \rightarrow M_R^{*+1} \xrightarrow{\eta} M_R^* \xrightarrow{c} M_C^* \xrightarrow{rB^{-1}} M_R^{*+2} \xrightarrow{\eta} \dots$$

is exact, and we note that the 2-adic CR -module $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ is always Bott exact for a spectrum or space X . To compare CR -modules, we shall often use:

Lemma 2.3. *For Bott exact 2-adic CR -modules M and N , a map $f: M \rightarrow N$ is an isomorphism if and only if $f: M_C \rightarrow N_C$ is an isomorphism.*

Proof. For the “if” part, we note that f gives a map of Bott exact couples with $f: M_C \cong N_C$. Using the map of second derived couples with $f: M_C^{(2)} \cong N_C^{(2)}$, we easily see that $f: \eta^2 M_R \cong \eta^2 N_R$; then using the map of first derived couples with $f: M_C^{(1)} \cong N_C^{(1)}$, we easily see that $f: \eta M_R \cong \eta N_R$; and finally using the original map of exact couples, we easily see that $f: M_R \cong N_R$. \square

2.4. The free 2-adic CR -modules. For each integer n and $L = C, R$, there is a *monogenic free* 2-adic CR -module $F^L(g, n)$ on a generator $g \in F^L(g, n)_L^n$ having the universal property that, for each 2-adic CR -module M and $y \in N_L^n$, there is a unique map $f: F^L(g, n) \rightarrow M$ with $f(g) = y$. The 2-adic CR modules $F^C(g, n)$ and $F^R(g, n)$ are given more explicitly by: $F^C(g, n)_C^{n-2i} = \hat{\mathbb{Z}}_2 \oplus \hat{\mathbb{Z}}_2 = \langle B^i g \rangle \oplus \langle B^i g^* \rangle$, $F^C(g, n)_C^{n-2i-1} = 0$, $F^C(g, n)_R^{n-2i} = \hat{\mathbb{Z}}_2 = \langle rB^i g \rangle$, $F^C(g, n)_R^{n-2i-1} = 0$, $F^R(g, n)_C^{n-2i} = \hat{\mathbb{Z}}_2 = \langle B^i c g \rangle$, $F^R(g, n)_C^{n-2i-1} = 0$, $F^R(g, n)_R^{n-8i} = \hat{\mathbb{Z}}_2 = \langle B_R^i g \rangle$, $F^R(g, n)_R^{n-8i-1} = \mathbb{Z}/2 = \langle B_R^i \eta g \rangle$, $F^R(g, n)_R^{n-8i-2} = \mathbb{Z}/2 = \langle B_R^i \eta^2 g \rangle$, $F^R(g, n)_R^{n-8i-4} = \hat{\mathbb{Z}}_2 = \langle B_R^i \xi g \rangle$, and $F^R(g, n)_R^{n-8i-k} = 0$ for $k = 3, 5, 6, 7$. We note that $F^C(g, n)$ and $F^R(g, n)$ are Bott exact for all n . In general, a free 2-adic CR -module on a finite set of generators may be constructed as a direct sum of the corresponding monogenic free 2-adic CR -modules. To test for this freeness, we may use:

Lemma 2.5. *For a Bott exact 2-adic CR -module M (e.g. for some $M = K_{CR}^*(X; \hat{\mathbb{Z}}_2)$), if M_C^* is a free module over $\hat{K}^* = \hat{\mathbb{Z}}_2[B, B^{-1}]$ on the generators $\{ca_i\}_i \amalg \{b_j\}_j \amalg \{b_j^*\}_j$ for finite sets of elements $\{a_i\}_i$ in M_R^* and $\{b_j\}_j$ in M_C^* , then M is a free 2-adic CR -module on the generators $\{a_i\}_i$ and $\{b_j\}_j$.*

Proof. The canonical map to M from the specified 2-adic CR -module is an isomorphism by Lemma 2.3. \square

To describe the multiplicative structure of $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ for a space X , we introduce:

2.6. The 2-adic CR -algebras. By a *2-adic CR -algebra* $A = \{A_C, A_R\}$, we mean a 2-adic CR -module with continuous bilinear multiplications $A_L^m \times A_L^n \rightarrow A_L^{m+n}$ and elements $1 \in A_L^0$ for $m, n \in \mathbb{Z}$ and $L = C, R$ such that:

- (i) the multiplication in A_C^* and A_R^* is graded commutative and associative with identity 1;
- (ii) $B(zw) = (Bz)w = z(Bw)$ and $(zw)^* = z^*w^*$ for $z \in A_C^m$ and $w \in A_C^n$;
- (iii) $B_R(xy) = (B_R x)y = x(B_R y)$, $\eta(xy) = (\eta x)y = x(\eta y)$, and $\xi(xy) = (\xi x)y = x(\xi y)$ for $x \in A_R^m$ and $y \in A_R^n$;
- (iv) $c1 = 1$ and $c(xy) = (cx)(cy)$ for $x \in A_R^m$ and $y \in A_R^n$;
- (v) $r((cx)z) = x(rz)$ and $r(z(cx)) = (rz)x$ for $x \in A_R^m$ and $z \in A_C^n$.

Equivalently, a 2-adic CR -algebra A consists of a 2-adic CR -module with a commutative associative multiplication $A \hat{\otimes}_{CR} A \rightarrow A$ with identity $\underline{e} \rightarrow A$ for $\underline{e} = F^R(1, 0) \cong K_{CR}^*(\text{pt}; \hat{\mathbb{Z}}_2)$, where $\hat{\otimes}_{CR}$ is the (symmetric monoidal) complete tensor product for 2-adic CR -modules [11, 2.6].

2.7. Augmentations and nilpotency. For a 2-adic CR -algebra A , an *augmentation* is a map $A \rightarrow \underline{e}$ of 2-adic CR -algebras which is left inverse to the identity $\underline{e} \rightarrow A$. When A is augmented, we let $\tilde{A} = \{\tilde{A}_C, \tilde{A}_R\}$ denote the augmentation ideal, and for $m \geq 1$ we let $\tilde{A}(m)$ denote the m -th power of \tilde{A} given by the image of the m -fold product $\tilde{A} \hat{\otimes}_{CR} \dots \hat{\otimes}_{CR} \tilde{A} \rightarrow \tilde{A}$. Thus, $\tilde{A}(m)_C$ is the image of the m -fold product $\tilde{A}_C^* \hat{\otimes} \dots \hat{\otimes} \tilde{A}_C^* \rightarrow \tilde{A}_C^*$, while $\tilde{A}(m)_R$ is the image of the m -fold product $\tilde{A}_R^* \hat{\otimes} \dots \hat{\otimes} \tilde{A}_R^* \rightarrow \tilde{A}_R^*$ plus the realification of $\tilde{A}(m)_C$. The *indecomposables* of A are given by the 2-adic CR -module $\hat{Q}A = \tilde{A}/\tilde{A}(2)$. We call A *nilpotent* when $\tilde{A}(m) = 0$ for sufficiently large m and call A *pronilpotent* when $\cap_m \tilde{A}(m) = 0$ or equivalently when $A \cong \lim_m A/\tilde{A}(m)$. For a space X , the cohomology $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ has a canonical augmentation $K_{CR}^*(X; \hat{\mathbb{Z}}_2) \rightarrow \underline{e}$ induced by the basepoint $\text{pt} \subset X$ with the usual augmentation ideal $\tilde{K}_{CR}^*(X; \hat{\mathbb{Z}}_2) = \{\tilde{K}^*(X; \hat{\mathbb{Z}}_2), \widetilde{KO}^*(X; \hat{\mathbb{Z}}_2)\}$. Moreover, when X is connected, the cohomology $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ is pronilpotent since it is the inverse limit of the cohomologies $K_{CR}^*(X_\alpha; \hat{\mathbb{Z}}_2)$ for the finite connected subspaces $X_\alpha \subset X$, where each $K_{CR}^*(X_\alpha; \hat{\mathbb{Z}}_2)$ is nilpotent.

3. THE 2-ADIC ϕCR -ALGEBRAS

To capture some additional features of the 2-adic CR -algebras $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ for spaces X , we now introduce the 2-adic ϕCR -algebras. These structures are often surprisingly rigid and will allow us to construct convenient bases for $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ in some important general cases, for instance, when X is a simply-connected compact Lie group.

3.1. The 2-adic ϕCR -algebras. By a *2-adic ϕCR -algebra* A , we mean a 2-adic CR -algebra with continuous functions $\phi: A_C^0 \rightarrow A_R^0$ and $\phi: A_C^{-1} \rightarrow A_R^0$ such that:

- (i) $c\phi a = a^*a$ and $c\phi x = B^{-1}x^*x$ for $a \in A_C^0$ and $x \in A_C^{-1}$;
- (ii) $\phi(a + b) = \phi a + \phi b + r(a^*b)$ and $\phi(x + y) = \phi x + \phi y + rB^{-1}(x^*y)$ for $a, b \in A_C^0$ and $x, y \in A_C^{-1}$;
- (iii) $\phi(ab) = (\phi a)(\phi b)$, $\phi(ax) = (\phi a)(\phi x)$, and $\phi B^{-1}(xy) = (\phi x)(\phi y)$ for $a, b \in A_C^0$ and $x, y \in A_C^{-1}$;
- (iv) $\phi(1) = 1$, $\phi(ka) = k^2\phi a$, $\phi(a^*) = \phi a$, $\phi(kx) = k^2\phi x$, and $\phi(x^*) = -\phi x$ for $a \in A_C^0$, $x \in A_C^{-1}$, and $k \in \hat{\mathbb{Z}}_2$.

For convenience, we extend the operation ϕ periodically to give $\phi: A_C^{2i} \rightarrow A_R^0$ and $\phi: A_C^{2i-1} \rightarrow A_R^0$ with $\phi w = \phi B^i w$ for all i and elements w . For a space X , the cohomology $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ has a natural 2-adic ϕCR -algebra structure with $\phi: K^*(X; \hat{\mathbb{Z}}_2) \rightarrow KO^0(X; \hat{\mathbb{Z}}_2)$ as in [11, Section 3]. In particular, $\underline{e} \cong K_{CR}^*(\text{pt}; \hat{\mathbb{Z}}_2)$ is a 2-adic ϕCR -algebra with $\phi(k1) = k^21$ for $k \in \hat{\mathbb{Z}}_2$. For a 2-adic ϕCR -algebra

A , an *augmentation* is a map $A \rightarrow \underline{e}$ of 2-adic ϕCR -algebras which is left inverse to the identity, and we retain the other notation and terminology of 2.7. Thus, for a space X , the ϕCR -algebra $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ has a canonical augmentation and is pronilpotent whenever X is connected. To capture some other needed features, we introduce:

3.2. Special 2-adic ϕCR -algebras. A 2-adic ϕCR -algebra A is called *special* when:

- (i) A is augmented and pronilpotent;
- (ii) $z^2 = 0$ for $z \in A_C^n$ with n odd;
- (iii) $y^2 = 0$ for $y \in A_R^n$ with $n \equiv 1, -3 \pmod{8}$;
- (iv) $\phi cx = 0$ for $x \in A_R^n$ with $n \equiv -1, -5 \pmod{8}$.

For a connected space X , the cohomology $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ is a special 2-adic ϕCR -algebra by [11, Section 3].

3.3. Simple systems of generators. Let A be a special 2-adic ϕCR -algebra. By a *simple system of generators of odd degree* for A , we mean finite ordered sets of odd-degree elements $\{x_i\}_i$ in \tilde{A}_R and $\{z_j\}_j$ in \tilde{A}_C such that A_C is an exterior algebra over $\hat{K}^* = \hat{\mathbb{Z}}_2[B, B^{-1}]$ on the generators $\{cx_i\}_i \amalg \{z_j\}_j \amalg \{z_j^*\}_j$. Such a simple system determines *associated products*

$$x_{i_1} \dots x_{i_m} (\phi z_{j_1}) \dots (\phi z_{j_n}) \in A_R,$$

$$(cx_{i_1}) \dots (cx_{i_m}) (cz_{j_1}) \dots (cz_{j_n}) w_{k_1} \dots w_{k_q} \in A_C$$

where: $i_1 < \dots < i_m$ with $m \geq 0$; $j_1 < \dots < j_n$ with $n \geq 0$; $k_1 < \dots < k_q$ with $q \geq 1$; each w_{k_t} is z_{k_t} or $z_{k_t}^*$ with $w_{k_q} = z_{k_q}$; and $\{k_1, \dots, k_q\}$ is disjoint from $\{j_1, \dots, j_n\}$ in each complex product.

Proposition 3.4. *If A is a Bott exact special 2-adic ϕCR -algebra with a simple system of generators of odd degree, then A is a free 2-adic CR -module on the associated products.*

Proof. This follows by Lemma 2.5. □

When the cohomology $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ of a connected space X has a simple system of generators of odd degree, this result will determine the 2-adic CR -algebra structure of the cohomology, provided that we can compute the squares of the real simple generators of degree $\equiv -1, -5 \pmod{8}$, since the squares of the other simple generators and of their ϕ 's must vanish. For a simply-connected compact Lie group G , we shall see that the cohomology $K_{CR}^*(G; \hat{\mathbb{Z}}_2)$ must always have a simple system of generators of odd degree by Theorem 10.3 below.

4. THE UNIVERSAL 2-ADIC ϕCR -ALGEBRA FUNCTOR \hat{L}

We must now go beyond simple systems of generators and develop functorial descriptions of cohomologies $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ using universal special 2-adic ϕCR -algebras. Our results will apply, for instance, when X is a suitable infinite loop space (Theorem 6.7) or a simply-connected compact Lie group (Theorem 10.3). We start by introducing the algebraic modules that will generate our universal algebras.

4.1. The 2-adic Δ -modules. By a 2-adic Δ -module $N = \{N_C, N_R, N_H\}$, we mean a triad of 2-profinite abelian groups N_C , N_R , and N_H with continuous additive operations

$$\begin{aligned} t: N_C &\cong N_C, & c: N_R &\rightarrow N_C, & r: N_C &\rightarrow N_R, \\ c': N_H &\rightarrow N_C, & q: N_C &\rightarrow N_H \end{aligned}$$

satisfying the relations

$$\begin{aligned} t^2 = 1, & \quad cr = 1 + t, & rc = 2, & \quad tc = c, & \quad rt = r, \\ c'q = 1 + t, & \quad qc' = 2, & tc' = c', & \quad qt = q \end{aligned}$$

as in [10, 4.5]. For $z \in N_C$, the element tz is sometimes written as z^* or $\psi^{-1}z$. For a 2-adic CR -module N and integer n , we obtain a 2-adic Δ -module $\Delta^n N = \{N_C^n, N_R^n, N_R^{n-4}\}$ with $c' = B^{-2}c: N_R^{n-4} \rightarrow N_C^n$ and $q = rB^2: N_C^n \rightarrow N_R^{n-4}$. In particular, we obtain a 2-adic Δ -module $K_\Delta^n(X; \hat{\mathbb{Z}}_2) = \Delta^n K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ for a space X . We say that a 2-adic Δ -module N is *torsion-free* when N_C , N_R , and N_H are torsion-free, and we say that N is *exact* when the sequence

$$\dots \longrightarrow N_C \xrightarrow{(r,q)} N_R \oplus N_H \xrightarrow{c-c'} N_C \xrightarrow{1-t} N_C \xrightarrow{(r,q)} N_R \oplus N_H \longrightarrow \dots$$

is exact (see [10, 4.5]). It is straightforward to show:

Lemma 4.2. *A 2-adic Δ -module $N = \{N_C, N_R, N_H\}$ is torsion-free and exact if and only if:*

- (i) $c: N_R \rightarrow N_C$ and $c': N_H \rightarrow N_C$ are monic;
- (ii) N_C is torsion-free with $\ker(1+t) = \text{im}(1-t)$ for $t: N_C \rightarrow N_C$;
- (iii) $cN_R + c'N_H = \ker(1-t)$ and $cN_R \cap c'N_H = \text{im}(1+t)$.

The 2-adic Δ -module

$$K_\Delta^{-1}(X; \hat{\mathbb{Z}}_2) = \{K^{-1}(X; \hat{\mathbb{Z}}_2), KO^{-1}(X; \hat{\mathbb{Z}}_2), KO^{-5}(X; \hat{\mathbb{Z}}_2)\}$$

of a space X has additional operations θ which we now include in:

4.3. The 2-adic $\theta\Delta$ -modules. By a 2-adic $\theta\Delta$ -module $M = \{M_C, M_R, M_H\}$, we mean a 2-adic Δ -module with continuous additive operations $\theta: M_C \rightarrow M_C$, $\theta: M_R \rightarrow M_R$, and $\theta: M_H \rightarrow M_R$ satisfying the following relations for elements $z \in M_C$, $x \in M_R$, and $y \in M_H$:

$$\theta cx = c\theta x, \quad \theta c'y = c\theta y, \quad \theta tz = t\theta z, \quad \theta qz = \theta rz, \quad \theta\theta rz = \theta r\theta z.$$

In general, θrz may differ from $r\theta z$, and we let $\bar{\phi}: M_C \rightarrow M_R$ be the difference operation with $\bar{\phi}z = \theta rz - r\theta z$ for $z \in M_C$. Using the above relations, we easily deduce:

$$\begin{aligned} \bar{\phi}cx &= 0, & \bar{\phi}c'x &= 0, & \bar{\phi}tz &= \bar{\phi}z, \\ 2\bar{\phi}z &= 0, & c\bar{\phi}z &= 0, & \theta\bar{\phi}z &= 0. \end{aligned}$$

For a space X , the cohomology $K_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ has a natural 2-adic $\theta\Delta$ -module structure by [11, Section 3] with the operations

$$\begin{aligned} \theta &= -\lambda^2: K^{-1}(X; \hat{\mathbb{Z}}_2) \longrightarrow K^{-1}(X; \hat{\mathbb{Z}}_2), \\ \theta &= -\lambda^2: KO^{-1}(X; \hat{\mathbb{Z}}_2) \longrightarrow KO^{-1}(X; \hat{\mathbb{Z}}_2), \\ \theta &= -\lambda^2: KO^{-5}(X; \hat{\mathbb{Z}}_2) \longrightarrow KO^{-1}(X; \hat{\mathbb{Z}}_2). \end{aligned}$$

Moreover, this structure interacts with the 2-adic ϕCR -algebra structure of $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ in several ways.

Lemma 4.4. *For a space X , we have:*

- (i) $\eta\phi z = \bar{\phi}z$ for $z \in K^{-1}(X; \hat{\mathbb{Z}}_2)$;
- (ii) $x^2 = \eta\theta x$ for $x \in KO^{-1}(X; \hat{\mathbb{Z}}_2)$;
- (iii) $y^2 = B_R\eta\theta y$ for $y \in KO^{-5}(X; \hat{\mathbb{Z}}_2)$.

Proof. This follows from [11, Section 3]. □

We shall take account of these relations in our universal algebras. For a 2-adic $\theta\Delta$ -module M and a special 2-adic ϕCR -algebra A , an *admissible* map $\alpha: M \rightarrow A$ consists of a 2-adic Δ -module map $\alpha: M \rightarrow \Delta^{-1}\tilde{A}$ such that:

- (i) $\eta\phi\alpha z = \alpha\bar{\phi}z$ in A_C^{-1} for each $z \in M_C$;
- (ii) $(\alpha x)^2 = \eta\alpha\theta x$ in A_R^{-2} for each $x \in M_R$;
- (iii) $(\alpha y)^2 = B_R\eta\alpha\theta y$ in A_R^{-10} for each $y \in M_H$.

We say that a special 2-adic ϕCR -algebra A with an admissible map $\alpha: M \rightarrow A$ is *universal* if, for each special 2-adic ϕCR -algebra B with admissible map $g: M \rightarrow B$, there exists a unique ϕCR -algebra map $\bar{g}: A \rightarrow B$ such that $\bar{g}\alpha = g$.

Lemma 4.5. *For each 2-adic $\theta\Delta$ -module M , there exists a universal special 2-adic ϕCR -algebra $\hat{L}M$ with admissible map $\alpha: M \rightarrow \hat{L}M$.*

This will be proved later in 11.6. By universality, $\hat{L}M$ is unique up to isomorphism and is natural in M , so that we have a functor \hat{L} from the category of 2-adic $\theta\Delta$ -modules to the category of special 2-adic ϕCR -algebras. We believe that the ϕCR -algebra $\hat{L}M$ can be given canonical operations θ satisfying all the formulae of [11, Section 3] and that this provides a strengthened version of \hat{L} that is right adjoint to $\Delta^{-1}(\tilde{})$. However, for simplicity, we rely on the present basic functor \hat{L} . We can describe the algebra $(\hat{L}M)_C$ explicitly using the 2-adic exterior algebra $\hat{\Lambda}M_C$ with $\hat{\Lambda}M_C = \lim_{\beta} \hat{\Lambda}M_{C\beta}$ where $M_{C\beta}$ ranges over the finite 2-adic quotients of M_C (ignoring θ).

Lemma 4.6. *For a 2-adic $\theta\Delta$ -module M , the canonical map $\hat{\Lambda}M_C \rightarrow (\hat{L}M)_C$ is an algebra isomorphism.*

This will be proved later in 11.8. We must impose extra conditions on M to ensure that $\hat{L}M$ is Bott exact and hence topologically relevant.

4.7. The robust 2-adic $\theta\Delta$ -modules. We say that a 2-adic $\theta\Delta$ -module M is *profinite* when it is the inverse limit of an inverse system of finite 2-adic $\theta\Delta$ -modules, and we let $M/\bar{\phi}$ denote the 2-adic Δ -module $\{M_C, M_R/\bar{\phi}M_C, M_H\}$. We call M *robust* when:

- (i) M is profinite;
- (ii) $M/\bar{\phi}$ is torsion-free and exact (4.1);
- (iii) $\ker \bar{\phi} = cM_R + c'M_H + 2M_C$.

When M is obtained from $K_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ for a space X , the profiniteness condition will usually hold automatically since $K_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2) = \lim_{\alpha, i} K_{\Delta}^{-1}(X_{\alpha}; \hat{\mathbb{Z}}_2)/2^i$ for the system of finite subcomplexes $X_{\alpha} \subset X$ and $i \geq 1$. The following key lemma will be proved later in 12.2.

Lemma 4.8. *If M is a robust 2-adic $\theta\Delta$ -module, then the special 2-adic ϕCR -algebra $\hat{L}M$ is Bott exact; in fact, $\hat{L}M$ is the inverse limit of an inverse system of finitely generated free 2-adic CR -modules.*

This leads to a crucial comparison theorem.

Theorem 4.9. *For a connected space X and a robust 2-adic $\theta\Delta$ -module M , suppose that $g: M \rightarrow \tilde{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ is a 2-adic $\theta\Delta$ -module map that induces an isomorphism $\hat{\Lambda}M_C \cong K^*(X; \hat{\mathbb{Z}}_2)$. Then g induces an isomorphism $\hat{L}M \cong K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ of special 2-adic ϕCR -algebras.*

Proof. Since g gives an admissible map $M \rightarrow K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ by Lemma 4.4, the result follows by Lemmas 2.3, 4.6, and 4.8. \square

When M is finitely generated in this theorem, we may easily choose a simple system of odd-degree generators (3.3) for $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ from M_C , M_R , and M_H . However, the present description of $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ as $\hat{L}M$ is more natural and includes the full multiplicative structure. To check whether such a description is possible for a given space X , we may use:

4.10. Determination of M from $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$. For a connected space X , we may take the indecomposables $\hat{Q}K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ as in 2.7 with the operations θ of 4.3 to produce a 2-adic $\theta\Delta$ -module

$$\hat{Q}K_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2) = \{\hat{Q}K^{-1}(X; \hat{\mathbb{Z}}_2), \hat{Q}KO^{-1}(X; \hat{\mathbb{Z}}_2), \hat{Q}KO^{-5}(X; \hat{\mathbb{Z}}_2)\}$$

together with a natural quotient map $\tilde{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2) \rightarrow \hat{Q}K_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$. Now by Lemma 4.11 below, whenever Theorem 4.9 applies to X , there is a canonical isomorphism $M \cong \hat{Q}K_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ and the map $g: M \rightarrow \tilde{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ in the theorem corresponds to a splitting of $\tilde{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2) \rightarrow \hat{Q}K_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$. When X is an H -space, we may often obtain the required splitting by mapping $\hat{Q}K_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ to the primitives in $\tilde{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$. For instance, this applies when X is a suitable infinite loop space or simply-connected compact Lie group (see Theorems 6.7 and 10.3). Finally, we note that the 2-adic $\theta\Delta$ -module $\hat{Q}K_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ will automatically be robust by Proposition 3.4 whenever $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ has a simple system of odd-degree generators with no real generators of degree $\equiv 1, -3 \pmod{8}$. We have used:

Lemma 4.11. *For a $\theta\Delta$ -module M , the canonical map $M \rightarrow \Delta^{-1}\hat{Q}\hat{L}M$ is an isomorphism.*

This will be proved later in 11.10.

5. STABLE 2-ADIC ADAMS OPERATIONS AND $K/2_*$ -LOCAL SPECTRA

We now bring stable Adams operations into our united 2-adic K -cohomology theory and use this theory to classify the needed $K/2_*$ -local spectra. We first recall some terminology from [8, 2.6].

5.1. Stable 2-adic Adams modules. By a *finite stable 2-adic Adams module* A , we mean a finite abelian 2-group with automorphisms $\psi^k: A \cong A$ for the odd $k \in \mathbb{Z}$ such that:

- (i) $\psi^1 = 1$ and $\psi^j\psi^k = \psi^{jk}$ for the odd $j, k \in \mathbb{Z}$;
- (ii) when n is sufficiently large, the condition $j \equiv k \pmod{2^n}$ implies $\psi^j = \psi^k$.

By a *stable 2-adic Adams module* A , we mean the topological inverse limit of an inverse system of finite stable 2-adic Adams modules. Such an A has an underlying 2-profinite abelian structure with continuous automorphisms $\psi^k: A \cong A$ for the odd $k \in \mathbb{Z}$ (and in fact for $k \in \hat{\mathbb{Z}}_2^\times$). We note that the operations ψ^{-1} and ψ^3 on A determine all of the other stable Adams operations ψ^k as in [5, 6.4]. Our main examples of stable 2-adic Adams modules are the cohomologies $K^n(X; \hat{\mathbb{Z}}_2)$ and $KO^n(X; \hat{\mathbb{Z}}_2)$ for a spectrum or space X and integer n with the usual stable Adams operations ψ^k . We let $\hat{\mathcal{A}}$ denote the abelian category of stable 2-adic Adams modules, and for $i \in \mathbb{Z}$ we let $\bar{S}^i: \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$ be the functor with $\bar{S}^i A$ equal to A as a group but with ψ^k on $\bar{S}^i A$ equal to $k^i \psi^k$ on A for the odd $k \in \mathbb{Z}$. We note that $\bar{S}^i A = A$ in $\hat{\mathcal{A}}$ for all i when $2A = 0$.

5.2. Stable 2-adic Adams CR -modules. By a *stable 2-adic Adams CR -module* M , we mean a 2-adic CR -module consisting of stable 2-adic Adams modules $\{M_C^*, M_R^*\}$ such that the operations $B: \bar{S}M_C^* \cong M_C^{*-2}$, $t: M_C^* \cong M_C^*$, $B_R: \bar{S}^4 M_R^* \cong M_R^{*-8}$, $\eta: M_R^* \rightarrow M_R^{*-1}$, $c: M_R^* \rightarrow M_C^*$, and $r: M_C^* \rightarrow M_R^*$ are all maps in $\hat{\mathcal{A}}$, where $\psi^{-1} = t$ in M_C^* and $\psi^{-1} = 1$ in M_R^* . For a spectrum or space X , the 2-adic K_{CR} -cohomology

$$K_{CR}^*(X; \hat{\mathbb{Z}}_2) = \{K^*(X; \hat{\mathbb{Z}}_2), KO^*(X; \hat{\mathbb{Z}}_2)\}$$

has a natural stable 2-adic Adams CR -module structure with the usual operations.

5.3. Stable 2-adic Adams Δ -modules. By a *stable 2-adic Adams Δ -module* N , we mean a 2-adic Δ -module consisting of stable 2-adic Adams modules $\{N_C, N_R, N_H\}$ such that the operations $t: N_C \cong N_C$, $c: N_R \rightarrow N_C$, $r: N_C \rightarrow N_R$, $c': N_H \rightarrow N_C$, and $q: N_C \rightarrow N_H$ are all maps in $\hat{\mathcal{A}}$, where $\psi^{-1} = t$ in N_C and $\psi^{-1} = 1$ in both N_R and N_H . For a stable 2-adic Adams CR -module M and integer n , we obtain a stable 2-adic Adams Δ -module

$$\Delta^n M = \{M_C^n, M_R^n, \bar{S}^{-2} M_R^{n-4}\}$$

as in 4.1. Thus, for a spectrum or space X and integer n , we now obtain a stable 2-adic Adams Δ -module

$$K_{\Delta}^n(X; \hat{\mathbb{Z}}_2) = \Delta^n K_{CR}^*(X; \hat{\mathbb{Z}}_2) = \{K^n(X; \hat{\mathbb{Z}}_2), KO^n(X; \hat{\mathbb{Z}}_2), \bar{S}^{-2} KO^{n-4}(X; \hat{\mathbb{Z}}_2)\}.$$

To give another example, we say that a 2-profinite abelian group G with involution $t: G \cong G$ is *positively torsion-free* when G is torsion-free with $\ker(1+t) = \text{im}(1-t)$. By [5, Proposition 3.8], this is equivalent to saying that G factors as a (possibly infinite) product of $\hat{\mathbb{Z}}_2$'s with $t = 1$ and $\hat{\mathbb{Z}}_2 \oplus t\hat{\mathbb{Z}}_2$'s. For a positively torsion-free stable 2-adic Adams module A , we may use the operation

$\psi^{-1}: A \cong A$ to construct a torsion-free exact stable 2-adic Adams Δ -module $\{A, A^+, A_+\}$ with $A^+ = \ker(1 - \psi^{-1})$, $A_+ = \text{coker}(1 - \psi^{-1})$, $t = \psi^{-1}$, $c = 1$, $r = 1 + \psi^{-1}$, $c' = 1 + \psi^{-1}$, and $q = 1$.

We let $\hat{\mathcal{A}}\mathcal{C}\mathcal{R}$ (resp. $\hat{\mathcal{A}}\Delta$) denote the abelian category of stable 2-adic Adams CR -modules (resp. Δ -modules), and we note that the functor $\Delta^n: \hat{\mathcal{A}}\mathcal{C}\mathcal{R} \rightarrow \hat{\mathcal{A}}\Delta$ for $n \in \mathbb{Z}$ has a left adjoint $CR^n: \hat{\mathcal{A}}\Delta \rightarrow \hat{\mathcal{A}}\mathcal{C}\mathcal{R}$ with $CR^n(N)_C^n = N_C$, with $CR^n(N)_C^{n-1} = 0$, and with

$$CR^n(N)_R^{n-i} = \begin{cases} N_R & \text{for } i = 0 \\ N_R/r & \text{for } i = 1 \\ \bar{S}N_C/c' & \text{for } i = 2 \\ 0 & \text{for } i = 3, 7 \\ \bar{S}^2N_H & \text{for } i = 4 \\ \bar{S}^2N_H/q & \text{for } i = 5 \\ \bar{S}^3N_C/c & \text{for } i = 6 \end{cases}$$

as in [10, 4.10]. We easily see that $CR^n(N)$ is Bott exact whenever N is torsion-free and exact. Our next lemma will often allow us to work in the simpler category $\hat{\mathcal{A}}\Delta$ instead of $\hat{\mathcal{A}}\mathcal{C}\mathcal{R}$.

Lemma 5.4. *For $n \in \mathbb{Z}$, the adjoint functors $CR^n: \hat{\mathcal{A}}\Delta \rightarrow \hat{\mathcal{A}}\mathcal{C}\mathcal{R}$ and $\Delta^n: \hat{\mathcal{A}}\mathcal{C}\mathcal{R} \rightarrow \hat{\mathcal{A}}\Delta$ restrict to equivalences between the full subcategories of all torsion-free exact $N \in \hat{\mathcal{A}}\Delta$ and all Bott exact $M \in \hat{\mathcal{A}}\mathcal{C}\mathcal{R}$ with M_C^n positively torsion-free and $M_C^{n-1} = 0$.*

Proof. For $M \in \hat{\mathcal{A}}\mathcal{C}\mathcal{R}$ as above, we see that $\Delta^n M$ is a torsion-free exact Δ -module by [10, 4.4 and 4.7] with an adjunction isomorphism $CR^n \Delta^n M \rightarrow M$ by Lemma 2.3. The corresponding result for $N \in \hat{\mathcal{A}}\Delta$ is obvious. \square

When E is a spectrum with $K^n(E; \hat{\mathbb{Z}}_2)$ positively torsion-free and $K^{n-1}(E; \hat{\mathbb{Z}}_2) = 0$ for some n , we now have $K_{CR}^*(E; \hat{\mathbb{Z}}_2) \cong CR^n(N)$ in $\hat{\mathcal{A}}\mathcal{C}\mathcal{R}$ for the torsion-free exact module $N = \Delta^n K_{CR}^*(E; \hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{A}}\Delta$, and we have the following existence theorem for such spectra in the stable homotopy category.

Theorem 5.5. *For each torsion-free exact $N \in \hat{\mathcal{A}}\Delta$ and $n \in \mathbb{Z}$, there exists a $K/2_*$ -local spectrum $\mathcal{E}^n N$ with $K_{CR}^*(\mathcal{E}^n N; \hat{\mathbb{Z}}_2) \cong CR^n(N)$ in $\hat{\mathcal{A}}\mathcal{C}\mathcal{R}$. Moreover, $\mathcal{E}^n N$ is unique up to (noncanonical) equivalence.*

Proof. This follows by Lemma 5.4 and [10, Theorem 5.3]. \square

The spectrum $\mathcal{E}^n N$ in the theorem will be endowed with an isomorphism $K_{CR}^*(\mathcal{E}^n N; \hat{\mathbb{Z}}_2) \cong CR^n(N)$ in $\hat{\mathcal{A}}\mathcal{C}\mathcal{R}$. Thus, for an arbitrary spectrum E , a map $g: E \rightarrow \mathcal{E}^n N$ induces a map $g^*: CR^n(N) \rightarrow K_{CR}^*(E; \hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{A}}\mathcal{C}\mathcal{R}$. Each algebraic map of this sort must come from a topological map by:

Theorem 5.6. *For a torsion-free exact $N \in \hat{A}\Delta$, $n \in \mathbb{Z}$, and an arbitrary spectrum E , if $\gamma: CR^n(N) \rightarrow K_{CR}^*(E; \hat{\mathbb{Z}}_2)$ is a map in \hat{ACR} , then there exists a map of spectra $g: E \rightarrow \mathcal{E}^n N$ with $g^* = \gamma$.*

Proof. Let $\tau_2 E$ denote the 2-torsion part of E given by the homotopy fiber of its localization away from 2. By Pontrjagin duality [10, Theorem 3.1], the map γ corresponds to an ACR -module map $K_*^{CR}(\tau_2 E) \rightarrow K_*^{CR}(\tau_2 \mathcal{E}^n N)$ in the sense of [5], where $K_*^{CR}(\tau_2 \mathcal{E}^n N)$ is CR -exact with $K_*(\tau_2 \mathcal{E}^n N)$ divisible. This ACR -module map prolongs canonically to an $ACRT$ -module map $K_*^{CRT}(\tau_2 E) \rightarrow K_*^{CRT}(\tau_2 \mathcal{E}^n N)$ by [5, Theorem 7.14], and the results of [5, 9.8 and 7.11] now show that this prolonged algebraic map must come from a topological map $\tau_2 E \rightarrow \tau_2 \mathcal{E}^n N$, which gives the desired $g: E \rightarrow \mathcal{E}^n N$. \square

The map g in this theorem is generally not unique (see [10, 5.4]).

6. ON THE UNITED 2-ADIC K -COHOMOLOGIES OF INFINITE LOOP SPACES

In preparation for our work on $K/2_*$ -localizations of spaces, we functorially determine the united 2-adic K -cohomologies of the needed infinite loop spaces (see Theorem 6.7). We must first introduce:

6.1. The 2-adic Adams Δ -modules. By a *2-adic Adams Δ -module* M , we mean a 2-adic $\theta\Delta$ -module (4.3) consisting of stable 2-adic Adams modules $\{M_C, M_R, M_H\}$ such that the operations $t: M_C \cong M_C$, $c: M_R \rightarrow M_C$, $r: M_C \rightarrow M_R$, $c': M_H \rightarrow M_C$, $q: M_C \rightarrow M_H$, $\theta: M_C \rightarrow M_C$, $\theta: M_R \rightarrow M_R$, and $\theta: M_H \rightarrow M_R$ are all maps in \hat{A} , where $\psi^{-1} = t$ in M_C and $\psi^{-1} = 1$ in both M_R and M_H . We let $\hat{\mathcal{M}}\Delta$ denote the abelian category of 2-adic Adams Δ -modules. We say that M is *θ -nilpotent* when it has $\theta^i = 0$ for sufficiently large i , and we say that M is *θ -pronilpotent* when it is the inverse limit of an inverse system of θ -nilpotent 2-adic Adams Δ -modules. Thus, M is θ -pronilpotent if and only if $M \cong \lim_i M/\theta^i$ where M/θ^i is the quotient module of M in $\hat{\mathcal{M}}\Delta$ with

$$\begin{aligned} (M/\theta^i)_C &= M_C/\theta^i M_C, \\ (M/\theta^i)_R &= M_R/(\theta^i M_R + \theta^i M_H + r\theta^i M_C), \\ (M/\theta^i)_H &= M_H/q\theta^i M_C \end{aligned}$$

for $i \geq 1$. More simply, M is θ -pronilpotent if and only if $\cap_i \theta^i M_C = 0$ and $\cap_i \theta^i M_R = 0$. It is not hard to show that whenever M is θ -pronilpotent, M must be profinite (i.e. M must be the inverse limit of an inverse system of finite 2-adic Adams Δ -modules). For a space X , the cohomology

$$\tilde{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2) = \{\tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2), \widetilde{KO}^{-1}(X; \hat{\mathbb{Z}}_2), \bar{S}^{-2}\widetilde{KO}^{-5}(X; \hat{\mathbb{Z}}_2)\}$$

has a natural 2-adic Adams Δ -module structure by 4.3, 5.3, and [11, 3.16] where:

Lemma 6.2. *If X is a connected space with $H^1(X; \hat{\mathbb{Z}}_2) = 0$, then the 2-adic Adams Δ -module $\tilde{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ is θ -pronilpotent.*

Proof. The condition $\cap_i \theta^i \tilde{K}^0(\Sigma X; \hat{\mathbb{Z}}_2) = 0$ holds by [6, 5.4 and 5.5] since $H^2(\Sigma X; \hat{\mathbb{Z}}_2) = 0$, and a similar proof shows $\cap_i \theta^i \widetilde{K\mathcal{O}}^0(\Sigma X; \hat{\mathbb{Z}}_2) = 0$ since $H^1(\Sigma X; \mathbb{Z}/2) = 0$. This proof uses the fact that the λ -ideal $\widetilde{K\mathcal{O}}^0 Y$ is γ -nilpotent for a connected finite CW complex Y by [10, Theorem 6.7] and the fact that the real line bundles over Y are classified by $H^1(Y; \mathbb{Z}/2)$. \square

6.3. The functor \tilde{F} . We shall construct a functor $\tilde{F}: \hat{\mathcal{A}}\Delta \rightarrow \hat{\mathcal{M}}\Delta$ where $\hat{\mathcal{A}}\Delta$ is the abelian category of stable 2-adic Adams Δ -modules (5.3) and $\hat{\mathcal{M}}\Delta$ is that of 2-adic Adams Δ -modules (6.1). This functor will carry each $N \in \hat{\mathcal{A}}\Delta$ to a universal θ -pronilpotent target module $\tilde{F}N \in \hat{\mathcal{M}}\Delta$. For $N \in \hat{\mathcal{A}}\Delta$, we first let $N_{RH} \in \hat{\mathcal{A}}$ denote the pushout of $N_R \xleftarrow{r} N_C \xrightarrow{q} N_H$ with a map $\bar{c}: N_{RH} \rightarrow N_C$ induced by c and c' , and with a map $\bar{r}: N_C \rightarrow N_{RH}$ induced by r or q . We also let $N_{C+} \in \hat{\mathcal{A}}$ denote $N_C/(1-t)N_C$ and let $N_{C\phi} \in \hat{\mathcal{A}}$ denote $N_C/(cN_R + c'N_H + 2N_C)$. We next let

$$\rho N = \{N_C, N_{RH} \oplus N_{C\phi}, N_{C+}\}$$

be the stable 2-adic Adams Δ -module with operations given by $tz = tz$, $c(x, w) = \bar{c}x$, $rz = (\bar{r}z, [z])$, $c'[z] = (1+t)z$, and $qz = [z]$. We then obtain a stable 2-adic Adams Δ -module

$$\tilde{F}N = N \times \rho N \times \rho N \times \dots$$

with components

$$\begin{aligned} \tilde{F}_C N &= N_C \times N_C \times N_C \times \dots, \\ \tilde{F}_R N &= N_R \times N_{RH} \times N_{C\phi} \times N_{RH} \times N_{C\phi} \times \dots, \\ \tilde{F}_H N &= N_H \times N_{C+} \times N_{C+} \times \dots \end{aligned}$$

We finally define operations $\theta: \tilde{F}_C N \rightarrow \tilde{F}_C N$, $\theta: \tilde{F}_R N \rightarrow \tilde{F}_R N$, and $\theta: \tilde{F}_H N \rightarrow \tilde{F}_H N$ respectively by the formulae

$$\begin{aligned} \theta(z_1, z_2, z_3, \dots) &= (0, z_1, z_2, z_3, \dots), \\ \theta(x_1, x_2, z_2, x_3, z_3, \dots) &= (0, [x_1], 0, x_2, 0, x_3, 0, \dots), \\ \theta(y_1, z_2, z_3, \dots) &= (0, [y_1], 0, \bar{r}z_2, 0, \bar{r}z_3, 0, \dots). \end{aligned}$$

This gives a natural 2-adic Adams Δ -module $\tilde{F}N$ and hence a functor $\tilde{F}: \hat{\mathcal{A}}\Delta \rightarrow \hat{\mathcal{M}}\Delta$. We let $\iota: N \rightarrow \tilde{F}N$ be the map in $\hat{\mathcal{A}}\Delta$ with $\iota_C(z) = (z, 0, 0, \dots)$, $\iota_R(x) = (x, 0, 0, \dots)$, and $\iota_H(y) = (y, 0, 0, \dots)$, and we show:

Theorem 6.4. *For a stable 2-adic Adams Δ -module $N \in \hat{A}\Delta$, the 2-adic Adams Δ -module $\tilde{F}N \in \hat{\mathcal{M}}\Delta$ is θ -pronilpotent and the map $\iota: N \rightarrow \tilde{F}N$ has the universal property that, for each θ -pronilpotent $M \in \hat{\mathcal{M}}\Delta$ and map $f: N \rightarrow M$ in $\hat{A}\Delta$, there exists a unique map $\bar{f}: \tilde{F}N \rightarrow M$ in $\hat{\mathcal{M}}\Delta$ with $\bar{f}\iota = f$.*

Proof. $\tilde{F}N$ is θ -pronilpotent since it is the inverse limit of its quotient modules

$$\tilde{F}N/\theta^{i+1} \cong N \times \rho N \times \cdots \times \rho N.$$

For $i \geq 1$, we define a map $f^{(i)}: \rho N \rightarrow M$ in $\hat{A}\Delta$ by

$$\begin{aligned} f_C^{(i)} &= \theta^i f_C: N_C \longrightarrow M_C, \\ f_R^{(i)} &= (\theta^i f_R, \theta^i f_H) + \bar{\phi}\theta^{i-1} f_C: N_{RH} \oplus N_{C\phi} \longrightarrow M_R, \\ f_H^{(i)} &= q\theta^i f_C: N_{C+} \longrightarrow M_H. \end{aligned}$$

We then define $\bar{f}: \tilde{F}N \rightarrow M$ as the inverse limit of the maps

$$f + f^{(1)} + \cdots + f^{(i)}: N \times \rho N \times \cdots \times \rho N \longrightarrow M/\theta^{i+1}$$

in $\hat{\mathcal{M}}\Delta$, and we check that $\bar{f}\iota = f$. The uniqueness condition for \bar{f} follows since the 2-adic Adams Δ -modules $\tilde{F}N/\theta^{i+1} = N \times \rho N \times \cdots \times \rho N$ are generated by ιN . \square

To show the robustness (4.7) of $\tilde{F}N$ for suitable N , we need:

6.5. The functor $\bar{\rho}: \hat{A}\Delta \rightarrow \hat{A}\Delta$. For $N \in \hat{A}\Delta$, we let $\bar{\rho}N = \{N_C, N_{RH}, N_{C+}\}$ be the stable 2-adic Adams Δ -module with operations given by $tz = z$, $cx = \bar{c}x$, $rz = \bar{r}z$, $c'[z] = (1+t)z$, and $qz = [z]$. Thus, $\bar{\rho}N$ is the quotient of $\rho N = \{N_C, N_{RH} \oplus N_{C\phi}, N_{C+}\}$ by $N_{C\phi}$. If N is torsion-free and exact, then $\bar{\rho}N$ is also torsion-free and exact by Lemma 4.2 since it is isomorphic to the module $\{N_C, N_R + N_H, N_R \cap N_H\}$ with c and c' treated as inclusions.

Lemma 6.6. *If $N \in \hat{A}\Delta$ is torsion-free and exact, then $\tilde{F}N \in \hat{\mathcal{M}}\Delta$ is robust.*

Proof. We check that $\bar{\phi}: \tilde{F}_C N \rightarrow \tilde{F}_R N$ is given by

$$\bar{\phi}(z_1, z_2, z_3, \dots) = (0, 0, [z_1], 0, [z_2], 0, \dots)$$

for $z_i \in N_C$ and $[z_i] \in N_{C\phi}$. Thus, $\ker \bar{\phi} = c\tilde{F}_R N + c'\tilde{F}_H N + 2\tilde{F}_C N$ and $\tilde{F}N/\bar{\phi} \cong N \times \bar{\rho}N \times \bar{\rho}N \times \cdots$. Hence, $\tilde{F}N/\bar{\phi}$ is torsion-free and exact by 6.5 as required. \square

Our main result in this section is:

Theorem 6.7. *If E is a 0-connected spectrum with $H^1(E; \hat{\mathbb{Z}}_2) = 0 = H^2(E; \hat{\mathbb{Z}}_2)$, with $K^0(E; \hat{\mathbb{Z}}_2) = 0$, and with $K^{-1}(E; \hat{\mathbb{Z}}_2)$ positively torsion-free (5.3), then there is a natural isomorphism $\hat{L}\tilde{F}K_{\Delta}^{-1}(E; \hat{\mathbb{Z}}_2) \cong K_{CR}^*(\Omega^{\infty}E; \hat{\mathbb{Z}}_2)$.*

Proof. Since $\tilde{K}_{\Delta}^{-1}(\Omega^{\infty}E; \hat{\mathbb{Z}}_2)$ is θ -pronilpotent by Lemma 6.2, the infinite suspension map $\sigma: K_{\Delta}^{-1}(E; \hat{\mathbb{Z}}_2) \rightarrow \tilde{K}_{\Delta}^{-1}(\Omega^{\infty}E; \hat{\mathbb{Z}}_2)$ induces a map $\bar{\sigma}: \tilde{F}K_{\Delta}^{-1}(E; \hat{\mathbb{Z}}_2) \rightarrow \tilde{K}_{\Delta}^{-1}(\Omega^{\infty}E; \hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{M}}\Delta$, where $\tilde{F}K_{\Delta}^{-1}(E; \hat{\mathbb{Z}}_2)$ is robust by Lemmas 5.4 and 6.6. Thus, $\bar{\sigma}$ induces an isomorphism $\hat{L}\tilde{F}K_{\Delta}^{-1}(E; \hat{\mathbb{Z}}_2) \cong K_{CR}^*(\Omega^{\infty}E; \hat{\mathbb{Z}}_2)$ by Theorem 4.9, since it induces an isomorphism of the complex components by [6, Theorem 8.3]. \square

7. STRONG 2-ADIC ADAMS Δ -MODULES

Our main results on $K/2_*$ -localizations in Section 8 will involve a space X with $K_{CR}^*(X; \hat{\mathbb{Z}}_2) \cong \hat{L}M$ for a 2-adic Adams Δ -module M that is strong (7.11) in the sense that it is robust (7.1), ψ^3 -splittable (7.2), and regular (7.8). In this section, we provide the required algebraic definitions and explanations of these notions. We first recall:

7.1. The robust modules. We say that a 2-adic Adams Δ -module M is *robust* when it is robust in the sense of 4.7, ignoring stable Adams operations. When M is robust, the underlying 2-adic Δ -module $M/\bar{\phi}$ satisfies the conditions of Lemma 4.2 and may be factored as a (possibly infinite) product of monogenic free 2-adic Δ -modules

$$\begin{aligned} F^C(z) &= \{\hat{\mathbb{Z}}_2 \oplus t\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2\} = \{\langle z \rangle \oplus \langle tz \rangle, \langle rz \rangle, \langle qz \rangle\}, \\ F^R(x) &= \{\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2\} = \{\langle cx \rangle, \langle x \rangle, \langle qcx \rangle\}, \\ F^H(y) &= \{\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2\} = \{\langle c'y \rangle, \langle rc'y \rangle, \langle y \rangle\} \end{aligned}$$

by an argument using the factorization of positively torsion-free groups in 5.3. We let $\text{gen}_C M$, $\text{gen}_R M$, and $\text{gen}_H M$ respectively denote the number of complex, real, and quaternionic monogenic free factors of $M/\bar{\phi}$. These numbers do not depend on the factorization since they respectively equal the $\mathbb{Z}/2$ -dimensions of the Pontrjagin duals $(M_{C\phi})^{\#}$, $(M_R/(\bar{\phi}M_C + rM_C))^{\#}$, and $(M_H/qM_C)^{\#}$. Using the factorization of $M/\bar{\phi}$, we find that

$$\text{gen } M_C = 2 \text{gen}_C M + \text{gen}_R M + \text{gen}_H M$$

where $\text{gen } M_C$ denotes the number of $\hat{\mathbb{Z}}_2$ factors in the 2-profinite abelian group M_C .

7.2. The ψ^3 -splittable modules. For a 2-adic Adams Δ -module $M \in \hat{\mathcal{M}}\Delta$, we consider the stable 2-adic Adams Δ -module $\bar{M} = M/\bar{\phi} \in \hat{\mathcal{A}}\Delta$, and we say that M is ψ^3 -splittable when the quotient map $M \twoheadrightarrow \bar{M}$ has a right inverse $s: \bar{M} \rightarrow M$ in $\hat{\mathcal{A}}\Delta$. We call such a map s a ψ^3 -splitting of M , and we note that it corresponds to a left inverse $s': M_R/rM_C \rightarrow \bar{\phi}M_C$ of the canonical map $\bar{\phi}M_C \rightarrow M_R/rM_C$ in the category $\hat{\mathcal{A}}$ of stable 2-adic Adams modules (5.1), or equivalently in the category of profinite $\mathbb{Z}/2$ -modules with automorphisms ψ^3 . We deduce that M is automatically ψ^3 -splittable in some important cases:

Lemma 7.3. *If M is a robust 2-adic Adams Δ -module with $\text{gen}_C M = 0$ or $\text{gen}_R M = 0$, then M is ψ^3 -splittable.*

Proof. Since M_C is positively torsion-free, the map $cr = 1 + t: M_{C+} \rightarrow M_C$ is monic, and hence $c: rM_C \rightarrow M_C$ is also monic. Thus, $\bar{\phi}M_C \cap rM_C = 0$ and there is a short exact sequence

$$0 \longrightarrow \bar{\phi}M_C \longrightarrow M_R/rM_C \longrightarrow M_R/(\bar{\phi}M_C + rM_C) \longrightarrow 0$$

in $\hat{\mathcal{A}}$. Since $\text{gen}_C M = 0$ or $\text{gen}_R M = 0$, this has $\bar{\phi}M_C = 0$ or $M_R/(\bar{\phi}M_C + rM_C) = 0$, and hence the map $\bar{\phi}M_C \rightarrow M_R/rM_C$ has an obvious left inverse in $\hat{\mathcal{A}}$. \square

We shall use the ψ^3 -splittability condition to construct:

7.4. θ -resolutions of modules. Let $M \in \hat{\mathcal{M}}\Delta$ be a 2-adic Adams Δ -module that is θ -pronilpotent, robust, and ψ^3 -splittable. These conditions will hold when M is strong (7.11). For a ψ^3 -splitting $s: \bar{M} \rightarrow M$ in $\hat{\mathcal{A}}\Delta$, we shall construct an associated θ -resolution

$$0 \longrightarrow \tilde{F}\bar{\rho}\bar{M} \xrightarrow{\bar{d}} \tilde{F}\bar{M} \xrightarrow{\bar{s}} M \longrightarrow 0$$

of M in $\hat{\mathcal{M}}\Delta$, with $\bar{\rho}\bar{M} = \{\bar{M}_C, \bar{M}_{RH}, \bar{M}_{C+}\}$ as in 6.5, where $\bar{s}: \tilde{F}\bar{M} \rightarrow M$ is induced by s via Theorem 6.4. To specify \bar{d} , we use the commutative square

$$\begin{array}{ccc} \bar{\rho}\bar{M} & \xrightarrow{\theta} & \bar{M} \\ \downarrow \sigma & & \downarrow s \\ \rho\bar{M} & \xrightarrow{s^{(1)}} & M \end{array}$$

in $\hat{\mathcal{A}}\Delta$ with $\rho\bar{M} = \{\bar{M}_C, \bar{M}_{RH} \oplus \bar{M}_{C\phi}, \bar{M}_{C+}\}$ as in 6.3, where $s^{(1)}$ is given by the proof of Theorem 6.4, where $\theta = \{\theta, (\theta, \theta), q\theta\}$, and where $\sigma = \{1, (1, \theta_\phi), 1\}$, using the map $\theta_\phi: \bar{M}_{RH} \rightarrow \bar{M}_{C\phi} = M_{C\phi}$ given by the composition of the sequence

$$\bar{M}_{RH} \xrightarrow{s} M_{RH} \xrightarrow{(\theta, \theta)} M_R \cong \bar{M}_R \oplus M_{C\phi} \xrightarrow{\text{proj}} M_{C\phi}$$

in which the isomorphism is the inverse of $(s, \bar{\phi}): \bar{M}_R \oplus M_{C\phi} \cong M_R$. The commutative square now gives a map

$$d = (\theta, -\sigma, 0, 0, \dots): \bar{\rho}\bar{M} \rightarrow \tilde{F}\bar{M}$$

in $\hat{A}\Delta$ with $\bar{s}d = 0$, and this induces the required map $\bar{d}: \tilde{F}\bar{\rho}\bar{M} \rightarrow \tilde{F}\bar{M}$ in $\hat{\mathcal{M}}\Delta$ with $\bar{s}\bar{d} = 0$.

Lemma 7.5. *If $M \in \hat{\mathcal{M}}\Delta$ is θ -pronilpotent and robust with a ψ^3 -splitting $s: \bar{M} \rightarrow M$, then the θ -resolution $0 \rightarrow \tilde{F}\bar{\rho}\bar{M} \xrightarrow{\bar{d}} \tilde{F}\bar{M} \xrightarrow{\bar{s}} M \rightarrow 0$ is exact in $\hat{\mathcal{M}}\Delta$.*

Proof. We easily check that $0 \rightarrow \bar{\phi}(\tilde{F}\bar{\rho}\bar{M})_C \rightarrow \bar{\phi}(\tilde{F}\bar{M})_C \rightarrow \bar{\phi}M_C \rightarrow 0$ is exact and that $\bar{s}/\bar{\phi}: \tilde{F}\bar{M}/\bar{\phi} \rightarrow M/\bar{\phi}$ is onto. Hence, it suffices to show that the map $\tilde{F}\bar{\rho}\bar{M}/\bar{\phi} \rightarrow \ker(\bar{s}/\bar{\phi})$ is an isomorphism. This follows by [10, Lemma 4.8] since the 2-adic Δ -modules $\tilde{F}\bar{\rho}\bar{M}/\bar{\phi}$ and $\ker(\bar{s}/\bar{\phi})$ are exact by Lemma 6.6 and [10, 4.8] and since the map $(\tilde{F}\bar{\rho}\bar{M}/\bar{\phi})_C \rightarrow \ker(\bar{s}/\bar{\phi})_C$ is clearly an isomorphism. \square

To formulate our regularity condition for M , we use:

7.6. The 2-adic Adams modules. These are the unstable versions of the stable modules in 5.1, previously discussed in [8, 2.8]. By a *finite 2-adic Adams module* A , we mean a finite abelian 2-group with endomorphisms $\psi^k: A \rightarrow A$ for $k \in \mathbb{Z}$ such that:

- (i) $\psi^1 = 1$ and $\psi^j\psi^k = \psi^{jk}$ for $j, k \in \mathbb{Z}$;
- (ii) when n is sufficiently large, the condition $j \equiv k \pmod{2^n}$ implies $\psi^j = \psi^k$.

By a *2-adic Adams module* A , we mean the topological inverse limit of an inverse system of finite 2-adic Adams modules. Such an A has an underlying 2-profinite abelian group with continuous endomorphisms $\psi^k: A \rightarrow A$ for $k \in \mathbb{Z}$ (and in fact for $k \in \hat{\mathbb{Z}}_2$). For a space X , the cohomology $K^1(X; \hat{\mathbb{Z}}_2)$ is a 2-adic Adams module with the usual Adams operations ψ^k for $k \in \mathbb{Z}$ as in [6, Example 5.2]. We note that the operations ψ^2 and ψ^k , for k odd, in $K^1(X; \hat{\mathbb{Z}}_2)$ correspond via Bott periodicity to θ and to $k^{-1}\psi^k$ in $K^{-1}(X; \hat{\mathbb{Z}}_2)$. In general, for a θ -pronilpotent 2-adic Adams Δ -module M , we obtain a 2-adic Adams module M^C having the same group as M_C but having $\psi^0 = 0$ and having ψ^{k2^i} equal to $k^{-1}\psi^k\theta^i$ on M_C for k odd and $i \geq 0$.

7.7. Linear and strictly nonlinear modules. As in [8, Section 4] and [7, Section 2], a 2-adic Adams module H is called *linear* when it has $\psi^k = k$ for all $k \in \mathbb{Z}$, and H is called *quasilinear* when $2H \subset \psi^2 H$. Each 2-adic Adams module A has a largest linear quotient module

$$\text{Lin } A = A/((\psi^2 - 2)A + (\psi^{-1} + 1)A + (\psi^3 - 3)A)$$

and also has a largest quasilinear submodule $A_{ql} \subset A$ by Lemma 13.1 below. A 2-adic Adams module A is called *strictly nonlinear* when $A_{ql} = 0$. This implies that A is torsion-free with $\cap_i (\psi^2)^i A = 0$,

and A will be strictly nonlinear by 13.2 and [7, 2.5] whenever it is torsion-free with $(\psi^2)^i A \subset 2^{i+1}A$ for some $i \geq 1$.

7.8. Regular modules. As in [8, 4.4], we say that a 2-adic Adams module A is *regular* when the kernel of $A \rightarrow \text{Lin } A$ is strictly nonlinear. This implies that $\cap_i (\psi^2)^i A = 0$, and A will be regular whenever it is an extension of a strictly nonlinear submodule by a linear quotient module. We also say that a 2-adic Adams Δ -module M is *regular* when it is θ -pronilpotent with M^C regular as a 2-adic Adams module. Thus, for a connected space X with $H^1(X; \hat{\mathbb{Z}}_2) = 0$, the 2-adic Adams Δ -module $\tilde{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ is regular if and only if $\tilde{K}^1(X; \hat{\mathbb{Z}}_2)$ is regular as a 2-adic Adams module. The following two lemmas will often guarantee regularity for our modules.

Lemma 7.9. *Let X be a connected space with $H^1(X; \hat{\mathbb{Z}}_2) = 0$, with $H^m(X; \hat{\mathbb{Z}}_2) = 0$ for sufficiently large m , and with $\tilde{K}^1(X; \hat{\mathbb{Z}}_2)$ torsion-free. Then $\tilde{K}^1(X; \hat{\mathbb{Z}}_2)$ is regular with $\psi^2: \tilde{K}^1(X; \hat{\mathbb{Z}}_2) \rightarrow \tilde{K}^1(X; \hat{\mathbb{Z}}_2)$ monic, and hence $\tilde{K}_{\Delta}^{-1}(X; \hat{\mathbb{Z}}_2)$ is regular with $\theta: \tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2) \rightarrow \tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2)$ monic.*

The proof is in 13.5.

Lemma 7.10. *For a regular 2-adic Adams module A , each submodule is regular, and each torsion-free quotient module is regular when A is finitely generated over $\hat{\mathbb{Z}}_2$.*

The proof is in 13.4. Combining the preceding definitions, we finally introduce:

7.11. Strong modules. We say that a 2-adic Adams Δ -module $M \in \hat{\mathcal{M}}\Delta$ is *strong* when:

- (i) M is robust (7.1);
- (ii) M is ψ^3 -splittable (7.2);
- (iii) M is regular (7.8);

Such an M is automatically θ -pronilpotent (and hence profinite) since it is regular.

8. ON THE $K/2_*$ -LOCALIZATIONS OF OUR SPACES

We recall that the $K/2_*$ -localizations of spaces or spectra are the same as the $K^*(-; \hat{\mathbb{Z}}_2)$ -localizations since the $K/2_*$ -equivalences are the same as the $K^*(-; \hat{\mathbb{Z}}_2)$ -equivalences. In this section, we give our key result (Theorem 8.6) on the $K/2_*$ -localization of a connected space X with $K_{CR}^*(X; \hat{\mathbb{Z}}_2) \cong \hat{L}M$ for a strong 2-adic Adams Δ -module M . We first consider:

8.1. Building blocks for $K/2_*$ -localizations. For a torsion-free exact stable 2-adic Adams Δ -module $N \in \hat{\mathcal{A}}\Delta$, we let $\mathcal{E}N$ denote the $K/2_*$ -local spectrum $\mathcal{E}^{-1}N$ of Theorem 5.5 with an isomorphism $K_{CR}^*(\mathcal{E}N; \hat{\mathbb{Z}}_2) \cong CR^{-1}N$ in the category $\hat{\mathcal{A}}\mathcal{C}\mathcal{R}$ of stable 2-adic Adams CR -modules. As in [8, 3.5], we let $\tilde{\mathcal{E}}N \rightarrow \mathcal{E}N \rightarrow \bar{P}^2\mathcal{E}N$ denote the Postnikov fiber sequence of spectra with $\pi_i\tilde{\mathcal{E}}N \cong \pi_i\mathcal{E}N$ for $i > 2$, with $\pi_i\tilde{\mathcal{E}}N = 0$ for $i < 2$, and with $\pi_2\tilde{\mathcal{E}}N \cong \hat{t}_2\pi_2\mathcal{E}N$, where $\hat{t}_2\pi_2\tilde{\mathcal{E}}N \subset \pi_2\mathcal{E}N$ denotes the Ext-2-completion of the torsion subgroup of $\pi_2\mathcal{E}N$. We now obtain a simply-connected infinite loop space $\Omega^\infty\tilde{\mathcal{E}}N$ which is $K/2_*$ -local by [8, Theorem 3.8]. These $\Omega^\infty\tilde{\mathcal{E}}N$, with their companions $\Omega^\infty\tilde{\mathcal{E}}\bar{\rho}N$, will serve as our building blocks for $K/2_*$ -localizations of spaces, where $\bar{\rho}N$ denotes the torsion-free exact stable 2-adic Adams Δ -module $\bar{\rho}N = \{N_C, N_R + N_H, N_R \cap N_H\}$ of 6.5.

8.2. Strict homomorphisms and isomorphisms. For a 2-adic Adams Δ -module $M \in \hat{\mathcal{M}}\Delta$ and a connected space X , a *strict homomorphism* (resp. *strict isomorphism*) $\hat{L}M \rightarrow K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ is a homomorphism (resp. isomorphism) of special 2-adic ϕCR -algebras induced by a map $M \rightarrow \tilde{K}_\Delta^{-1}(X; \hat{\mathbb{Z}}_2)$ of 2-adic Adams Δ -modules. For instance, there is a strict isomorphism

$$\hat{L}\tilde{F}N \cong K_{CR}^*(\Omega^\infty\tilde{\mathcal{E}}N; \hat{\mathbb{Z}}_2)$$

for each torsion-free exact stable 2-adic Adams Δ -module $N \in \hat{\mathcal{A}}\Delta$ by Theorem 6.7, and we have:

Lemma 8.3. *For a torsion-free exact module $N \in \hat{\mathcal{A}}\Delta$ and a connected space X with $H^1(X; \hat{\mathbb{Z}}_2) = 0 = H^2(X; \hat{\mathbb{Z}}_2)$, each strict homomorphism $\hat{L}\tilde{F}N \rightarrow K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ is induced by a (possibly non-unique) map $X \rightarrow \Omega^\infty\tilde{\mathcal{E}}N$.*

Proof. A strict homomorphism $\hat{L}\tilde{F}N \rightarrow K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ corresponds successively to: a map $\tilde{F}N \rightarrow \tilde{K}_\Delta^{-1}(X; \hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{M}}\Delta$, a map $N \rightarrow \tilde{K}_\Delta^{-1}(X; \hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{A}}\Delta$, and a map $CR^{-1}N \rightarrow K_{CR}^*(\Sigma^\infty X; \hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{A}}\mathcal{C}\mathcal{R}$. By Theorem 5.6, this last map is induced by a map $\Sigma^\infty X \rightarrow \mathcal{E}N$, which lifts uniquely to a map $\Sigma^\infty X \rightarrow \tilde{\mathcal{E}}N$, and we now check that the adjoint map $X \rightarrow \Omega^\infty\tilde{\mathcal{E}}N$ induces the original strict homomorphism. \square

8.4. The key construction. For a strong 2-adic Adams Δ -module $M \in \hat{\mathcal{M}}\Delta$, we may take a θ -resolution (7.4)

$$0 \longrightarrow \tilde{F}\bar{\rho}\bar{M} \xrightarrow{\bar{d}} \tilde{F}\bar{M} \xrightarrow{\bar{s}} M \longrightarrow 0$$

using the torsion-free exact module $\bar{M} = M/\bar{\phi} \in \hat{\mathcal{A}}\Delta$. We may then apply Lemma 8.3 to give a map $f: \Omega^\infty\tilde{\mathcal{E}}\bar{M} \rightarrow \Omega^\infty\tilde{\mathcal{E}}\bar{\rho}\bar{M}$ inducing the $K_{CR}^*(-; \hat{\mathbb{Z}}_2)$ -homomorphism $f^* = \hat{L}\bar{d}: \hat{L}\tilde{F}\bar{\rho}\bar{M} \rightarrow \hat{L}\tilde{F}\bar{M}$. Any such f will be called a *companion map* of M , and its homotopy fiber $\text{Fib } f$ will be $K/2_*$ -local since

$\Omega^\infty \tilde{\mathcal{E}}M$ and $\Omega^\infty \tilde{\mathcal{E}}\bar{\rho}M$ are. As in [8, 4.6] and 8.1, we let

$$\widetilde{\text{Fib}}f \longrightarrow \text{Fib}f \longrightarrow \bar{P}^2 \text{Fib}f$$

denote the Postnikov fiber sequence with $\pi_i \widetilde{\text{Fib}}f \cong \pi_i \text{Fib}f$ for $i > 2$, with $\pi_i \widetilde{\text{Fib}}f = 0$ for $i < 2$, and with $\pi_i \widetilde{\text{Fib}}f \cong \hat{t}_2 \pi_2 \widetilde{\text{Fib}}f$. We note that $\bar{P}^2 \text{Fib}f$ is an infinite loop space which is $K/2_*$ -local by [8, Theorem 3.8], and we conclude that $\widetilde{\text{Fib}}f$ is also $K/2_*$ -local. Moreover, we have $K_{CR}^*(\widetilde{\text{Fib}}f; \hat{\mathbb{Z}}_2) \cong \hat{L}M$ by:

Theorem 8.5. *For a strong 2-adic Adams Δ -module $M \in \hat{\mathcal{M}}\Delta$ and any companion map $f: \Omega^\infty \tilde{\mathcal{E}}\bar{M} \rightarrow \Omega^\infty \tilde{\mathcal{E}}\bar{\rho}\bar{M}$, there is a strict isomorphism $\hat{L}M \cong K_{CR}^*(\widetilde{\text{Fib}}f; \hat{\mathbb{Z}}_2)$.*

Thus, $\hat{L}M$ is topologically realizable for each strong $M \in \hat{\mathcal{M}}\Delta$. This theorem will be proved in 14.7 and leads immediately to our key result on $K/2_*$ -localizations of spaces.

Theorem 8.6. *If X is a connected space with a strict isomorphism $\hat{L}M \cong K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ for a strong 2-adic Adams Δ -module $M \in \hat{\mathcal{M}}\Delta$, then there is an equivalence $X_{K/2} \simeq \widetilde{\text{Fib}}f$ for some companion map $f: \Omega^\infty \tilde{\mathcal{E}}\bar{M} \rightarrow \Omega^\infty \tilde{\mathcal{E}}\bar{\rho}\bar{M}$ of M , where the equivalence induces the canonical isomorphism $K_{CR}^*(\widetilde{\text{Fib}}f; \hat{\mathbb{Z}}_2) \cong \hat{L}M \cong K_{CR}^*(X; \hat{\mathbb{Z}}_2)$. Moreover, $H^1(X; \hat{\mathbb{Z}}_2) = 0 = H^2(X; \hat{\mathbb{Z}}_2)$.*

Proof. The last statement follows by [6, 5.4]. For the first, we take a θ -resolution $0 \rightarrow \tilde{F}\bar{\rho}\bar{M} \xrightarrow{\bar{d}} \tilde{F}\bar{M} \xrightarrow{\bar{s}} M \rightarrow 0$ of M and apply Lemma 8.3 to give a map $h: X \rightarrow \Omega^\infty \tilde{\mathcal{E}}\bar{M}$ with $h^* = \hat{L}\bar{s}: \hat{L}\tilde{F}\bar{M} \rightarrow \hat{L}M$. We then apply Lemma 8.3 again to give a map $k: \text{Cof}h \rightarrow \Omega^\infty \tilde{\mathcal{E}}\bar{\rho}\bar{M}$ with

$$k^* = \hat{L}\bar{d}: \hat{L}\tilde{F}\bar{\rho}\bar{M} \longrightarrow K_{CR}^*(\text{Cof}h; \hat{\mathbb{Z}}_2) \subset \hat{L}\tilde{F}\bar{M}.$$

Composing k with the cofiber map, we obtain a companion map $f: \Omega^\infty \tilde{\mathcal{E}}\bar{M} \rightarrow \Omega^\infty \tilde{\mathcal{E}}\bar{\rho}\bar{M}$ of M such that h lifts to a map $u: X \rightarrow \widetilde{\text{Fib}}f$ which is a $K/2_*$ -equivalence by Theorem 8.5. Since $\widetilde{\text{Fib}}f$ is $K/2_*$ -local, this gives the desired equivalence $X_{K/2} \simeq \widetilde{\text{Fib}}f$. \square

In this theorem, M is uniquely determined by the space X since there is a canonical isomorphism $M \cong \hat{Q}K_\Delta^{-1}(X; \hat{\mathbb{Z}}_2)$ in $\hat{\mathcal{M}}\Delta$ by 4.10 and [11, Section 3].

9. ON THE v_1 -PERIODIC HOMOTOPY GROUPS OF OUR SPACES

The p -primary v_1 -periodic homotopy groups $v_1^{-1}\pi_*X$ of a space X at a prime p were defined by Davis and Mahowald [15] and have been studied extensively (see [13]). In this section, we apply the preceding result (Theorem 8.6) on the $K/2_*$ -localizations of our spaces to approach their v_1 -periodic homotopy groups at $p = 2$ using:

9.1. The functor Φ_1 . As in [4], [9], [16], and [19], there is a v_1 -stabilization functor Φ_1 from the homotopy category of spaces to that of spectra such that:

- (i) for a space X , there is a natural isomorphism $v_1^{-1}\pi_*X \cong \pi_*\tau_2\Phi_1X$ where $\tau_2\Phi_1X$ is the 2-torsion part of Φ_1X (given by the fiber of its localization away from 2);
- (ii) Φ_1X is $K/2_*$ -local for each space X ;
- (iii) for a spectrum E , there is a natural equivalence $\Phi_1(\Omega^\infty E) \simeq E_{K/2}$;
- (iv) Φ_1 preserves fiber squares.

Various other properties of Φ_1 are described in [10, Section 2], and the isomorphism $v_1^{-1}\pi_*X \cong \pi_*\tau_2\Phi_1X$ may be applied as in [10, Theorem 3.2] to show:

Theorem 9.2. *For a space X , there is a natural long exact sequence*

$$\begin{aligned} \dots &\longrightarrow KO^{n-3}(\Phi_1X; \hat{\mathbb{Z}}_2) \xrightarrow{\psi^3-9} KO^{n-3}(\Phi_1X; \hat{\mathbb{Z}}_2) \longrightarrow (v_1^{-1}\pi_nX)^\# \\ &\longrightarrow KO^{n-2}(\Phi_1X; \hat{\mathbb{Z}}_2) \xrightarrow{\psi^3-9} KO^{n-2}(\Phi_1X; \hat{\mathbb{Z}}_2) \longrightarrow \dots \end{aligned}$$

where $(-)^{\#}$ gives the Pontrjagin dual.

This may be used to calculate $v_1^{-1}\pi_*X$ from $KO^*(\Phi_1X; \hat{\mathbb{Z}}_2)$ up to extension. To approach $KO^*(\Phi_1X; \hat{\mathbb{Z}}_2)$ or $K^*(\Phi_1X; \hat{\mathbb{Z}}_2)$, we work with:

9.3. $K/2_*$ -durable spaces. Following [8, 7.8], we say that a space X is $K/2_*$ -durable when the $K/2_*$ -localization $X \rightarrow X_{K/2}$ induces an equivalence $\Phi_1X \simeq \Phi_1X_{K/2}$ (or equivalently induces an isomorphism $v_1^{-1}\pi_*X \cong v_1^{-1}\pi_*X_{K/2}$), and we recall that each connected H -space is $K/2_*$ -durable. For such X , we may apply our key result on $K/2_*$ -localizations (Theorem 8.6) to deduce:

Theorem 9.4. *If X is a connected $K/2_*$ -durable space (e.g. H -space) with a strict isomorphism $\hat{L}M \cong K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ for a strong module $M \in \hat{\mathcal{M}}\Delta$, then there is a (co)fiber sequence of spectra $\Phi_1X \rightarrow \mathcal{E}\bar{M} \xrightarrow{\epsilon} \mathcal{E}\bar{\rho}\bar{M}$ such that $\epsilon^*: K_{CR}^*(\mathcal{E}\bar{\rho}\bar{M}; \hat{\mathbb{Z}}_2) \rightarrow K_{CR}^*(\mathcal{E}\bar{M}; \hat{\mathbb{Z}}_2)$ is given by $CR^{-1}\theta: CR^{-1}\bar{\rho}\bar{M} \rightarrow CR^{-1}\bar{M}$.*

Here, the map $\theta: \bar{\rho}\bar{M} \rightarrow \bar{M}$ is given by

$$\theta = (\theta, \theta, \theta): \{\bar{M}_C, \bar{M}_R + \bar{M}_H, \bar{M}_R \cap \bar{M}_H\} \longrightarrow \{\bar{M}_C, \bar{M}_R, \bar{M}_H\}$$

in $\hat{\mathcal{A}}\Delta$. This theorem will be proved below in 9.9 and may be used to calculate $K^*(\Phi_1X; \hat{\mathbb{Z}}_2)$ and $KO^*(\Phi_1X; \hat{\mathbb{Z}}_2)$ since it immediately implies:

Theorem 9.5. For X as in Theorem 9.4, there is a $K^*(-; \hat{\mathbb{Z}}_2)$ cohomology exact sequence

$$0 \longrightarrow K^{-2}(\Phi_1 X; \hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_C \xrightarrow{\theta} \bar{M}_C \longrightarrow K^{-1}(\Phi_1 X; \hat{\mathbb{Z}}_2) \longrightarrow 0,$$

and there is a $KO^*(-; \hat{\mathbb{Z}}_2)$ cohomology exact sequence

$$\begin{aligned} 0 \longrightarrow KO^{-8}(\Phi_1 X; \hat{\mathbb{Z}}_2) &\longrightarrow \bar{M}_C / (\bar{M}_R + \bar{M}_H) \xrightarrow{\theta} \bar{M}_C / \bar{M}_R \longrightarrow KO^{-7}(\Phi_1 X; \hat{\mathbb{Z}}_2) \longrightarrow 0 \\ &\longrightarrow \bar{M}_H / (\bar{M}_R \cap \bar{M}_H) \longrightarrow KO^{-6}(\Phi_1 X; \hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_R \cap \bar{M}_H \xrightarrow{\theta} \bar{M}_H \longrightarrow KO^{-5}(\Phi_1 X; \hat{\mathbb{Z}}_2) \\ &\longrightarrow 0 \longrightarrow 0 \longrightarrow KO^{-4}(\Phi_1 X; \hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_C / (\bar{M}_R \cap \bar{M}_H) \xrightarrow{\theta} \bar{M}_C / \bar{M}_H \\ &\longrightarrow KO^{-3}(\Phi_1 X; \hat{\mathbb{Z}}_2) \longrightarrow (\bar{M}_R + \bar{M}_H) / (\bar{M}_R \cap \bar{M}_H) \xrightarrow{\theta} \bar{M}_R / (\bar{M}_R \cap \bar{M}_H) \\ &\longrightarrow KO^{-2}(\Phi_1 X; \hat{\mathbb{Z}}_2) \longrightarrow \bar{M}_R + \bar{M}_H \xrightarrow{\theta} \bar{M}_R \longrightarrow KO^{-1}(\Phi_1 X; \hat{\mathbb{Z}}_2) \longrightarrow 0 \end{aligned}$$

In these sequences, θ may be replaced by $\lambda^2 = -\theta$. Also, for $i, k \in \mathbb{Z}$ with k odd, the Adams operation ψ^k in $K^{2i-1}(\Phi_1 X; \hat{\mathbb{Z}}_2)$, $K^{2i-2}(\Phi_1 X; \hat{\mathbb{Z}}_2)$, $KO^{2i-1}(\Phi_1 X; \hat{\mathbb{Z}}_2)$, or $KO^{2i-2}(\Phi_1 X; \hat{\mathbb{Z}}_2)$ agrees with $k^{-i}\psi^k$ in the adjacent \bar{M} terms.

Thus, for X as in Theorem 9.4, we may essentially calculate $v_1^{-1}\pi_* X$ from \bar{M} (up to extension problems) using Theorems 9.2 and 9.5. By [10, 7.6], this approach to $v_1^{-1}\pi_* X$ may be extended to various other important spaces X using:

9.6. The $\hat{K}\Phi_1$ -goodness condition. For a space X , we let $\Phi_1: \tilde{K}_{CR}^*(X; \hat{\mathbb{Z}}_2) \longrightarrow K_{CR}^*(\Phi_1 X; \hat{\mathbb{Z}}_2)$ denote the v_1 -stabilization homomorphism of [10, 7.1], and we recall that it induces a homomorphism $\Phi_1: \hat{Q}K_{\Delta}^n(X; \hat{\mathbb{Z}}_2)/\theta \longrightarrow K_{\Delta}^n(\Phi_1 X; \hat{\mathbb{Z}}_2)$ in $\hat{A}\Delta$ for $n = -1, 0$ by [10, 7.4], where $\hat{Q}K_{\Delta}^n(X; \hat{\mathbb{Z}}_2)/\theta$ is as in 4.10 and 6.1. Following [10, 7.5], we say that a space X is $\hat{K}\Phi_1$ -good when the complex v_1 -stabilization homomorphism $\Phi_1: \hat{Q}K^n(X; \hat{\mathbb{Z}}_2)/\theta \rightarrow K^n(\Phi_1 X; \hat{\mathbb{Z}}_2)$ is an isomorphism for $n = -1, 0$. Our next theorem will provide initial examples of $\hat{K}\Phi_1$ -good spaces from which other examples may be built.

Theorem 9.7. If X is a connected $K/2_*$ -durable space (e.g. H -space) with a strict isomorphism $\hat{L}M \cong K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ for a strong module $M \in \hat{M}\Delta$ such that $\theta: \bar{M}_C \rightarrow \bar{M}_C$ is monic, then X is $\hat{K}\Phi_1$ -good with $K^0(\Phi_1 X; \hat{\mathbb{Z}}_2) = 0$, with $K^{-1}(\Phi_1 X; \hat{\mathbb{Z}}_2) = \bar{M}_C/\theta$, and with $K_{\Delta}^{-1}(\Phi_1 X; \hat{\mathbb{Z}}_2) \cong \bar{M}/\theta$.

To prove Theorems 9.4 and 9.7, we first consider the spectrum $\tilde{\mathcal{E}}N$ for a torsion-free exact module $N \in \hat{A}\Delta$ and note that $\Phi_1\Omega^{\infty}\tilde{\mathcal{E}}N \simeq (\tilde{\mathcal{E}}N)_{K/2} \simeq \mathcal{E}N$.

Lemma 9.8. The space $\Omega^{\infty}\tilde{\mathcal{E}}N$ is $\hat{K}\Phi_1$ -good, and the v_1 -stabilization gives a natural isomorphism

$$\Phi_1: \hat{Q}K_{\Delta}^{-1}(\Omega^{\infty}\tilde{\mathcal{E}}N; \hat{\mathbb{Z}}_2)/\theta \cong K_{\Delta}^{-1}(\mathcal{E}N; \hat{\mathbb{Z}}_2)$$

Proof. By [10, 7.1], the homomorphism $\Phi_1: K_{\Delta}^{-1}(\Omega^{\infty}\tilde{\mathcal{E}}N; \hat{\mathbb{Z}}_2) \rightarrow K_{\Delta}^{-1}(\mathcal{E}N; \hat{\mathbb{Z}}_2)$ is left inverse to the infinite suspension homomorphism, and the lemma now follows by Theorem 6.7 together with 4.11, and 6.3. \square

9.9. Proof of Theorem 9.4. Applying the functor Φ_1 to the fiber sequence of Theorem 8.6, we obtain a (co)fiber sequence of spectra

$$\Phi_1 X_{K/2} \longrightarrow \Phi_1 \Omega^{\infty} \tilde{\mathcal{E}}\bar{M} \xrightarrow{\Phi_1 f} \Phi_1 \Omega^{\infty} \tilde{\mathcal{E}}\bar{\rho}\bar{M}$$

for some companion map f of M . We then deduce that $\Phi_1 f$ corresponds to a map $\mathcal{E}\bar{M} \rightarrow \mathcal{E}\bar{\rho}\bar{M}$ having the desired properties by Lemmas 9.8 and 5.4. \square

9.10. Proof of Theorem 9.7. The results on $K^*(\Phi_1 X; \hat{\mathbb{Z}}_2)$ and $K_{\Delta}^{-1}(\Phi_1 X; \hat{\mathbb{Z}}_2)$ follow from Theorem 9.5. Since $K^*(X; \hat{\mathbb{Z}}_2) \cong \hat{\Lambda}M_C$ by Lemma 4.6, we obtain isomorphisms $\hat{Q}K^0(X; \hat{\mathbb{Z}}_2)/\theta = 0$ and $\hat{Q}K^{-1}(X; \hat{\mathbb{Z}}_2)/\theta \cong M_C/\theta$, and we deduce that $\Phi_1: \hat{Q}K^n(X; \hat{\mathbb{Z}}_2)/\theta \cong K^n(\Phi_1 X; \hat{\mathbb{Z}}_2)$ for $n = -1, 0$ by Lemma 9.8 and naturality. \square

10. APPLICATIONS TO SIMPLY-CONNECTED COMPACT LIE GROUPS

We now apply the preceding results to a simply-connected compact Lie group G . We first use the representation theory of G to functorially determine the united 2-adic K -cohomology ring $K_{CR}^*(G; \hat{\mathbb{Z}}_2) = \{K^*(G; \hat{\mathbb{Z}}_2), KO^*(G; \hat{\mathbb{Z}}_2)\}$ in Theorem 10.3. Then, with slight restrictions on the group, we show that G is $\hat{K}\Phi_1$ -good, and we use the representation theory of G to give expressions for the $K/2_*$ -localization $G_{K/2}$, for the v_1 -stabilization $\Phi_1 G$, and for the cohomology $KO^*(\Phi_1 G; \hat{\mathbb{Z}}_2)$. Our results are summarized in Theorem 10.6 and permit calculations of the 2-primary v_1 -periodic homotopy $v_1^{-1}\pi_*G$ using Theorem 9.2 as accomplished very successfully by Davis [14]. In this section, we assume some general familiarity with the representation rings of our Lie groups as described in [12, Sections II.6 and VI.4] and [14, Theorem 2.3].

10.1. The representation ring $R_{\Delta}G$. For a simply-connected compact Lie group G , we let RG be the complex representation ring and let $R_R G, R_H G \subset RG$ be the real and quaternionic parts of RG with the usual λ -ring structures on RG and $R_R G \oplus R_H G$. We also let $t = \psi^{-1}: RG \cong RG$, $c: R_R G \subset RG$, $r: RG \rightarrow R_R G$, $c': R_H G \subset RG$, and $q: RG \rightarrow R_H G$ be the usual operations satisfying the Δ -module relations of 4.1. These structures are compatible in the expected ways and combine to give a $\Delta\lambda$ -ring $R_{\Delta}G = \{RG, R_R G, R_H G\}$ in the sense of [10, 6.2]. We let $\tilde{R}_{\Delta}G = \{\tilde{R}G, \tilde{R}_R G, \tilde{R}_H G\}$ be the *augmentation ideal* of $R_{\Delta}G$ given by the kernel $\tilde{R}G$ of the complex

augmentation $\dim: RG \rightarrow \mathbb{Z}$, where $\tilde{R}_R G = R_R G \cap \tilde{R}G$ and $\tilde{R}_H G = R_H G \cap \tilde{R}G$. We also let $QR_{\Delta}G = \{QRG, QR_R G, QR_H G\}$ be the *indecomposables* of $R_{\Delta}G$ given by

$$\begin{aligned} QRG &= \tilde{R}G/(\tilde{R}G)^2, \\ QR_R G &= \tilde{R}_R G/((\tilde{R}_R G)^2 + (\tilde{R}_H G)^2 + r(\tilde{R}G)^2), \\ QR_H G &= \tilde{R}_H G/((\tilde{R}_R G)(\tilde{R}_H G) + q(\tilde{R}G)^2). \end{aligned}$$

It is straightforward to show that $\tilde{R}_{\Delta}G$ and $QR_{\Delta}G$ inherit $\Delta\lambda$ -ring structures (without identities) from $R_{\Delta}G$. Since $QR_{\Delta}G$ is a $\Delta\lambda$ -ring with trivial multiplication, it is equipped with additive operations $t: QRG \cong QRG$, $c: QR_R G \rightarrow QRG$, $r: QRG \rightarrow QR_R G$, $c': QR_H G \rightarrow QRG$, $q: QRG \rightarrow QR_H G$, $\theta = -\lambda^2: QRG \rightarrow QRG$, $\theta = -\lambda^2: QR_R G \rightarrow QR_R G$, $\theta = -\lambda^2: QR_H G \rightarrow QR_R G$, $\psi^k: QRG \rightarrow QRG$, $\psi^k: QR_R G \rightarrow QR_R G$, and $\psi^k: QR_H G \rightarrow QR_H G$ for the odd $k \in \mathbb{Z}$. We now let $\hat{Q}R_{\Delta}G = \{\hat{Q}RG, \hat{Q}R_R G, \hat{Q}R_H G\}$ be the 2-adic completion of $QR_{\Delta}G$ with the induced additive operations on the components $\hat{Q}RG = \hat{\mathbb{Z}}_2 \otimes QRG$, $\hat{Q}R_R G = \hat{\mathbb{Z}}_2 \otimes QR_R G$, and $\hat{Q}R_H G = \hat{\mathbb{Z}}_2 \otimes QR_H G$.

Lemma 10.2. *For a simply-connected compact Lie group G , $\hat{Q}R_{\Delta}G$ is a robust 2-adic Adams Δ -module.*

This will be proved later in 10.9. To determine the cohomology ring $K_{CR}^*(G; \hat{\mathbb{Z}}_2) = \{K^*(G; \hat{\mathbb{Z}}_2), KO^*(G; \hat{\mathbb{Z}}_2)\}$ from the representation theory of G , we now let $\beta: \hat{Q}R_{\Delta}G \rightarrow \tilde{K}_{\Delta}^{-1}(G; \hat{\mathbb{Z}}_2)$ be the 2-adic Adams Δ -module homomorphism induced by the composition of the canonical homomorphisms $\tilde{R}_{\Delta}G \rightarrow \tilde{K}_{\Delta}^0(BG; \hat{\mathbb{Z}}_2) \rightarrow \tilde{K}_{\Delta}^{-1}(G; \hat{\mathbb{Z}}_2)$.

Theorem 10.3. *For a simply-connected compact Lie group G , there is a natural strict isomorphism $\bar{\beta}: \hat{L}(\hat{Q}R_{\Delta}G) \cong K_{CR}^*(G; \hat{\mathbb{Z}}_2)$.*

Proof. This follows by Lemma 10.2 and Theorem 4.9 since $\beta: \hat{Q}RG \rightarrow K^{-1}(G; \hat{\mathbb{Z}}_2)$ induces an isomorphism $\hat{L}(\hat{Q}RG) \cong K^*(G; \hat{\mathbb{Z}}_2)$ by [18]. \square

We note that $K_{CR}^*(G; \hat{\mathbb{Z}}_2)$ has a simple system of generators (3.3) consisting of the $\beta\tilde{z}_{\gamma} \in K^{-1}(G; \hat{\mathbb{Z}}_2)$, the $\beta\tilde{x}_{\alpha} \in KO^{-1}(G; \hat{\mathbb{Z}}_2)$, and the $\beta\tilde{y}_{\beta} \in KO^{-5}(G; \hat{\mathbb{Z}}_2)$ obtained from the analysis of $\hat{Q}R_{\Delta}G$ below in 10.8. Thus, by Proposition 3.4, $K_{CR}^*(G; \hat{\mathbb{Z}}_2)$ is a free 2-adic CR -module on the associated products. However, our description of $K_{CR}^*(G; \hat{\mathbb{Z}}_2)$ as $\hat{L}(\hat{Q}R_{\Delta}G)$ is more natural and includes the full multiplicative structure. Moreover, it will let us apply our main results to G .

Lemma 10.4. *For a simply-connected compact Lie group G , the 2-adic Adams Δ -module $\hat{Q}R_\Delta G$ is regular with $\theta: \hat{Q}RG \rightarrow \hat{Q}RG$ monic.*

Proof. This follows by Lemmas 7.9 and 7.10 since $\beta: \hat{Q}RG \rightarrow \tilde{K}^{-1}(G; \hat{\mathbb{Z}}_2)$ is monic by Theorem 10.3. \square

Thus, $\hat{Q}R_\Delta G$ is strong (robust, ψ^3 -splittable, and regular) if and only if it is ψ^3 -splittable, and this is usually the case by:

Lemma 10.5. *For a simply-connected compact simple Lie group G , the 2-adic Adams Δ -module $\hat{Q}R_\Delta G$ is ψ^3 -splittable (and hence strong) if and only if G is not E_6 or $Spin(4k+2)$ with k not a 2-power.*

This will be proved later in 10.11 using work of Davis [14]. For a simply-connected compact Lie group G , we now let $\hat{Q}_\Delta = \{\hat{Q}, \hat{Q}_R, \hat{Q}_H\}$ briefly denote the associated stable 2-adic Adams Δ -module $\overline{\hat{Q}_\Delta RG} = (\hat{Q}_\Delta RG)/\bar{\phi}$. This agrees with the notation of [10, 9.2] and [14], since our $\hat{Q}_\Delta = \{\hat{Q}, \hat{Q}_R, \hat{Q}_H\}$ is the 2-adic completion of their $Q_\Delta = \{Q, Q_R, Q_H\}$. Our main results now give the following omnibus theorem, whose four parts may be expanded in the obvious ways to match the cited theorems.

Theorem 10.6. *Let G be a simply-connected compact Lie group such that the 2-adic Adams Δ -module $\hat{Q}_\Delta RG$ is ψ^3 -splittable (see Lemma 10.5), and let $\hat{Q}_\Delta = \{\hat{Q}, \hat{Q}_R, \hat{Q}_H\}$ be the associated stable 2-adic Adams Δ -module. Then:*

- (i) *the $K/2_*$ -localization $G_{K/2}$ is the homotopy fiber of a map $\Omega^\infty \tilde{\mathcal{E}}\hat{Q}_\Delta \rightarrow \Omega^\infty \tilde{\mathcal{E}}\bar{\rho}\hat{Q}_\Delta$ with low dimensional modifications as in Theorem 8.6;*
- (ii) *the 2-adic v_1 -stabilization $\Phi_1 G$ is the homotopy fiber of a map of spectra $\mathcal{E}\hat{Q}_\Delta \rightarrow \mathcal{E}\bar{\rho}\hat{Q}_\Delta$ as in Theorem 9.4;*
- (iii) *there is an exact sequence*

$$0 \longrightarrow KO^{-8}(\Phi_1 G; \hat{\mathbb{Z}}_2) \longrightarrow \hat{Q}/(\hat{Q}_R + \hat{Q}_H) \xrightarrow{\theta} \hat{Q}/\hat{Q}_R \longrightarrow \dots$$

continuing as in Theorem 9.5;

- (iv) *G is $\hat{K}\Phi_1$ -good at the prime 2 as in Theorem 9.7.*

This permits calculations of the 2-primary v_1 -periodic homotopy $v_1^{-1}\pi_* G$ using Theorem 9.2.

Remark 10.7. Slightly extending the present work, it is straightforward to show that the above conclusions remain valid for the exotic 2-compact group $DI(4)$ of Dwyer-Wilkerson [17]. In particular, parts (i)–(iv) remain valid with $DI(4)$ in place of G , with $\hat{Q}K_{\Delta}^0(BDI(4); \hat{\mathbb{Z}}_2)$ in place of $\hat{Q}_{\Delta}RG$, and with $\hat{Q}K_{\Delta}^0(BDI(4); \hat{\mathbb{Z}}_2)/\bar{\phi}$ in place of \hat{Q}_{Δ} .

We devote the rest of this section to proving Lemmas 10.2 and 10.5 and start by giving:

10.8. Generators for representation rings. For a simply-connected compact Lie group G , standard results summarized in [14, Theorem 2.3] show that RG is a finitely generated polynomial ring $\mathbb{Z}[z_{\gamma}, z_{\gamma}^*, x_{\alpha}, y_{\beta}]_{\gamma, \alpha, \beta}$ on certain basic complex representations z_{γ} together with their conjugates $z_{\gamma}^* = tz_{\gamma}$, certain basic real representations x_{α} , and certain basic quaternionic representations y_{β} . Moreover, in terms of these generators, the $\mathbb{Z}/2$ -graded ring $\{R_{RG}, R_{HG}\}$ is characterized by the fact that its quotient $\{R_{RG}/rRG, R_{HG}/qRG\}$ is a $\mathbb{Z}/2$ -graded polynomial algebra $\mathbb{Z}/2[x_{\alpha}, \bar{\phi}z_{\gamma}, y_{\beta}]_{\alpha, \gamma, \beta}$ on the real generators x_{α} and $\bar{\phi}z_{\gamma}$ (with $c\bar{\phi}z_{\gamma} = z_{\gamma}^*z_{\gamma}$) and the quaternionic generators y_{β} . Consequently, the indecomposables $QR_{\Delta}G = \{QRG, QR_{RG}, QR_{HG}\}$ may be expressed as

$$\begin{aligned} QRG &= \mathbb{Z}\{\tilde{z}_{\gamma}, \tilde{z}_{\gamma}^*, c\tilde{x}_{\alpha}, c'\tilde{y}_{\beta}\}_{\gamma, \alpha, \beta}, \\ QR_{RG} &= \mathbb{Z}\{r\tilde{z}_{\gamma}, \tilde{x}_{\alpha}, rc'\tilde{y}_{\beta}\}_{\gamma, \alpha, \beta} \oplus \mathbb{Z}/2\{\bar{\phi}\tilde{z}_{\gamma}\}_{\gamma}, \\ QR_{HG} &= \mathbb{Z}\{q\tilde{z}_{\gamma}, qc\tilde{x}_{\alpha}, \tilde{y}_{\beta}\}_{\gamma, \alpha, \beta} \end{aligned}$$

where \tilde{w} denotes $w - \dim w$ for $w \in RG$. Thus, the 2-adic indecomposables $\hat{Q}R_{\Delta}G = \{\hat{Q}RG, \hat{Q}R_{RG}, \hat{Q}R_{HG}\}$ may be expressed similarly using $\hat{\mathbb{Z}}_2$ in place of \mathbb{Z} , and the stable 2-adic indecomposables $\hat{Q}_{\Delta} = \{\hat{Q}, \hat{Q}_R, \hat{Q}_H\}$ may be expressed as

$$\begin{aligned} \hat{Q} &= \hat{\mathbb{Z}}_2\{\tilde{z}_{\gamma}, \tilde{z}_{\gamma}^*, c\tilde{x}_{\alpha}, c'\tilde{y}_{\beta}\}_{\gamma, \alpha, \beta}, \\ \hat{Q}_R &= \hat{\mathbb{Z}}_2\{r\tilde{z}_{\gamma}, \tilde{x}_{\alpha}, rc'\tilde{y}_{\beta}\}_{\gamma, \alpha, \beta}, \\ \hat{Q}_H &= \hat{\mathbb{Z}}_2\{q\tilde{z}_{\gamma}, qc\tilde{x}_{\alpha}, \tilde{y}_{\beta}\}_{\gamma, \alpha, \beta}. \end{aligned}$$

10.9. Proof of Lemma 10.2. Since $QR_{\Delta}G$ is a $\Delta\lambda$ -ring with trivial multiplication, it is straightforward to check all of the required relations for operations (see 4.3 and 6.1). In particular, we deduce $\theta\theta r = \theta r\theta$ from the relations $\lambda^4 r = r\lambda^4 + \bar{\phi}\lambda^2$, $\lambda^4 = -\lambda^2\lambda^2$, $\bar{\phi} = \lambda^2 r - r\lambda^2$, $2\bar{\phi} = 0$, and $\theta = -\lambda^2$, which hold generally in $\Delta\lambda$ -rings with trivial multiplication [10, 6.2]. We next observe that $\hat{Q}RG$, $\hat{Q}R_{RG}$, and $\hat{Q}R_{HG}$ are stable 2-adic Adams modules by [6, 6.2], since QRG and $QR_{RG} \oplus QR_{HG}$ are γ -nilpotent and finitely generated abelian (because they have trivial multiplications and have finite generating sets of elements \tilde{w} for representations w). Thus, $\hat{Q}R_{\Delta}G$ is a 2-adic Adams Δ -module, and it must be robust by the analysis of 10.8. \square

To check the ψ^3 -splittability of $\hat{Q}R_\Delta G$, we let $hG = \ker(1-t)/\text{im}(1+t)$ be the augmented algebra over $\mathbb{Z}/2$ obtained from RG using the involution $t = \psi^{-1}: RG \cong RG$. This is a polynomial algebra $hG \cong \mathbb{Z}/2[c\tilde{x}_\alpha, \tilde{z}_\gamma^* \tilde{z}_\gamma, c'\tilde{y}_\beta]_{\alpha,\gamma,\beta}$ which is $\mathbb{Z}/2$ -graded since there is an isomorphism

$$c + c': R_R G / rRG \oplus R_H G / qRG \cong hG,$$

and we let $Q_R hG \cong \mathbb{Z}/2\{c\tilde{x}_\alpha, \tilde{z}_\gamma^* \tilde{z}_\gamma\}_{\alpha,\gamma}$ denote the real (degree 0) indecomposables. We define a homomorphism $s: QRG \rightarrow Q_R hG$ by $s[u] = [u^*u]$ for $u \in \tilde{R}G$ and note that $sQRG = \mathbb{Z}/2\{\tilde{z}_\gamma^* \tilde{z}_\gamma\}_\gamma$. We view s as a homomorphism of ψ^3 -modules (abelian groups with endomorphisms ψ^3) as in [14, 2.4].

Lemma 10.10. *For a simply-connected compact Lie group G , $\hat{Q}R_\Delta G$ is ψ^3 -splittable if and only if the ψ^3 -submodule $sQRG \subset Q_R hG$ is a direct summand.*

Proof. By 7.2 and the proof of Lemma 7.3, $\hat{Q}R_\Delta G$ is ψ^3 -splittable if and only if the ψ^3 -submodule $\bar{\phi}\hat{Q}RG \subset \hat{Q}R_R G / r\hat{Q}RG$ (or equivalently $\bar{\phi}QRG \subset QR_R G / rQRG$) is a direct summand. The lemma now follows since $\bar{\phi}QRG$ corresponds to $sQRG$ under the isomorphism $c: QR_R G / rQRG \cong Q_R hG$. \square

10.11. **Proof of Lemma 10.5.** By Lemma 10.10 and Davis [14, Theorem 1.3], the following conditions are equivalent: $\hat{Q}R_\Delta G$ is ψ^3 -splittable; the ψ^3 -submodule $sQRG \subset Q_R hG$ is a direct summand; G satisfies the Technical Condition of [14, Definition 2.4]; G is not E_6 or $Spin(4k+2)$ with k not a 2-power. \square

11. PROOFS OF BASIC LEMMAS FOR \hat{L}

We shall prove Lemmas 4.5, 4.6, and 4.11 showing the basic properties of the functor $\hat{L}: \theta\Delta\hat{M}od \rightarrow \phi\mathcal{CR}\hat{A}lg$, where $\theta\Delta\hat{M}od$ is the category of 2-adic $\theta\Delta$ -modules (4.3) and $\phi\mathcal{CR}\hat{A}lg$ is that of special 2-adic $\phi\mathcal{C}R$ -algebras (3.2). We first introduce an intermediate category of modules.

11.1. **The 2-adic $\eta\Delta$ -modules.** By a 2-adic $\eta\Delta$ -module $N = \{N_C, N_R, N_H, N_S\}$, we mean a 2-adic Δ -module $\{N_C, N_R, N_H\}$, with operations t, c, r, c' , and q as in 4.1, together with a 2-profinite abelian group N_S and continuous additive operations $\bar{\phi}: N_C \rightarrow N_R$, $\eta: N_R \rightarrow N_S$, $()^{[2]}: N_R \rightarrow N_S$, and $()^{[2]}: N_H \rightarrow N_S$ satisfying the following relations for elements $z \in N_C$, $x \in N_R$, and $y \in N_H$:

$$\begin{aligned} \bar{\phi}cx &= 0, & \bar{\phi}c'y &= 0, & \bar{\phi}tz &= \bar{\phi}z, & 2\bar{\phi}z &= 0, & c\bar{\phi}z &= 0, \\ (\bar{\phi}z)^{[2]} &= 0, & 2\eta x &= 0, & \eta rz &= 0, & (qz)^{[2]} &= (rz)^{[2]} &= \eta\bar{\phi}z. \end{aligned}$$

We let $\eta\Delta\hat{Mod}$ denote the category of 2-adic $\eta\Delta$ -modules.

11.2. A functorial interpretation of admissible maps. Let $J: \theta\Delta\hat{Mod} \rightarrow \eta\Delta\hat{Mod}$ be the functor carrying a 2-adic $\theta\Delta$ -module M to the 2-adic $\eta\Delta$ -module $JM = \{M_C, M_R, M_H, M_R/rM_C\}$ having the original operations t, c, r, c', q , and $\bar{\phi}$ together with operations $\eta: M_R \rightarrow M_R/rM_C$, $()^{[2]}: M_R \rightarrow M_R/rM_C$, and $()^{[2]}: M_H \rightarrow M_R/rM_C$ given by $\eta x = [x]$, $x^{[2]} = [\theta x]$, and $y^{[2]} = [\theta y]$ for $x \in M_R$ and $y \in M_H$. Let $I: \phi CR\hat{Alg} \rightarrow \eta\Delta\hat{Mod}$ be the functor carrying a special 2-adic ϕCR -algebra A to the 2-adic $\eta\Delta$ -module $IA = \{\tilde{A}_C^{-1}, \tilde{A}_R^{-1}, \tilde{A}_R^{-5}, \tilde{A}_R^{-2}\}$ having the operations t, c, r, c' , and q of $\Delta^{-1}\tilde{A}$ (see 4.1) together with operations $\bar{\phi}: \tilde{A}_C^{-1} \rightarrow \tilde{A}_R^{-1}$, $\eta: \tilde{A}_R^{-1} \rightarrow \tilde{A}_R^{-2}$, $()^{[2]}: \tilde{A}_R^{-1} \rightarrow \tilde{A}_R^{-2}$, and $()^{[2]}: \tilde{A}_R^{-5} \rightarrow \tilde{A}_R^{-2}$ given by $\bar{\phi}z = \eta\phi z$, $\eta x = \eta x$, $x^{[2]} = x^2$, and $y^{[2]} = B_R^{-1}y^2$ for $z \in A_C^{-1}$, $x \in A_R^{-1}$, and $y \in A_R^{-5}$. We now easily see:

Lemma 11.3. *For $M \in \theta\Delta\hat{Mod}$ and $A \in \phi CR\hat{Alg}$, an admissible map $f: M \rightarrow A$ (see 4.4) is equivalent to a map $f: JM \rightarrow IA$ in $\eta\Delta\hat{Mod}$.*

To construct the functor \hat{L} , we need:

Lemma 11.4. *The functor $I: \phi CR\hat{Alg} \rightarrow \eta\Delta\hat{Mod}$ has a left adjoint $\hat{V}: \eta\Delta\hat{Mod} \rightarrow \phi CR\hat{Alg}$.*

Proof. This follows by the Special Adjoint Functor Theorem (see [20]) since I preserves small limits and since $\phi CR\hat{Alg}$ has a small cogenerating set by Lemma 11.5 below. \square

A special 2-adic ϕCR -algebra A will be called *finite* when the groups \tilde{A}_C^m and \tilde{A}_R^m are finite for all m .

Lemma 11.5. *Each special 2-adic ϕCR -algebra A is the inverse limit of its finite quotients in $\phi CR\hat{Alg}$.*

Proof. This is similar to the corresponding result for topological rings in [23, 5.1.2]. For a 2-adic CR -submodule $G \subset \tilde{A}$ with \tilde{A}/G finite, we must obtain a special 2-adic ϕCR -ideal H of A with $H \subset G$ and \tilde{A}/H finite. We first obtain an ideal M of A_R (closed under B_R, B_R^{-1}, η , and ξ) with $M \subset G_R$ and \tilde{A}_R/M finite as in [23]. We next obtain an ideal N of A_C (closed under B, B^{-1} , and t) with $N \subset G_C \cap r^{-1}M \cap \phi^{-1}M^0$ and \tilde{A}_C/N finite as in [23]. The desired ideal H is now given by $H_C = N$ and $H_R = M \cap c^{-1}N$. \square

11.6. Proof of Lemma 4.5. Using Lemmas 11.3 and 11.4, we obtain the desired universal algebra $\hat{L}M$ from the functor $\hat{L} = \hat{V}J: \theta\Delta\hat{\mathcal{M}}od \rightarrow \phi\mathcal{CR}\hat{\mathcal{M}}od$. \square

A 2-adic $\eta\Delta$ -module N is called *sharp* when $\eta: N_R/rN_C \rightarrow N_S$ is an isomorphism, and we may now derive the properties of \hat{L} from the corresponding properties of \hat{V} on such sharp modules,

Lemma 11.7. *For a sharp 2-adic $\eta\Delta$ -module N , the canonical map $\hat{\Lambda}N_C \rightarrow (\hat{V}N)_C$ is an algebra isomorphism.*

Proof. Let $W: \phi\mathcal{CR}\hat{\mathcal{A}}lg \rightarrow \mathcal{C}\hat{\mathcal{A}}lg$ be the forgetful functor carrying each $A \in \phi\mathcal{CR}\hat{\mathcal{A}}lg$ to its complex part $A_C \in \mathcal{C}\hat{\mathcal{A}}lg$ where $\mathcal{C}\hat{\mathcal{A}}lg$ is the category of *special 2-adic C -algebras*, which are defined similarly to *special 2-adic $\phi\mathcal{C}R$ -algebras* (3.2) but using only complex terms and their operations. The functor W has a right adjoint $H: \mathcal{C}\hat{\mathcal{A}}lg \rightarrow \phi\mathcal{CR}\hat{\mathcal{A}}lg$ where $(HX)_C = X$ and $(HX)_R = \{x \in X | tx = x\}$ with $c = 1$, $r = 1 + t$, $\eta = 0$, $\phi z = z^*z$ for $z \in X^0$, and $\phi w = B^{-1}w^*w$ for $w \in X^{-1}$. For each $N \in \eta\Delta\hat{\mathcal{M}}od$ and each $X \in \mathcal{C}\hat{\mathcal{A}}lg$, a map $N \rightarrow IHX$ in $\eta\Delta\hat{\mathcal{M}}od$ corresponds to a map $N_C \rightarrow \tilde{X}^{-1}$ respecting t , which in turn corresponds to a map $\hat{\Lambda}N_C \rightarrow X$ in $\mathcal{C}\hat{\mathcal{A}}lg$. Hence, since $W\hat{V}$ is left adjoint to IH , the canonical map $\hat{\Lambda}N_C \rightarrow W\hat{V}N$ is an isomorphism. \square

11.8. Proof of Lemma 4.6. For a 2-adic $\theta\Delta$ -module M , the canonical map $\hat{\Lambda}M_C \rightarrow (\hat{L}M)_C$ is an isomorphism by 11.6 and Lemma 11.7. \square

Let $\hat{Q}: \phi\mathcal{CR}\hat{\mathcal{A}}lg \rightarrow \phi\mathcal{CR}\hat{\mathcal{M}}od$ be the functor carrying each $A \in \phi\mathcal{CR}\hat{\mathcal{A}}lg$ to its indecomposables $\hat{Q}A \in \phi\mathcal{CR}\hat{\mathcal{M}}od$ where $\phi\mathcal{CR}\hat{\mathcal{M}}od$ is the category of *special 2-adic $\phi\mathcal{C}R$ -modules*, which may be defined as the augmentation ideals of the special 2-adic $\phi\mathcal{C}R$ -algebras having trivial multiplication.

Lemma 11.9. *For a sharp 2-adic $\eta\Delta$ -module N , the canonical map $\{N_C, N_R, N_H\} \rightarrow \Delta^{-1}\hat{Q}\hat{V}N$ is an isomorphism.*

Proof. The functor \hat{Q} has a right adjoint $E: \phi\mathcal{CR}\hat{\mathcal{M}}od \rightarrow \phi\mathcal{CR}\hat{\mathcal{A}}lg$ where $EX = \underline{e} \oplus X$. Since $\hat{Q}\hat{V}: \eta\Delta\hat{\mathcal{M}}od \rightarrow \phi\mathcal{CR}\hat{\mathcal{M}}od$ is left adjoint to IE , a detailed analysis shows that $\hat{Q}\hat{V}N$ is a special 2-adic $\phi\mathcal{C}R$ -module with $(\hat{Q}\hat{V}N)_C^{-1} = N_C$, $(\hat{Q}\hat{V}N)_R^{-1} = N_R$, and $(\hat{Q}\hat{V}N)_R^{-5} = N_H$. \square

11.10. Proof of Lemma 4.11. For a 2-adic $\theta\Delta$ -module M , the canonical map $M \rightarrow \Delta^{-1}\hat{Q}\hat{L}M$ is an isomorphism by 11.6 and Lemma 11.9. \square

12. PROOF OF THE BOTT EXACTNESS LEMMA FOR \hat{L}

We must now prove Lemma 4.8 showing Bott exactness of $\hat{L}M$ for a robust 2-adic $\theta\Delta$ -module M . This lemma will follow easily from the corresponding result for $\eta\Delta$ -modules (Lemma 12.1), whose

proof will extend through most of this section. We say that a 2-adic $\eta\Delta$ -module N is *profinutely sharp* when it is the inverse limit of an inverse system of finite sharp 2-adic $\eta\Delta$ -modules. This obviously implies that N is sharp. We call N *robust* when:

- (i) N is profinitely sharp;
- (ii) the 2-adic Δ -module $\{N_C, N_R/\bar{\phi}N_C, N_H\}$ is torsion-free and exact (4.1);
- (iii) $\ker \bar{\phi} = cN_R + c'N_H + 2N_C$.

Lemma 12.1. *If N is a robust 2-adic $\eta\Delta$ -module, then the special 2-adic ϕCR -algebra $\hat{V}N$ is Bott exact; in fact, $\hat{V}N$ is the inverse limit of an inverse system of finitely generated free 2-adic CR -modules.*

This will be proved in 12.9.

12.2. Proof of Lemma 4.8. For a robust 2-adic $\theta\Delta$ -module M , the 2-adic $\eta\Delta$ -module JM is also robust, and hence $\hat{L}M$ has the required properties by 11.6 and Lemma 12.1. \square

Before proving Lemma 12.1, we must analyze the robust 2-adic $\eta\Delta$ -modules, and we start with:

12.3. The complex 2-adic $\eta\Delta$ -modules. The functor $(-)_C: \eta\Delta\hat{M}od \rightarrow \hat{A}b$ from the 2-adic $\eta\Delta$ -modules to the 2-profinite abelian groups has a left adjoint $C: \hat{A}b \rightarrow \eta\Delta\hat{M}od$ with $C(G)_C = G \oplus G = G \oplus tG$, $C(G)_R = G \oplus G/2 = rG \oplus \bar{\phi}G$, $C(G)_H = G = qG$, and $C(G)_S = G/2 = (\bar{\phi}G)^{[2]}$ for $G \in \hat{A}b$. If G is torsion-free, then $C(G)$ is obviously robust. For an arbitrary $N \in \eta\Delta\hat{M}od$ and $G \in \hat{A}b$, we may describe the possible maps $N \rightarrow C(G)$ as follows. Let $f: N_C \rightarrow G$ and $g: N_S \rightarrow G/2$ be maps such that the diagram

$$\begin{array}{ccc} N_R \oplus N_H & \xrightarrow{fc+fc'} & G \\ \downarrow (\cdot)^{[2]} & & \downarrow 1 \\ N_S & \xrightarrow{g} & G/2 \end{array}$$

commutes. Then there is a map $F(f, g): N \rightarrow C(G)$ with components $(f, ft): N_C \rightarrow G \oplus G$, $(fc, g\eta): N_R \rightarrow G \oplus G/2$, $fc': N_H \rightarrow G$, and $g: N_S \rightarrow G/2$. Moreover, each map $N \rightarrow C(G)$ is of the above form for some f and g . When N is robust, the compatibility condition on $f: N_C \rightarrow G$ and $g: N_S \rightarrow G/2$ may be expressed by the commutativity of the diagram

$$\begin{array}{ccc} N_C^+ & \xrightarrow{f} & G \\ \downarrow \pi & & \downarrow 1 \\ N_S & \xrightarrow{g} & G/2 \end{array}$$

where $N_C^+ = \{z \in N_C | tz = z\}$ and π is the composition of $(c, c'): N_R/\bar{\phi}N_C \amalg_{N_C} N_H \cong N_C^+$ and $()^{[2]}: N_R/\bar{\phi}N_C \amalg_{N_C} N_H \rightarrow N_S$. Letting $N_C^- = \{z \in N_C | tz = -z\}$, we now have:

Lemma 12.4. *If $\tilde{N} \subset N$ is an inclusion of robust 2-adic $\eta\Delta$ -modules such that N_C/\tilde{N}_C is torsion-free and $\tilde{N}_C^- = N_C^-$, then each map $\tilde{N} \rightarrow C(G)$ for $G \in \hat{A}b$ may be extended to a map $N \rightarrow C(G)$ of 2-adic $\eta\Delta$ -modules.*

Proof. For a given map $F(\tilde{f}, \tilde{g}): \tilde{N} \rightarrow C(G)$, we first extend $\tilde{g}: \tilde{N}_S \rightarrow G/2$ to a map $g: N_S \rightarrow G/2$. Since $\tilde{N}_C/\tilde{N}_C^+ \cong \tilde{N}_C^-$, $N_C/N_C^+ \cong N_C^-$, and $\tilde{N}_C^- = N_C^-$, we see that N_C is the pushout of the inclusions $N_C^+ \leftarrow \tilde{N}_C^+ \rightarrow \tilde{N}_C$. Thus, the maps $g\pi: N_C^+ \rightarrow G/2$ and $[\tilde{f}]: \tilde{N}_C \rightarrow G/2$ induce a map $f': N_C \rightarrow G/2$, and we obtain a commutative diagram

$$\begin{array}{ccc} \tilde{N}_C & \xrightarrow{\tilde{f}} & G \\ \downarrow c & & \downarrow 1 \\ N_C & \xrightarrow{f'} & G/2 \end{array}$$

Since N_C/\tilde{N}_C is projective in $\hat{A}b$, we may now choose a lifting $f: N_C \rightarrow G$ in the diagram, and this gives the desired extension $F(f, g): N_C \rightarrow C(G)$ of $F(\tilde{f}, \tilde{g})$. \square

Lemma 12.5. *For a robust 2-adic $\eta\Delta$ -module N , there exists a decomposition $N \cong C(G) \oplus P$ where G is torsion-free and P is robust with $t = 1$ on P_C .*

Proof. By the factorization of positively torsion-free groups in 5.3, there exists a decomposition $N_C \cong (G \oplus tG) \oplus H$ with $t = 1$ on H , and we let $i: C(G) \rightarrow N$ be the induced map. Then i is monic since $i: \{G \oplus tG, G, G\} \rightarrow \{N_C, N_R/\bar{\phi}N_C, N_H\}$ is monic by [10, Lemma 4.8], and since $i: G/2 \rightarrow \bar{\phi}N_C$ and $\eta: \bar{\phi}N_C \rightarrow N_S$ are monic by the proof of Lemma 7.3. Thus, $i: C(G) \rightarrow N$ has a left inverse by Lemma 12.4, and the result follows. \square

12.6. The t -trivial robust 2-adic $\eta\Delta$ -modules. A robust 2-adic $\eta\Delta$ -module N will be called *t -trivial* when $t = 1$ on N_C . Such an N must have $\bar{\phi} = 0: N_C \rightarrow N_R$ since $N_C = cN_R + c'N_H$ by the exactness of $\{N_C, N_R/\bar{\phi}N_C, N_H\}$. Moreover, it must also have $(rN_C)^{[2]} = 0$, $(qN_C)^{[2]} = 0$, and $c + c': N_R/rN_C \oplus N_H/qN_C \cong N_C/2$ by [10, Lemma 4.7]. Hence, the operations $()^{[2]}: N_R \rightarrow N_S$ and $()^{[2]}: N_H \rightarrow N_S$ induce operations $\bar{\theta}: N_R/rN_C \rightarrow N_R/rN_C$ and $\bar{\theta}: N_H/qN_C \rightarrow N_R/rN_C$, where the $\bar{\theta}$ -module N_R/rN_C is profinite since N is profinitely sharp. In this way, a t -trivial robust 2-adic $\eta\Delta$ -module N corresponds to a torsion-free group $G \in \hat{A}b$ together with a decomposition $(G/2)_R \oplus (G/2)_H = G/2$ equipped with operations $\bar{\theta}: (G/2)_R \rightarrow (G/2)_R$ and $\bar{\theta}: (G/2)_H \rightarrow (G/2)_R$

such that the $\bar{\theta}$ -module $(G/2)_R$ is profinite. We say that a 2-adic $\eta\Delta$ -module N is of *finite type* when N_C , N_R , N_H , and N_S are finitely generated over $\hat{\mathbb{Z}}_2$, and we now easily deduce:

Lemma 12.7. *A t -trivial robust 2-adic $\eta\Delta$ -module may be expressed as the inverse limit of an inverse system of t -trivial robust quotient modules of finite type.*

A similar result obviously holds for the robust 2-adic $\eta\Delta$ -modules $C(G)$ with G torsion-free, and this will allow us to focus on the robust modules of finite type since we have:

Lemma 12.8. *If a 2-adic $\eta\Delta$ -module N is the inverse limit of an inverse system $\{N_\alpha\}_\alpha$ of quotient modules, then $\hat{V}N \cong \lim_\alpha \hat{V}N_\alpha$.*

Proof. For a finite special 2-adic ϕCR -algebra F , there is a canonical isomorphism $\text{Hom}(\lim_\alpha \hat{V}N_\alpha, F) \cong \text{Hom}(\hat{V}N, F)$. Hence, the map $\hat{V}N \rightarrow \lim_\alpha \hat{V}N_\alpha$ is an isomorphism by Lemma 11.5. \square

12.9. Proof of Lemma 12.1. It now suffices to show that $\hat{V}N$ is a free 2-adic CR -module when $N = C(G) \oplus P$ for a finitely generated free $\hat{\mathbb{Z}}_2$ -module G and a t -trivial robust 2-adic $\eta\Delta$ -module P of finite type. By 7.1, we may choose finite ordered sets of elements $\{z_k\}_k$ in G , $\{x_i\}_i$ in P_R , and $\{y_j\}_j$ in P_H such that G is a free $\hat{\mathbb{Z}}_2$ -module on $\{z_k\}_k$ and $\{P_C, P_R, P_H\}$ is a free 2-adic Δ -module on $\{x_i\}_i$ and $\{y_j\}_j$. Since P_S is a free $\mathbb{Z}/2$ -module on the generators $\{\eta x_i\}_i$, there are expressions $x_i^{[2]} = r_i$ and $y_j^{[2]} = s_j$ for each i and j where the r_i and s_j are $\mathbb{Z}/2$ -linear combinations of these generators. We may now obtain $\hat{V}N$ as the free augmented 2-adic CR -algebra on the generators $x_i \in (\widehat{VN})_R^{-1}$, $y_j \in (\widehat{VN})_R^{-5}$, $z_k \in (\widehat{VN})_C^{-1}$, and $\phi z_k \in (\widehat{VN})_R^0$ subject to the relations $x_i^2 = r_i$, $y_j^2 = B_R s_j$, $z_k^2 = 0$, $z_k^* z_k = B_C \phi z_k$, and $(\phi z_k)^2 = 0$ for each i , j , and k . It follows by a straightforward analysis that $\hat{V}N$ is a free 2-adic CR -module on the associated products (3.3) of $\{x_i\}_i$, $\{y_j\}_j$, and $\{z_k\}_k$. \square

13. PROOFS FOR REGULAR MODULES

We first show that our strict nonlinearity condition (7.7) for 2-adic Adams modules agrees with that of [7, 2.4], and we then prove Lemmas 7.9 and 7.10 for regular modules. For a 2-adic Adams module A , we let $TA \subset A$ be given by the pullback square

$$\begin{array}{ccc} TA & \longrightarrow & (A/\psi^2 A)\backslash 2 \\ \downarrow \subset & & \downarrow \subset \\ A & \longrightarrow & A/\psi^2 A \end{array}$$

where $(A/\psi^2 A)\backslash 2$ is the kernel of $2: A/\psi^2 A \rightarrow A/\psi^2 A$. Since the square is also a pushout, A is quasilinear if and only if $TA = A$. Now let $T^\infty A$ be the intersection of the submodules $T^i A \subset A$ for $i > 0$.

Lemma 13.1. $T^\infty A$ is the largest quasilinear submodule of A , and hence $A_{qI} = T^\infty A$.

Proof. Using the inverse limit of the pullback squares for $T^i A$ with $i > 0$, we find that $T^\infty A$ contains each quasilinear submodule of A and that $T(T^\infty A) = T^\infty A$. \square

13.2. Strict nonlinearity conditions. Our definition of strict nonlinearity in 7.7 is equivalent to our earlier definition in [7, 2.3 and 2.4]. In fact, for a 2-adic Adams module A , the largest quasilinear submodule A_{qI} remains unchanged in the earlier category of 2-adic ψ^2 -modules, since it is still given by $T^\infty A$. To prove Lemma 7.10, we need:

Lemma 13.3. *For a strictly nonlinear 2-adic Adams module A , each submodule is strictly nonlinear. Moreover, when A is finitely generated over $\hat{\mathbb{Z}}_2$, each torsion-free quotient module is strictly nonlinear.*

Proof. The first statement is clear, and we shall prove the second by working in the earlier category $\hat{\mathcal{N}}$ of 2-adic ψ^2 -modules that are ψ^2 -pronilpotent. Let $0 \rightarrow \tilde{A} \rightarrow A \rightarrow \bar{A} \rightarrow 0$ be a short exact sequence in $\hat{\mathcal{N}}$ with A strictly nonlinear and finitely generated over $\hat{\mathbb{Z}}_2$ and with \bar{A} torsion-free. To show that \bar{A} is strictly nonlinear, it suffices to show that $\text{Hom}_{\hat{\mathcal{N}}}(H, \bar{A}) = 0$ for each torsion-free quasilinear $H \in \hat{\mathcal{N}}$ that is finitely generated over $\hat{\mathbb{Z}}_2$. Since \bar{A} is torsion-free, it now suffices to show that $\text{Hom}_{\hat{\mathcal{N}}}(H, \bar{A})$ is finite for such H . Hence, since $\text{Hom}_{\hat{\mathcal{N}}}(H, A) = 0$ by strict nonlinearity, it suffices to show that $\text{Ext}_{\hat{\mathcal{N}}}^1(H, \tilde{A})$ is finite for such H . This finiteness follows using the exact sequence

$$0 \longrightarrow \text{Hom}_{\hat{\mathcal{N}}}(H, \tilde{A}) \longrightarrow \text{Hom}_{\hat{A}b}(H, \tilde{A}) \longrightarrow \text{Hom}_{\hat{A}b}(H, \tilde{A}) \longrightarrow \text{Ext}_{\hat{\mathcal{N}}}^1(H, \tilde{A}) \longrightarrow 0$$

with $\text{Hom}_{\hat{\mathcal{N}}}(H, \tilde{A}) = 0$ by strict nonlinearity, where $\hat{A}b$ is the category of 2-profinite abelian groups. \square

13.4. Proof of Lemma 7.10. This result follows easily from 7.8 and Lemma 13.3. \square

13.5. Proof of Lemma 7.9. By [8, Lemma 5.5], there is an exact sequence

$$0 \longrightarrow \tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2) \longrightarrow \tilde{K}^1(X; \hat{\mathbb{Z}}_2) \longrightarrow H^3(X; \hat{\mathbb{Z}}_2)$$

of 2-adic Adams modules with $H^3(X; \hat{\mathbb{Z}}_2)$ linear and $\tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2)$ torsion-free, where X^3 is the 3-skeleton of X . Hence, it suffices to show that $\tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2)$ is strictly nonlinear with monic ψ^2 . Since $H^m(X; \hat{\mathbb{Z}}_2) = 0$ for sufficiently large m , the map $\tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2) \rightarrow \tilde{K}^1(X^m/X^3; \hat{\mathbb{Z}}_2)$ is monic for such m . Thus by skeletal induction, the operator ψ^2 on $Q \otimes \tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2)$ is annihilated by the polynomial $f(x) = (x - 2^2)(x - 2^3) \dots (x - 2^k)$ for sufficiently large k . It follows that

$Q \otimes \tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2)$ is the direct sum of the eigenspaces E_i of ψ^2 with eigenvalues 2^i for $2 \leq i \leq k$, and hence ψ^2 is monic on $\tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2)$ as desired. Moreover, the projection to E_i is given by the operator $f_i(\psi^2)/f_i(2^i)$ on $Q \otimes \tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2)$ where $f_i(x) = f(x)/(x - 2^i)$. This implies that $2^v \tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2)$ is contained in $\bigoplus_{i=2}^k E_i \cap \tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2)$ where 2^v is the highest power of 2 dividing an integer $f_i(2^i)$ for some i . Since the above direct sum is strictly nonlinear, so is $2^v \tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2)$ by Lemma 13.3, and hence so is $\tilde{K}^1(X/X^3; \hat{\mathbb{Z}}_2)$. \square

14. PROOF OF THE REALIZABILITY THEOREM FOR $\hat{L}M$

We shall prove Theorem 8.5 giving a strict isomorphism $\hat{L}M \cong K_{C_R}^*(\widetilde{\text{Fib}}f; \hat{\mathbb{Z}}_2)$ for a companion map $f: \Omega^\infty \tilde{\mathcal{E}}\bar{M} \rightarrow \Omega^\infty \tilde{\mathcal{E}}\bar{\rho}\bar{M}$ of a strong 2-adic Adams Δ -module M . For this, it will suffice by our comparison theorem (4.9) to obtain an isomorphism $\hat{L}M_C \cong K^*(\widetilde{\text{Fib}}f; \hat{\mathbb{Z}}_2)$ of the complex components. We do this by adapting our proof of the corresponding odd primary result (Theorem 4.7) in [8]. First, to determine the 2-adic K -cohomology of the loops on $\Omega^\infty \tilde{\mathcal{E}}\bar{M}$ or $\Omega^\infty \tilde{\mathcal{E}}\bar{\rho}\bar{M}$, we may replace Theorem 11.2 of [8] by the following two theorems.

Theorem 14.1. *If $X = \Omega^\infty E$ for a 1-connected spectrum E with $H^2(E; \hat{\mathbb{Z}}_2) = 0$, with $K^0(E; \hat{\mathbb{Z}}_2) = 0$, and with $K^1(E; \hat{\mathbb{Z}}_2)$ torsion-free, then $K^1(\Omega X; \hat{\mathbb{Z}}_2) = 0$ and $K^0(\Omega X; \hat{\mathbb{Z}}_2)$ is torsion-free.*

Proof. This follows by [6, Theorem 8.3]. \square

Theorem 14.2. *If X is a 1-connected H -space with $K^1(\Omega X; \hat{\mathbb{Z}}_2) = 0$ and $K^0(\Omega X; \hat{\mathbb{Z}}_2)$ torsion-free, then $\sigma: U(\hat{Q}K^1(X; \hat{\mathbb{Z}}_2) \downarrow H^3(X; \hat{\mathbb{Z}}_2)) \cong K^0(\Omega X; \hat{\mathbb{Z}}_2)$.*

Proof. This follows from [7, Theorem 10.2]. \square

When X is $\Omega^\infty \tilde{\mathcal{E}}\bar{M}$ or $\Omega^\infty \tilde{\mathcal{E}}\bar{\rho}\bar{M}$, we shall determine $H^3(X; \hat{\mathbb{Z}}_2)$ from the united 2-adic K -cohomology of X . For any 1-connected space X , we let $\alpha_R: \widetilde{KO}^{-1}(X; \hat{\mathbb{Z}}_2) \rightarrow H^3(X; \hat{\mathbb{Z}}_2)$ be the homomorphism induced by the Postnikov section $KO\hat{\mathbb{Z}}_2 \rightarrow P^4 KO\hat{\mathbb{Z}}_2$. Using the indecomposables $\hat{Q}KO^*(X; \hat{\mathbb{Z}}_2)$ of 2.7 and 4.10, we have:

Lemma 14.3. *If X is a 1-connected space with $H^2(X; \hat{\mathbb{Z}}_2) = 0$, then $\alpha_R: \widetilde{KO}^{-1}(X; \hat{\mathbb{Z}}_2) \rightarrow H^3(X; \hat{\mathbb{Z}}_2)$ factors through $\hat{Q}KO^{-1}(X; \hat{\mathbb{Z}}_2)$ and vanishes on the following subgroups: $\bar{\phi}\tilde{K}^{-1}(X; \hat{\mathbb{Z}}_2)$, $(\theta - 2)\widetilde{KO}^{-1}(X; \hat{\mathbb{Z}}_2)$, $(\theta - rB^{-2}c)\widetilde{KO}^{-5}(X; \hat{\mathbb{Z}}_2)$, and $(\psi^3 - 9)\widetilde{KO}^{-1}(X; \hat{\mathbb{Z}}_2)$.*

Proof. The map α_R factors through $\hat{Q}KO^{-1}(X; \hat{\mathbb{Z}}_2)$ by a suspension argument using the isomorphism $H^3(X; \hat{\mathbb{Z}}_2) \cong H^2(\Omega X; \hat{\mathbb{Z}}_2)$. Since X is 1-connected with $H^2(X; \hat{\mathbb{Z}}_2) = 0$, there is a natural

isomorphism $H^3(X; \hat{\mathbb{Z}}_2) \cong (\pi_2(\tau_2 X))^\#$ by [8, Lemma 11.4]. Thus, it suffices by naturality to prove the desired vanishing results when X is $S^2 \cup_{2^k} e^3$ for $k \geq 1$, and these results now follow from the elementary case $X = S^3$ since the collapsing map $S^2 \cup_{2^k} e^3 \rightarrow S^3$ induces epimorphisms of the cohomologies $\widetilde{K}^{-1}(-; \hat{\mathbb{Z}}_2)$, $\widetilde{KO}^{-1}(-; \hat{\mathbb{Z}}_2)$, and $\widetilde{KO}^{-5}(-; \hat{\mathbb{Z}}_2)$. \square

For a 1-connected space X with $H^2(X; \hat{\mathbb{Z}}_2) = 0$, the above α_R now induces a homomorphism $\bar{\alpha}_R: \text{Lin}^\Delta \hat{Q}K_\Delta^{-1}(X; \hat{\mathbb{Z}}_2) \rightarrow H^3(X; \hat{\mathbb{Z}}_2)$ where $\hat{Q}K_\Delta^{-1}(X; \hat{\mathbb{Z}}_2)$ is the 2-adic Adams Δ -module of indecomposables given by 4.10 and 6.1, and where Lin^Δ carries a 2-adic Adams Δ -module M to the group

$$\text{Lin}^\Delta M = M_R / (\bar{\varphi}M_C + (\theta - 2)M_R + (\theta - rc')M_H + (\psi^3 - 9)M_R).$$

To determine $H^3(X; \hat{\mathbb{Z}}_2)$ when X is $\Omega^\infty \tilde{\mathcal{E}}\bar{M}$ or $\Omega^\infty \tilde{\mathcal{E}}\bar{\rho}\bar{M}$, we may replace Proposition 11.3 of [8] by:

Proposition 14.4. *If N is a torsion-free exact stable 2-adic Adams Δ -module, then*

$$\bar{\alpha}_R: \text{Lin}^\Delta \hat{Q}K_\Delta^{-1}(\Omega^\infty \tilde{\mathcal{E}}N; \hat{\mathbb{Z}}_2) \cong H^3(\Omega^\infty \tilde{\mathcal{E}}N; \hat{\mathbb{Z}}_2).$$

Proof. Since there is a stable isomorphism $\bar{\alpha}_R: KO^{-1}(\tilde{\mathcal{E}}N; \hat{\mathbb{Z}}_2) / (\psi^3 - 9) \cong H^3(\tilde{\mathcal{E}}N; \hat{\mathbb{Z}}_2)$ by [10, Theorem 3.2] and [8, Lemma 11.4], the proposition follows using Theorem 6.7 and Lemma 4.11. \square

For any θ -pronilpotent 2-adic Adams Δ -module M , we obtain a homomorphism $r: M^C \rightarrow \text{Lin}^\Delta M$ of 2-adic Adams modules with M^C as in 7.6 and $\text{Lin}^\Delta M$ linear. Such a homomorphism is called *properly torsion-free* [7, 4.5] when its source is torsion-free and its kernel is strictly nonlinear (7.7). We shall need:

Lemma 14.5. *If M is a strong 2-adic Adams Δ -module, then $r: M^C \rightarrow \text{Lin}^\Delta M$ is properly torsion-free.*

Proof. Since M is strong, M^C is torsion-free and $\ker(M^C \rightarrow \text{Lin} M^C)$ is strictly nonlinear. Using the maps $r: \text{Lin} M^C \rightarrow \text{Lin}^\Delta M$ and $c: \text{Lin}^\Delta M \rightarrow \text{Lin} M^C$ with $cr = 2$, we see that $2 \ker(M^C \rightarrow \text{Lin}^\Delta M)$ is contained in $\ker(M^C \rightarrow \text{Lin} M)$. Thus, $\ker(M^C \rightarrow \text{Lin}^\Delta M)$ is strictly nonlinear by Lemma 13.3. \square

As in [8, Section 11], for a strong 2-adic Adams Δ -module M and a companion map f , we obtain a ladder of p -complete fiber sequences

$$\begin{array}{ccccc} \widetilde{\text{Fib}} f & \longrightarrow & X & \xrightarrow{\bar{f}} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \text{Fib } f & \longrightarrow & \Omega^\infty \tilde{\mathcal{E}}\bar{M} & \xrightarrow{f} & \Omega^\infty \tilde{\mathcal{E}}\bar{\rho}\bar{M} \end{array}$$

such that:

- (i) X and Y satisfy the hypotheses of Theorems 14.1 and 14.2;
- (ii) the vertical maps from X and Y are $K^*(-; \hat{\mathbb{Z}}_2)$ -equivalences;
- (iii) $H^3(Y; \hat{\mathbb{Z}}_2) = 0$ and the sequence $H^3(\Omega^\infty \tilde{\mathcal{E}}\bar{M}; \hat{\mathbb{Z}}_2) \rightarrow H^3(\Omega^\infty \tilde{\mathcal{E}}\bar{M}; \hat{\mathbb{Z}}_2) \rightarrow H^3(X; \hat{\mathbb{Z}}_2) \rightarrow 0$ is exact.

Lemma 14.6. *There is a canonical isomorphism $H^3(X; \hat{\mathbb{Z}}_2) \cong \text{Lin}^\Delta M$.*

Proof. Since $f^*: K_{CR}^*(\Omega^\infty \tilde{\mathcal{E}}\bar{M}; \hat{\mathbb{Z}}_2) \rightarrow K_{CR}^*(\Omega^\infty \tilde{\mathcal{E}}\bar{M}; \hat{\mathbb{Z}}_2)$ is equivalent to $\hat{L}\bar{d}: \hat{L}\tilde{F}\bar{\rho}\bar{M} \rightarrow \hat{L}\tilde{F}\bar{M}$ for the θ -resolution map \bar{d} , the homomorphism $f^*: H^3(\Omega^\infty \tilde{\mathcal{E}}\bar{M}; \hat{\mathbb{Z}}_2) \rightarrow H^3(\Omega^\infty \tilde{\mathcal{E}}\bar{M}; \hat{\mathbb{Z}}_2)$ is equivalent to $\text{Lin}^\Delta \bar{d}: \text{Lin}^\Delta \tilde{F}\bar{\rho}\bar{M} \rightarrow \text{Lin}^\Delta \tilde{F}\bar{M}$ by Proposition 14.4. Hence, there is an isomorphism of cokernels $H^3(X; \hat{\mathbb{Z}}_2) \cong \text{Lin}^\Delta M$. \square

14.7. Proof of Theorem 8.5. The proof of Theorem 4.7 in [8] is now easily adapted to give Theorem 8.5. In more detail, Propositions 11.5 and 11.6 of [8] remain valid in our setting using Lemmas 14.5 and 14.6 together with the short exact sequence

$$0 \longrightarrow (\tilde{F}M^C \downarrow 0) \longrightarrow (\tilde{F}M^C \downarrow \text{Lin}^\Delta M) \longrightarrow (M^C \downarrow \text{Lin}^\Delta M) \longrightarrow 0$$

induced by the θ -resolution. Propositions 11.7 and 11.8 likewise remain valid, and thus $\hat{\Lambda}M^C \cong K^*(\widetilde{\text{Fib}}f; \hat{\mathbb{Z}}_2)$, so that Theorem 8.5 follows by Theorem 4.9. \square

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