KUNNETH THEOREMS AND UNSTABLE OPERATIONS IN 2-ADIC KO-COHOMOLOGY

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Abstract. We develop Kunneth theorems and obtain detailed results on unstable operations in 2-adic KO-cohomology and, more generally, in united 2-adic K-cohomology. These results are needed for work on the K-localizations of H-spaces at the prime 2 and should be of independent interest. Our proofs of relations for unstable operations rely on Atiyah's Real K-theory and on a convenient mod 2 simplification of 2-adic KO-cohomology.

1. Introduction

In [9], we determined the K-localizations of finite H-spaces and related spaces at an odd prime, and we recently found that this work can be extended to the prime 2 with some modifications and restrictions. For this, we must use various basic results on Kunneth theorems and unstable operations, particularly in 2-adic KO-cohomology, that are not covered in the literature, and we provide the required background in this paper.

In [2], Atiyah proved a Kunneth theorem for the K-cohomology of a product of finite CW-complexes but noted that the corresponding theorem fails for KO-cohomology because of difficulties at the prime 2. As suggested in [6] and [4], these difficulties may be overcome by using united K-theory. For a space X, the united 2-adic K-cohomology

$$\widehat{K}^*_{CRT}X \;=\; K^*_{CRT}(X;\hat{\mathbb{Z}}_2) \;=\; \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2), KT^*(X;\hat{\mathbb{Z}}_2)\}$$

consists of the complex, real, and self-conjugate 2-adic K-cohomologies of X together with the realification, complexification, and other "stable KO-module operations" relating these cohomologies (see [5, Sections 2]). This has much better homological algebraic properties than 2-adic KO-cohomology alone, and we now obtain a Kunneth short exact sequence

$$0 \longrightarrow \widehat{K}^*_{CRT}X \hat{\otimes}_{CRT} \widehat{K}^*_{CRT}Y \longrightarrow \widehat{K}^*_{CRT}(X \times Y) \longrightarrow \hat{\mathrm{Tor}}_1^{CRT}(\widehat{K}^{*+1}_{CRT}X, \widehat{K}^*_{CRT}Y) \longrightarrow 0$$

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for arbitrary spaces X and Y in Theorem 2.12. This Kunneth theorem can often be simplified by eliminating the self-conjugate components and using the (partial) united 2-adic K-cohomology

$$\widehat{K}^*_{CR}X \;=\; K^*_{CR}(X;\hat{\mathbb{Z}}_2) \;=\; \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\}$$

instead of $\widehat{K}^*_{CRT}X$, as we show in Theorem 2.8 under the common condition that $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is CR-exact. In this case, $KO^*(X \times Y;\hat{\mathbb{Z}}_2)$ depends on the 2-adic K-cohomology as well as the 2-adic KO-cohomology of X and Y.

A major purpose of this paper is to study the unstable operations in the united 2-adic KO-cohomology, $K^*_{CR}(X;\hat{\mathbb{Z}}_2) = \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\}$, of a space X. The basic unstable operations in $K^*(X;\hat{\mathbb{Z}}_2)$ are just the exterior squares

$$\theta = -\lambda^2\colon\ K^0(X;\hat{\mathbb{Z}}_2) \longrightarrow K^0(X;\hat{\mathbb{Z}}_2)$$
$$\theta = -\lambda^2\colon\ K^{-1}(X;\hat{\mathbb{Z}}_2) \longrightarrow K^{-1}(X;\hat{\mathbb{Z}}_2)$$

which give it the structure of a $\mathbb{Z}/2$-graded 2-adic θ-ring (see 3.4), and which combine with the stable Adams operations to give it the structure of a $\mathbb{Z}/2$-graded 2-adic λ-ring (see [7, Theorem 6.2]). We find that the other basic unstable operations in $\{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\}$ are the exterior squares

$$\theta = -\lambda^2\colon\ KO^n(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^0(X;\hat{\mathbb{Z}}_2) \quad \text{for } n = 0, -4$$
$$\theta = -\lambda^2\colon\ KO^n(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^{-1}(X;\hat{\mathbb{Z}}_2) \quad \text{for } n = -1, -5$$
$$\theta = -\lambda^2\colon\ KO^n(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^1(X;\hat{\mathbb{Z}}_2) \quad \text{for } n = 1, -3$$

and the multiplicative operations

$$\phi\colon\ K^0(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^0(X;\hat{\mathbb{Z}}_2)$$
$$\phi\colon\ K^{-1}(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^0(X;\hat{\mathbb{Z}}_2)$$

of Seymour [16], which are all constructed in [10, Section 6] using Atiyah's Real K-theory. These operations satisfy a large assortment of algebraic relations which we present in Section 3. Some of the relations are well-known or are easily proved using the methods of Real K-theory, while others seem to defy those methods. To establish these difficult relations, we use a mod 2 simplification j^*X of the cohomology $KO^*(X;\hat{\mathbb{Z}}_2)$ as explained below.

For a space X, we let j^*X denote the quotient of $KO^*(X;\hat{\mathbb{Z}}_2)$ by the realification ideal $rK^*(X;\hat{\mathbb{Z}}_2)$. We find that j^*X is an algebra over the 8-periodic exterior algebra $j^* = \mathbb{Z}/2[\eta, B_R, B_R^{-1}]$ with $|\eta| = -1$ and $|B_R| = -8$, and we obtain a Kunneth isomorphism $j^*X \hat{\otimes}_{j^*} j^*Y \cong j^*(X \times Y)$ in Theorem 5.2 for an arbitrary space Y when $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is CR-exact with $K^*(X;\hat{\mathbb{Z}}_2)$ torsion-free.

For many H-spaces X, this applies to show that j^*X is actually a Hopf algebra over j^*. In particular, $j^*\underline{\widehat{KO}}_m$ and $j^*\underline{\widehat{K}}_m$ are such Hopf algebras, where $\underline{\widehat{KO}}_m$ and $\underline{\widehat{K}}_m$ are the infinite loop spaces representing the cohomologies $\widetilde{KO}^m(-;\hat{\mathbb{Z}}_2)$ and $\widetilde{K}^m(-;\hat{\mathbb{Z}}_2)$. To prove a relation $\alpha = \beta$ for unstable operations in $\{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\}$, we first check that the possible error $\alpha - \beta$ is additive and has trivial complexification. We then deduce that this error is represented by a primitive element $\epsilon \in Pj^*\underline{\widehat{KO}}_m$ or $\epsilon \in Pj^*\underline{\widehat{K}}_m$. We finally show that this ϵ must vanish by using our knowledge of $Pj^*\underline{\widehat{KO}}_m$ and $Pj^*\underline{\widehat{K}}_m$, and we thereby deduce $\alpha = \beta$. This method allows us to establish all of the difficult relations for our unstable operations.

For a space X such as $\underline{\widehat{KO}}_m$ or $\underline{\widehat{K}}_m$, we may approach j^*X by first studying a mod 2 simplification h^*X of $K^*(X;\hat{\mathbb{Z}}_2)$, where $h^nX = \ker(1-t)/\operatorname{im}(1+t)$ for the conjugation $t = \psi^{-1}$ on $K^n(X;\hat{\mathbb{Z}}_2)$. We find that h^*X is an algebra over the 4-periodic algebra $h^* = \mathbb{Z}/2[B^2, B^{-2}]$ and that h^*X has a natural differential $\partial\colon\ h^*X \to h^{*-1}X$ satisfying a Leibniz formula (see Theorem 5.9). Moreover, this differential vanishes if and only if $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is CR-exact, which happens if and only if j^*X is free over j^*. In this case, we have $j^*X/\eta \cong h^*X$, and we may view h^*X as a first approximation to the algebra j^*X. For many H-spaces X, we can go even farther and view h^*X as a first approximation to the Hopf algebra j^*X. We use this approach to derive the needed properties of $j^*\underline{\widehat{KO}}_m$ and $j^*\underline{\widehat{K}}_m$, and we believe that the functors j^* and h^* are of independent interest.

The paper is divided into the following sections:

1. Introduction
2. Kunneth theorems for united 2-adic K-cohomology
3. Unstable operations in united 2-adic K-cohomology
4. Proofs of the Kunneth theorems
5. The mod 2 K-cohomological functors j^* and h^*
6. On the j^* groups of some infinite loop spaces
7. Proofs of relations for unstable operations

The present work will allow us to determine the structure of the united 2-adic K-cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2) = \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\}$ for many H-spaces X, and this will allow us to derive the promised 2-primary extensions of our results in [9] on K-localizations of spaces. It will also allow us to complete the work of [10] on the v_1-periodic homotopy theory of compact Lie groups.

2. Kunneth theorems for united 2-adic K-cohomology

In this section, we first discuss the 2-adic version of Atiyah's original Kunneth theorem [3] and then develop our Kunneth theorems for the united 2-adic K-cohomology theories

$$K^*_{CR}(X;\hat{\mathbb{Z}}_2) = \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\},$$
$$K^*_{CRT}(X;\hat{\mathbb{Z}}_2) = \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2), KT^*(X;\hat{\mathbb{Z}}_2)\},$$

of spectra or spaces X. We note that the Kunneth theorem for $K^*_{CRT}(-;\hat{\mathbb{Z}}_2)$ is the strongest, but may be somewhat cumbersome since it involves the self-conjugate theory $KT^*(-;\hat{\mathbb{Z}}_2)$. For simplicity, we shall focus mainly on the version for $K^*_{CR}(-;\hat{\mathbb{Z}}_2)$. The results of this section immediately extend to an odd prime p, but these extensions add nothing valuable beyond the Kunneth theorem for $K^*(-;\hat{\mathbb{Z}}_p)$ since $K^*_{CR}(X;\hat{\mathbb{Z}}_p)$ and $K^*_{CRT}(X;\hat{\mathbb{Z}}_p)$ are naturally determined by $K^*(X;\hat{\mathbb{Z}}_p)$ as in [5, Theorem 4.6].

2.1. **The 2-profinite abelian groups.** Recall that a *2-profinite abelian group* is the topological inverse limit of an inverse system of finite 2-torsion abelian groups. For instance, each finitely generated $\hat{\mathbb{Z}}_2$-module is a 2-profinite abelian group with the canonical topology. In general, the 2-profinite abelian groups correspond to the 2-torsion abelian groups by Pontrjagin duality, and they form an abelian category $\hat{\mathcal{A}b}$. For 2-profinite abelian groups $A, B \in \hat{\mathcal{A}b}$, let $A\hat{\otimes}B \in \hat{\mathcal{A}b}$ be the *complete tensor product* with $A\hat{\otimes}B = \lim_{\alpha,\beta} A_\alpha \otimes B_\beta$ where A_α and B_β range over the finite 2-torsion quotients of A and B (see e.g. [15, p.184]). Note that $A\hat{\otimes}B \cong A \otimes_{\hat{\mathbb{Z}}_2} B$ when A and B are finitely generated $\hat{\mathbb{Z}}_2$-modules. The operation $\hat{\otimes}$ gives a symmetric monoidal structure to the abelian category $\hat{\mathcal{A}b}$. For $A \in \hat{\mathcal{A}b}$, the functor $A\hat{\otimes}-\colon \hat{\mathcal{A}b} \to \hat{\mathcal{A}b}$ is right exact, and we call A *flat* when $A\hat{\otimes}-$ is exact. Using Pontrjagin duality, we see that the following conditions on A are equivalent:

(i) A is flat in $\hat{\mathcal{A}b}$;
(ii) A is torsion-free;
(iii) A is projective in $\hat{\mathcal{A}b}$.

Moreover, there are enough projectives in $\hat{\mathcal{A}b}$, and the complete tensor product $\hat{\otimes}$ has left derived functors $\hat{\mathrm{T}}\mathrm{or}_n(A,B) \in \hat{\mathcal{A}b}$ for $A, B \in \hat{\mathcal{A}b}$ and $n \geq 0$. As usual, $\hat{\mathrm{T}}\mathrm{or}_0(A,B) \cong A\hat{\otimes}B$ and $\hat{\mathrm{T}}\mathrm{or}_n(A,B) = 0$ for $n > 1$.

2.2. **The 2-adic K^*-modules.** By a *2-adic K^*-module* M, we mean a $\mathbb{Z}$-graded 2-profinite abelian group with operation $B\colon M^* \cong M^{*-2}$. We let $\hat{\mathcal{K}}^*$ denote the abelian category of these modules. For a spectrum or space X, we obtain a 2-adic K^*-module $K^*(X;\hat{\mathbb{Z}}_2)$ using the Bott periodicity B and the limit topology on $K^*(X;\hat{\mathbb{Z}}_2) \cong \lim_\alpha K^*(X_\alpha;\hat{\mathbb{Z}}_2)$ where X_α ranges over the finite subcomplexes of X. For $M,N,P \in \hat{\mathcal{K}}^*$, a *$K^*$-pairing* $f\colon (M,N) \to P$ consists of homomorphisms $f\colon M^m \hat{\otimes} N^n \to P^{m+n}$ for $m,n \in \mathbb{Z}$ such that $Bf = f(B\hat{\otimes}1) = f(1\hat{\otimes}B)$ on each $M^m\hat{\otimes}N^n$. The *complete tensor product* $M\hat{\otimes}_{K^*}N \in \hat{\mathcal{K}}^*$ may be defined as the target of the universal K^*-pairing $(M,N) \to M\hat{\otimes}_{K^*}N$, and we note that

$$(M\hat{\otimes}_{K^*}N)^0 \;\cong\; (M^0\hat{\otimes}N^0) \oplus (M^1\hat{\otimes}N^1),$$
$$(M\hat{\otimes}_{K^*}N)^1 \;\cong\; (M^1\hat{\otimes}N^0) \oplus (M^0\hat{\otimes}N^1).$$

The operation $\hat{\otimes}_{K^*}$ in the abelian category $\hat{\mathcal{K}}^*$ inherits the basic properties of $\hat{\otimes}$ in $\hat{\mathcal{A}b}$ (see 2.1), and $\hat{\otimes}_{K^*}$ has left derived functors $\hat{\mathrm{Tor}}_n^{K^*}(M,N) \in \hat{\mathcal{K}}^*$ for $M,N \in \hat{\mathcal{K}}^*$ and $n \geq 0$. As usual, $\hat{\mathrm{Tor}}_0^{K^*}(M,N) \cong M\hat{\otimes}_{K^*}N$ and $\hat{\mathrm{Tor}}_n^{K^*}(M,N) = 0$ for $n > 1$. For spectra X and Y, the product homomorphisms

$$\mu\colon\; K^m(X;\hat{\mathbb{Z}}_2)\hat{\otimes}K^n(Y;\hat{\mathbb{Z}}_2) \;\longrightarrow\; K^{m+n}(X\wedge Y;\hat{\mathbb{Z}}_2)$$

give a product homomorphism

$$\mu\colon\; K^*(X;\hat{\mathbb{Z}}_2)\hat{\otimes}_{K^*}K^*(Y;\hat{\mathbb{Z}}_2) \;\longrightarrow\; K^*(X\wedge Y;\hat{\mathbb{Z}}_2)$$

of 2-adic K^*-modules. This also holds for spaces X and Y when $X \wedge Y$ is replaced by $X \times Y$. We can now state a 2-adic version of Atiyah's Kunneth theorem [2], where we write $K^*(X;\hat{\mathbb{Z}}_2)$ as $\widehat{K}^*X$.

Theorem 2.3. *For spectra X and Y, there is a natural short exact sequence*

$$0 \longrightarrow \widehat{K}^*X\hat{\otimes}_{K^*}\widehat{K}^*Y \xrightarrow{\;\mu\;} \widehat{K}^*(X\wedge Y) \longrightarrow \hat{\mathrm{Tor}}_1^{K^*}(\widehat{K}^{*+1}X, \widehat{K}^*Y) \longrightarrow 0$$

of 2-adic K^*-modules. This also holds for spaces X and Y when $X\wedge Y$ is replaced by $X \times Y$.

This will be proved in 4.7, and we now develop our Kunneth theorems for the united 2-adic K-cohomology theories.

2.4. **The 2-adic CR-modules.** By a *2-adic CR-module*, we mean a CR-module over $\hat{\mathcal{A}b}$ in the sense of [10, Section 4.1]. Thus, a 2-adic CR-module consists of $\mathbb{Z}$-graded 2-profinite abelian groups M_C and M_R with operations

$$B\colon\; M_C^* \cong M_C^{*-2}, \qquad t\colon\; M_C^* \cong M_C^*, \qquad B_R\colon\; M_R^* \cong M_R^{*-8},$$
$$\eta\colon\; M_R^* \to M_R^{*-1}, \qquad c\colon\; M_R^* \to M_C^*, \qquad r\colon\; M_C^* \to M_R^*,$$

satisfying the relations

$$2\eta = 0, \qquad \eta^3 = 0, \qquad \eta B_R = B_R\eta, \qquad \eta r = 0, \qquad c\eta = 0,$$

$$t^2 = 1, \quad tB = -Bt, \quad rt = r, \quad tc = c, \quad cB_R = B^4 c,$$
$$rB^4 = B_R r, \quad cr = 1 + t, \quad rc = 2, \quad rBc = \eta^2, \quad rB^{-1}c = 0.$$

These operations are patterned after the usual periodicity, conjugation, Hopf, complexification, and realification operations in K-theory. For $z \in M_C^*$, the element tz is sometimes written as $\psi^{-1}z$ or z^*. We let $\hat{\mathcal{CR}}$ denote the abelian category of 2-adic CR-modules. For a spectrum or space X, we obtain a 2-adic CR-module

$$K_{CR}^*(X; \hat{\mathbb{Z}}_2) \;=\; \{K^*(X; \hat{\mathbb{Z}}_2), KO^*(X; \hat{\mathbb{Z}}_2)\}$$

using the operations coming from the standard maps of the spectra K and KO [5, Section 1.9] and using the limit topology on $K_{CR}^*(X; \hat{\mathbb{Z}}_2) \cong \lim_\alpha K_{CR}^*(X_\alpha; \hat{\mathbb{Z}}_2)$ where X_α ranges over the finite subcomplexes of X.

2.5. **Bott exactness and CR-exactness.** As in [5, Section 4.7] or [10, Section 4.1], we say that a 2-adic CR-module $M \in \hat{\mathcal{CR}}$ is *Bott exact* when the Bott sequence

$$\dots \longrightarrow M_R^{*+1} \xrightarrow{\eta} M_R^* \xrightarrow{c} M_C^* \xrightarrow{rB^{-1}} M_R^{*+2} \xrightarrow{\eta} \dots$$

is exact, and we say that M is *CR-exact* when it is Bott exact and the chain complex

$$\dots \longrightarrow M_R^{*+1}/r \xrightarrow{\eta} M_R^*/r \xrightarrow{\eta} M_R^{*-1}/r \xrightarrow{\eta} \dots$$

is exact. For a short exact sequence $0 \to L \to M \to N \to 0$ of 2-adic CR-modules, we note that if any two of the three modules are Bott exact or CR-exact, then so is the third. We also note that a Bott exact CR-module M is automatically CR-exact when M_C is concentrated in even or odd degrees. Thus, for a spectrum or space X, the 2-adic CR-module $K_{CR}^*(X; \hat{\mathbb{Z}}_2)$ is always CR-exact when $\tilde{K}^*(X; \hat{\mathbb{Z}}_2)$ is concentrated in even or odd degrees.

2.6. **Complete tensor products of 2-adic CR-modules.** For 2-adic CR-modules $M, N, P \in \hat{\mathcal{CR}}$, a *$CR$-pairing* $f\colon (M, N) \to P$ consists of homomorphisms $f_C\colon M_C^m \hat{\otimes} N_C^n \to P_C^{m+n}$ and $f_R\colon M_R^m \hat{\otimes} N_R^n \to P_R^{m+n}$ for $m, n \in \mathbb{Z}$ such that:

(i) $Bf_C = f_C(B \hat{\otimes} 1) = f_C(1 \hat{\otimes} B)$ on each $M_C^m \hat{\otimes} N_C^n$;

(ii) $B_R f_R = f_R(B_R \hat{\otimes} 1) = f_R(1 \hat{\otimes} B_R)$ on each $M_R^m \hat{\otimes} N_R^n$;

(iii) $\eta f_R = f_R(\eta \hat{\otimes} 1) = f_R(1 \hat{\otimes} \eta)$ on each $M_R^m \hat{\otimes} N_R^n$;

(iv) $tf_C = f_C(t \hat{\otimes} t)$ on each $M_C^m \hat{\otimes} N_C^n$;

(v) $cf_R = f_C(c \hat{\otimes} c)$ on each $M_R^m \hat{\otimes} N_R^n$;

(vi) $rf_C(c \hat{\otimes} 1) = f_R(1 \hat{\otimes} r)$ on each $M_R^m \hat{\otimes} N_C^n$ and $rf_C(1 \hat{\otimes} c) = f_R(r \hat{\otimes} 1)$ on each $M_C^m \hat{\otimes} N_R^n$.

The *complete tensor product* $M\hat{\otimes}_{CR}N \in \hat{\mathcal{CR}}$ may be defined as the target of the universal CR-pairing $(M,N) \to M\hat{\otimes}_{CR}N$. We note that $(M\hat{\otimes}_{CR}N)_C \cong M_C\hat{\otimes}_{K^*}N_C$ while $(M\hat{\otimes}_{CR}N)_R$ depends on both real and complex components; it is generated by terms $M_R^m\hat{\otimes}N_R^n$ together with terms $r(M_C^m\hat{\otimes}N_C^n)$. The operation $\hat{\otimes}_{CR}$ gives a symmetric monoidal structure to the abelian category $\hat{\mathcal{CR}}$. For $M \in \hat{\mathcal{CR}}$, the functor $M\hat{\otimes}_{CR}\colon \hat{\mathcal{CR}} \to \hat{\mathcal{CR}}$ is right exact, and we call M *flat* when $M\hat{\otimes}_{CR}$ is exact.

Lemma 2.7. *For a 2-adic CR-module M, the following are equivalent:*

(i) *M is flat in $\hat{\mathcal{CR}}$;*

(ii) *M is CR-exact with M_C^* torsion-free;*

(iii) *M is projective in $\hat{\mathcal{CR}}$.*

The abelian category $\hat{\mathcal{CR}}$ has enough projectives.

This will be proved in 4.3. The complete tensor product $\hat{\otimes}_{CR}$ in $\hat{\mathcal{CR}}$ now has left derived functors $\hat{\mathrm{Tor}}_n^{CR}(M,N) \in \hat{\mathcal{CR}}$ for $M,N \in \hat{\mathcal{CR}}$ and $n \geq 0$. As usual, $\hat{\mathrm{Tor}}_0^{CR}(M,N) \cong M\hat{\otimes}_{CR}N$ and $\hat{\mathrm{Tor}}_n^{CR}(M,N) = 0$ for $n > 1$ when M or N is CR-exact. For spectra X and Y, the product homomorphisms

$$\mu\colon\ K^m(X;\hat{\mathbb{Z}}_2)\hat{\otimes}K^n(Y;\hat{\mathbb{Z}}_2) \longrightarrow K^{m+n}(X\wedge Y;\hat{\mathbb{Z}}_2)$$

$$\mu\colon\ KO^m(X;\hat{\mathbb{Z}}_2)\hat{\otimes}KO^n(Y;\hat{\mathbb{Z}}_2) \longrightarrow KO^{m+n}(X\wedge Y;\hat{\mathbb{Z}}_2)$$

give a CR-pairing by [5, Section 1], and they consequently determine a product homomorphism

$$\mu\colon\ K_{CR}^m(X;\hat{\mathbb{Z}}_2)\hat{\otimes}_{CR}K_{CR}^n(Y;\hat{\mathbb{Z}}_2) \longrightarrow K_{CR}^{m+n}(X\wedge Y;\hat{\mathbb{Z}}_2)$$

of 2-adic CR-modules. This also holds for spaces X and Y when $X\wedge Y$ is replaced by $X\times Y$. We can now state our Kunneth theorem for the cohomology $K_{CR}^*(-;\hat{\mathbb{Z}}_2)$, which we write as $\widehat{K}_{CR}^*(-)$.

Theorem 2.8. *For spectra X and Y with $\widehat{K}_{CR}^*X$ (or $\widehat{K}_{CR}^*Y$) CR-exact, there is a natural short exact sequence*

$$0 \longrightarrow \widehat{K}_{CR}^*X\hat{\otimes}_{CR}\widehat{K}_{CR}^*Y \xrightarrow{\ \mu\ } \widehat{K}_{CR}^*(X\wedge Y) \longrightarrow \hat{\mathrm{Tor}}_1^{CR}(\widehat{K}_{CR}^{*+1}X,\widehat{K}_{CR}^*Y) \longrightarrow 0$$

of 2-adic CR-modules. This also holds for spaces X and Y when $X\wedge Y$ is replaced by $X\times Y$.

This will be proved in 4.6. Finally, we develop an unrestricted general Kunneth theorem for $K_{CRT}^*(-;\hat{\mathbb{Z}}_2)$. Since this is basically similar to the version over CR, we shall be quite brief and shall assume familiarity with [5].

2.9. **The 2-adic CRT-modules.** A *2-adic CRT-module* $M = \{M_C, M_R, M_T\}$ consists of $\mathbb{Z}$-graded 2-profinite abelian groups M_C, M_R, and M_T with the operations and relations of [5, Section 2.1], but with cohomological indexing. We let $\hat{\mathcal{C}}\mathcal{RT}$ denote the abelian category of 2-adic CRT-modules. For a spectrum or space X, we obtain a 2-adic CRT-module

$$K^*_{CRT}(X;\hat{\mathbb{Z}}_2) \;=\; \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2), KT^*(X;\hat{\mathbb{Z}}_2)\}$$

using the operations coming from the standard maps of the spectra K, KO, and KT [5, Section 1.9]. We say that a 2-adic CRT-module M is *CRT-exact* when the three chain complexes of [5, Section 2.3] (including the Bott sequence) are exact for M. For a short exact sequence $0 \to L \to M \to N \to 0$ of 2-adic CRT-modules, if any two of the modules are CRT-exact, then so is the third. For a spectrum or space X, the 2-adic CRT-module $K^*_{CRT}(X;\hat{\mathbb{Z}}_2)$ is always CRT-exact.

2.10. **Complete tensor products of 2-adic CRT-modules.** For 2-adic CRT-modules $M, N, P \in \hat{\mathcal{C}}\mathcal{RT}$, a CRT-pairing $f\colon (M,N) \to P$ consists of homomorphisms $f_C\colon M^m_C \hat{\otimes} N^n_C \to P^{m+n}_C$, $f_R\colon M^m_R \hat{\otimes} N^n_R \to P^{m+n}_R$, and $f_T\colon M^m_T \hat{\otimes} N^n_T \to P^{m+n}_T$, for $m, n \in \mathbb{Z}$ such that:

(i) $Bf_C = f_C(B\hat{\otimes}1) = f_C(1\hat{\otimes}B)$ on each $M^m_C\hat{\otimes}N^n_C$;
(ii) $B_R f_R = f_R(B_R\hat{\otimes}1) = f_R(1\hat{\otimes}B_R)$ on each $M^m_R\hat{\otimes}N^n_R$;
(iii) $B_T f_T = f_T(B_T\hat{\otimes}1) = f_T(1\hat{\otimes}B_T)$ on each $M^m_T\hat{\otimes}N^n_T$;
(iv) $\eta f_R = f_R(\eta\hat{\otimes}1) = f_R(1\hat{\otimes}\eta)$ on each $M^m_R\hat{\otimes}N^n_R$;
(v) $\eta f_T = f_T(\eta\hat{\otimes}1) = f_T(1\hat{\otimes}\eta)$ on each $M^m_T\hat{\otimes}N^n_T$;
(vi) $tf_C = f_C(t\hat{\otimes}t)$ on each $M^m_C\hat{\otimes}N^n_C$;
(vii) $t_T f_T = f_T(t_T\hat{\otimes}t_T)$ on each $M^m_T\hat{\otimes}N^n_T$ where $t_T = \psi_T^{-1}$;
(viii) $\epsilon f_R = f_T(\epsilon\hat{\otimes}\epsilon)$ on each $M^m_R\hat{\otimes}N^n_R$;
(ix) $\zeta f_T = f_C(\zeta\hat{\otimes}\zeta)$ on each $M^m_T\hat{\otimes}N^n_T$;
(x) $\tau f_T(1\hat{\otimes}\epsilon) = f_R(\tau\hat{\otimes}1)$ on each $M^m_T\hat{\otimes}N^n_R$ and $\tau f_T(\epsilon\hat{\otimes}1) = (-1)^m f_R(1\hat{\otimes}\tau)$ on each $M^m_R\hat{\otimes}N^n_T$;
(xi) $\gamma f_C(1\hat{\otimes}\zeta) = f_T(\gamma\hat{\otimes}1)$ on each $M^m_C\hat{\otimes}N^n_T$ and $\gamma f_C(\zeta\hat{\otimes}1) = (-1)^m f_T(1\hat{\otimes}\gamma)$ on each $M^m_T\hat{\otimes}N^n_C$;
(xii) $\epsilon\tau f_T = f_T(\epsilon\tau\hat{\otimes}1) + (-1)^m f_T(1\hat{\otimes}\epsilon\tau) + \eta f_T$ on each $M^m_T\hat{\otimes}N^n_T$.

The *complete tensor product* $M\hat{\otimes}_{CRT}N \in \hat{\mathcal{C}}\mathcal{RT}$ may be defined as the target of the universal CRT-pairing $(M,N) \to M\hat{\otimes}_{CRT}N$. The operation $\hat{\otimes}_{CRT}$ gives a symmetric monoidal structure to the abelian category $\hat{\mathcal{C}}\mathcal{RT}$. For $M \in \hat{\mathcal{C}}\mathcal{RT}$, the functor $M\hat{\otimes}_{CRT}\colon \hat{\mathcal{C}}\mathcal{RT} \to \hat{\mathcal{C}}\mathcal{RT}$ is right exact, and we call M *flat* when $M\hat{\otimes}_{CRT}$ is exact.

Lemma 2.11. *For a 2-adic CRT-module M, the following are equivalent:*

(i) *M is flat in $\hat{\mathcal{CRT}}$;*

(ii) *M is CRT-exact with M_C^* torsion-free;*

(iii) *M is projective in $\hat{\mathcal{CRT}}$.*

The abelian category $\hat{\mathcal{CRT}}$ has enough projectives.

This will be proved in 4.10. The complete tensor product $\hat{\otimes}_{CRT}$ in $\hat{\mathcal{CRT}}$ now has left derived functors $\hat{\mathrm{Tor}}_n^{CRT}(M,N) \in \hat{\mathcal{CRT}}$ for $M,N \in \hat{\mathcal{CRT}}$ and $n \geq 0$. As usual, $\hat{\mathrm{Tor}}_0^{CRT}(M,N) \cong M\hat{\otimes}_{CRT}N$ and $\hat{\mathrm{Tor}}_n^{CRT}(M,N) = 0$ for $n > 1$ when M or N is CRT-exact.

For spectra X and Y, the product homomorphisms

$$\mu\colon\ K^m(X;\hat{\mathbb{Z}}_2)\hat{\otimes}K^n(Y;\hat{\mathbb{Z}}_2) \longrightarrow K^{m+n}(X\wedge Y;\hat{\mathbb{Z}}_2)$$
$$\mu\colon\ KO^m(X;\hat{\mathbb{Z}}_2)\hat{\otimes}KO^n(Y;\hat{\mathbb{Z}}_2) \longrightarrow KO^{m+n}(X\wedge Y;\hat{\mathbb{Z}}_2)$$
$$\mu\colon\ KT^m(X;\hat{\mathbb{Z}}_2)\hat{\otimes}KT^n(Y;\hat{\mathbb{Z}}_2) \longrightarrow KT^{m+n}(X\wedge Y;\hat{\mathbb{Z}}_2)$$

give a CRT-pairing, where conditions (i)–(xi) follow by [5, Section 1] and (xii) follows by Lemma 4.11 below, and they consequently determine a product homomorphism

$$\mu\colon\ K^m_{CRT}(X;\hat{\mathbb{Z}}_2)\hat{\otimes}_{CRT}K^n_{CRT}(Y;\hat{\mathbb{Z}}_2) \longrightarrow K^{m+n}_{CRT}(X\wedge Y;\hat{\mathbb{Z}}_2)$$

of 2-adic CRT-modules. This also holds for spaces X and Y when $X\wedge Y$ is replaced by $X\times Y$. We can finally state our Kunneth theorem for $K^*_{CRT}(-;\hat{\mathbb{Z}}_2)$, where we write $K^*_{CRT}(X;\hat{\mathbb{Z}}_2)$ as $\widehat{K}^*_{CRT}X$.

Theorem 2.12. *For spectra X and Y, there is a natural short exact sequence*

$$0 \longrightarrow \widehat{K}^*_{CRT}X\hat{\otimes}_{CRT}\widehat{K}^*_{CRT}Y \xrightarrow{\ \mu\ } \widehat{K}^*_{CRT}(X\wedge Y) \longrightarrow \hat{\mathrm{Tor}}_1^{CRT}(\widehat{K}^{*+1}_{CRT}X, \widehat{K}^*_{CRT}Y) \longrightarrow 0$$

of 2-adic CRT-modules. This also holds for spaces X and Y when $X\wedge Y$ is replaced by $X\times Y$.

This will be proved in 4.12.

3. Unstable operations in united 2-adic K-cohomology

In this section, we discuss the unstable operations in the united 2-adic K-cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2) = \{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\}$ of a space X and determine their algebraic relations. We first consider:

3.1. **The CR-algebra operations.** For a space X, the cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ has commutative associative multiplication $K^*_{CR}(X;\hat{\mathbb{Z}}_2)\hat{\otimes}_{CR}K^*_{CR}(X;\hat{\mathbb{Z}}_2) \to K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ induced by the diagonal $X \to X \times X$ using the homomorphism μ of Theorem 2.8. In particular, for elements $x \in KO^m(X;\hat{\mathbb{Z}}_2)$, $y \in KO^n(X;\hat{\mathbb{Z}}_2)$, $z \in K^s(X;\hat{\mathbb{Z}}_2)$, and $w \in K^t(X;\hat{\mathbb{Z}}_2)$, there are products $xy \in KO^{m+n}(X;\hat{\mathbb{Z}}_2)$ and $zw \in K^{s+t}(X;\hat{\mathbb{Z}}_2)$ satisfying the usual relations including:

(i) $xy = (-1)^{mn}yz$ and $zw = (-1)^{st}wz$;

(ii) $c(xy) = (cx)(cy)$;

(iii) $r((cx)z) = x(rz)$;

(iv) $(zw)^* = z^*w^*$.

For odd dimensional elements, we obtain additional squaring properties which will be refined in Proposition 3.11 below.

Proposition 3.2. *For a space X and elements $x \in KO^m(X;\hat{\mathbb{Z}}_2)$ and $z \in K^s(X;\hat{\mathbb{Z}}_2)$, we have:*

(i) $x^2 = 0$ *when* $m \equiv 1, -3 \mod 8$*;*

(ii) $x^4 = 0$ *when* $m \equiv -1, -5 \mod 8$*;*

(iii) $z^2 = 0$ *when s is odd.*

Proof. Part (i) follows by [11, p.66]. Part (ii) follows since $x^4 = (\eta\lambda^2 x)^2 = \eta^2(\eta\lambda^2\lambda^2 x) = 0$ by [11, p.67] when $m = -1$ and by [12, Proposition 2.2] when $m = -5$. Part (iii) follows by, e.g., [7, Theorem 1.11]. ◻

The algebraic structure of $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is shaped by two important families of unstable operations which we call the ϕ's and θ's.

3.3. **The operations ϕ.** By [10, Theorem 6.5], for a space X, there are natural operations

$$\phi\colon\ K^0(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^0(X;\hat{\mathbb{Z}}_2)$$

$$\phi\colon\ K^{-1}(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^0(X;\hat{\mathbb{Z}}_2)$$

such that the following relations hold for elements $a, b \in K^0(X;\hat{\mathbb{Z}}_2)$ and $x, y \in K^{-1}(X;\hat{\mathbb{Z}}_2)$:

(i) $\phi(a+b) = \phi a + \phi b + r(a^*b)$;

(ii) $\phi(x+y) = \phi x + \phi y + rB^{-1}(x^*y)$;

(iii) $\phi(a^*) = \phi(a)$ and $\phi(x^*) = -\phi(x)$;

(iv) $\phi(ab) = (\phi a)(\phi b)$;

(v) $\phi(ax) = (\phi a)(\phi x)$;

(vi) $\phi B^{-1}(xy) = (\phi x)(\phi y)$;

(vii) $\phi(1) = 1$;

(viii) $\phi(ka) = k^2\phi a$ and $\phi(kx) = k^2\phi x$ for $k \in \hat{\mathbb{Z}}_2$;

(ix) $c\phi(a) = a^*a$ and $c\phi(x) = B^{-1}(x^*x)$.

For convenience, we extend the operation ϕ periodically to give $\phi\colon K^{2i}(X;\hat{\mathbb{Z}}_2) \to KO^0(X;\hat{\mathbb{Z}}_2)$ and $\phi\colon K^{2i-1}(X;\hat{\mathbb{Z}}_2) \to KO^0(X;\hat{\mathbb{Z}}_2)$ with $\phi w = \phi B^i w$ for all $i \in \mathbb{Z}$ and elements w. This corrects an inconsistent extension of ϕ used in [10, Theorem 6.5]. We note that the operation ϕ on $K^{-1}(X;\hat{\mathbb{Z}}_2)$ was implicitly introduced by Seymour [16]. We now consider:

3.4. **The complex operations θ.** For a space X, we let

$$\theta = -\lambda^2\colon\ K^0(X;\hat{\mathbb{Z}}_2) \longrightarrow K^0(X;\hat{\mathbb{Z}}_2)$$

$$\theta = -\lambda^2\colon\ K^{-1}(X;\hat{\mathbb{Z}}_2) \longrightarrow K^{-1}(X;\hat{\mathbb{Z}}_2)$$

be the specified exterior power operations and let $\psi^2\colon K^0(X;\hat{\mathbb{Z}}_2) \to K^0(X;\hat{\mathbb{Z}}_2)$ be the Adams operation with $\psi^2(a) = a^2 + 2\theta a$. As in [7], we have the following θ-ring relations for elements $a, b \in K^0(X;\hat{\mathbb{Z}}_2)$ and $x, y \in K^{-1}(X;\hat{\mathbb{Z}}_2)$:

(i) $\theta(a+b) = \theta a + \theta b - ab$;

(ii) $\theta(x+y) = \theta x + \theta y$;

(iii) $\theta(a^*) = (\theta a)^*$ and $\theta(x^*) = (\theta x)^*$;

(iv) $\theta(ab) = (\theta a)b^2 + a^2(\theta b) + 2(\theta a)(\theta b)$;

(v) $\theta(ax) = (\psi^2 a)(\theta x)$;

(vi) $\theta(B^{-1}xy) = B^{-1}(\theta x)(\theta y)$;

(vii) $\theta(1) = 0$;

(viii) $\theta(ka) = k(\theta a) - \binom{k}{2}a^2$ and $\theta(kx) = k(\theta x)$ for $k \in \hat{\mathbb{Z}}_2$.

For convenience, we extend the operation θ periodically to give $\theta\colon K^{2i}(X;\hat{\mathbb{Z}}_2) \to K^{2j}(X;\hat{\mathbb{Z}}_2)$ and $\theta\colon K^{2i-1}(X;\hat{\mathbb{Z}}_2) \to K^{2j-1}(X;\hat{\mathbb{Z}}_2)$ with $\theta w = B^{-j}\theta B^i w$ for all $i, j \in \mathbb{Z}$ and elements w. We note that $K^0(X;\hat{\mathbb{Z}}_2)$ and $K^{-1}(X;\hat{\mathbb{Z}}_2)$ have additional exterior power operations, but these are captured by the action of the stable 2-adic Adams operations as shown in [7]. Since we are primarily interested in the unstable operations, we now turn to:

3.5. **The real operations θ.** For a space X, we let

$$\theta = -\lambda^2\colon\ KO^n(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^0(X;\hat{\mathbb{Z}}_2) \quad \text{for } n = 0, -4$$

$$\theta = -\lambda^2\colon\ KO^n(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^{-1}(X;\hat{\mathbb{Z}}_2) \quad \text{for } n = -1, -5$$

$$\theta = -\lambda^2\colon\ KO^n(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^1(X;\hat{\mathbb{Z}}_2) \quad \text{for } n = 1, -3$$

be the specified exterior power operations of [10, Section 6]. To simplify formulae, we now treat the cohomology $KO^*(X;\hat{\mathbb{Z}}_2)$ as $\mathbb{Z}/8$-graded with $B_R = 1$, and we periodically extend the above operations θ to $KO^*(X;\hat{\mathbb{Z}}_2)$. We likewise treat the cohomology $K^*(X;\hat{\mathbb{Z}}_2)$ as $\mathbb{Z}/8$-graded with $B^4 = 1$, and we periodically extend the preceding operations ϕ and θ to $K^*(X;\hat{\mathbb{Z}}_2)$. For convenience, we also use the Adams operations

$$\psi^2\colon\ KO^{m-4i}(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^m(X;\hat{\mathbb{Z}}_2)$$

defined for $m \equiv 0, \pm 2 \mod 8$ and $i \in \mathbb{Z}$ by

$$\psi^2(x) \ = \ \begin{cases} x^2 + 2\theta x & \text{for } m \equiv 0 \mod 8 \\ r\theta c x & \text{for } m \equiv \pm 2 \mod 8 \end{cases}$$

where the target dimensions of the complex operations θ are determined by the context. There are elementary relations:

(i) $\psi^2(x+y) = \psi^2 x + \psi^2 y \quad$ for $m \equiv 0, -2 \mod 8$,

(ii) $\psi^2(x+y) = \psi^2 x + \psi^2 y + \eta^2 xy \quad$ for $m \equiv 2 \mod 8$,

(iii) $c\psi^2 x = \psi^2 c x \quad$ for $m \equiv 0, \pm 2 \mod 8$,

which may be derived using the θ-ring formula $\psi^2 a = \theta a - \theta(-a)$ and the CR-module formula $\eta^2 = rBc$. We devote the rest of this section to a detailed study of the algebraic relations involving the real 2-adic operations θ, starting with sum and product formulae.

Proposition 3.6. *For a space X and elements $x, y \in KO^n(X;\hat{\mathbb{Z}}_2)$, we have*

$$\theta(x+y) \ = \ \begin{cases} \theta x + \theta y - xy & \text{for } n \equiv 0, -4 \mod 8 \\ \theta x + \theta y & \text{for } n \equiv -1, -5 \mod 8 \\ \theta x + \theta y + \eta xy & \text{for } n \equiv 1, -3 \mod 8. \end{cases}$$

Proof. This follows from [10, Theorem 6.4] since $\theta = -\lambda^2$. □

Proposition 3.7. *For a space X and elements $x \in KO^m(X;\hat{\mathbb{Z}}_2)$ and $y \in KO^n(X;\hat{\mathbb{Z}}_2)$, we have*

$$\theta(xy) \ = \ \begin{cases} (\theta x)y^2 + x^2(\theta y) + 2(\theta x)(\theta y) & \text{for } m, n \equiv 0, -4 \mod 8 \\ (\psi^2 x)(\theta y) & \text{for } m \text{ even and } n \text{ odd} \\ (\theta x)(\psi^2 y) & \text{for } m \text{ odd and } n \text{ even} \\ (\theta x)(\theta y) & \text{for } m, n \text{ odd and } m+n \equiv 0, -4 \mod 8 \\ r(\theta c x)(\theta c y)) - x^2 y^2 & \text{for } m, n \equiv \pm 2 \mod 8. \end{cases}$$

This will be proved in 7.7. For brevity, we have not specified the target dimensions of the operations ψ^2 and θ in these formulae, since they are sufficiently determined by the context.

Proposition 3.8. *For a space X and elements $x \in KO^n(X;\hat{\mathbb{Z}}_2)$, $k \in \hat{\mathbb{Z}}_2$, and $1 \in KO^0(X;\hat{\mathbb{Z}}_2)$, we have $\theta(1) = 0$ and*

$$\theta(kx) \;=\; \begin{cases} k\theta x - \binom{k}{2}x^2 & \text{for } n \equiv 0, -4 \mod 8 \\ k\theta x & \text{for } n \text{ odd.} \end{cases}$$

Proof. This follows from [10, Theorem 6.4] since $\theta = -\lambda^2$. □

Proposition 3.9. *For a space X and element $x \in KO^n(X;\hat{\mathbb{Z}}_2)$, we have*

$$\theta(\eta x) \;=\; \begin{cases} \eta\psi^2 x & \text{for } n \equiv 0, -4 \mod 8 \\ \eta\theta x & \text{for } n \equiv 1, -3 \mod 8 \\ 0 & \text{for } n \equiv \pm 2 \mod 8. \end{cases}$$

This will be proved in 7.3. In the case $n \equiv 0, -4 \mod 8$, we can equivalently write $\theta(\eta x) = \eta x^2$ since $\psi^2 x = x^2 + 2\theta x$. We next consider the operation

$$\xi \;=\; rB^2c\colon\, KO^*(X;\hat{\mathbb{Z}}_2) \longrightarrow KO^{*-4}(X;\hat{\mathbb{Z}}_2).$$

Proposition 3.10. *For a space X and element $x \in KO^n(X;\hat{\mathbb{Z}}_2)$, we have*

$$\theta(\xi x) \;=\; \begin{cases} 2\theta x - x^2 & \text{for } n \equiv 0, -4 \mod 8 \\ 2\theta x & \text{for } n \text{ odd.} \end{cases}$$

This will be proved in 7.5.

The operations θ play an important role in formulae for squares of elements in $KO^*(X;\hat{\mathbb{Z}}_2)$.

Proposition 3.11. *For a space X and element $x \in KO^n(X;\hat{\mathbb{Z}}_2)$, we have*

$$x^2 \;=\; \begin{cases} \eta\theta x & \text{for } n \equiv -1, -5 \mod 8 \\ 0 & \text{for } n \equiv 1, -3 \mod 8 \\ -r\theta cx & \text{for } n \equiv \pm 2 \mod 8. \end{cases}$$

This will be proved in 7.2 using results of Crabb and Minami. A similar result is:

Proposition 3.12. *For a space X and element $x \in KO^n(X;\hat{\mathbb{Z}}_2)$, we have*

$$\phi cx \;=\; \begin{cases} x^2 & \text{for } n \equiv 0, -4 \mod 8 \\ 0 & \text{for } n \equiv -1, -5 \mod 8 \\ \eta\theta x & \text{for } n \equiv 1, -3 \mod 8 \\ r\theta cx & \text{for } n \equiv \pm 2 \mod 8. \end{cases}$$

This will be proved in 7.4. Our last three propositions will give formulae for the commutation of θ with the operations c, r, and ϕ.

Proposition 3.13. *For a space X and element $x \in KO^n(X;\hat{\mathbb{Z}}_2)$ with $n \not\equiv \pm 2 \mod 8$, we have $\theta cx = c\theta x$.*

Proof. This follows by [10, Theorem 6.4]. □

Proposition 3.14. *For a space X and element $z \in K^n(X;\hat{\mathbb{Z}}_2)$, we have*

$$\theta r z = \begin{cases} r\theta z - \phi z & \text{for } n \equiv 0, -4 \mod 8 \\ r\theta z + \eta\phi z & \text{for } n \equiv -1, -5 \mod 8 \\ r\theta z & \text{for } n \equiv 1, -3 \mod 8. \end{cases}$$

Proof. This follows by [10, Theorems 6.4 and 6.5]. □

Proposition 3.15. *For a space X and element $z \in K^n(X;\hat{\mathbb{Z}}_2)$, we have*

$$\theta\phi z = \begin{cases} r(z^2\theta z^* + (\theta z)(\theta z^*)) & \text{for } n = 0 \\ \phi\theta z & \text{for } n = -1. \end{cases}$$

This will be proved in 7.6. Although we are primarily interested in unstable operations, we finally consider:

3.16. **The stable 2-adic Adams operations.** For a space X, the cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ of 3.1 is endowed with the usual stable Adams operations $\psi^k\colon K^*(X;\hat{\mathbb{Z}}_2) \to K^*(X;\hat{\mathbb{Z}}_2)$ and $\psi^k\colon KO^*(X;\hat{\mathbb{Z}}_2) \to KO^*(X;\hat{\mathbb{Z}}_2)$ for units $k \in \mathbb{Z}^\times_{(2)}$ or more generally for units $k \in \hat{\mathbb{Z}}^\times_2$ (see, e.g., [10]). Moreover, for elements $x \in KO^m(X;\hat{\mathbb{Z}}_2)$, $y \in KO^n(X;\hat{\mathbb{Z}}_2)$, $z \in K^s(X;\hat{\mathbb{Z}}_2)$, and $w \in K^t(X;\hat{\mathbb{Z}}_2)$, we have the usual relations including: $\psi^k(xy) = (\psi^k x)(\psi^k y)$, $\psi^k(wz) = (\psi^k w)(\psi^k z)$, $\psi^k B_R x = k^4 B_R \psi^k x$, $\psi^k B z = kB\psi^k z$, $\psi^k c x = c\psi^k x$, $\psi^k r z = r\psi^k z$, and $\psi^k \eta x = \eta\psi^k x$. We also have the following formulae for the commutation of ψ^k with the operations θ and ϕ:

(i) $\psi^k\theta x = \theta\psi^k x$ for $m = 0, -1, 1$;
(ii) $\psi^k\theta x = \theta(k^{-2}\psi^k x)$ for $m = -4, -5, -3$;
(iii) $\psi^k\theta z = \theta\psi^k z$ for $s = 0, -1$;
(iv) $\psi^k\phi z = \phi\psi^k z$ for $s = 0$ and $\psi^k\phi z = k^{-1}\phi\psi^k z$ for $s = -1$.

These will be proved in 7.8.

4. Proofs of the Kunneth theorems

The rest of this paper will be devoted to proving results of Sections 2 and 3. In this section, we prove the Kunneth theorems 2.3, 2.8, and 2.12, together with the lemmas 2.7 and 2.11 for united 2-adic K-theory. We start with the CR-results. To use the work of [5], we need:

4.1. **Pontrjagin duality for 2-adic CR-modules.** Recall that Pontrjagin duality gives contravariant equivalences $(-)^\sharp : \tilde{\mathcal{A}}b^{\mathrm{op}} \leftrightarrows \hat{\mathcal{A}}b : (-)^\sharp$ between the category $\tilde{\mathcal{A}}b$ of 2-torsion abelian groups and the category $\hat{\mathcal{A}}b$ of 2-profinite abelian groups. As in [10, Theorem 3.1], this prolongs to contravariant equivalences $(-)^\sharp : \tilde{\mathcal{C}}\mathcal{R}^{\mathrm{op}} \leftrightarrows \hat{\mathcal{C}}\mathcal{R} : (-)^\sharp$ between the category $\tilde{\mathcal{C}}\mathcal{R}$ of 2-torsion CR-modules [5, Section 4.7] and the category $\hat{\mathcal{C}}\mathcal{R}$ of 2-adic CR-modules, where the former are indexed homologically. More explicitly, a 2-torsion CR-module $M \in \tilde{\mathcal{C}}\mathcal{R}$ corresponds to a 2-adic CR-module $M^\sharp \in \hat{\mathcal{C}}\mathcal{R}$ with $(M^\sharp)^n_C = (M^C_n)^\sharp$ and $(M^\sharp)^n_R = (M^R_{n+4})^\sharp$ for $n \in \mathbb{Z}$, where the operations B,t, B_R,η, c, and r in $M^\sharp$ correspond respectively to B, t, B_R, η, rB^2, and $B^{-2}c$ in M. We note that M is Bott exact or CR-exact if and only if its Pontrjagin dual $M^\sharp$ has this property. For a spectrum X, there are now natural isomorphisms of 2-adic CR-modules

$$K^*_{CR}(X;\hat{\mathbb{Z}}_2) \;\cong\; K^{CR}_*(X;\mathbb{Z}_{2^\infty})^\sharp \;\cong\; K^{CR}_{*-1}(\tau_2 X)^\sharp$$

by [10, Theorem 3.1], where $\tau_2 X$ is the 2-torsion part of X given by the homotopy fiber of $X \to X[1/2]$. To prove Lemma 2.7 on projectivity and flatness in $\hat{\mathcal{C}}\mathcal{R}$, we need:

4.2. **Free 2-adic CR-modules.** For a 2-profinite abelian group $G \in \hat{\mathcal{A}}b$ and integer n, there are free 2-adic CR-modules $F_C(G,n)$ and $F_R(G,n)$ whose maps into a 2-adic CR-module M correspond to the maps $G \to M^n_C$ and $G \to M^n_R$. We may obtain $F_C(G,n)$ and $F_R(G,n)$ explicitly by tensoring G with the free CR-modules of [5, Secton 2.4]. When G is torsion-free, $F_C(G,n)$ and $F_R(G,n)$ are projective in $\hat{\mathcal{C}}\mathcal{R}$ since G is projective in $\hat{\mathcal{A}}b$, and they are flat in $\hat{\mathcal{C}}\mathcal{R}$ since G is flat in $\hat{\mathcal{A}}b$ and since

$$\begin{aligned}
(F_C(G,n)\hat{\otimes}_{CR}M)^*_C \;&\cong\; (G\hat{\otimes}M^{*-n}_C) \oplus (G\hat{\otimes}M^{*-n}_C)\\
(F_C(G,n)\hat{\otimes}_{CR}M)^*_R \;&\cong\; G\hat{\otimes}M^{*-n}_C\\
(F_R(G,n)\hat{\otimes}_{CR}M)^*_C \;&\cong\; G\hat{\otimes}M^{*-n}_C\\
(F_R(G,n)\hat{\otimes}_{CR}M)^*_R \;&\cong\; G\hat{\otimes}M^{*-n}_R
\end{aligned}$$

for 2-adic CR-modules $M \in \hat{\mathcal{C}}\mathcal{R}$.

4.3. **Proof of Lemma 2.7.** There are enough projectives in $\hat{\mathcal{C}}\mathcal{R}$ given by finite direct sums of $F_C(G,n)$'s and $F_R(G,n)$'s for various torsion-free $G \in \hat{\mathcal{A}}b$ and $n \in \mathbb{Z}$. The implication $(iii) \Rightarrow (i)$ now follows since each projective $M \in \hat{\mathcal{C}}\mathcal{R}$ is a quotient (and hence direct summand) of a flat projective module of the above sort. The implication $(i) \Rightarrow (ii)$ follows by applying $M\hat{\otimes}_{CR}$ to the exact sequences

$$\dots \longrightarrow F_R(\hat{\mathbb{Z}}_2,n{-}1) \xrightarrow{[\eta]} F_R(\hat{\mathbb{Z}}_2,n) \xrightarrow{[r]} F_C(\hat{\mathbb{Z}}_2,n) \xrightarrow{[B^{-1}c]} F_R(\hat{\mathbb{Z}}_2,n{-}2) \xrightarrow{[\eta]} \dots$$

$$0 \longrightarrow F_C(\hat{\mathbb{Z}}_2,n) \xrightarrow{[2]} F_C(\hat{\mathbb{Z}}_2,n).$$

The remaining implication $(ii) \Rightarrow (iii)$ follows from the Pontrjagin dual result in $\tilde{\mathcal{C}}\mathcal{R}$, which in turn follows from [5, Proposition 4.9]. ☐

The proof of the Kunneth theorem 2.8 will depend on two lemmas.

Lemma 4.4. *If X and Y are spectra with $\widehat{K}^*_{CR}X$ (or $\widehat{K}^*_{CR}Y$) flat in $\hat{\mathcal{C}}\mathcal{R}$, then $\mu\colon\ \widehat{K}^*_{CR}X \hat{\otimes}_{CR} \widehat{K}^*_{CR}Y \to \widehat{K}^*_{CR}(X \wedge Y)$ is an isomorphism.*

Proof. When Y is the sphere spectrum S, this follows since $\widehat{K}^*_{CR}S = F_R(\hat{\mathbb{Z}}_2, 0)$ is the unit for $\hat{\otimes}_{CR}$. When Y is a finite spectrum, it follows by induction on the number of cells in Y since $\widehat{K}^*_{CR}X$ is flat. The general result then follows by a limit argument. ☐

Lemma 4.5. *For a spectrum X, there exists a spectrum P and a map $f\colon\ X \to P$ such that $\widehat{K}^*_{CR}P$ is flat in $\hat{\mathcal{C}}\mathcal{R}$ and $f^*\colon\ \widehat{K}^*_{CR}P \to \widehat{K}^*_{CR}X$ is onto.*

Proof. Let $f\colon\ X \to P$ be a map from X to a (possibly infinite) product P of spectra $\Sigma^i K\hat{\mathbb{Z}}_2$ and $\Sigma^i KO\hat{\mathbb{Z}}_2$ such that $f^*\colon\ \widehat{K}^*_{CR}P \to \widehat{K}^*_{CR}X$ is onto. This P is a finite wedge of suspensions of spectra KG and KOG for various products $G = \prod \hat{\mathbb{Z}}_2$, and we must show that $\widehat{K}^*_{CR}$ carries such KG and KOG to flat modules in $\hat{\mathcal{C}}\mathcal{R}$. For this, it suffices by 2.5 and Lemma 2.7 to show that $K^*(KG; \hat{\mathbb{Z}}_2)$ and $K^*(KOG; \hat{\mathbb{Z}}_2)$ are torsion-free and concentrated in even degrees. This follows by 4.1 since $\tau_2 KG$ and $\tau_2 KOG$ are (possibly infinite) wedges of spectra $\Sigma^{-1}K\mathbb{Z}_{2^\infty}$ and $\Sigma^{-1}KO\mathbb{Z}_{2^\infty}$ which have suitable K-homologies by [5, Theorem 8.2]. ☐

4.6. **Proof of Theorem 2.8.** By Lemmas 4.5 and 2.7, we may choose a spectrum P and a map $f\colon\ X \to P$ with homotopy cofiber P' such that $f^*\colon\ \widehat{K}^*_{CR}P \to \widehat{K}^*_{CR}X$ is onto and such that $\widehat{K}^*_{CR}P$ and $\widehat{K}^*_{CR}P'$ are flat in $\hat{\mathcal{C}}\mathcal{R}$. Thus, by Lemma 4.4, we obtain a short exact Kunneth sequence from the long exact $\widehat{K}^*_{CR}$-sequence of $X \wedge Y \to P \wedge Y \to P' \wedge Y$, and we deduce the required naturality by the argument of Atiyah [2]. The result for spaces follows easily from the result for spectra. ☐

4.7. **Proof of Theorem 2.3.** Lemmas 4.4 and 4.5 remain valid when $\widehat{K}^*_{CR}$, $\hat{\mathcal{C}}\mathcal{R}$, and $\hat{\otimes}_{CR}$, are replaced by $\widehat{K}^*$, $\hat{\mathcal{K}}^*$, and $\hat{\otimes}_{K^*}$. Hence, Theorem 2.3 follows as in 4.6. ☐

We now proceed to prove the corresponding CRT-results assuming some familiarity with the work of [5] and [10].

4.8. **Pontrjagin duality for 2-adic CRT-modules.** As in [10, Theorem 3.1], the Pontrjagin duality equivalences $(-)^\sharp : \tilde{\mathcal{A}}b^{\mathrm{op}} \leftrightarrows \hat{\mathcal{A}}b : (-)^\sharp$ prolong to equivalences $(-)^\sharp : \tilde{\mathcal{C}}\mathcal{R}\mathcal{T}^{\mathrm{op}} \leftrightarrows \hat{\mathcal{C}}\mathcal{R}\mathcal{T} : (-)^\sharp$ between the category $\tilde{\mathcal{C}}\mathcal{R}\mathcal{T}$ of 2-torsion CRT-modules [5, Section 2.1] and the category $\hat{\mathcal{C}}\mathcal{R}\mathcal{T}$ of

2-adic CRT-modules, where the former are indexed homologically. More explicitly, a 2-torsion CRT-module $M \in \tilde{\mathcal{C}}\mathcal{R}\mathcal{T}$ corresponds to a 2-adic CRT-module $M^\sharp \in \hat{\mathcal{C}}\mathcal{R}\mathcal{T}$ with $(M^\sharp)^n_C = (M^C_n)^\sharp$, $(M^\sharp)^n_R = (M^R_{n+4})^\sharp$, and $(M^\sharp)^n_T = (M^T_{n+3})^\sharp$ for $n \in \mathbb{Z}$, where the operations B, t, B_R, B_T, t_T, η, ϵ, τ, ζ, and γ in $M^\sharp$ correspond to B, t, B_R, B_T, t_T, η, τ, ϵ, γB^2, and $B^{-2}\zeta$ in M. We note that M is CRT-exact if and only if its Pontrjagin dual $M^\sharp$ has this property. For a spectrum X, there are now natural isomorphisms of 2-adic CRT-modules

$$K^*_{CRT}(X;\hat{\mathbb{Z}}_2) \;\cong\; K^{CRT}_*(X;\mathbb{Z}_{2^\infty})^\sharp \;\cong\; K^{CRT}_{*-1}(\tau_2 X)^\sharp$$

by [10, Theorem 3.1].

4.9. **Free 2-adic CRT-modules.** For a 2-profinite abelian group $G \in \hat{\mathcal{A}}b$ and integer n, there are free 2-adic CRT-modules $F_C(G,n)$, $F_R(G,n)$, $F_T(G,n)$ whose maps into a 2-adic CRT-module M correspond to the maps $G \to M^n_C$, $G \to M^n_R$, and $G \to M^n_T$ as in 4.2. Moreover, when G is torsion-free, the modules $F_C(G,n)$, $F_R(G,n)$, and $F_T(G,n)$ are projective and flat in $\hat{\mathcal{C}}\mathcal{R}\mathcal{T}$.

4.10. **Proof of Lemma 2.11.** The above proof of Lemma 2.7 is easily modified to prove Lemma 2.11 using 4.8, 4.9, and [5, Theorem 3.3]. □

Before proving the general Kunneth theorem 2.12 for united 2-adic K-cohomology, we note the following technical lemma which was used to construct the product homomorphism in that theorem.

Lemma 4.11. *The relation*

$$(\epsilon\tau)\circ\mu \;=\; \mu\circ(\epsilon\tau\wedge 1) + \mu\circ(1\wedge\epsilon\tau) + \eta\circ\mu$$

holds in $[KT\wedge KT, KT]_1$ *where* $\mu\colon\ KT\wedge KT \to KT$ *is the multiplication map.*

This will be proved in 4.15.

4.12. **Proof of Theorem 2.12.** Lemmas 4.4 and 4.5 remain valid when $\widehat{K}^*_{CR}$, $\hat{\mathcal{C}}\mathcal{R}$, and $\hat{\otimes}_{CR}$ are replaced by $\widehat{K}^*_{CRT}$, $\hat{\mathcal{C}}\mathcal{R}\mathcal{T}$, and $\hat{\otimes}_{CRT}$. Hence, Theorem 2.12 follows as in 4.6. □

To prove Lemma 4.11, we must show that the difference map

$$D \;=\; (\epsilon\tau)\circ\mu - \mu\circ(\epsilon\tau\wedge 1) - \mu\circ(1\wedge\epsilon\tau) - \eta\circ\mu$$

is trivial in $[KT\wedge KT, KT]_1$, and we first consider:

4.13. **The action of $\epsilon\tau$ on π_*KT.** By [5, Section 1.6], the graded commutative ring π_*KT has $\pi_{4i}KT = \mathbb{Z} = \langle B_T^i \rangle$, $\pi_{4i+1}KT = \mathbb{Z}/2 = \langle B_T^i \eta \rangle$, $\pi_{4i+3}KT = \mathbb{Z} = \langle B_T^i \omega \rangle$, and $\pi_{4i+2}KT = 0$ for $i \in \mathbb{Z}$, where the product is determined by the relations $\eta^2 = 0$, $\eta\omega = 0$, and $\omega^2 = 0$. Moreover, by [5, Sections 1.6 and 1.7], the map $\epsilon\tau \in [KT, KT]_1$ induces a homomorphism $\epsilon\tau_* \colon \pi_*KT \to \pi_{*+1}KT$ with $\epsilon\tau_*(B_T^i) = \eta B_T^i$ for i even, with $\epsilon\tau_*(B_T^i) = 0$ for i odd, with $\epsilon\tau_*(\eta B_T^i) = 0$ for all i, and with $\epsilon\tau_*(B_T^i \omega) = 2B_T^{i+1}$ for all i.

Lemma 4.14. *The compositions $D \circ (1 \wedge e)$ and $q \circ D$ are trivial where $1 \wedge e \colon KT \wedge S \to KT \wedge KT$ is the canonical map and where $q \colon KT \to KT_Q$ is the rationalization.*

Proof. It is straightforward to verify that $D \circ (1 \wedge e)$ is trivial since $(\epsilon\tau) \circ e = \eta \in \pi_1 KT$ by 4.13. To verify that $q \circ D$ is trivial, it suffices to show that $D_*(x \wedge y)$ is trivial in $\pi_{i+j+1}KT$ for each $x \in \pi_i KT$ and $y \in \pi_j KT$. This follows by a calculation using 4.13 together with the relation $(f \wedge g)_*(x \wedge y) = (-1)^{ni} f_* x \wedge g_* y$ in $\pi_{i+j+m+n}(KT \wedge KT)$ for maps $f \in [KT, KT]_m$ and $g \in [KT, KT]_n$. ☐

4.15. **Proof of Lemma 4.11.** As in [5, Section 1.2], we let $C(\eta^2) = S \cup_{\eta^2} e^3$ and recall that the unit map $S \to KT$ may be extended to a map $w \colon C(\eta^2) \to KT$ that induces a KO-module equivalence $KO \wedge C(\eta^2) \simeq KT$. Since $\epsilon\tau \in [KT, KT]_1$ is a KO-module map, the desired triviality of $D \in [KT \wedge KT, KT]_1$ will follow from the triviality of $D \circ (w \wedge w) \in [C(\eta^2) \wedge C(\eta^2), KT]_1$. By [5, Section 1.3], we have

$$C(\eta^2) \vee \Sigma^3 C(\eta^2) \simeq C(\eta^2) \wedge C(\eta^2)$$

$$[C(\eta^2), KT]_1 \cong \mathbb{Z}/2 \oplus \mathbb{Z}$$

$$[\Sigma^3 C(\eta^2), KT]_1 \cong \mathbb{Z} \oplus \mathbb{Z}$$

where the wedge summand $C(\eta^2)$ is mapped by $1 \wedge e \colon C(\eta^2) \to C(\eta^2) \wedge C(\eta^2)$. Consequently, the triviality of $D \circ (w \wedge w)$ will follow from the triviality of the maps $D \circ (w \wedge w) \circ (1 \wedge e) \in [C(\eta^2), KT]_1$ and $q \circ D \circ (w \wedge w) \in [C(\eta^2) \wedge C(\eta^2), KT_Q]_1$, which in turn follows from Lemma 4.14. ☐

5. The mod 2 K-cohomological functors j^* and h^*

In preparation for our proofs of relations for unstable operations, we now introduce mod 2 simplifications, j^*X and h^*X, for the cohomologies $KO^*(X; \hat{\mathbb{Z}}_2)$ and $K^*(X; \hat{\mathbb{Z}}_2)$ of a spectrum or space X. These simplifications will help us to handle the purely real parts of these cohomologies. We show that j^*X and h^*X have convenient multiplicative properties, and we obtain Kunneth theorems for them. We also show that h^*X has a natural differential satisfying a Leibniz formula and show that this differential vanishes if and only if $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is CR-exact. Although we approach j^*X and h^*X directly, we note that they may equivalently be derived from the Bott exact couple for $KO^*(X;\hat{\mathbb{Z}}_2)$ and $K^*(X;\hat{\mathbb{Z}}_2)$ (see Proposition 5.7).

5.1. **The functor j^*.** For a spectrum or space X, we let j^*X be the graded profinite $\mathbb{Z}/2$-module

$$j^*X \;=\; KO^*(X;\hat{\mathbb{Z}}_2)/rK^*(X;\hat{\mathbb{Z}}_2) \;\cong\; \eta KO^{*+1}(X;\hat{\mathbb{Z}}_2).$$

We may view j^*X as a module over the graded commutative $\mathbb{Z}/2$-algebra $j^* = j^*S$ with $j^{-8k} = \mathbb{Z}/2 = \langle B_R^k\rangle$ and $j^{-8k-1} = \mathbb{Z}/2 = \langle \eta B_R^k\rangle$ for $k \in \mathbb{Z}$ and with $j^n = 0$ for $n \not\equiv 0, -1 \mod 8$. For spectra X and Y, there is a product homomorphism

$$\mu\colon\; j^*X\hat{\otimes}_{j^*}j^*Y \longrightarrow j^*(X\wedge Y)$$

induced by the KO-cohomological product homomorphisms $j^mX\hat{\otimes}j^nY \to j^{m+n}(X\wedge Y)$ for $m,n \in \mathbb{Z}$. Likewise, for spaces X and Y, there is a product homomorphism

$$\mu\colon\; j^*X\hat{\otimes}_{j^*}j^*Y \to j^*(X\times Y)$$

and j^*X has a commutative graded algebra structure over j^*. For a spectrum or space X, we see from 2.5 that $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is CR-exact if and only if j^*X is free as a j^*-module.

Theorem 5.2. *If X and Y are spectra with $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ CR-exact and $K^*(X;\hat{\mathbb{Z}}_2)$ torsion-free (or with $K^*_{CR}(Y;\hat{\mathbb{Z}}_2)$ CR-exact and $K^*(Y;\hat{\mathbb{Z}}_2)$ torsion-free), then $\mu\colon\; j^*X\hat{\otimes}_{j^*}j^*Y \to j^*(X\wedge Y)$ is an isomorphism. This also holds for spaces X and Y when $X\wedge Y$ is replaced by $X\times Y$.*

Proof. We may assume that X and Y are spectra. Then

$$\mu\colon\; K^*_{CR}(X;\hat{\mathbb{Z}}_2)\hat{\otimes}_{CR}K^*_{CR}(Y;\hat{\mathbb{Z}}_2) \longrightarrow K^*_{CR}(X\wedge Y;\hat{\mathbb{Z}}_2)$$

is an isomorphism by Theorem 2.8, and it suffices to show that the natural map

$$(M_R/r)\hat{\otimes}_{j^*}(N_R/r) \longrightarrow (M\hat{\otimes}_{CR}N)_R/r$$

is an isomorphism whenever M and N are 2-adic CR-modules with M CR-exact and M_C torsion-free. By 4.3, we may assume that M is $F_R(G,n)$ or $F_C(G,n)$ for some torsion-free 2-profinite G and integer n, and the result then follows easily using 4.2. □

5.3. **The functor h^*.** For a spectrum or space X, we let h^*X be the graded profinite $\mathbb{Z}/2$-module with

$$h^nX \;=\; \frac{\ker(1-t)}{\operatorname{im}(1+t)}$$

for the involution $t = \psi^{-1}\colon\ K^n(X;\hat{\mathbb{Z}}_2) \to K^n(X;\hat{\mathbb{Z}}_2)$. We may view h^*X as a module over the graded commutative $\mathbb{Z}/2$-algebra $h^* = h^*S$ with $h^{-4k} = \mathbb{Z}/2 = \langle B^{2k}\rangle$ for $k \in \mathbb{Z}$ and with $h^n = 0$ for $n \not\equiv 0 \mod 4$. For spectra X and Y, there is a product homomorphism

$$\mu\colon\ h^*X\hat{\otimes}_{h^*}h^*Y \longrightarrow h^*(X\wedge Y)$$

induced by the K-cohomological product homomorphisms $\mu\colon\ h^mX\hat{\otimes}h^nY \longrightarrow h^{m+n}(X\wedge Y)$ for $m,n \in \mathbb{Z}$. Likewise, for spaces X and Y, there is a product homomorphism

$$\mu\colon\ h^*X\hat{\otimes}_{h^*}h^*Y \longrightarrow h^*(X\times Y)$$

and h^*X has a commutative graded algebra structure over h^*.

Theorem 5.4. *If X and Y are spectra with $K^*(X;\hat{\mathbb{Z}}_2)$ (or $K^*(Y;\hat{\mathbb{Z}}_2)$) torsion-free, then $\mu\colon\ h^*X\hat{\otimes}_{h^*}h^*Y \longrightarrow h^*(X\wedge Y)$ is an isomorphism. This also holds for spaces X and Y when $X\wedge Y$ is replaced by $X\times Y$.*

Proof. We may assume that X and Y are spectra. Then

$$\mu\colon\ K^*(X;\hat{\mathbb{Z}}_2)\hat{\otimes}K^*(Y;\hat{\mathbb{Z}}_2) \longrightarrow K^*(X\wedge Y;\hat{\mathbb{Z}}_2)$$

is an isomorphism by Theorem 2.3, and we may proceed algebraically. Let $\hat{I}nv$ be the category of 2-profinite abelian groups with involution t (i.e., with endomorphism t having $t^2 = 1$). Then, by [5, Proposition 3.8] and 4.1, each torsion-free $N \in \hat{I}nv$ has a direct sum decomposition

$$M \;\cong\; I \oplus J \oplus (G \oplus tG)$$

for 2-profinite abelian groups I, J, and G, where $t = 1$ on I and $t = -1$ on J. This may be used to decompose $K^*(X;\hat{\mathbb{Z}}_2)$. We also note that, for each $N \in \hat{I}nv$, the object $(G\oplus tG)\hat{\otimes}N \in \hat{I}nv$, with diagonal t action, is isomorphic to $(G\hat{\otimes}N)\oplus t(G\hat{\otimes}N) \in \hat{I}nv$. This allows us to ignore the $G\oplus tG$ components of $K^*(X;\hat{\mathbb{Z}}_2)$, and the result follows easily. □

5.5. **The complexification and realification maps.** For a spectrum or space X, we let $c\colon\ j^*X \to h^*X$ and $r'\colon\ h^*X \to j^{*+3}X$ be the *complexification* and *realification* maps given by $c[x] = [cx]$ for $x \in KO^*(X;\hat{\mathbb{Z}}_2)$ and by $r'[z] = [y]$ for $z \in K^*(X;\hat{\mathbb{Z}}_2)$ with $tz = z$ and for $y \in KO^{*+3}(X;\hat{\mathbb{Z}}_2)$ with $\eta y = rB^{-1}z$, where such a y exists by Bott exactness since $c(rB^{-1}z) = (1+t)B^{-1}z = 0$. For spectra X and Y with $m,n \in \mathbb{Z}$, the maps c and r' have the usual multiplicative properties:

$$c\mu = \mu(c\hat{\otimes}c)\colon\ j^mX\hat{\otimes}j^nY \longrightarrow h^{m+n}(X\wedge Y)$$

$$\mu(1\hat{\otimes}r') = r'\mu(c\hat{\otimes}1)\colon\ j^mX\hat{\otimes}h^nY \longrightarrow j^{m+n+3}(X\wedge Y)$$
$$\mu(r'\hat{\otimes}1) = r'\mu(1\hat{\otimes}c)\colon\ h^mX\hat{\otimes}j^nY \longrightarrow j^{m+n+3}(X\wedge Y).$$

These also hold for spaces X and Y when $X\wedge Y$ is replaced by $X\times Y$. Moreover, the map $c\colon j^*X\to h^*X$ is an algebra homomorphism, while $r'\colon h^*X\to j^{*+3}X$ is a left or right j^*X-module homomorphism. Other basic properties are given by:

Proposition 5.6. *For a spectrum or space X, the operations $c\colon j^*X\to h^*X$ and $r'\colon h^*X\to j^{*+3}X$ satisfy:*

(i) $cB_R = B^4c$ *and* $B_Rr' = r'B^4$*;*
(ii) $c\eta = 0$, $r'c = 0$, *and* $\eta r' = 0$*;*
(iii) $r'B^2c = \eta$*;*
(iv) $B^2cr' = cr'B^2$.

Proof. The relations (i)–(iii) follow easily from the CR-module relations in 2.4. The relation (iv) requires additional input from united K-theory and will be proved in 5.14. □

Proposition 5.7. *For a spectrum or space X, the sequence*

$$\ldots\longrightarrow j^{*+1}X \xrightarrow{\eta} j^*X \xrightarrow{c} h^*X \xrightarrow{r'} j^{*+3}X \xrightarrow{\eta} \ldots$$

is exact.

Proof. This sequence is equivalent to the derived couple of the Bott exact couple

$$\ldots\longrightarrow KO^{*+1}(X;\hat{\mathbb{Z}}_2) \xrightarrow{\eta} KO^*(X;\hat{\mathbb{Z}}_2) \xrightarrow{c} K^*(X;\hat{\mathbb{Z}}_2) \xrightarrow{rB^{-1}} KO^{*+2}(X;\hat{\mathbb{Z}}_2) \xrightarrow{\eta} \ldots$$

□

5.8. **The differential in h^*X.** For a spectrum or space X, we define a differential $\partial\colon h^*X\to h^{*-1}X$ by $\partial = B^2cr' = cr'B^2$. Of course, this is equivalent to the differential cr' of the above derived couple. It has the properties $\partial^2 = 0$, $B^2\partial = \partial B^2$, $\partial c = 0$, and $r'\partial = 0$ by Proposition 5.6, and it obeys a Leibniz rule by:

Theorem 5.9. *For spectra X and Y and for elements $x\in h^mX$ and $y\in h^nY$, we have $\partial(xy) = (\partial x)y + x(\partial y)$ in $h^{m+n-1}(X\wedge Y)$. This also holds for spaces X and Y when $X\wedge Y$ is replaced by $X\times Y$, and holds for a space $X = Y$ when $X\wedge Y$ is replaced by X.*

This will be proved in 5.15. Our main interest in ∂ stems from its connection with the CR-exactness condition.

Proposition 5.10. *For a spectrum or space X, the cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is CR-exact if and only if $\partial = 0$ in h^*X. This happens if and only if j^*X is free as a j^*-module.*

Proof. The following conditions are successively equivalent: (i) $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is CR-exact; (ii) $\ker\eta \subset \operatorname{im}\eta$ in j^*X; (iii) $\operatorname{im} r' \subset \ker c$ in j^*X; (iv) $cr' = 0$ in h^*X; and (v) $\partial = 0$ in h^*X. The proposition now follows from 5.1. $\square$

When the cohomology $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is CR-exact, we may view h^*X as a first approximation to j^*X by:

Proposition 5.11. *If X is a spectrum or space such that $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is CR-exact, then the complexification map*

$$c + B^{-2}c\colon\ (j^*X)/\eta \oplus (j^{*-4}X)/\eta \longrightarrow h^*X$$

is an isomorphism.

Proof. There is a short exact sequence

$$0 \longrightarrow (j^*X)/\eta \xrightarrow{\ c\ } h^*X \xrightarrow{\ r'\ } (j^{*+3}X)\backslash\eta \longrightarrow 0$$

by Proposition 5.7 where $(j^{*+3}X)\backslash\eta$ denotes the kernel of $\eta\colon\ j^{*+3}X \to j^{*+2}X$. When $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is CR-exact, there is an isomorphism $\eta\colon\ (j^{*+4}X)/\eta \cong (j^{*+3}X)\backslash\eta$ which factors as the composition of $B^2c\colon\ (j^{*+4}X)/\eta \to h^*X$ and $r'\colon\ h^*X \to (j^{*+3}X)\backslash\eta$ by 5.6 (iii). This gives a splitting of the short exact sequence, and the result follows using $B_R\colon\ j^{*+4}X \cong j^{*-4}X$. $\square$

We devote the rest of this section to proving 5.6(iv) and Theorem 5.9 using inputs from united K-theory. For a spectrum or space X, we let h^*_TX denote the graded profinite $\mathbb{Z}/2$-module

$$h^*_TX \ = \ KT^*(X;\hat{\mathbb{Z}}_2)/(\gamma K^{*-1}(X;\hat{\mathbb{Z}}_2) + \epsilon r K^*(X;\hat{\mathbb{Z}}_2)),$$

and we let $\zeta\colon\ h^*_TX \to h^*X$ be the homomorphism induced by $\zeta\colon\ KT^*(X;\hat{\mathbb{Z}}_2) \to K^*(X;\hat{\mathbb{Z}}_2)$ using the relations $t\zeta = \zeta$, $\zeta\gamma = 0$, and $\zeta\epsilon r = 1 + t$ of [5, Section 1.9].

Lemma 5.12. *For a spectrum or space X, $\zeta\colon\ h^*_TX \to h^*X$ is an isomorphism.*

Proof. The homomorphism

$$\zeta\colon\ KT^*(X;\hat{\mathbb{Z}}_2)/\gamma \longrightarrow K^*(X;\hat{\mathbb{Z}}_2)\backslash(1-t)$$

is an isomorphism since $K^*_{CRT}(X;\hat{\mathbb{Z}}_2)$ is CRT-exact by 2.9. The lemma now follows since the homomorphism $\epsilon r\colon\ K^*(X;\hat{\mathbb{Z}}_2) \longrightarrow KT^*(X;\hat{\mathbb{Z}}_2)/\gamma$ composes with the above isomorphism ζ to give $1 + t$. $\square$

We may now replace h^*X by h^*_TX in the exact sequence of Proposition 5.7. Let $\epsilon\colon\ j^*X \to h^*_TX$ and $\tau B_T^{-1}\colon\ h^*_TX \to j^{*+3}X$ be the homomorphisms induced by $\epsilon\colon\ KO^*(X;\hat{\mathbb{Z}}_2) \to KT^*(X;\hat{\mathbb{Z}}_2)$ and $\tau B_T^{-1}\colon\ KT^*(X;\hat{\mathbb{Z}}_2) \to KO^{*+3}(X;\hat{\mathbb{Z}}_2)$ using the relations $\tau B_T^{-1}\gamma = rB^{-2}$ and $\tau B_T^{-1}\epsilon = 0$.

Lemma 5.13. *The composition of* $\epsilon\colon\ j^*X \to h^*_TX$ *with* $\zeta\colon\ h^*_TX \cong h^*X$ *is* $c\colon\ j^*X \to h^*X$*, and the composition of* $\zeta\colon\ h^*_TX \cong h^*X$ *with* $r'\colon\ h^*X \to j^{*+3}X$ *is* $\tau B_T^{-1}\colon\ h^*_TX \to j^{*+3}X$*.*

Proof. This follows from the equality $\zeta\epsilon = c\colon\ KO^*(X;\hat{\mathbb{Z}}_2) \to K^*(X;\hat{\mathbb{Z}}_2)$ and the equality $rB^{-1}\zeta = \eta\tau B_T^{-1}\colon\ KT^*(X;\hat{\mathbb{Z}}_2) \to KO^{*+2}(X;\hat{\mathbb{Z}}_2)$ of [5]. □

5.14. **Proof of 5.6(iv).** By Lemma 5.13, the operations $\epsilon\tau B_T^{-1}$ and B_T in h^*_TX correspond to the operations cr' and B^2 in h^*X under the isomorphism $\zeta\colon\ h^*_TX \cong h^*X$. Thus, it suffices to show $B_T(\epsilon\tau B_T^{-1}) = (\epsilon\tau B_T^{-1})B_T$ in h^*_TX. This follows from the relations $B_T\epsilon\tau = \epsilon\tau B_T + \eta B_T$ and $\eta = \gamma B\zeta$ in $KT^*(X;\hat{\mathbb{Z}}_2)$. □

5.15. **Proof of Theorem 5.9.** As in 5.14, the operation $\epsilon\tau$ in h^*_TX corresponds to the operation ∂ in h^*X under the isomorphism $\zeta\colon\ h^*_TX \cong h^*X$. Thus, for spectra X and Y and for elements $x \in h^m_TX$ and $y \in h^n_TY$, it suffices to show $\epsilon\tau(xy) = (\epsilon\tau x)y + x(\epsilon\tau y)$ in $h^{m+n-1}_T(X \wedge Y)$. This follows since, for elements $z \in KT^m(X;\hat{\mathbb{Z}}_2)$ and $w \in KT^n(Y;\hat{\mathbb{Z}}_2)$, we have

$$\epsilon\tau(zw) = (\epsilon\tau z)w + (-1)^m z(\epsilon\tau w) + \eta zw$$

in $KT^{m+n-1}(X \wedge Y;\hat{\mathbb{Z}}_2)$ by Lemma 4.11. □

6. On the j^* groups of some infinite loop spaces

In this section, we complete our preparations for proofs of relations for unstable operations. For a spectrum E and integer m, we let $\underline{E}_m$ denote the infinite loop space $\Omega_0^\infty\Sigma^m E$ given by the base component of $\Omega^\infty\Sigma^m E$. Thus, for a connected space X, there is a natural isomorphism $\widetilde{E}^m(X) \cong [X, \underline{E}_m]$ which specializes to isomorphisms $\widetilde{KO}^m(X;\hat{\mathbb{Z}}_2) \cong [X, \underline{\widehat{KO}}_m]$ and $\widetilde{K}^m(X;\hat{\mathbb{Z}}_2) \cong [X, \underline{\widehat{K}}_m]$ where $\widehat{KO}$ and $\widehat{K}$ denote the 2-adic completions of KO and K. Our main results in this section (Theorems 6.1 and 6.2) will show the sparseness of primitive elements in the j^* groups of $\underline{\widehat{KO}}_m$ and $\underline{\widehat{K}}_m$. After stating these results, we turn to their proofs, which require some extensive preliminaries and need not detain the reader.

Theorem 6.1. *For each integer* m*, the cohomologies* $K^*_{CR}(\underline{\widehat{KO}}_m;\hat{\mathbb{Z}}_2)$ *and* $K^*_{CR}(\underline{\widehat{K}}_m;\hat{\mathbb{Z}}_2)$ *are CR-exact with* $K^*(\underline{\widehat{KO}}_m;\hat{\mathbb{Z}}_2)$ *and* $K^*(\underline{\widehat{K}}_m;\hat{\mathbb{Z}}_2)$ *torsion-free.*

The proof is in 6.5. Letting L_m denote the infinite loop space $\widehat{\underline{KO}}_m$ or $\widehat{\underline{K}}_m$, we may now apply the Kunneth theorem for j^* to show that j^*L_m is a 2-profinite Hopf algebra over $j^* = j^*(pt)$. Using the spectrum $B^\infty L_m$, we let $\sigma\colon\ j^*B^\infty L_m \to Pj^*L_m$ denote the infinite cohomology suspension from $j^*B^\infty L_m$ to the primitives in j^*L_m. We now give the needed sparseness results for $Pj^*\underline{KO}_m$ and $Pj^*\underline{K}_m$.

Theorem 6.2. *We have:*

(i) $Pj^i\widehat{\underline{KO}}_0 = 0$ *for* $i \not\equiv 0, -1 \mod 8$ *and* $\eta\colon\ Pj^i\widehat{\underline{KO}}_0 \to Pj^{i-1}\widehat{\underline{KO}}_0$ *is monic for* $i \equiv 0 \mod 8$ *with* $\eta = 0$ *on* $Pj^{i-1}\widehat{\underline{KO}}_0$*;*

(ii) $Pj^i\widehat{\underline{KO}}_1 = 0$ *for* $i \not\equiv 1, 0 \mod 8$ *and* $\eta\colon\ Pj^i\widehat{\underline{KO}}_1 \to Pj^{i-1}\widehat{\underline{KO}}_1$ *is monic for* $i \equiv 1 \mod 8$ *with* $\eta = 0$ *on* $Pj^{i-1}\widehat{\underline{KO}}_1$*;*

(iii) $Pj^i\widehat{\underline{KO}}_2 = 0$ *for* $i \not\equiv 2, 1 \mod 8$ *and* $\eta\colon\ Pj^i\widehat{\underline{KO}}_2 \to Pj^{i-1}\widehat{\underline{KO}}_2$ *is monic for* $i \equiv 2 \mod 8$ *with* $\eta = 0$ *on* $Pj^{i-1}\widehat{\underline{KO}}_2$*; in fact,* $\sigma\colon\ j^*B^\infty\widehat{\underline{KO}}_2 \cong Pj^*\widehat{\underline{KO}}_2$*;*

(iv) $Pj^i\widehat{\underline{KO}}_3 = 0$ *for* $i \not\equiv 3, 2, -1, -2 \mod 8$ *and* $\eta\colon\ Pj^i\widehat{\underline{KO}}_3 \to Pj^{i-1}\widehat{\underline{KO}}_3$ *is monic for* $i \equiv 3, -1 \mod 8$ *with* $\eta = 0$ *on* $Pj^{i-1}\widehat{\underline{KO}}_3$*;*

(v) $Pj^i\widehat{\underline{KO}}_{-4} = 0$ *for* $i \not\equiv 0, -1, -4, -5 \mod 8$ *and* $\eta\colon\ Pj^i\widehat{\underline{KO}}_{-4} \to Pj^{i-1}\widehat{\underline{KO}}_{-4}$ *is monic for* $i \equiv 0, -4 \mod 8$ *with* $\eta = 0$ *on* $Pj^{i-1}\widehat{\underline{KO}}_{-4}$*;*

(vi) $Pj^i\widehat{\underline{KO}}_{-3} = 0$ *for* $i \not\equiv 1, 0, -3, -4 \mod 8$ *and* $\eta\colon\ Pj^i\widehat{\underline{KO}}_{-3} \to Pj^{i-1}\widehat{\underline{KO}}_{-3}$ *is monic for* $i \equiv 1, -3 \mod 8$ *with* $\eta = 0$ *on* $Pj^{i-1}\widehat{\underline{KO}}_{-3}$*;*

(vii) $Pj^i\widehat{\underline{KO}}_{-2} = 0$ *for* $i \not\equiv -2, -3 \mod 8$ *and* $\eta\colon\ Pj^i\widehat{\underline{KO}}_{-2} \to Pj^{i-1}\widehat{\underline{KO}}_{-2}$ *is monic for* $i \equiv -2 \mod 8$ *with* $\eta = 0$ *on* $Pj^{i-1}\widehat{\underline{KO}}_{-2}$*; in fact,* $\sigma\colon\ j^*B^\infty\widehat{\underline{KO}}_{-2} \cong Pj^*\widehat{\underline{KO}}_{-2}$*;*

(viii) $Pj^i\widehat{\underline{KO}}_{-1} = 0$ *for* $i \not\equiv -1, -2 \mod 8$ *and* $\eta\colon\ Pj^i\widehat{\underline{KO}}_{-1} \to Pj^{i-1}\widehat{\underline{KO}}_{-1}$ *is monic for* $i \equiv -1 \mod 8$ *with* $\eta = 0$ *on* $Pj^{i-1}\widehat{\underline{KO}}_{-1}$*;*

(ix) $Pj^i\widehat{\underline{K}}_{-1} = 0$ *for* $i \not\equiv 1, 0, -1 \mod 8$ *and* $\alpha^*\colon\ Pj^i\widehat{\underline{K}}_{-1} \cong Pj^iS^1$ *for* $i \equiv 1 \mod 8$ *where* $\alpha\colon\ S^1 \to \widehat{\underline{K}}_{-1}$ *is obtained from the inclusion* $U(1) \subset U$*;*

(x) $Pj^i\widehat{\underline{K}}_0 = 0$ *for* $i \not\equiv 0, -1 \mod 8$ *and* $\eta\colon\ Pj^i\widehat{\underline{K}}_0 \to Pj^{i-1}\widehat{\underline{K}}_0$ *is monic for* $i \equiv 0 \mod 8$ *with* $\eta = 0$ *on* $Pj^{i-1}\widehat{\underline{K}}_0$.

The proof is in 6.10. In preparation, we first recall a result from [7], using notation and terminology of that paper. For a 2-profinite abelian group G, we let $\tilde{I}G = G \times G \times G \times \dots$ denote the 2-adic ψ^2-module with $\psi^2(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$.

Theorem 6.3. *Let E be a 0-connected spectrum with $K^*(E;\hat{\mathbb{Z}}_2)$ torsion-free. Then $K^*(\Omega^\infty E;\hat{\mathbb{Z}}_2)$ is torsion-free, and if $K^0(E;\hat{\mathbb{Z}}_2)=0$, then there is a natural isomorphism*

$$K^*(\Omega^\infty E;\hat{\mathbb{Z}}_2) \;\cong\; \hat{\Lambda}\tilde{I}K^1(E;\hat{\mathbb{Z}}_2)\hat{\otimes}\hat{\Lambda}H^1(E;\hat{\mathbb{Z}}_2)\hat{\otimes}JH^2(E;\hat{\mathbb{Z}}_2)$$

of $\mathbb{Z}/2$-graded 2-adic λ-rings and Hopf algebras. Also, if $K^1(E;\hat{\mathbb{Z}}_2)=0$ and $H^1(E;\hat{\mathbb{Z}}_2)=0$, then $K^1(\Omega^\infty E;\hat{\mathbb{Z}}_2)=0$ and there is a natural isomorphism

$$K^0(\Omega^\infty E;\hat{\mathbb{Z}}_2) \;\cong\; W(K^0(E;\hat{\mathbb{Z}}_2)\downarrow H^2(E;\hat{\mathbb{Z}}_2))$$

of 2-adic λ-rings and Hopf algebras.

Proof. This is obtained from [7, Theorem 8.3]. □

The above result applies to the spectra $B^\infty\widehat{\underline{KO}}_m$ and $B^\infty\widehat{\underline{K}}_m$ by:

Theorem 6.4. *For each m, $K^*(B^\infty\widehat{\underline{KO}}_m;\hat{\mathbb{Z}}_2)$ and $K^*(B^\infty\widehat{\underline{K}}_m;\hat{\mathbb{Z}}_2)$ are torsion-free with $K^{m+1}(B^\infty\widehat{\underline{KO}}_m;\hat{\mathbb{Z}}_2)=0$ and $K^{m+1}(B^\infty\widehat{\underline{K}}_m;\hat{\mathbb{Z}}_2)=0$. Moreover, for each m, $K^m(B^\infty\widehat{\underline{KO}}_m;\hat{\mathbb{Z}}_2)$ has $\psi^{-1}=1$ with $c\colon KO^m(B^\infty\widehat{\underline{KO}}_m;\hat{\mathbb{Z}}_2)\cong K^m(B^\infty\widehat{\underline{KO}}_m;\hat{\mathbb{Z}}_2)$, and $K^m(B^\infty\widehat{\underline{K}}_m;\hat{\mathbb{Z}}_2)$ is of the form $G\oplus\psi^{-1}G$ (under the ψ^{-1} action) for some 2-profinite abelian group G.*

Proof. Since all Postnikov spectra and all uniquely 2-divisible spectra are $K/2_*$-acyclic, the maps $B^\infty\widehat{\underline{KO}}_m\to\Sigma^m\widehat{KO}\leftarrow\Sigma^m KO$ and $B^\infty\widehat{\underline{K}}_m\to\Sigma^m\hat{K}\leftarrow\Sigma^m K$ are $K/2_*$-equivalences. Hence, they are also $K\hat{\mathbb{Z}}_2^*$-equivalences and $KO\hat{\mathbb{Z}}_2^*$-equivalences. Thus, it suffices to prove the version of the theorem with m replaced by 0, with $B^\infty\widehat{\underline{KO}}_m$ replaced by KO, and with $B^\infty\widehat{\underline{K}}_m$ replaced by K. This version follows using our knowledge of $K_*^{CR}(KO\mathbb{Z}_{2^\infty})$ and $K_*^{CR}(K\mathbb{Z}_{2^\infty})$ from [5, Theorem 8.2] and using the Pontrjagin duality of [10, Theorem 3.1] between $K_*^{CR}(E\mathbb{Z}_{2^\infty})$ and $K^*_{CR}(E;\hat{\mathbb{Z}}_2)$ for a spectrum E. □

6.5. **Proof of Theorem 6.1.** Let L_m denote $\widehat{\underline{KO}}_m$ or $\widehat{\underline{K}}_m$. Then $K^*(L_m;\hat{\mathbb{Z}}_2)$ is torsion-free by Theorems 6.3 and 6.4. Moreover, for m even, $K^*_{CR}(L_m;\hat{\mathbb{Z}}_2)$ is CR-exact since $K^1(L_m;\hat{\mathbb{Z}}_2)=0$ by the above theorems. Now let m be odd, and let $\tilde{L}_m$ be the 1-connected cover of L_m. By Theorems 6.3 and 6.4, $h^*\tilde{L}_m\cong\hat{\Lambda}M$ is a 2-profinite exterior Hopf algebra on primitives $M\subset h^*\tilde{L}_m$ which are concentrated in a single degree modulo 4. Since the differential ∂ of 5.8 must carry primitives to primitives, it must vanish on M and hence vanish on $h^*L_m\cong\hat{\Lambda}M$ by Theorem 5.9. Thus, $K^*_{CR}(\tilde{L}_m;\hat{\mathbb{Z}}_2)$ is CR-exact by Proposition 5.10. Moreover, $K^*_{CR}(K(\pi_1L_m,1);\hat{\mathbb{Z}}_2)$ is CR-exact since π_1L_m is $\hat{\mathbb{Z}}_2$, $\mathbb{Z}/2$, or 0. Finally, since $L_m\simeq\tilde{L}\times K(\pi_1L_m,1)$, we conclude that $K^*_{CR}(L_m;\hat{\mathbb{Z}}_2)$ is CR-exact by Theorems 5.4, 5.9, and 5.10. □

To prove Theorem 6.2, we shall use:

6.6. **The functor $\bar{j}^*$.** For a spectrum or space X, we let $\bar{j}^*X$ be the graded profinite $\mathbb{Z}/2$-module

$$\bar{j}^*X \ = \ j^*X/\eta j^{*+1}X.$$

We may view $\bar{j}^*X$ as a module over the graded commutative $\mathbb{Z}/2$-algebra $\bar{j}^* = \bar{j}^*S$ with $\bar{j}^{-8k} = \mathbb{Z}/2 = \langle B_R^k \rangle$ for $k \in \mathbb{Z}$ and $\bar{j}^n = 0$ for $n \not\equiv 0 \mod 8$. For spaces X and Y, there is a product homomorphism

$$\mu\colon\ \bar{j}^*X \hat{\otimes}_{\bar{j}^*} \bar{j}^*Y \longrightarrow \bar{j}^*(X \times Y)$$

which is an isomorphism by Theorem 5.2 when $K_{CR}^*(X;\hat{\mathbb{Z}}_2)$ is CR-exact with $K^*(X;\hat{\mathbb{Z}}_2)$ torsion-free. By Theorem 6.1, this applies when $X = L_m$ where L_m denotes the infinite loop space $\underline{\widehat{KO}}_m$ or $\underline{\widehat{K}}_m$. Thus, $\bar{j}^*L_m$ is a 2-profinite Hopf algebra over $\bar{j}^*$, and there is a short exact sequence

$$0 \longrightarrow \bar{j}^{*+1}L_m \xrightarrow{\ \eta\ } j^*L_m \xrightarrow{\ q\ } \bar{j}^*L_m \longrightarrow 0$$

where q is the quotient map. Furthermore, we have:

Lemma 6.7. *For each m, there is an exact sequence*

$$0 \longrightarrow P\bar{j}^{*+1}L_m \xrightarrow{\ \eta\ } Pj^*L_m \xrightarrow{\ q\ } P\bar{j}^*L_m$$

Proof. This follows by comparing the preceding short exact sequence with the short exact sequence

$$0 \longrightarrow \bar{j}^{*+1}(L_m \times L_m) \xrightarrow{\ \eta\ } j^*(L_m \times L_m) \xrightarrow{\ q\ } \bar{j}^*(L_m \times L_m) \longrightarrow 0$$

whose terms are expanded using the Kunneth theorems for j^* and $\bar{j}^*$ (5.2 and 6.6). □

Lemma 6.8. *The following results hold for each m and n:*

(i) *if $P\bar{j}^{n+1}L_m = 0$ and $P\bar{j}^nL_m = 0$, then $Pj^nL_m = 0$;*
(ii) *if $P\bar{j}^{n+1}L_m = 0$, then $\eta = 0$ on $Pj^{n+1}L_m$ and $\eta\colon\ Pj^nL_m \to Pj^{n-1}L_m$ is monic;*
(iii) *if $\sigma\colon\ \bar{j}^*B^\infty L_m \cong P\bar{j}^*L_m$, then $\sigma\colon\ j^*B^\infty L_m \cong Pj^*L_m$.*

Proof. These results are obtained from Lemma 6.7 using the factorization $\eta = \eta q\colon\ Pj^*L_m \to Pj^{*-1}L_m$ and using the short exact sequence

$$0 \longrightarrow \bar{j}^{*+1}B^\infty L_m \xrightarrow{\ \eta\ } j^*B^\infty L_m \xrightarrow{\ q\ } \bar{j}^*B^\infty L_m \longrightarrow 0$$

□

To obtain Theorem 6.2, we may now use:

Theorem 6.9. *We have:*

(i) $P\bar{j}^i\widehat{\underline{KO}}_0 = 0$ *for* $i \not\equiv 0 \mod 8$;

(ii) $P\bar{j}^i\widehat{\underline{KO}}_1 = 0$ *for* $i \not\equiv 1 \mod 8$;

(iii) $P\bar{j}^i\widehat{\underline{KO}}_2 = 0$ *for* $i \not\equiv 2 \mod 8$ *and* $\sigma\colon \bar{j}^*B^\infty\widehat{\underline{KO}}_2 \cong P\bar{j}^*\widehat{\underline{KO}}_2$;

(iv) $P\bar{j}^i\widehat{\underline{KO}}_3 = 0$ *for* $i \not\equiv 3, -1 \mod 8$;

(v) $P\bar{j}^i\widehat{\underline{KO}}_{-4} = 0$ *for* $i \not\equiv 0, -4 \mod 8$;

(vi) $P\bar{j}^i\widehat{\underline{KO}}_{-3} = 0$ *for* $i \not\equiv 1, -3 \mod 8$;

(vii) $P\bar{j}^i\widehat{\underline{KO}}_{-2} = 0$ *for* $i \not\equiv -2 \mod 8$ *and* $\sigma\colon \bar{j}^*B^\infty\widehat{\underline{KO}}_{-2} = P\bar{j}^*\widehat{\underline{KO}}_{-2}$;

(viii) $P\bar{j}^i\widehat{\underline{KO}}_{-1} = 0$ *for* $i \not\equiv -1 \mod 8$;

(ix) $P\bar{j}^i\widehat{\underline{K}}_{-1} = 0$ *for* $i \not\equiv 1, 0 \mod 8$ *and* $\alpha^*\colon P\bar{j}^i\widehat{\underline{K}}_{-1} \cong P\bar{j}^iS^1$ *for* $i \equiv 1 \mod 8$ *where* $\alpha\colon S^1 \to \widehat{\underline{K}}_{-1}$ *is obtained from the inclusion* $U(1) \subset U$;

(x) $P\bar{j}^i\widehat{\underline{K}}_0 = 0$ *for* $i \not\equiv 0 \mod 8$.

6.10. **Proof of Theorem 6.2.** This theorem follows easily from Lemma 6.8 and Theorem 6.9. □

We devote the rest of this section to proving Theorem 6.9. We may approach $P\bar{j}^*L_m$ via Ph^*L_m using:

Lemma 6.11. *Let X be a connected loop space such that $K^*_{CR}(X;\hat{\mathbb{Z}}_2)$ is CR-exact and $K^*(X;\hat{\mathbb{Z}}_2)$ torsion-free. Then there is an isomorphism*

$$c + B^{-2}c\colon\ P\bar{j}^*X \oplus P\bar{j}^{*-4}X \cong Ph^*X.$$

Proof. This follows since there is a Hopf algebra isomorphism

$$c + B^{-2}c\colon\ \bar{j}^*X \oplus \bar{j}^{*-4}X \cong h^*X.$$

by 5.5 and Proposition 5.11. □

The components of Ph^*X are now given by $P\bar{j}^*X \cong Ph^*X \cap c\bar{j}^*X$ and $P\bar{j}^{*-4}X \cong Ph^*X \cap B^{-2}c\bar{j}^{*-4}X$. This lemma may be combined with:

Lemma 6.12. *We have $Ph^*K(\mathbb{Z}/2^r, 1) = 0$ for $r \geq 1$, $Ph^*K(\hat{\mathbb{Z}}_2, 2) = 0$, $Ph^1K(\hat{\mathbb{Z}}_2, 1) = \mathbb{Z}/2$, and $Ph^iK(\hat{\mathbb{Z}}_2, 1) = 0$ for $i \not\equiv 1 \mod 4$. Moreover, $h^0K(\hat{\mathbb{Z}}_2, 2) = \mathbb{Z}/2$ and $h^iK(\hat{\mathbb{Z}}_2, 2) = 0$ for $i \not\equiv 0 \mod 4$.*

Proof. By Theorem 6.3, for a finite 2-torsion group G, we have $K^1(K(G,1);\hat{\mathbb{Z}}_2) = 0$ and

$$K^0(K(G,1);\hat{\mathbb{Z}}_2)\ \cong\ J(G^\sharp)\ \cong\ \hat{\mathbb{Z}}_2(G^\sharp)$$

where $\hat{\mathbb{Z}}_2(G^\sharp)$ is the group ring on $G^\sharp$ with ψ^{-1} inverting elements of $G^\sharp$. Thus, $Ph^*K(G,1) = 0$ since $h^0K(G,1) \cong \mathbb{Z}/2(G/2)^\sharp$ and since $h^iK(G,1) = 0$ for $i \not\equiv 0 \mod 8$. Now the results for

$K(\mathbb{Z}/2^r,1)$ follow immediately; the results for $K(\hat{\mathbb{Z}}_2,2)$ follow by limit arguments since $h^*K(\hat{\mathbb{Z}}_2,2)\cong h^*K(\mathbb{Z}_{2^\infty},1)$; and the results for $K(\hat{\mathbb{Z}}_2,1)$ follow since $h^*K(\hat{\mathbb{Z}}_2,1)\cong h^*S^1$. □

6.13. **Proof of Theorem 6.9 except for parts (iii) and (vii).** Let L_m denote $\widehat{\underline{KO}}_m$ or $\widehat{\underline{K}}_m$, and let $\tilde{L}_m$ denote the 1-connected cover of L_m. For m odd, we have $Ph^*L_m\cong Ph^*\tilde{L}_m\oplus Ph^*K(\pi_1L_m,1)$ by Theorem 6.3, and we easily determine these summands using Lemma 6.12 and Theorem 6.4. We then extract the groups $P\bar{j}^*L_m$ from Ph^*L_m using Lemma 6.11 together with the operations θ of 3.5 (when $L_m=\widehat{\underline{KO}}_m$) and ϕ of 3.3 (when $L_m=\widehat{\underline{K}}_m$). This gives parts (ii), (iv), (vi), (viii), and (ix) of Theorem 6.9. Next, let $L_m=\widehat{\underline{KO}}_m$ with $m=0$ or $m=4$. Then $K^1(L_m;\hat{\mathbb{Z}}_2)=0$ and $K^0(L_m;\hat{\mathbb{Z}}_2)$ is torsion-free with $\psi^{-1}=1$ by Theorems 6.3 and 6.4. Hence,

$$\sigma\colon\ \hat{Q}K^{*+1}(\tilde{L}_{m+1};\mathbb{Z}/2)\ \cong\ PK^*(L_m;\mathbb{Z}/2)\ \cong\ Ph^*L_m$$

by [8, Theorem 10.2]. We may now determine Ph^*L_m by Theorems 6.3 and 6.4, and we then extract the groups $P\bar{j}^*L_m$ from Ph^*L_m using Lemma 6.11 together with the operator θ of 3.5. This gives parts (i) and (v) of Theorem 6.9. Finally, let $L_0=\widehat{\underline{K}}_0$ and let $\tilde{L}_0$ be the 2-connected cover of L_0. Using the splitting $L_0\simeq\tilde{L}_0\times K(\hat{\mathbb{Z}}_2,2)$ of [13], we obtain $Ph^*L_0\cong Ph^*\tilde{L}_0$ by Theorem 5.4 and Lemma 6.12. We may now determine Ph^*L_0 and extract $P\bar{j}^*L_0$ as above to give part (x) of Theorem 6.9. □

Before proving parts (iii) and (vii) of Theorem 6.9, we must make some algebraic preparations. For a module M over F_2 $(=\mathbb{Z}/2)$ with involution $t\colon\ M\cong M$, we let $h(M)$ be the homology of M with respect of the differential $d=1+t$. For such modules M and N, there is a pairing $h(M)\otimes h(N)\to h(M\otimes N)$ since $d(x\otimes y)=dx\otimes y+x\otimes dy+dx\otimes dy$, and we obtain $h(M)\otimes h(N)\cong h(M\otimes N)$ since such modules decompose as direct sums of components $F_2\oplus tF_2$ and F_2. Thus, if A is an abelian F_2-Hopf algebra, then $h(A)$ inherits this structure, where $t\colon\ A\cong A$ is the involution given by the antipode (see [17]). As in [8, Appendix A], we let $W_\infty=F_2[x_0,x_1,\ldots]$ be the infinite Witt Hopf algebra. Then:

Lemma 6.14. *$h(W_\infty)$ is an exterior Hopf algebra on the generator x_0.*

Proof. We give W_∞ the grading with $|x_i|=2^i$ for $i\geq 0$, and we consider the short exact sequence of graded abelian Hopf algebras $W_0\rightarrowtail W_\infty\twoheadrightarrow W'_\infty$ of [8, Appendix A], where $W_0=F_2[x_0]$ and where $W'_\infty=F_2[x'_0,x'_1,\ldots]$ is the infinite Witt Hopf algebra with $|x'_i|=2^{i+1}$ for $i\geq 0$. Then W_∞ is a free graded W_0-module on any graded set of elements in W_∞ which project to a graded basis for W'_∞, as shown by an argument using the faithfulness of $F_2\otimes_{W_0}-$. We may now obtain an F_2-basis for W_∞ consisting of 1 and x_0 followed by elements of the form $\{g_\alpha,tg_\alpha\}_\alpha$ in higher dimensions. This is accomplished by copying partial bases of W_∞ to W'_∞, then lifting these copies back to W_∞, and then multiplying these lifts by powers of x_0, with special arguments for the lowest lifts. □

Next, recall that the abelian F_2-Hopf algebras form an abelian category [18, Corollary 4.16], and let $B \rightarrowtail A \twoheadrightarrow C$ be a short exact sequence of such Hopf algebras. Suppose A is irreducible (see e.g. [7, Section 10.5]); suppose $B \cong W_\infty$; and suppose $C \cong (J\hat{\mathbb{Z}}_2/2)^\sharp$. Then:

Lemma 6.15. *The map $h(B) \to h(A)$ is an isomorphism.*

Proof. Following Radford [14], we let $B = B^{(0)} \subset B^{(1)} \subset B^{(2)} \subset \dots$ be the natural filtration of A by "wedges" of B, and we recall that there is a natural isomorphism of left B-modules $B \otimes C_s/C_{s-1} \cong B^{(s)}/B^{(s-1)}$ for each s, where $F_2 = C_0 \subset C_1 \subset C_2 \subset \dots$ is the coradical filtration of C. Since the above filtrations all respect the action of t, we obtain a spectral sequence $\{E_r^s A\}$ of left $h(B)$-modules converging to $h(A)$ with $E_1^s A = h(B^{(s)}/B^{(s-1)})$ for $s \geq 0$. We find that the spectral sequence $\{E_r^s C\}$ collapses to $h(C) = F_2$ with $C_s/C_{s-1} = F_2$ for $s \geq 0$ and with $d_1\colon E_1^{2m} C \cong E_1^{2m-1} C$ for $m \geq 1$. Using the natural chain isomorphism $F_2 \otimes_{h(B)} E_1^* A \cong E_1^* C$ together with our knowledge of $h(B)$ from Lemma 6.14, we deduce that $d_1\colon E_1^{2m} A \cong E_1^{2m-1} A$ for $m \geq 1$. Hence, the spectral sequence $\{E_r^s A\}$ collapses to give $h(A) \cong h(B)$. □

For a 2-profinite abelian group A with involution $t\colon A \cong A$, we let $h^\pm A = \{h^+ A, h^- A\}$ be the $\mathbb{Z}/2$-graded profinite F_2-module with $h^+ A = \ker(1-t)/\operatorname{im}(1+t)$ and with $h^- A = \ker(1+t)/\operatorname{im}(1-t)$. When A is torsion-free, we obtain a natural isomorphism $h^+ A \oplus h^- A \cong h(A/2)$ using the decomposition of A described in the proof of Theorem 5.4. Moreover, for such torsion-free groups A and B, we obtain a natural isomorphism $h^\pm A \hat{\otimes} h^\pm B \cong h^\pm(A \hat{\otimes} B)$ which agrees with the natural isomorphism $h(A/2) \hat{\otimes} h(B/2) \cong h(A/2 \hat{\otimes} B/2)$. We now consider the torsion-free 2-profinite Hopf algebra $W(f)$ of [7, Sections 7.6 and 7.9], where $f\colon M \to H$ is a map of torsion-free 2-profinite abelian groups. Using the involutions induced by $t = -1$ on M and H, we obtain a $\mathbb{Z}/2$-graded profinite F_2-Hopf algebra $h^\pm W(f)$ with a natural map $h^\pm M \to Ph^\pm W(f)$.

Lemma 6.16. *If $f\colon M \to H$ is a map of torsion-free 2-profinite abelian groups with $H = 0$ or $H = \hat{\mathbb{Z}}_2$, then the natural map $h^\pm M \to Ph^\pm W(f)$ is an isomorphism.*

Proof. Since M and $W(f)$ are torsion-free, it suffices to show that $h(M/2) \to Ph(W(f)/2)$ is an isomorphism. When $f\colon \hat{\mathbb{Z}}_2 \to 0$, this follows by [7, Section 7.6] and Lemma 6.14. When $f\colon \hat{\mathbb{Z}}_2 \to \hat{\mathbb{Z}}_2$, it follows by [7, Section 7.7] and Lemma 6.15. When $f\colon 0 \to \hat{\mathbb{Z}}_2$, it follows by retraction from the preceding case with $f = 0$. The general result now follows by limit arguments since the given map

$f\colon M\to 0$ or $f\colon M\to\hat{\mathbb{Z}}_2$ may be decomposed as (possibly infinite) products of maps $\hat{\mathbb{Z}}_2\to 0$, $\hat{\mathbb{Z}}_2\to\hat{\mathbb{Z}}_2$, and $0\to\hat{\mathbb{Z}}_2$. □

6.17. **Proof of parts (iii) and (vii) of Theorem 6.9.** For $L_m=\widehat{\underline{KO}}_m$ with $m=\pm 2$, we have $\sigma\colon h^*B_\infty L_m\cong Ph^*L_m$ by Theorem 6.3 and Lemma 6.16, and hence $\sigma\colon \bar{j}^*B^\infty L_m\cong P\bar{j}^*L_m$ by Lemma 6.11. The desired results now follow since $\bar{j}^*B^\infty L_m=0$ for $i\not\equiv m \mod 8$ by Theorem 6.4. □

7. Proofs of relations for unstable operations

In this section, we prove Propositions 3.7, 3.9, 3.10, 3.11, 3.12, and 3.15, giving all of the remaining relations for unstable operations in $\{K^*(X;\hat{\mathbb{Z}}_2), KO^*(X;\hat{\mathbb{Z}}_2)\}$. Before focusing on specific relations, we describe:

7.1. **The basic method of proof.** Suppose that we seek to prove a relation $\alpha=\beta$ for natural operations $\alpha,\beta\colon KO^m(X;\hat{\mathbb{Z}}_2)\to KO^n(X;\hat{\mathbb{Z}}_2)$ defined for spaces X. We may assume that X is connected since our operations α and β will act coordinatewise on $KO^*(\coprod_\alpha X_\alpha;\hat{\mathbb{Z}}_2)\cong\prod_\alpha KO^*(X_\alpha;\hat{\mathbb{Z}}_2)$ for a disjoint union of spaces $\coprod_\alpha X_\alpha$. We check:

(i) $\alpha=\beta$ when X is a point;
(ii) $\alpha-\beta$ is additive for all X;
(iii) $c\alpha=c\beta\colon KO^m(X;\hat{\mathbb{Z}}_2)\to K^n(X;\hat{\mathbb{Z}}_2)$ for all X.

By (i) and (ii), it now suffices to show $\alpha=\beta\colon \widetilde{KO}^m(X;\hat{\mathbb{Z}}_2)\to\widetilde{KO}^n(X;\hat{\mathbb{Z}}_2)$ for connected spaces X. Equivalently, using the universal example $X=\widehat{\underline{KO}}_m$, it suffices to show $\alpha\iota=\beta\iota$ for the canonical class $\iota\in\widetilde{KO}^m(\widehat{\underline{KO}}_m;\hat{\mathbb{Z}}_2)$. This stable class ι is primitive in the sense that $\mu^*\iota=p_1^*\iota+p_2^*\iota$ in $\widetilde{KO}^m(\widehat{\underline{KO}}_m\times\widehat{\underline{KO}}_m;\hat{\mathbb{Z}}_2)$ where p_1, p_2, and μ are the projection and addition maps from $\widehat{\underline{KO}}_m\times\widehat{\underline{KO}}_m$ to $\widehat{\underline{KO}}_m$. Hence, the element $\alpha\iota-\beta\iota\in\widetilde{KO}^n(\widehat{\underline{KO}}_m;\hat{\mathbb{Z}}_2)$ is also primitive since $\alpha-\beta$ is additive. Moreover, $\alpha\iota-\beta\iota$ is in the image of η since it is in the kernel of c, and thus $\alpha\iota-\beta\iota$ corresponds to a unique element $\epsilon\in Pj^{n+1}\widehat{\underline{KO}}_m$ with $\eta\epsilon=\alpha\iota-\beta\iota$ by Theorems 5.2 and 6.1. It now suffices to show that this error element $\epsilon\in Pj^{n+1}\widehat{\underline{KO}}_m$ is trivial, and this may usually be accomplished using our sparseness results on $Pj^{n+1}\widehat{\underline{KO}}_m$ from Theorem 6.2. In a few cases, we shall rely on other methods involving Real K-theory.

7.2. **Proof of Proposition 3.11.** For a space X and element $x\in KO^n(X;\hat{\mathbb{Z}}_2)$, we have $x^2=\eta\lambda^2x=\eta\theta x$ for $n\equiv -1,-5 \mod 8$ by Crabb [11, p.67], Minami [12, Proposition 2.2], and [10, Section 6.9]. We likewise have $x^2=0$ for $n\equiv 1,-3 \mod 8$ by Crabb [11, p.66] and [10, Section

6.9]. For an element $x \in KO^n(X;\hat{\mathbb{Z}}_2)$, we must show $x^2 = -rB^{-2}\theta Bcx$ when $n = 2$ and show $x^2 = -rB^2\theta B^{-1}cx$ when $n = -2$, using the operation θ on $K^0(X;\hat{\mathbb{Z}}_2)$. We verify 7.1(i)–(iii) using the relations $\theta(y+z) = \theta y + \theta z - yz$ and $\theta(y) + \theta(-y) = -y^2$ in $K^0(X;\hat{\mathbb{Z}}_2)$. Hence, there is an error element $\epsilon \in Pj^{2n+1}\underline{\widehat{KO}}_n$ with an isomorphism $\sigma\colon j^{2n+1}B^\infty\underline{\widehat{KO}}_n \cong Pj^{2n+1}\underline{\widehat{KO}}_n$ by Theorem 6.2. Since the v_1-stabilization homomorphism Φ of [10, Section 7.1] is left inverse to σ, it also gives an isomorphism $\Phi\colon Pj^{2n+1}\underline{\widehat{KO}}_n \cong j^{2n+1}B^\infty\underline{\widehat{KO}}_n$, and there is a monomorphism $\eta\colon j^{2n+1}B^\infty\underline{\widehat{KO}}_n \to KO^{2n}(B^\infty\underline{\widehat{KO}}_n;\hat{\mathbb{Z}}_2)$ by 5.1. We now deduce that $\epsilon = 0$ since $\eta\Phi\epsilon = 0$ by [10, Section 7.4], and the result follows. □

7.3. **Proof of Proposition 3.9.** For a (possibly nonconnected) pointed space X and element $x \in \widetilde{KO}^n(X;\hat{\mathbb{Z}}_2)$ with $n = 1$ or $n = -3$, we must show $\theta\eta x = \eta\theta x$ in $\widetilde{KO}^0(X;\hat{\mathbb{Z}}_2)$. When X is a finite complex, this follows as in [10, Section 6.9] since

$$\eta\colon \widetilde{KO}^1(X;\hat{\mathbb{Z}}_2) \oplus \widetilde{KO}^{-3}(X;\hat{\mathbb{Z}}_2) \longrightarrow \widetilde{KO}^0(X;\hat{\mathbb{Z}}_2) \oplus \widetilde{KO}^{-4}(X;\hat{\mathbb{Z}}_2)$$

may be obtained by tensoring the λ-homomorphism

$$i^*\colon \widetilde{KR}(\Sigma^{1,0}\wedge X) \oplus \widetilde{KH}(\Sigma^{1,0}\wedge X) \longrightarrow \widetilde{KR}(\Sigma^{0,0}\wedge X) \oplus \widetilde{KH}(\Sigma^{0,0}\wedge X)$$

with the λ-ring $\hat{\mathbb{Z}}_2$ where $i\colon \Sigma^{0,0}\wedge X \to \Sigma^{1,0}\wedge X$ is the standard inclusion. The general result now follows by an inverse limit argument. For a pointed space X and element $x \in \widetilde{KO}^n(X;\hat{\mathbb{Z}}_2)$ with $n = 0$ or $n = -4$, we must show $\theta\eta x = \eta\psi^2 x$. This follows from the relation $\theta\eta = \eta\theta$ on $\widetilde{KO}^{n+1}(\Sigma X;\hat{\mathbb{Z}}_2)$. Finally, for a space X and element $x \in KO^n(X;\hat{\mathbb{Z}}_2)$ with $n = \pm 2$, we must show $\theta\eta x = 0$. We verify 7.1(i)–(iii) using Propositions 3.6 and 3.13, and we obtain an error element $\epsilon \in Pj^n\underline{\widehat{KO}}_n$. As in 7.2, there is a monomorphism $\eta\Phi\colon Pj^n\underline{\widehat{KO}}_n \to KO^{n-1}(B^\infty\underline{\widehat{KO}}_n;\hat{\mathbb{Z}}_2)$, and we deduce that $\epsilon = 0$ since $\eta\Phi\epsilon = 0$. This shows $\theta\eta x = 0$. □

7.4. **Proof of Proposition 3.12.** For a space X and element $x \in KO^n(X;\hat{\mathbb{Z}}_2)$, we must show $\phi cx = x^2$ when $n = 0$ and show $\phi B^{-2}cx = B_R^{-1}x^2$ when $n = -4$. The case $n = 0$ follows since $\phi cx = r\theta cx - \theta rcx = 2\theta x - \theta(2x) = x^2$ by Propositions 3.13 and 3.14, and the case $n = -4$ follows similarly. For $x \in KO^n(X;\hat{\mathbb{Z}}_2)$, we must show $\phi cx = 0$ when $n = -1$ and show $\phi B^{-2}cx = 0$ when $n = -5$. We easily verify 7.1(i)–(iii) and obtain an error element $\epsilon \in Pj^1\underline{\widehat{KO}}_n$. The results now follow since $\epsilon = 0$ by Theorem 6.2. For $x \in KO^n(X;\hat{\mathbb{Z}}_2)$, we must show $\phi Bcx = \eta\theta x$ when $n = 1$ and show $\phi B^{-1}cx = \eta\theta x$ when $n = -3$. We easily verify 7.1(i)–(iii), and we also obtain $\eta\phi Bcx = \eta^2\theta x$ and $\eta\phi B^{-1}cx = \eta^2\theta x$ by Propositions 3.9, 3.13, and 3.14. Hence, there is an error element $\epsilon \in Pj^1\underline{\widehat{KO}}_n$ with $\eta\epsilon = 0$ in $Pj^0\underline{\widehat{KO}}_n$. The results now follow since $\epsilon = 0$ by Theorem 6.2. Finally, for $x \in KO^n(X;\hat{\mathbb{Z}}_2)$, we must show $\phi Bcx = r\theta Bcx$ when $n = 2$ and show $\phi B^{-1}cx = r\theta B^{-1}cx$ when $n = -2$. The case $n = 2$ follows since $\phi Bcx = r\theta Bcx - \theta rBcx = r\theta Bcx - \theta\eta^2 x = r\theta Bcx$ by Propositions 3.9 and 3.14, and the case $n = -2$ follows similarly. □

7.5. **Proof of Proposition 3.10.** Since $\xi = rB^2c$, the result now follows easily from Propositions 3.12 and 3.14. □

7.6. **Proof of Proposition 3.15.** For a space X and element $z \in K^0(X;\hat{\mathbb{Z}}_2)$, we must show $\theta\phi z = r(z^2(\theta z^*) + (\theta z)(\theta z^*))$ in $KO^0(X;\hat{\mathbb{Z}}_2)$. We verify the 7.1(i)–(iii) conditions using 3.3, 3.4, and 3.13, and we obtain an error element $\epsilon \in Pj^1\underline{\widehat{K}}_0$. The result now follows since $\epsilon = 0$ by Theorem 6.2. For $z \in K^{-1}(X;\hat{\mathbb{Z}}_2)$, we must show $\theta\phi z = \phi\theta z$. We verify the 7.1(i)–(iii) conditions using 3.3, 3.4, 3.6, 3.13, and 3.14, and we obtain an error element $\epsilon \in Pj^1\underline{\widehat{K}}_{-1}$. Using the map $\alpha\colon S^1 \to \underline{\widehat{K}}_{-1}$ of 6.2(ix), we obtain $\alpha^*\epsilon = 0$ in Pj^1S^1 since the relation $\theta\phi = \phi\theta$ holds on $\widetilde{K}^{-1}(S^1;\hat{\mathbb{Z}}_2)$. The result now follows since $\epsilon = 0$ by Theorem 6.2. □

7.7. **Proof of Proposition 3.7.** It suffices to prove the corresponding results for external cohomology products of spaces X and Y. Thus, for a pair of elements $x \in KO^m(X;\hat{\mathbb{Z}}_2)$ and $y \in KO^n(Y;\hat{\mathbb{Z}}_2)$, we must verify the given formula for $\theta(xy) \in KO^q(X \times Y;\hat{\mathbb{Z}}_2)$ with $q = m + n$ or $q = m + n + 4$ depending on the dimensions m and n. Using the method of 7.1 with two variables, we first check the given formula when X or Y is a point; we next check the additivity in each variable of the difference in the sides of the given formula; and we then check the complexification of the given formula. This is all straightforward using the known results in Section 3. We then obtain an error element $\epsilon \in j^{q+1}(\underline{\widehat{KO}}_m \times \underline{\widehat{KO}}_n)$ which is primitive with respect to the multiplication of $\underline{\widehat{KO}}_m$ and of $\underline{\widehat{KO}}_n$. Thus, ϵ belongs to the intersection of $Pj^*\underline{\widehat{KO}}_m\hat{\otimes}_{j^*}j^*\underline{\widehat{KO}}_n$ and $j^*\underline{\widehat{KO}}_m\hat{\otimes}_{j^*}Pj^*\underline{\widehat{KO}}_n$ in $j^*\underline{\widehat{KO}}_m\hat{\otimes}_{j^*}j^*\underline{\widehat{KO}}_n$. By Theorem 6.2, we may extend $Pj^*\underline{\widehat{KO}}_m$ and $Pj^*\underline{\widehat{KO}}_n$ to free j^*-submodules $\bar{P}j^*\underline{\widehat{KO}}_m \subset j^*\underline{\widehat{KO}}_m$ and $\bar{P}j^*\underline{\widehat{KO}}_n \subset j^*\underline{\widehat{KO}}_n$ on generators in dimensions congruent to m and n mod 4. Moreover, these j^*-submodules are direct summands since $j^*\underline{\widehat{KO}}_m/\bar{P}j^*\underline{\widehat{KO}}_m$ and $j^*\underline{\widehat{KO}}_n/\bar{P}j^*\underline{\widehat{KO}}_n$ are η-acyclic. Hence, the error element ϵ belongs to the free j^*-submodule $\bar{P}j^*\underline{\widehat{KO}}_m\hat{\otimes}_{j^*}\bar{P}j^*\underline{\widehat{KO}}_n \subset j^*\underline{\widehat{KO}}_m\hat{\otimes}_{j^*}j^*\underline{\widehat{KO}}_n$ on generators in dimensions congruent to $m + n$ mod 4. Since ϵ is of dimension congruent to $m + n + 1$ mod 4, this implies $\epsilon = 0$, and the result follows. □

7.8. **Proofs of the relations 3.16(i)–(iv).** The relation 3.16(iii) follows since Adams operations commute with exterior power operations in p-adic λ-rings. The remaining relations follow by verifying the 7.1(i)–(iii) conditions and showing the vanishing of the resulting error elements as in the preceding proofs. □

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