

# Groupoids, the Phragmen-Brouwer Property, and the Jordan Curve Theorem

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## Abstract

We publicise a proof of the Jordan Curve Theorem which relates it to the Phragmen-Brouwer Property, and whose proof uses the van Kampen theorem for the fundamental groupoid on a set of base points.

## 1 Introduction

This article extracts from [Bro88] a proof of the Jordan Curve Theorem based on the use of groupoids, the van Kampen Theorem for the fundamental groupoid on a set of base points, and the use of the Phragmen-Brouwer Property. In the process, we give two results on the Phragmen-Brouwer Property (Propositions 4.1, 4.3), which may not have been published other than in [Bro88]. There is a renewed interest in such classical results<sup>1</sup>, as shown in the article [Sie] which revisits proofs of the Schoenflies Theorem.

There are many books containing a further discussion of this area. For more on the Phragmen-Brouwer property, see [Why42] and [Wil49]. Wilder lists five other properties which he shows for a connected and locally connected metric space are each equivalent to the PBP. The proof we give of the Jordan Curve Theorem is adapted from [Mun75]. Because he does not have our van Kampen theorem for non-connected spaces, he is forced into rather special covering space arguments to prove his replacements for Corollary 3.3 and for Proposition 4.1.

The intention is to make these methods more widely available, since the above book has been out of print for at least ten years. A new edition is in preparation under the title ‘Topology and groupoids’ which better reflects the character of the book.

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<sup>1</sup>See also the web site on the Jordan Curve Theorem: <http://www.maths.ed.ac.uk/~aar/jordan/>

I mention in the same spirit that the results from [Bro88] on orbit spaces have been made available in [BH02]. A further example of the use of groupoid methods, this time in combinatorial group theory, is in [Bra04], which gives a new result combining the Kurosch theorem and a theorem of Higgins which generalises Grusko’s theorem.

Note that we use groupoids not to give a nice proof of theorems on the fundamental group of a space with base point, but because we maintain that theorems in this area are about *the fundamental groupoid on a set of base points*, where that set is chosen in a way appropriate to the geometry of the situation at hand. The set of objects of a groupoid gives a spatial component to group theory which allows for more powerful and more easily understood modelling of geometry, and hence for more computational power.

## 2 The groupoid van Kampen theorem

We assume as known the notion of the fundamental groupoid  $\beta_1 X; J$  of a topological space  $X$  on a set  $J$ : it consists of homotopy classes rel end points of paths in  $X$  joining points of  $J$  “  $X$ . We say the pair  $X, J$  is *connected* if  $J$  meets each path component of  $X$ . The following theorem was proved in [Bro67, 6.7.2].

**Theorem 2.1 (van Kampen Theorem)** *Let the space  $X$  be the union of open subsets  $U, V$  with intersection  $W$ , let  $J$  be a set and suppose the pairs  $U, J, V, J, W, J$  are connected. Then the pair  $X, J$  is connected and the following diagram of morphisms induced by inclusion is a pushout in the category of groupoids:*

$$\begin{array}{ccc} \beta_1 W; J & \longrightarrow & \beta_1 V; J \\ \downarrow & & \downarrow \\ \beta_1 U; J & \longrightarrow & \beta_1 X; J \end{array}$$

This has been generalised to arbitrary unions in [BR84].

## 3 Pushouts of groupoids

In order to apply Theorem 2.1, we need some combinatorial groupoid theory. This was set up in [Hig05], [Bro88], and here we just quote some first main facts, for example how to compute an object group  $H_x$  of a groupoid  $H = G/R$  given as the quotient of a groupoid  $G$  by a totally disconnected graph  $R = \{R_x\}_{x \in \text{Ob}G}$  of relations: of course  $G/R$  is defined by the obvious universal property, and has the same object set as  $G$ .

Recall from [Bro88, 8.3.3] that:

**Proposition 3.1** (a) *If  $G$  is a connected groupoid, and  $x \in \text{Ob}G$ , then there is a retraction  $r : G \rightarrow G_x$  obtained by choosing for each  $y \in \text{Ob}G$  an element  $\phi_y \in G_{x,y}$ , with  $\phi_x = 1_x$ .*  
 (b) *If further  $R = \{R_y\}_{y \in \text{Ob}G}$  is a family of subsets of the object groups  $G_y$  of  $G$ , then*

the object group  $G = R_x$  is isomorphic to the object group  $G_x$  factored by the relations  $r \circ \alpha$  for all  $\alpha \in R_y, y \in \text{Ob}G$ .

We assume as understood the notion of free groupoid on a (directed) graph. If  $G, H$  are groupoids then their free product  $G \Lambda H$  is given by the pushout of groupoids

$$\begin{array}{ccc} \text{Ob}G \sqcup \text{Ob}H & \xrightarrow{i} & H \\ j \downarrow & & \downarrow \\ G & \longrightarrow & G \Lambda H \end{array}$$

where  $\text{Ob}G \sqcup \text{Ob}H$  is regarded as the subgroupoid of identities of both  $G, H$  on this object set, and  $i, j$  are the inclusions. We assume, as may be proved from the results of [Bro88, Chapter 8]:

**Proposition 3.2** *If  $G, H$  are free groupoids, then so also is  $G \Lambda H$ .*

If  $J$  is a set, then by the category of groupoids over  $J$  we mean the category whose objects are groupoids with object set  $J$  and whose morphisms are morphisms of groupoids which are the identity on  $J$ .

**Proposition 3.3** *Suppose given a pushout of connected groupoids over  $J$*

$$\begin{array}{ccc} C & \xrightarrow{i} & A \\ j \downarrow & & \downarrow u \\ B & \xrightarrow{v} & G \end{array} \tag{1}$$

Let  $p$  be a chosen element of  $J$ . Let  $r : A \rightarrow A_p, s : B \rightarrow B_p$  be retractions obtained by choosing elements  $f_x \in A_p, x, f_x \in B_p, x$ , for all  $x \in J$ , with  $f_p = 1, f_p = 1$ . Let  $f_x = u f_x \Gamma_1 v f_x$  in  $G_p$ , and let  $F$  be the free group on the elements  $f_x, x \in J$ , with the relation  $f_p = 1$ . Then the object group  $G_p$  is isomorphic to the quotient of the free product group

$$A_p \Lambda B_p \Lambda F$$

by the relations

$$r f_x s j f_y \Gamma_1 f_y \Gamma_1 = 1 \tag{2}$$

for all  $x, y \in J$  and all  $f_l \in C_{x, y}$ .

**Proof** We first remark that the pushout (1) implies that the groupoid  $G$  may be presented as the quotient of the free product groupoid  $A \Lambda B$  by the relations  $r f_l j f_l \Gamma_1$  for all  $f_l \in C$ . The problem is to interpret this fact in terms of the object group at  $p$  of  $G$ .

To this end, let  $T, S$  be the tree subgroupoids of  $A, B$  respectively generated by the elements  $f_x, f_x, x \in J$ . The elements  $f_x, f_x, x \in J$ , define isomorphisms

$$\rho : A \rightarrow A_p \Lambda T, \quad \sigma : B \rightarrow B_p \Lambda S$$

where if  $g \in G_x, y$  then

$$g = f_y r g f_x^{-1}, \quad g = f_y s g f_x^{-1}.$$

So  $G$  is isomorphic to the quotient of the groupoid

$$H = A_p \wedge T \wedge B_p \wedge S$$

by the relations

$$i f_j f_i^{-1} = 1$$

for all  $f_i \in C$ . By Proposition 3.1, the object group  $G_p$  is isomorphic to the quotient of the group  $H_p$  by the relations

$$r' i f_j r' j f_i^{-1} = 1$$

for all  $f_i \in C$ .

Now if  $J^0 = J \cap f_p g$ , then  $T, S$  are free groupoids on the elements  $f_x, f_x^{-1}, x \in J^0$ , respectively. By Proposition 3.2, and as the reader may readily prove,  $T \wedge S$  is the free groupoid on all the elements  $f_x, f_x^{-1}, x \in J^0$ . It follows from [Bro88, 8.2.3] (and from Proposition 3.1), that  $T \wedge S_p$  is the free group on the elements  $r f_x, f_x^{-1} f_x^{-1}, x \in J^0$ . Let  $f_p^{-1} \in F$ . Since

$$r' i f_j r' i f_j^{-1}, \quad r' j f_i r' j f_i^{-1},$$

the result follows. □

**Remark 3.4** The above formula is given in essence in van Kampen's paper [Kam33], since he needed the case of non connected intersection for applications in algebraic geometry. However his proof is difficult to follow, and a modern proof of the case of connected intersection was given by Crowell in [Cro59]. □

There is a consequence of the above computation which we shall use in the next section in proving the Jordan Curve Theorem.

First, if  $F$  and  $H$  are groups, recall that we say that  $F$  is a *retract* of  $H$  if there are morphisms  $\alpha: F \rightarrow H, \beta: H \rightarrow F$  such that  $\beta \alpha = 1$ . This implies that  $F$  is isomorphic to a subgroup of  $H$ .

**Corollary 3.5** *Under the situation of Proposition 3.3, the free group  $F$  is a retract of  $G_p$ . Hence if  $J \cap \text{Ob} C$  has more than one element, then the group  $G_p$  is not trivial, and if  $J$  has more than two elements, then  $G_p$  is not abelian.*

**Proof** Let  $M = A_p \wedge B_p \wedge F$ , and let  $\alpha: F \rightarrow M$  be the inclusion. Let  $\beta: M \rightarrow F$  be the retraction which is trivial on  $A_p$  and  $B_p$  and is the identity on  $F$ . Let  $q: M \rightarrow G_p$  be the quotient morphism. Then it is clear that  $\beta$  preserves the relations (2), and so  $\beta$  defines uniquely a morphism  $\alpha: G_p \rightarrow F$  such that  $\alpha q = \beta$ . Let  $\alpha = q^{-1} \beta$ . Then  $\alpha \beta = 1$ . So  $F$  is a retract of  $G_p$ .

The concluding statements are clear. □

We use the last two statements of the Corollary in sections 4 and 5 respectively.

### 4 The Phragmen-Brouwer Property

A topological space  $X$  is said to *have the Phragmen-Brouwer Property* (here abbreviated to PBP) if  $X$  is connected and the following holds: if  $D$  and  $E$  are disjoint, closed subsets of  $X$ , and if  $a$  and  $b$  are points in  $X \setminus D \cap E$  which lie in the same component of  $X \setminus D$  and in the same component of  $X \setminus E$ , then  $a$  and  $b$  lie in the same component of  $X \setminus (D \cup E)$ . To express this more succinctly, we say a subset  $D$  of a space  $X$  separates the points  $a$  and  $b$  if  $a$  and  $b$  lie in distinct components of  $X \setminus D$ . Thus the PBP is that: if  $D$  and  $E$  are disjoint closed subsets of  $X$  and  $a, b$  are points of  $X$  not in  $D \cup E$  such that neither  $D$  nor  $E$  separate  $a$  and  $b$ , then  $D \cup E$  does not separate  $a$  and  $b$ .

A standard example of a space not having the PBP is the circle  $S^1$ , since we can take  $D = \{1\}$ ,  $E = \{-1\}$ ,  $a = i$ ,  $b = -i$ . This example is typical, as the next result shows. But first we remark that our criterion for the PBP will involve fundamental groups, that is will involve paths, and so we need to work with path-components rather than components. However, if  $X$  is locally path-connected, then components and path-components of open sets of  $X$  coincide, and so for these spaces we can replace in the PBP ‘component’ by ‘path-component’. This explains the assumption of locally path-connected in the results that follow.

**Proposition 4.1** *Let  $X$  be a path-connected and locally path-connected space whose fundamental group (at any point) does not have the integers  $\mathbb{Z}$  as a retract. Then  $X$  has the PBP.*

**Proof** Suppose  $X$  does not have the PBP. Then there are disjoint, closed subsets  $D$  and  $E$  of  $X$  and points  $a$  and  $b$  of  $X \setminus D \cap E$  such that  $D \cup E$  separates  $a$  and  $b$  but neither  $D$  nor  $E$  separates  $a$  and  $b$ . Let  $U \subset X \setminus D$ ,  $V \subset X \setminus E$ ,  $W \subset X \setminus (D \cup E)$ ,  $U \cup V = W$ . Let  $J$  be a subset of  $W$  such that  $a, b \in J$  and  $J$  meets each path-component of  $W$  in exactly one point. Since  $D$  and  $E$  do not separate  $a$  and  $b$ , there are elements  $f \in \pi_1 U_{a,b}$  and  $g \in \pi_1 V_{a,b}$ . Since  $X$  is path-connected, the pairs  $U, J, V, J, W, J$  are connected. By the van Kampen Theorem 2.1 the following diagram of morphisms induced by inclusions is a pushout of groupoids:

$$\begin{array}{ccc}
 \pi_1 WJ & \xrightarrow{i_1} & \pi_1 UJ \\
 \downarrow i_2 & & \downarrow u_1 \\
 \pi_1 VJ & \xrightarrow{u_2} & \pi_1 XJ.
 \end{array}$$

Since  $U$  and  $V$  are path-connected and  $J$  has more than one element, it follows from Corollary 3.5 that  $\pi_1 XJ$  has the integers  $\mathbb{Z}$  as a retract. □

As an immediate application we obtain:

**Proposition 4.2** *The following spaces have the PBP: the sphere  $S^n$  for  $n \geq 1$ ;  $S^2 \setminus \text{fag}$  for a  $\neq S^2$ ;  $S^n \setminus \Lambda$  if  $\Lambda$  is a finite set in  $S^n$  and  $n \geq 2$ . □*

In each of these cases the fundamental group is trivial.

An important step in our proof of the Jordan Curve Theorem is to show that if  $A$  is an arc in  $\mathbb{S}^2$ , that is a subspace of  $\mathbb{S}^2$  homeomorphic to the unit interval  $\mathbb{I}$ , then the complement of  $A$  is path-connected. This follows from the following more general result.

**Proposition 4.3** *Let  $X$  be a path-connected and locally path-connected Hausdorff space such that for each  $x$  in  $X$  the space  $X \setminus \{x\}$  has the PBP. Then any arc in  $X$  has path-connected complement.*

**Proof** Suppose  $A$  is an arc in  $X$  and  $X \setminus A$  is not path-connected. Let  $a$  and  $b$  lie in distinct path-components of  $X \setminus A$ .

By choosing a homeomorphism  $\mathbb{I} \rightarrow A$  we can speak unambiguously of the mid-point of  $A$  or of any subarc of  $A$ . Let  $x$  be the mid-point of  $A$ , so that  $A$  is the union of sub-arcs  $A^0$  and  $A^{00}$  with intersection  $\{x\}$ . Since  $X$  is Hausdorff, the compact sets  $A^0$  and  $A^{00}$  are closed in  $X$ . Hence  $A^0 \setminus \{x\}$  and  $A^{00} \setminus \{x\}$  are disjoint and closed in  $X \setminus \{x\}$ . Also  $A \setminus \{x\}$  separates  $a$  and  $b$  in  $X \setminus \{x\}$  and so one at least of  $A^0 \setminus \{x\}, A^{00} \setminus \{x\}$  separates  $a$  and  $b$  in  $X \setminus \{x\}$ . Write  $A_1$  for one of  $A^0 \setminus \{x\}, A^{00} \setminus \{x\}$  which does separate  $a$  and  $b$ . Then  $A_1$  is also an arc in  $X$ .

In this way we can find by repeated bisection a sequence  $A_i, i \geq 1$ , of sub-arcs of  $A$  such that for all  $i$  the points  $a$  and  $b$  lie in distinct path-components of  $X \setminus A_i$  and such that the intersection of the  $A_i$  for  $i \geq 1$  is a single point, say  $y$ , of  $X$ .

Now  $X \setminus \{y\}$  is path-connected, by definition of the PBP. Hence there is a path  $\gamma$  joining  $a$  to  $b$  in  $X \setminus \{y\}$ . But  $\gamma$  has compact image and hence lies in some  $X \setminus A_i$ . This is a contradiction.  $\square$

**Corollary 4.4** *The complement of any arc in  $\mathbb{S}^n$  is path-connected.*  $\square$

In this theorem the case  $n = 0$  is trivial, while the case  $n = 1$  needs a special argument that the complement of any arc in  $\mathbb{S}^1$  is an open arc. The case  $n \geq 2$  follows from the above results.

## 5 The Jordan Separation and Curve Theorems

We now prove one step along the way to the full Jordan Curve Theorem.

**Theorem 5.1 (The Jordan Separation Theorem)** *The complement of a simple closed curve in  $\mathbb{S}^2$  is not connected.*

**Proof** Let  $C$  be a simple closed curve in  $\mathbb{S}^2$ . Since  $C$  is compact and  $\mathbb{S}^2$  is Hausdorff,  $C$  is closed,  $\mathbb{S}^2 \setminus C$  is open, and so path-connectedness of  $\mathbb{S}^2 \setminus C$  is equivalent to connectedness.

Write  $C = A \cup B$  where  $A$  and  $B$  are arcs in  $C$  meeting only at  $a$  and  $b$  say. Let  $U = \mathbb{S}^2 \setminus A, V = \mathbb{S}^2 \setminus B, W = U \cap V, X = U \cup V$ . Then  $W = \mathbb{S}^2 \setminus C$  and  $X = \mathbb{S}^2 \setminus \{a, b\}$ . Also  $X$  is path-connected, and, by Corollary 4.4, so also are  $U$  and  $V$ .

Let  $x \in W$ . Suppose that  $W$  is path-connected. By the van Kampen Theorem ??, the following diagram of morphisms induced by inclusion is a pushout of groups:

$$\begin{array}{ccc}
 \beta_1 W, x & \longrightarrow & \beta_1 U, x \\
 \downarrow & & \downarrow i_\Lambda \\
 \beta_1 V, x & \xrightarrow{j_\Lambda} & \beta_1 X, x.
 \end{array}$$

Now  $\beta_1 X, x$  is isomorphic to the group  $\mathbb{Z}$  of integers. We derive a contradiction by proving that the morphisms  $i_\Lambda$  and  $j_\Lambda$  are trivial. We give the proof for  $i_\Lambda$ , as that for  $j_\Lambda$  is similar.

Let  $f: S^1 \rightarrow U$  be a map and let  $g: S^1 \rightarrow X$ . Choose a homeomorphism  $h: S^2 \rightarrow \mathbb{R}^2$  which takes  $b$  to 0 and such that  $hg$  maps  $S^1$  into  $\mathbb{R}^2 \setminus \{0\}$ . Since  $hgS^1$  is compact, there is only one unbounded component of  $\mathbb{R}^2 \setminus hgS^1$ , and we may assume this contains 0. Again since  $hgS^1$  is compact, there is an  $r > 0$  such that  $hgS^1 \cap B(0, r) = \emptyset$ . Choose a point  $y$  in  $\mathbb{R}^2$  such that  $|y| < r$ . Then there is a path  $\gamma$ , say, joining 0 to  $y$  in  $\mathbb{R}^2 \setminus hgS^1$ , since 0 lies in the unbounded component of this set.

Define  $G: S^1 \times I \rightarrow \mathbb{R}^2$  by

$$G(z, t) = \begin{cases} hg(z) \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2\gamma(2t)hg(z) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then  $G$  is well-defined. Also  $G$  never takes the value 0 (this explains the choices of  $\gamma$  and  $y$ ). So  $G$  gives a homotopy in  $\mathbb{R}^2 \setminus \{0\}$  from  $hg$  to the constant map at  $\gamma y$ . So  $hg$  is inessential and hence  $g$  is inessential. This completes the proof that  $i_\Lambda$  is trivial.  $\square$

As we shall see, the Jordan Separation Theorem is used in the proof of the Jordan Curve Theorem.

**Theorem 5.2 (Jordan Curve Theorem)** *If  $C$  is a simple closed curve in  $S^2$ , then the complement of  $C$  has exactly two components, each with  $C$  as boundary.*

**Proof** As in the proof of Theorem 5.1, write  $C$  as the union of two arcs  $A$  and  $B$  meeting only at  $a$  and  $b$  say, and let  $U \subset S^2 \setminus A$ ,  $V \subset S^2 \setminus B$ . Then  $U$  and  $V$  are path-connected and  $X = U \cup V \subset S^2 \setminus \{a, b\}$  has fundamental group isomorphic to  $\mathbb{Z}$ . Also  $W = U \cup V \subset S^2 \setminus C$  has at least two path-components, by the Jordan Separation Theorem 5.1.

If  $W$  has more than two path-components, then the fundamental group  $G$  of  $X$  contains a copy of the free group on two generators, by Corollary ??, and so  $G$  is non-abelian. This is a contradiction, since  $G \cong \mathbb{Z}$ . So  $W$  has exactly two path-components  $P$  and  $Q$ , say, and this proves the first part of Theorem 5.2.

Since  $C$  is closed in  $S^2$  and  $S^2$  is locally path-connected, the sets  $P$  and  $Q$  are open in  $S^2$ . It follows that if  $x \in \overline{P} \cap P$  then  $x \notin Q$ , and hence  $\overline{P} \cap P$  is contained in  $C$ . So also is  $\overline{Q} \cap Q$ , for similar reasons. We prove these sets are equal to  $C$ .

Let  $x \in C$  and let  $N$  be a neighbourhood of  $x$  in  $S^2$ . We prove  $N$  meets  $\overline{P} \cap P$ . Since  $\overline{P} \cap P$  is closed and  $N$  is arbitrary, this proves that  $x \in \overline{P} \cap P$ .

Write  $C$  in a possibly new way as a union of two arcs  $D$  and  $E$  intersecting in precisely two points and such that  $x$  is in the interior with respect to  $D$  of  $D$ . Choose points  $p$  in  $P$  and  $q$  in  $Q$ . Since  $\mathbb{S}^2 \setminus E$  is path-connected, there is a path  $\gamma$  joining  $p$  to  $q$  in  $\mathbb{S}^2 \setminus E$ . Then  $\gamma$  must meet  $D$ , since  $p$  and  $q$  lie in distinct path-components of  $\mathbb{S}^2 \setminus E$ . In fact if  $s = \sup\{t \in [0, t] \mid \gamma|_{[0, t]} \cap P = \emptyset\}$ , then  $s \in \overline{P} \cap P$ . It follows that  $N$  meets  $\overline{P} \cap P$ .

So  $\overline{P} \cap P \subset C$  and similarly  $\overline{Q} \cap Q \subset C$ . □

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