FINITE GENERATION OF TATE COHOMOLOGY

JON F. CARLSON, SUNIL K. CHEBOLU, AND JÁN MINÁČ

Dedicated to Professor Luchezar Avramov on his sixtieth birthday.

Abstract. Let G be a finite group and let k be a field of characteristic p. If M is a finitely generated indecomposable non-projective kG-module, we conjecture that the Tate cohomology $\hat{\mathrm{H}}^*(G, M)$ of G with coefficients in M is finitely generated over the Tate cohomology ring $\hat{\mathrm{H}}^*(G, k)$ if and only if the support variety $V_G(M)$ of M is equal to the entire maximal ideal spectrum $V_G(k)$. We prove various results all of which support this conjecture. It is also shown that all finitely generated kG-modules over a group G have finitely generated Tate cohomology if and only if G has periodic cohomology.

Contents

1. Introduction

Tate cohomology was introduced by Tate in his celebrated paper [14] where he proved the main theorem of class field theory in a remarkably simple way using Tate cohomology. After Cartan and Eilenberg's treatment [9] of Tate cohomology and Swan's basic results on free group actions on spheres [13], Tate cohomology became one of the basic tools in current mathematics. However, one of the most fundamental questions – when is the Tate cohomology of a kG-module module finitely generated over the Tate cohomology ring of G? – remains unanswered. This is the question we address in this paper.

Let G be a finite group and let k be a field of characteristic p. If M is a finitely generated kG-module, then a well-known result in group cohomology due to Evens and

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Venkov tells that $\mathrm{H}^*(G, M)$ is finitely generated as a graded module over $\mathrm{H}^*(G, k)$. Our goal is to investigate a similar finite-generation result for Tate cohomology. More precisely, if M is a finitely generated kG-module, then we want to known whether the Tate cohomology $\hat{\mathrm{H}}^*(G, M)$ of G with coefficients in M is finitely generated as a graded module over the Tate cohomology ring $\hat{\mathrm{H}}^*(G, k)$. We call this the finite generation problem for Tate cohomology. In section 2 we explain how we had naturally arrived at this problem.

Our main results split into two categories: modules for which we have an affirmative answer to the finite generation problem, and modules for which the Tate cohomology is not finitely generated. It turns out that Tate cohomology $\hat{\mathrm{H}}^*(G, M)$ is seldom finite generated – a striking contrast with the Evens-Venkov result for group cohomology mentioned earlier. However, there are still interesting non-trivial examples where we have finite generation. So both categories are important. We now summarize our results.

Our first result deals with finite groups G which have the property that the product of any two elements in negative degrees in the Tate cohomology ring is zero. We assure the reader that there are plenty of groups which satisfy this condition. For instance, this holds whenever the p-rank of the center of a Sylow p-subgroup of G is at least two. Now let ζ be a regular element in the cohomology ring $\mathrm{H}^*(G, k)$ of such a finite group G. Then our first result says that the Tate cohomology $\hat{\mathrm{H}}^*(G, L_\zeta)$ of the module L_ζ (kernel of the cocycle $\Omega^{|\zeta|}k \xrightarrow{\zeta} k$) is not finitely generated. More generally, we show in Theorem 3.13 that the same holds for any finitely generated non-projective module M whose support variety is contained in $V_G\langle\zeta\rangle$ – the hyper surface determined by ζ.

In our second result we consider finite groups G with p-rank (the rank of the maximal p-elementary abelian subgroup) at least two. Let M be a finitely generated periodic kG-module. That is, M satisfies $\Omega^t M \cong M$ for some t. (Such modules always exist.) We show in Proposition 4.1 that the Tate cohomology of $\mathrm{End}_k M$ is not finitely generated as a $\hat{\mathrm{H}}^*(G, k)$-module. Now recall that groups with p-rank one are precisely those groups which have periodic cohomology, and in this case all finitely generated kG-modules have finitely generated Tate cohomology. Thus we get a characterization of groups with periodic cohomology as groups G which have the property that $\hat{\mathrm{H}}^*(G, L)$ is finitely generated for all finitely generated kG-modules L. It is interesting to note that the groups which satisfy these properties for all p are are also characterized by the property that they admit a free action on some finite complex that has the homotopy type of a sphere; this is a result of Swan [13].

The above results suggest that finite generation is a rare phenomena. Nevertheless, there are surprising and interesting cases where we have an affirmative answer to the finite generation problem. We show that for any finite group G, the Tate cohomology of the middle term of the almost split sequence ending in k has finitely generated Tate cohomology. In fact, our result is more general. Consider any element ζ in the Tate cohomology ring with the property that multiplication by ζ has finite dimensional image in the Tate cohomology ring. Then we show in Theorem 6.1 that the Tate cohomology of the module determined by ζ has finitely generated Tate cohomology.

Motivated by all the aforementioned results, we make the following conjecture:

Conjecture 1.1. *Let G be a finite group and let M be an indecomposable non-projective finitely generated kG-module. Then $\hat{\mathrm{H}}^*(G, M)$ is finitely generated over $\hat{\mathrm{H}}^*(G, k)$ if and* *only if the support variety* $V_G(M)$ *of* M *is equal to entire maximal ideal spectrum* $V_G(k)$ *of the group cohomology ring.*

All the above results affirm this conjecture. For instance, in the result on periodic modules which failed to have finitely generated Tate cohomology, note that dimension of $V_G(M) = 1$ (since M is periodic), and that of $V_G(k)$ is at least 2 (since p-rank of G is at least 2). In particular, $V_G(M) \subsetneq V_G(k)$, and therefore our result is in support of this conjecture. On the other hand, modules which are middle terms of almost split sequences ending in k satisfy the condition $V_G(M) = V_G(k)$ [1, Proposition 5.2], and for these we have finite generation of Tate cohomology, as predicted. One consequence of this conjecture is that if M is a finitely generated non-projective indecomposable kG-module, whose complexity is less than the p-rank of G, then the Tate cohomology of M is not finitely generated.

When G is p-group, we know by the thick subcategory theorem [5] that $V_G(M) = V_G(k)$ if and only if M can build k by iterated cofiberings and retractions. Thus we have a purely homotopical avatar of our conjecture for p-groups: For M as above, $\hat{\mathrm{H}}^*(G, M)$ is finitely generated over $\hat{\mathrm{H}}^*(G, k)$ if and only if M can build k by iterated cofiberings and retractions.

The paper is organized as follows. We begin in Section 2 by explaining how we had naturally arrived at the problem of finite generation of Tate cohomology. Sections 3 and 4 deal with modules whose Tate cohomology is not finitely generated and contain proofs of the first two main results mentioned above. In Section 5 we give an example of a module which does not have the BFGS property but whose Tate cohomology, nevertheless, is not finitely generated. In Section 6 we prove the affirmative result mentioned above which provides a good source of modules whose Tate cohomology is finitely generated.

Throughout the paper G will denote a non-trivial finite group, and all kG-modules are assumed to be finitely generated. We use standard facts and notation of the stable module category of kG [7] and of support varieties [2, 8].

2. Universal ghosts in stmod(kG)

Here we explain briefly how we had naturally arrived at the problem of finite generation of Tate cohomology. More details can be found in [10, 11].

The following natural question was raised in [11]: when does the Tate cohomology functor detect trivial maps in the stable module category stmod(kG) of finitely generated kG-modules? A map $\phi \colon M \to N$ between finitely generated kG-modules is said to be a *ghost* if the induced map in Tate cohomology groups

$$\underline{\mathrm{Hom}}_{kG}(\Omega^i k, M) \longrightarrow \underline{\mathrm{Hom}}_{kG}(\Omega^i k, N)$$

is zero for each integer i. With this definition, the above question is equivalent to asking when every ghost map in stmod(kG) trivial. To address this question it turns out to be very convenient to have a universal ghost out of any finitely generated kG-module M in stmod(kG), i.e., a ghost map $\phi \colon M \to N$ in stmod(kG) such that every ghost out of M factors through N via ϕ. The point is that if the universal ghost vanishes then all ghosts vanish.

So our problem now boils down to finding a universal ghost out of M (if it exists) in stmod(kG). We now show that if the Tate cohomology $\hat{\mathrm{H}}^*(G, M)$ is finitely generated as a graded module over $\hat{\mathrm{H}}^*(G,k)$, then a universal ghost out of M can be constructed in the category $\mathrm{stmod}(kG)$. This is done as follows. Let $\{v_j\}$ be a finite set of homogeneous generators for $\hat{\mathrm{H}}^*(G,M)$ as a $\hat{\mathrm{H}}^*(G,k)$-module. These generators can be assembled into a map

$$\bigoplus_j \Omega^{|v_j|}\, k \longrightarrow M$$

in $\mathrm{stmod}(kG)$. This map can then be completed to a triangle

$$\bigoplus_j \Omega^{|v_j|}\, k \longrightarrow M \xrightarrow{\Psi_M} F_M. \tag{2.1}$$

By construction, it is clear that the first map in the above triangle is surjective on the functors $\underline{\mathrm{Hom}}_{kG}(\Omega^l k, -)$ for each l. Therefore, the second map Ψ_M must be a ghost. Thus we have the following proposition.

Proposition 2.1. *Suppose M is a finitely generated kG-module such that $\hat{\mathrm{H}}^*(G,M)$ is finitely generated as a graded module over $\hat{\mathrm{H}}^*(G,k)$. Then the map $\Psi_M \colon M \to F_M$ in the above triangle is a universal ghost out of M.*

Proof. Universality of Ψ_M is easy to see. For the last statement, since the sum is finite, $\oplus_j \Omega^{|v_j|}\, k$ is compact. And since the category of compact objects forms a triangulated subcategory of $\mathrm{StMod}(kG)$, F_M is compact as well. □

Thus, in view of this proposition, we were strongly motivated for a serious investigation of modules M for which the Tate cohomology $\hat{\mathrm{H}}^*(G,M)$ is finitely generated over $\hat{\mathrm{H}}^*(G,k)$.

3. MODULES WITH BFGS PROPERTY

The material will draw heavily on the methods introduced in the paper [3]. We wish to explore the following condition on cohomology which implies the absence of finite generation of Tate cohomology.

Definition 3.1. A graded module $T = \oplus_{n\in\mathbb{Z}} T^n$ over $\hat{\mathrm{H}}^*(G,k)$ is said to have the BFGS property (bounded finitely generated submodule) if for any m there is a number $N = N(m)$ such that the submodule of T generated by $\oplus_{n>m} T^n$ is contained in $\oplus_{n>N} T^n$.

In general, the number $N = N(m)$ will be negative even if m is positive. Of course, $N(m)$ is a function of the module T as well as the integer m. If L is a kG-module, we sometimes say that L has the BFGS property when $\hat{\mathrm{H}}^*(G,L)$ has the BFGS property. We state the following for emphasis even though its proof is obvious.

Lemma 3.2. *If a graded module $T = \oplus_{n\in\mathbb{Z}} T^n$ over $\hat{\mathrm{H}}^*(G,k)$ has the BFGS property, and if $T^n \neq \{0\}$ for arbitrarily small (meaning negative) values of n, then T is not a finitely generated module over $\hat{\mathrm{H}}^*(G,k)$.*

Proof. Any finite subset of T is contained in $\oplus_{n>m} T^n$ for some m, and by the hypothesis and the property BFGS, can not generate the entire module T. □

The graded modules over the Tate cohomology ring that we are interested in are of course $\hat{\mathrm{H}}^*(G,L)$, where L is a kG-module. In this case, we have the following corollary which will be used frequently. Recall that the thick subcategory generated by k is the smallest full subcategory of $\operatorname{stmod}(kG)$ that contains k and closed under exact triangles and direct summands.

Corollary 3.3. *Let L be a finitely generated non-projective kG-module in the thick subcategory generated by k. If $\hat{\mathrm{H}}^*(G,L)$ has the BFGS property, then it is not finitely generated over $\hat{\mathrm{H}}^*(G,K)$.*

Proof. A standard thick subcategory argument shows that any non-projective module L in the thick subcategory generated by k has non-vanishing Tate cohomology. It is shown in [6, Thm. 1.1] that $\hat{\mathrm{H}}^i(G,L) \neq 0$ for some i implies that it is also non-zero for infinitely many negative values of i. So the hypothesis of the above lemma is satisfied and that completes the proof. ☐

We begin with a general result, that we will use as a basic tool in a lot of what follows.

Lemma 3.4. *Suppose that we have an exact sequence*

$$\mathcal{E}: \quad 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of kG-modules where $\mathcal{E}$ represents an element ζ in $\operatorname{Ext}^1_{kG}(N,L)$. Cup product with the element ζ induces a homomorphism $\zeta : \hat{\mathrm{H}}^*(G,N) \longrightarrow \hat{\mathrm{H}}^*(G,L)$. Let $\mathcal{K}^*$ be the kernel of the multiplication by ζ, and let $\mathcal{J}^*$ be the cokernel of multiplication by ζ. Then we have an exact sequence of $\hat{\mathrm{H}}^*(G,k)$-modules

$$0 \longrightarrow \mathcal{J}^* \longrightarrow \hat{\mathrm{H}}^*(G,M) \longrightarrow \mathcal{K}^* \longrightarrow 0.$$

Proof. The proof is a straightforward consequence of the naturality of the long exact sequence on Tate cohomology. That is, we have a sequence

$$\cdots \xrightarrow{\zeta} \hat{\mathrm{H}}^n(G,L) \longrightarrow \hat{\mathrm{H}}^n(G,M) \longrightarrow \hat{\mathrm{H}}^n(G,N) \xrightarrow{\zeta} \hat{\mathrm{H}}^{n+1}(G,L) \longrightarrow \cdots$$

and we note that the collection of the maps ζ in the long exact sequence is a map of degree 1 of $\hat{\mathrm{H}}^*(G,k)$-modules

$$\zeta : \quad \hat{\mathrm{H}}^*(G,N) \longrightarrow \hat{\mathrm{H}}^*(G,L)[1].$$

(The symbol $\mathcal{X}[i]$ is meant to indicate the shift of the $\hat{\mathrm{H}}^*(G,k)$-module $\mathcal{X}$ by i degrees.) ☐

Proposition 3.5. *Suppose that*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0$$

is an exact sequence of graded $\hat{\mathrm{H}}^*(G,k)$-modules. Then $\mathcal{M}$ has the BFGS property if and only if both $\mathcal{L}$ and $\mathcal{N}$ have the BFGS property. Moreover, the full subcategory $\mathcal{C}$ consisting of all finitely generated kG-modules M such that $\hat{\mathrm{H}}^*(G,M)$ has the BFGS property is a thick subcategory of $\operatorname{stmod}(kG)$.

Proof. The first statement is fairly obvious. Now we argue that the subcategory $\mathcal{C}$ is thick. It is easy to see that $\mathcal{C}$ is closed under direct summands. To show that $\mathcal{C}$ is closed under exact triangles, consider an exact triangle

$$L \longrightarrow M \longrightarrow N \xrightarrow{\delta} \Omega^{-1}L$$

in the stable category such that for two of the objects, say L and N, the cohomology has the BFGS property. Then notice that both the kernel and cokernel of

$$\delta^* : \hat{\mathrm{H}}^*(G, N) \longrightarrow \hat{\mathrm{H}}^*(G, L)[1]$$

have the BFGS property. The first statement and Lemma 3.4 will tell us that the cohomology of M has the BFGS property. ☐

The thick subcategory property has several interesting consequences. One example is the following.

Corollary 3.6. *Let G be a finite p-group. Suppose that M is a finitely generated kG-module which has the BFGS property. Then $M \otimes L$ has the BFGS property for every finitely generated kG-module L. Moreover, M has the BFGS property if and only if M^* and $\mathrm{Hom}_k(M, M) \cong M \otimes M^*$ have the BFGS property.*

Proof. It is well-known [5, Corollary 3.5] that over a p-group every thick subcategory is a thick tensor ideal subcategory. That is, the thick subcategories of $\mathrm{stmod}(kG)$ are determined entirely by support varieties, with the consequence that $M \otimes L$ is always in the thick subcategory generated by M. The last statement follows from the fact that M, M^* and $M \otimes M^*$ all generate the same thick subcategory of $\mathrm{stmod}(kG)$. ☐

Now suppose that $\zeta \in \mathrm{H}^d(G, k)$ for $d > 0$ and that $\zeta \neq 0$. We have an exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^d(k) \xrightarrow{\zeta} k \longrightarrow 0$$

where ζ in the sequence is a homomorphism (uniquely) representing the cohomology element ζ. In the corresponding long exact sequence on Tate cohomology

$$\cdots \xrightarrow{\zeta} \hat{\mathrm{H}}^{n-1}(G, k) \longrightarrow \hat{\mathrm{H}}^n(G, L_\zeta) \longrightarrow \hat{\mathrm{H}}^n(G, \Omega^d(k)) \xrightarrow{\zeta} \hat{\mathrm{H}}^n(G, k) \longrightarrow \cdots,$$

the connecting homomorphism is multiplication by ζ. That is, it is degree n map:

$$\zeta : \hat{\mathrm{H}}^*(G, k)[-n] \quad \longrightarrow \quad \hat{\mathrm{H}}^*(G, k).$$

Here we are using the fact that $\hat{\mathrm{H}}^s(G, \Omega^n k) \cong \hat{\mathrm{H}}^{s-n}(G, k)$.

As a result, we have, as in Lemma 3.4, an exact sequence of $\hat{\mathrm{H}}^*(G, k)$-modules

$$0 \longrightarrow \mathcal{J}^*[-1] \longrightarrow \hat{\mathrm{H}}^*(G, L_\zeta) \longrightarrow \mathcal{K}^*[-d] \longrightarrow 0.$$

where $\mathcal{J}^*$ and $\mathcal{K}^*$ are the cokernel and kernel of multiplication by ζ, respectively.

Lemma 3.7. *Suppose that $\zeta \in \mathrm{H}^d(G, k)$ is a regular element on $\mathrm{H}^*(G, k)$. Then*

(1) $\mathcal{K}^m = \{0\}$ *for all* $m \geq 0$, *and*
(2) $\mathcal{J}^m = \{0\}$ *for all* $m < 0$.

Proof. The first statement is the definition that ζ is a regular element in $\mathrm{H}^*(G, k)$. The second statement is a consequence of Lemma 3.5 of [3]. For the sake of completeness we include a proof. For $t > 0$, let

$$\langle\ ,\ \rangle : \hat{\mathrm{H}}^{-t-1}(G, k) \otimes \hat{\mathrm{H}}^t(G, k) \quad \longrightarrow \quad \hat{\mathrm{H}}^{-1}(G, k) \ \cong\ k$$

be the Tate duality. Let $\zeta_1, \dots, \zeta_s$ be a k-basis for $\hat{\mathrm{H}}^{-m-1}(G,k)$. Then because multiplication by ζ,

$$\hat{\mathrm{H}}^{-m-1}(G,k) \longrightarrow \hat{\mathrm{H}}^{-m+d-1}(G,k)$$

is a monomorphism (since $-m-1 \geq 0$), the elements $\zeta\zeta_1, \dots, \zeta\zeta_s$ are linearly independent. So there must exist elements $\gamma_1, \dots, \gamma_s$ in $\hat{\mathrm{H}}^{m-d}(G,k)$ such that for all i and j, we have

$$\langle \gamma_i, \zeta\zeta_j \rangle \;=\; \langle \gamma_i\zeta, \zeta_j \rangle \;=\; \delta_{i,j}$$

where by $\delta_{i,j}$ we mean the usual Kronecker delta. A consequence of this is that the elements $\gamma_1\zeta, \dots, \gamma_s\zeta$ must be linearly independent and hence must form a basis for $\hat{\mathrm{H}}^{m}(G,k)$. This proves the lemma. □

We remind the reader that if there exists a regular sequence of length two, then any product of two homogeneous elements in negative degrees is zero in Tate cohomology. From this it follows by a theorem of Duflot that if the center of a Sylow p-subgroup of G has rank two or more, then the product of any two homogeneous elements in negative degrees of the Tate cohomology ring $\hat{\mathrm{H}}^{*}(G,k)$ is necessarily zero [3]. The same conclusion is valid if the p-rank of G is two or more and $H^*(G,k)$ is Cohen-Macaulay. Using the above result, we prove the following.

Proposition 3.8. *Suppose that $\hat{\mathrm{H}}^{*}(G,k)$ has the property that the product of any two elements in negative degrees is zero. If $\zeta \in \mathrm{H}^d(G,k)$ $(d > 0)$ is a regular element for $\mathrm{H}^*(G,k)$, then $\hat{\mathrm{H}}^{*}(G,L_\zeta)$ has the BFGS property. In particular it is not finitely generated as a module over $\hat{\mathrm{H}}^{*}(G,k)$.*

Proof. As before, let $\mathcal{K}^*$ be the kernel of the multiplication by ζ on $\hat{\mathrm{H}}^{*}(G,k)$ and let $\mathcal{J}^*$ be the cokernel of multiplication by ζ. Note that by Lemma 3.7, $\mathcal{K}^*$ and $\mathcal{J}^*$ both have the BFGS property. That is, $\mathcal{J}^*$ has this property because it has no non-zero elements in negative degrees. On the other hand $\mathcal{K}^*$ has the property because it has elements only in negative degrees and products of elements in negative degrees are zero. Now because, $\hat{\mathrm{H}}^{*}(G,L_\zeta)$ is an extension of $\mathcal{J}^*$ by $\mathcal{K}^*$, it too has the BFGS property. L_ζ clearly belongs to the thick subcategory generated by k, and moreover, since L_ζ is not projective (because the hypothesis implies $\Omega^l k \neq k$ for any l). So we are done by Corollary 3.3. □

We now give an example to illustrate the last Proposition.

Example 3.9. We consider the Klein four group V_4. The classification of the indecomposable kV_4-modules over a field k of characteristic 2 is well-known; see [2, Vol. 1, Thm. 4.3.2] for instance. Every odd-dimensional indecomposable kV_4-module is a syzygy $\Omega^i\, k$ of k. The Tate cohomology of such a module is clearly generated by one element, in particular, it does not have the BFGS property. Every even-dimensional indecomposable projective-free kV_4-module has the form L_{ζ^m} for some $\zeta \neq 0$ in $H^1(V_4,k)$ and $m \geq 1$. Since $H^*(V_4,k) \cong k[x,y]$ with $|x| = 1 = |y|$, all non-zero elements in the cohomology ring are regular. Therefore by Proposition 3.8, we know that the Tate cohomology of every even-dimensional indecomposable kV_4-module has the BFGS property. To summarize, we have shown that the Tate cohomology of an indecomposable projective-free kV_4-module M is finitely generated over $\hat{\mathrm{H}}^{*}(V_4,k)$ if and only if M is an odd-dimensional module, in which case $V_G(M) = V_G(k)$.

If ζ is a nilpotent element, then Proposition 3.8 is no longer true. In fact, we have:

Proposition 3.10. *Let G be any finite group and let η be a nilpotent element in $H^*(G,k)$. Then $\hat{\mathrm{H}}^*(G, L_\eta)$ does not have the BFGS property.*

The proof of this proposition is based on the following lemma which is well-known. We include a proof for the reader's convenience.

Lemma 3.11. *Let G be any finite group and let α and β be two homogeneous elements in $H^*(G,k)$. Then there is an exact triangle*

$$\Omega^{|\alpha|} L_\beta \to L_{\alpha\beta} \to L_\alpha$$

in $\mathrm{stmod}(kG)$.

Proof. This is immediate from the octahedral axiom which gives the following commutative diagram with exact rows and columns in $\mathrm{stmod}(kG)$

$$\begin{tikzcd}
\Omega L_\alpha \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & L_\alpha \arrow[d] \\
\Omega^{|\alpha|} L_\beta \arrow[r] \arrow[d, dotted] & \Omega^{|\alpha|+|\beta|} k \arrow[r, "\Omega^{|\alpha|}\beta"] \arrow[d, "="] & \Omega^{|\alpha|} k \arrow[d, "\alpha"] \\
L_{\alpha\beta} \arrow[r] \arrow[d, dotted] & \Omega^{|\alpha|+|\beta|} k \arrow[r, "\alpha\beta"] \arrow[d] & k \arrow[d] \\
L_\alpha \arrow[r] & 0 \arrow[r] & \Omega^{-1} L_\alpha
\end{tikzcd}$$

The dotted exact triangle is the desired one. ☐

Proof. (Proposition 3.10) Let t be the nilpotence degree of η, so that $\eta^t = 0$. By repeated application of the (Octahedral axiom) above lemma, we have for each $2 \leq i \leq t$, an exact triangle

$$\Omega^{(i-1)|\eta|} L_{\eta^{i-1}} \to L_{\eta^i} \to L_\eta.$$

So if we assume to the contrary that L_η has the BFGS property, then it follows that L_{η^i} also has the BFGS property for all i, since the modules which have the BFGS property form a thick subcategory. In particular, we have

$$L_{\eta^t} \cong L_0 \cong \Omega k \oplus \Omega^{|\eta^t|} k,$$

and hence also k has the BFGS property which is a contradiction. ☐

Lemma 3.12. *If for $\zeta \in \mathrm{H}^d(G,k)$, the module $\hat{\mathrm{H}}^*(G, L_\zeta)$ has the BFGS property, then so do $\hat{\mathrm{H}}^*(G, L_\zeta^*)$, $\hat{\mathrm{H}}^*(G, L_{\zeta^n})$ for any positive n, and $\widehat{\mathrm{Ext}}^*_{kG}(L_\zeta, L_\zeta)$.*

Proof. We need to remember that for $\zeta \in \mathrm{H}^d(G,k)$, we have an isomorphism $L_\zeta^* \cong \Omega^{-d-1} L_\zeta$. Using Lemma 3.11 we see that L_{ζ^n} and $L_\zeta \otimes L_\zeta^*$ are in the thick subcategory generated by L_ζ. Consequently the Lemma follows form Proposition 3.5 and the fact that

$$\widehat{\mathrm{Ext}}^*_{kG}(L_\zeta, L_\zeta) \;\cong\; \hat{\mathrm{H}}^*(G, \mathrm{Hom}_k(L_\zeta, L_\zeta)) \;\cong\; \hat{\mathrm{H}}^*(G, L_\zeta^* \otimes L_\zeta).$$

☐

We are now prepared to prove the main theorem.

Theorem 3.13. *Let $\zeta \in \mathrm{H}^*(G,k)$ be a regular element of degree d, and suppose that $\hat{\mathrm{H}}^*(G, L_\zeta)$ has the BFGS property. If M is a finitely generated non-projective kG module such that $V_G(M) \subseteq V_G\langle\zeta\rangle$, then $\hat{\mathrm{H}}^*(G,M)$ is not finitely generated as an $\hat{\mathrm{H}}^*(G,k)$-module.*

Proof. Because of the condition that $V_G(M) \subseteq V_G\langle\zeta\rangle$, we know that some power of ζ, say ζ^t, annihilates the cohomology of M. Hence it follows that

$$L_{\zeta^t} \otimes M \;\cong\; \Omega M \oplus \Omega^{td} M,$$

and M has the BFGS property if and only if $L_{\zeta^t} \otimes M$ has the BFGS property. However, the action of $\hat{\mathrm{H}}^*(G,k)$ on $\hat{\mathrm{H}}^*(G, L_{\zeta^t} \otimes M)$ factors through the map $\hat{\mathrm{H}}^*(G,k) \longrightarrow \widehat{\mathrm{Ext}}^*_{kG}(L_{\zeta^t}, L_{\zeta^t})$. By Lemma 3.12 the target of that map has the BFGS property.

Now let m be any integer. Without loss of generality we can assume that $m < 0$. Let

$$\mathcal{M} = \bigoplus_{n \geq m} \hat{\mathrm{H}}^n(G, L_{\zeta^t} \otimes M) \subseteq \left(\bigoplus_{n \geq m} \widehat{\mathrm{Ext}}^n_{kG}(L_{\zeta^t}, L_{\zeta^t}) \right) \left(\bigoplus_{n \geq m} \hat{\mathrm{H}}^n(G, L_{\zeta^t} \otimes M) \right).$$

From the BFGS property we know that there exists a number N such that

$$\hat{\mathrm{H}}^*(G,k) \cdot \bigoplus_{n \geq m} \widehat{\mathrm{Ext}}^n_{kG}(L_{\zeta^t}, L_{\zeta^t}) \subseteq \bigoplus_{n \geq N} \widehat{\mathrm{Ext}}^n_{kG}(L_{\zeta^t}, L_{\zeta^t}).$$

Hence we have that

$$\begin{aligned} \hat{\mathrm{H}}^*(G,k) \cdot \mathcal{M} &\subseteq \hat{\mathrm{H}}^*(G,k) \cdot \left(\bigoplus_{n \geq m} \widehat{\mathrm{Ext}}^n_{kG}(L_{\zeta^t}, L_{\zeta^t}) \right) \left(\bigoplus_{n \geq m} \hat{\mathrm{H}}^n(G, L_{\zeta^t} \otimes M) \right) \\ &\subseteq \left(\bigoplus_{n \geq N} \widehat{\mathrm{Ext}}^n_{kG}(L_{\zeta^t}, L_{\zeta^t}) \right) \left(\bigoplus_{n \geq m} \hat{\mathrm{H}}^n(G, L_{\zeta^t} \otimes M) \right) \\ &\subseteq \bigoplus_{n \geq N+m} \hat{\mathrm{H}}^n(G, L_{\zeta^t} \otimes M). \end{aligned}$$

Therefore, $\hat{\mathrm{H}}^n(G, L_{\zeta^t} \otimes M)$ has the BFGS property. □

Remark 3.14. If G is a p-group, then the proof of Theorem 3.13 follows from proposition on thick subcategories and the results of [5]. That is, from [5] we know that the thick subcategory generated by L_ζ is precisely the set of all finitely generated module M such that $V_G(M) \subseteq V_G(L_\zeta) = V_G\langle\zeta\rangle$. Now the theorem follows from Proposition 3.5.

The theorem has some obvious consequences.

Corollary 3.15. *Suppose that $\hat{\mathrm{H}}^*(G,k)$ has the property that the product of any two elements in negative degrees is zero. Let $\zeta \in \mathrm{H}^*(G,k)$ be a regular element of degree d. If M is a finitely generated non-projective kG-module such that $V_G(M) \subseteq V_G\langle\zeta\rangle$, then $\hat{\mathrm{H}}^*(G,M)$ in not finitely generated as an $\hat{\mathrm{H}}^*(G,k)$-module.*

Proof. By Proposition 3.8, $\hat{\mathrm{H}}^*(G, L_\zeta)$ has the BFGS property. Hence the theorem applies. □

Corollary 3.16. *Suppose that G is a group with an abelian Sylow p-subgroup. If M is a finitely generated non-projective kG-module and if $V_G(M)$ is a proper subvariety of $V_G(k)$ then $\hat{\mathrm{H}}^*(G, M)$ has the BFGS property and hence is not finitely generated as a module over $\hat{\mathrm{H}}^*(G, k)$.*

Proof. If $V_G(M)$ is a proper subvariety of $V_G(k)$, then $V_G(M) \subseteq V_G(\zeta)$ for some non-nilpotent element $\zeta \in \mathrm{H}^*(G, k)$. But because the Sylow subgroup of G is an abelian p-group, every non-nilpotent element in $\mathrm{H}^*(G, k)$ is regular, and moreover, any two element is negative degrees in $\hat{\mathrm{H}}^*(G, k)$ have zero product. So the proof is complete by the previous corollary. □

4. PERIODIC MODULES

In this section we show that for any group G with p-rank at least 2, there is a finitely generated module M with the property that $\hat{\mathrm{H}}^*(G, \mathrm{End}_k M)$ is not finitely generated as a $\hat{\mathrm{H}}^*(G, k)$-module. In fact, we will show in the next proposition that any periodic module has this property.

Proposition 4.1. *Suppose that the group G has p-rank at least 2. Let M be a non-projective periodic kG-module. Then $\hat{\mathrm{H}}^*(G, \mathrm{Hom}_k(M, M)) \cong \widehat{\mathrm{Ext}}^*_{kG}(M, M)$ is not finitely generated as a $\hat{\mathrm{H}}^*(G, k)$-module.*

Proof. Let $E = \langle x_1, \ldots, x_n\rangle$ be a maximal elementary abelian p-subgroup such that the restriction M_E is not a free module. There exists an element $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$ and a corresponding cyclic shifted subgroup $\langle u_\alpha\rangle$,

$$u_\alpha \;=\; 1 \;+\; \sum_{i=1}^{n} \alpha_i(x_i - 1)$$

such that the restriction of M to $\langle u_\alpha\rangle$ is not projective (see Section 5.8 of [2]). In particular, we observe that this implies that the identity homomorphism $\mathrm{Id}_M : M \longrightarrow M$ does not factor through a projective $k\langle u_\alpha\rangle$-module. Consequently, the map $k \longrightarrow \mathrm{Hom}_k(M, M)$ which sends $1 \in k$ to Id_M represents a non-zero class in $\hat{\mathrm{H}}^0(\langle u_\alpha\rangle, \mathrm{Hom}_k(M, M))$.

The next thing that we note is that the restriction map

$$\mathrm{res}_{G,\langle u_\alpha\rangle} : \hat{\mathrm{H}}^d(G, k) \longrightarrow \hat{\mathrm{H}}^d(\langle u_\alpha\rangle, k)$$

is the zero map if $d < 0$. The reason is that the restriction map

$$\mathrm{res}_{E,\langle u_\alpha\rangle} : \hat{\mathrm{H}}^d(E, k) \;\longrightarrow \hat{\mathrm{H}}^d(\langle u_\alpha\rangle, k)$$

is zero by [3] since the hypothesis requires that E have rank at least 2. Thus our claim follows by the transitivity of the restriction map.

Now suppose that M is periodic of period t. For every m we have that $\Omega^{mt} M \cong M$ and there exists an element

$$\zeta_m \in \widehat{\mathrm{Ext}}^{mt}_{kG}(M, M) \;\cong\; \hat{\mathrm{H}}^{mt}(G, \mathrm{Hom}_k(M, M))$$

such that ζ_m is not zero on restriction to $\langle u_\alpha\rangle$. That is, ζ_m is represented by a cocycle

$$k \quad\longrightarrow\quad \mathrm{Hom}_k(M, M) \cong \Omega^{mt}\,\mathrm{Hom}_k(M, M)$$

which does not factor through a projective module on restriction to $\langle u_\alpha\rangle$.

Suppose that $\hat{\mathrm{H}}^*(G, \mathrm{Hom}_k(M, M))$ is finitely generated as a module over $\hat{\mathrm{H}}^*(G, k)$. Then there exist generators $\mu_1, \ldots, \mu_r$ of $\hat{\mathrm{H}}^*(G, \mathrm{Hom}_k(M, M))$, having degrees $d_1, \ldots, d_r$, respectively. Choose an integer m such that $mt < \min\{d_i\}$. Then we must have that

$$\zeta_m = \sum_{i=1}^{r} \gamma_i \mu_i$$

for some $\gamma_i \in \hat{\mathrm{H}}^{mt-d_i}(G, k)$. But now, for every i, we have that $mt - d_i$ is negative. Hence $\mathrm{res}_{G,\langle u_\alpha \rangle}(\gamma_i) = 0$ for every i. Therefore, since restriction onto a shifted subgroup is a homomorphism we have that $\mathrm{res}_{G,\langle u_\alpha \rangle}(\zeta_m) = 0$. But this is a contradiction. $\square$

We know that every finite group with non-trivial Sylow p-subgroup admits a finitely generated non-projective and periodic kG-module. This is a consequence of the fact that any module M with the property that $V_G(M)$ has dimension 1, is periodic (see [2]). Consequently, we have the following result.

Theorem 4.2. *Let G be a finite group. Then the Tate cohomology of every finitely generated kG-modules is finitely generated over $\hat{\mathrm{H}}^*(G, k)$ if and only if the Sylow p-subgroup of G is either a cyclic group or a generalized Quaternion group.*

4.1. **Projective classes.** There is one other concept which ties up well with finite generation of Tate cohomology, and this is a ghost projective class in the $\mathrm{stmod}(kG)$. Consider the pair $(\mathcal{P}, \mathcal{G})$, where $\mathcal{P}$ is a class of objects isomorphic in $\mathrm{stmod}(kG)$ to finite direct sums of suspensions of k, and $\mathcal{G}$ is a class of all ghosts in $\mathrm{stmod}(kG)$. Recall that a ghost is a map of kG-modules that is zero in Tate cohomology. We say that $(\mathcal{P}, \mathcal{G})$ is a ghost projective class if the following 3 conditions are satisfied.

1. The class of all maps $X \to Y$ such that the composite $P \to X \to Y$ is zero for all P in $\mathcal{P}$ and all maps $P \to X$ is precisely $\mathcal{G}$.
2. The class of all objects P such that the composite $P \to X \to Y$ is zero for all maps $X \to Y$ in $\mathcal{G}$ and all maps $P \to X$ is precisely $\mathcal{P}$.
3. For each object X there is an exact triangle $P \to X \to Y$ with P in $\mathcal{P}$ and $X \to Y$ in $\mathcal{G}$.

The first question that comes to mind is whether the ghost projective class exists in $\mathrm{stmod}(kG)$. We answer this in the next theorem.

Theorem 4.3. *The following statements are equivalent.*

1. *The Tate cohomology of every finitely generated kG-modules is finitely generated over $\hat{\mathrm{H}}^*(G, k)$.*
2. *The ghost projective class exists in* $\mathrm{stmod}(kG)$.
3. *The Sylow p-subgroup of G is either a cyclic group or a generalized Quaternion group.*

Proof. In view of Theorem 4.2 it enough to prove the equivalence of the first two statements.

(1) $\implies$ (2): It is clear from the definition of a ghost that $\mathcal{P}$ and $\mathcal{G}$ are orthogonal, i.e., the composite $P \to M \xrightarrow{h} N$ is zero for all P in $\mathcal{P}$, for all h in $\mathcal{G}$, and all maps $P \to M$. So by [12, Lemma 3.2] it remains to show that for all finitely generated kG-modules M, there exists a triangle $P \to M \to N$ such that P is in $\mathcal{P}$ and $M \to N$ is in $\mathcal{G}$. The exact triangle (2.1) has this form.

(2) $\implies$ (1): Let M be a finitely generated kG-module. Since the ghost projective class exists, we have an exact triangle

$$\bigoplus \Omega^i k \xrightarrow{\oplus\theta_i} M \xrightarrow{\rho} N$$

in $\mathrm{stmod}(kG)$ where ρ is a ghost. We claim that the finite set $\{\theta_i\}$ generate $\hat{\mathrm{H}}^*(G, M)$ as a module over $\hat{\mathrm{H}}^*(G, k)$. To see this, consider any element γ in $\hat{\mathrm{H}}^t(G, M)$ represented by a cocycle $\gamma\colon \Omega^t k \to M$. Since ρ is a ghost, we get the following commutative diagram:

$$\begin{tikzcd}
\bigoplus \Omega^i k \arrow[r, "\oplus\theta_i"] & M \arrow[r, "\rho"] & N \\
& \Omega^t k \arrow[u, "\gamma"] \arrow[ul, dotted, "\oplus r_i"] \arrow[ur, "\rho\gamma=0"'] &
\end{tikzcd}$$

From this diagram, we infer that $\gamma = \sum r_i\theta_i$. This shows that $\hat{\mathrm{H}}^*(G, M)$ is finitely generated, as desired. ☐

5. A MODULE WITHOUT BFGS PROPERTY

The purpose of this section is to prove the following.

Proposition 5.1. *There exist a group G and module M with the property that $\hat{\mathrm{H}}^*(G, M)$ is not finitely generated as a module over $\hat{\mathrm{H}}^*(G, k)$ but does not have the BFGS property.*

The proof, of course, is contained in a single example. However, we are reasonably confident that the example is not that exotic. Consider the semi-dihedral group $G = SD_{16}$ of order 16 whose ordinary cohomology ring, with coefficients in a field of characteristic 2, has the form

$$\mathrm{H}^*(G, k) \cong k[z, y, x, w]/(z^3,\ zy,\ zx,\ x^2 + y^2w)$$

where the variables z, y, x, and w have degrees $1, 1, 3$, and 4 respectively. Note that every element of $\mathrm{H}^*(G, k)$ can be written as a linear combination of monomials of the form $z^a y^b x^c w^d$ where the non-negative integers a, b, c and d satisfy the relations

$$ab = 0, \quad ac = 0, \quad a = 0, 1, 2, \quad \text{and} \quad c = 0, 1. \tag{5.1}$$

For any n, we have Tate duality

$$\langle\ ,\ \rangle : \hat{\mathrm{H}}^n(G, k) \otimes \hat{\mathrm{H}}^{-n-1}(G, k) \longrightarrow \hat{\mathrm{H}}^{-1}(G, k) \cong k$$

which is the product in Tate cohomology. In particular, we have for any homogeneous element α, β and γ with $\alpha\beta$ in degree n and γ in degree $-n+1$ that

$$\langle \alpha\beta, \gamma\rangle \ =\ \langle \alpha, \beta\gamma\rangle = \alpha\beta\gamma.$$

For notation, for $n > 0$, let $\overline{f} \in \hat{\mathrm{H}}^{-n-1}(G, k)$ be the dual of $f \in \mathrm{H}^n(G, k)$ where f is a monomial in the form $z^a y^b x^c w^d$ with a, b, c, d satisfying the conditions (5.1) so that $a + b + 3c + 4d = n$. By this we mean that $\langle \overline{f}, f\rangle = 1$ and that $\langle \overline{f}, g\rangle = 0$ if g is any other monomial of the same form, satisfying the same conditions (5.1), but with different values of a, b, c and d.

It is shown in [3], using the hypercohomology spectral sequence, that $\overline{z}w = z^2$. Consequently, we have that

$$1 \ =\ \langle w\overline{z}, \overline{z^2}\rangle \ =\ \langle w, \overline{z}\overline{z^2}\rangle.$$

We conclude from this that $\overline{z}\,\overline{z^2} = \overline{w}$. In a similar way we see that

$$1 \;=\; \langle w^t\; \overline{w^{t-1}z}, \overline{z^2}\rangle \;=\; \langle w^t, \overline{w^{t-1}z}\; \overline{z^2}\rangle.$$

and hence, $\overline{w^{t-1}z}\;\overline{z^2} = \overline{w^t}$. Consequently the $\hat{\mathrm{H}}^*(G,k)$-submodule of $\hat{\mathrm{H}}^*(G,k)$ generated by $\overline{z^2}$ is an $\hat{\mathrm{H}}^*(G,k)$-module which fails to have the BFGS property.

Now to construct the module, let $y : \Omega k \longrightarrow k$ be a cocycle representing the element $y \in \mathrm{H}^1(G,k)$. Let $L = L_y$ be the kernel, so that we have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & \Omega k \arrow[r, "y"] & k \arrow[r] & 0. \end{tikzcd}$$

Hence in the stable category we have a triangle

$$\begin{tikzcd} \Omega k \arrow[r] & L \arrow[r] & \Omega k \arrow[r, "y"] & k \end{tikzcd}$$

and by Lemma 3.4, we have the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Cokernel}(y_*)[-1] \arrow[r] & \hat{\mathrm{H}}^*(G,L) \arrow[r] & \mathrm{Kernel}(y_*)[-1] \arrow[r] & 0 \end{tikzcd}$$

of $\hat{\mathrm{H}}^*(G,k)$-modules as before. It follows that the kernel of y_* is a quotient module of $\hat{\mathrm{H}}^*(G,L)$. This is a submodule of $\hat{\mathrm{H}}^*(G,k)$ and it contains the element $\overline{z^2}$. Hence it does not have the BFGS property and neither does $\hat{\mathrm{H}}^*(G,L)$. On the other hand L is periodic and hence $\hat{\mathrm{H}}^*(G, L\otimes L^*)$ is not finitely generated as a module over $\hat{\mathrm{H}}^*(G,k)$. It remains to show that $\hat{\mathrm{H}}^*(G,L)$ is not finitely generated.

Now notice that L has dimension 14 and also that L embeds in Ωk which in turn embeds in kG. Hence, the cokernel $\Omega^{-1}L$ of the embedding of L into kG, has dimension 2. Moreover, L is defined over the base field $\mathbb{F}_2$, since the cohomology element y is defined over $\mathbb{F}_2$. That is, $y \in \mathrm{H}^1(G,\mathbb{F}_2) \cong \mathrm{Hom}(G,\mathbb{F}_2)$, and corresponds to a maximal subgroup H which is the kernel of the homomorphism. This means that L is the induced module $L \cong {k_H}^{\uparrow G}$ from the subgroup H. It follows that $L \cong L^*$ and that $L\otimes L \cong L\oplus L$ by Frobenius reciprocity. Hence, $\hat{\mathrm{H}}^*(G,L)$ is finitely generated as a module over $\hat{\mathrm{H}}^*(G,k)$ if and only if $\hat{\mathrm{H}}^*(G, L\otimes L^*)$ is also finitely generated. This completes the proof.

6. Modules with finitely generated Tate cohomology

It is clear that any module M which is a direct sum of Heller translates $\Omega^n k$ has finitely generated Tate cohomology. This is simply because $\hat{\mathrm{H}}^*(G,M)$ is a direct sum of copies of $\hat{\mathrm{H}}^*(G,k)$ which have been suitably translated in degrees. Also any finitely generated modules over a group with periodic cohomology has finitely generated Tate cohomology. In this section we show that in general there are many more modules with this property. Note that every one of the modules which we discuss has the property that $V_G(M) = V_G(k)$, thereby adding some further evidence for the conjecture that for an indecomposable non-projective finitely generated kG-module M, $\hat{\mathrm{H}}^*(G,M)$ is finitely generated as an $\hat{\mathrm{H}}^*(G,k)$-module if and only if $V_G(M) = V_G(k)$.

We consider the Tate cohomology of modules M which can occur as the middle term of an exact sequence of the form

$$\begin{tikzcd} 0 \arrow[r] & \Omega^m k \arrow[r] & M \arrow[r] & \Omega^n k \arrow[r] & 0 \end{tikzcd}$$

for some values of m and n. Note that such a sequence represents an element ζ in

$$\mathrm{Ext}^1_{kG}(\Omega^n k, \Omega^m k) \cong \widehat{\mathrm{Ext}}^{n+1-m}_{kG}(k,k) \cong \hat{\mathrm{H}}^{n+1-m}(G,k).$$

For the purposes of examining the Tate cohomology of M there is no loss of generality in applying the shift operator Ω^{-m}. Consequently we can assume that the sequence has the form

$$0 \longrightarrow k \longrightarrow M \longrightarrow \Omega^n k \longrightarrow 0 \tag{6.1}$$

for some n and that $\zeta \in \hat{\mathrm{H}}^{n+1}(G,k)$.

The principal result of this section is the following.

Theorem 6.1. *Suppose that the cohomology of G is not periodic and that for the module M and cohomology element ζ as above, the map*

$$\zeta : \hat{\mathrm{H}}^{*}(G,k) \longrightarrow \hat{\mathrm{H}}^{*}(G,k)$$

given by multiplication by ζ has a finite dimensional image. Then the Tate cohomology $\hat{\mathrm{H}}^{*}(G,M)$ is finitely generated as a module over $\hat{\mathrm{H}}^{*}(G,k)$.

Before beginning the proof, we should note that there are many example of such modules. One example is the middle term of the almost split sequence for the trivial module k. The almost split sequence represents a generator ζ for $\hat{\mathrm{H}}^{-1}(G,k)$. Multiplication by this element annihilates $\hat{\mathrm{H}}^{r}(G,k)$ for all r except $r = 0$. Full details are provided in Corollary 6.3 which follows.

Other examples can be found whenever the depth of $\mathrm{H}^*(G,k)$ is two or more. In this situation, all products involving elements in negative degrees are zero. In addition the principal ideal generated by any element in negative cohomology contains no non-zero elements in positive degrees (see [3]). Hence, multiplication by an element ζ in negative cohomology has finite dimensional image and the middle term of a sequence representing ζ must have finitely generated cohomology by the theorem.

Proof. As before we have an exact sequence of $\hat{\mathrm{H}}^{*}(G,k)$-modules

$$0 \longrightarrow \mathcal{J}^* \longrightarrow \hat{\mathrm{H}}^{*}(G,M) \longrightarrow \mathcal{K}^*[-n] \longrightarrow 0.$$

by Lemma 3.4. Here $\mathcal{K}^*$ is the kernel of multiplication by ζ,

$$\zeta : \hat{\mathrm{H}}^{*}(G,k) \longrightarrow \hat{\mathrm{H}}^{*}(G,k),$$

and $\mathcal{J}^*$ is the cokernel of multiplication by ζ. By assumption, the image of multiplication by ζ has finite total dimension. This means that in all but a finite number of degrees r, the map

$$\zeta : \hat{\mathrm{H}}^{r}(G,k) \longrightarrow \hat{\mathrm{H}}^{r+n+1}(G,k)$$

is the zero map. From the Lemma we know that $\hat{\mathrm{H}}^{*}(G,M)$ is finitely generated as a module over $\hat{\mathrm{H}}(G,k)$ if and only if $\mathcal{K}^*$ has the same property.

First we view $\mathcal{K}^*$ as a module over the ordinary cohomology ring $\mathrm{H}^*(G,k)$. The elements in non-negative degrees form a submodule $\mathcal{M}^* = \sum_{i\geq 0} \mathcal{K}^i$, The submodule $\mathcal{M}^*$ is finitely generated over $\mathrm{H}^*(G,k)$ because it is a quotient of a finitely generated module. Let $\mu_1, \ldots, \mu_t$ be a set of generators for $\mathcal{M}^*$ as an $\mathrm{H}^*(G,k)$-module. We claim that $\mu_1, \ldots, \mu_t$ generate $\mathcal{K}^*$ as an $\hat{\mathrm{H}}^{*}(G,k)$-module. To this end, let $\mathcal{N}^*$ be the $\hat{\mathrm{H}}^{*}(G,k)$-submodule module of $\mathcal{K}^*$ generated by $\mu_1, \ldots, \mu_t$. We notice first that $\mathcal{K}^n \subseteq \mathcal{N}^*$ for $n \geq 0$. It remains only to show the same for $n < 0$.

Because $\mathcal{K}^*$ is a quotient of $\hat{\mathrm{H}}^*(G,k)$ by a finite dimensional submodule, we must have that the quotient map is an injection of $\hat{\mathrm{H}}^n(G,k)$ into $\mathcal{K}^n$ for n sufficiently large. For some sufficiently large n, we can find an element γ in $\mathrm{H}^n(G,k)$ which is a regular element for the ordinary cohomology ring $\mathrm{H}^*(G,k)$. For example, by Duflot's Theorem, any element whose restriction to the center of a Sylow p-subgroup of G is not nilpotent will serve this purpose (see [2] or [8]). Let θ be the image of γ in $\mathcal{K}^n$. We know that θ is not zero. We also know that multiplication by γ is a surjective map

$$\gamma : \hat{\mathrm{H}}^{m-n}(G,k) \longrightarrow \hat{\mathrm{H}}^m(G,k)$$

whenever $m < 0$. This fact is an easy consequence of Tate duality. Full details can be found in Lemma 3.5 of [3]. From this it follows that for any $m < 0$, we must have that $\hat{\mathrm{H}}^{m-n}(G,k)\theta = \mathcal{K}^m$. Since $\theta \in \mathcal{N}^*$, we get that $\mathcal{K}^m \subseteq \mathcal{N}^*$ for all $m < 0$. Hence, $\mathcal{K}^* = \mathcal{N}^*$ is finitely generated as a module over $\hat{\mathrm{H}}^*(G,k)$. □

Remark 6.2. The conclusion of Theorem 6.1 is also true for groups with periodic cohomology. However, for groups with periodic cohomology, the only element ζ in the Tate cohomology ring which satisfies the hypothesis of the theorem is the zero element. Consequently the modules given by the theorem in the periodic case are just direct sums of suspensions of k. Since we are interested in modules with finitely generated Tate cohomology that are not isomorphic to a direct sum of suspensions of k, we assumed in the statement of the theorem that G has non-periodic cohomology.

Corollary 6.3. *Let G be a group having p-rank at least 2. Then the middle term of the almost split sequence*

$$0 \longrightarrow \Omega^2 k \longrightarrow M \longrightarrow k \longrightarrow 0$$

ending in k has finitely generated Tate cohomology.

Proof. The almost split sequence corresponds to an element ζ in $\hat{\mathrm{H}}^{-1}(G,k)$. By Theorem 6.1 it suffices to show that multiplication by ζ on $\hat{\mathrm{H}}^*(G,k)$ has finite-dimensional image. In fact, we will see that ζ annihilates $\hat{\mathrm{H}}^r(G,k)$ for all r except $r = 0$. (Note that ζ does not annihilate $\hat{\mathrm{H}}^0(G,k) \cong k$ because of the Tate duality pairing.)

The element ζ is represented in the stable category by a map $\Omega k \longrightarrow \Omega^2 k$ which is obtained as shown in the commutative diagram below. Here the top row is the defining sequence for Ωk and the bottom row is the almost split sequence ending in k.

$$\begin{tikzcd}
0 \arrow[r] & \Omega k \arrow[r] \arrow[d, dotted, "\zeta"'] & P \arrow[r] \arrow[d, dotted] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^2 k \arrow[r] & M \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Recall that the products in the Tate cohomology ring correspond to composition of maps in the stable category. Therefore showing that ζ annihilates $\hat{\mathrm{H}}^r(G,k)$ for $r \neq 0$, is equivalent to showing that the composite

$$\Omega k \xrightarrow{\zeta} \Omega^2 k \xrightarrow{f} \Omega^i k$$

factors through a projective for all f and all $i \neq 2$. Since $i \neq 2$ and G has non-periodic cohomology, the map $f\colon \Omega^2 k \to \Omega^i k$ cannot be an split monomorphism. So by the defining property of an almost split sequence, we have a factorization $f = \tilde{f}\sigma$ of f as shown in the commutative diagram below

$$\begin{tikzcd}
0 \arrow[r] & \Omega k \arrow[r] \arrow[d, dotted, "\zeta"'] & P \arrow[r] \arrow[d, dotted] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^2 k \arrow[r, "\sigma"] \arrow[d, "f"'] & M \arrow[r] \arrow[ld, dotted, "\tilde{f}"] & k \arrow[r] & 0 \\
 & \Omega^i k & & &
\end{tikzcd}$$

From this commutative diagram it is clear that the desired composite $f\zeta$ factors through the projective module P. This completes the proof. $\square$

Remark 6.4. Using the same techniques as above, particularly Lemma 3.4, it can be shown that for many other modules M with the property that $V_G(M) = V_G(k)$, we get that $\hat{\mathrm{H}}^*(G, M)$ is finitely generated as a module over $\hat{\mathrm{H}}^*(G, k)$. Examples arise in the case that G is an elementary abelian group of order p^n for p odd. In this case, the cohomology ring $\mathrm{H}^*(G, k)$ has the form

$$\mathrm{H}^*(G, k) \;= k[\zeta_1, \dots \zeta_n] \otimes \Lambda(\eta_1, \dots, \eta_n)$$

where the ζ_i's are in degree 2 and the η_i's are in degree one and are nilpotent. It can be checked that L_η for η some product of the η_i's has the property that $\hat{\mathrm{H}}^*(G, L_\eta)$ is finitely generated as a module over $\hat{\mathrm{H}}^*(G, k)$. The same can be verified for L_γ where γ is any element in negative degree in $\hat{\mathrm{H}}^*(G, k)$.

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Department of Mathematics, University of Georgia, Athens, GA 30602, USA
E-mail address: `jfc@math.uga.edu`

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7, Canada
E-mail address: `schebolu@uwo.ca`

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7, Canada
E-mail address: `minac@uwo.ca`