# Schur–De-Rham complex and its cohomology *

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**Abstract**

We associate to a Young diagram $\lambda$ the Schur–De-Rham complex $\mathbf{S}_\lambda$. We show that this complex is exact when the $p$–core of $\lambda$ is nontrivial and compute its cohomology when the $p$–core is trivial and the $p$–quotient of $\lambda$ consists of a single diagram.

*Keywords:* Young diagram; Schur complex; cohomology.

## 1 Introduction

One can associate to a map $f : F \longrightarrow G$ of free $R$–modules the Koszul complex by a well known construction. Under some regularity assumptions this complex is a free resolution of an $R$–module $S^*(G/\mathrm{im}(f))$. This construction was generalized in [ABW] by putting it into the context of representation theory (see also [La], [Ni]). The resulting complex: the Schur complex $\mathbf{S}_\lambda(f)$ (depending additionally on a Young diagram $\lambda$) provided a free resolution for a much larger class of $R$–modules and became a standard tool in representation theory.

The Koszul complex however, turned out to be useful also in the situation when it was exact (eg. for $f$ being an identity map). As an example we may point out the whole string of works concerning the so–called Koszul duality

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([BGG], [GKM], [MW] etc.). Another striking application of the Koszul complex (for $f = \mathrm{id}$) appeared in the work [FLS] on homological algebra in the category of functors. The main result of that work was a computation of some Ext–groups in this category. The crucial part of the proof was an analysis of hiperExt spectral sequences with coefficients in the Koszul complex and in the De–Rham complex (which is the same graded space as the Koszul complex but has a differential going into another direction).

The ideas of that important paper have inspired many further works on the functor category (eg. [B1], [FS], [FFSS], [K1], [K2], [K3]). In particular, there was established a tight connection between the category of functors and the category of $GL_n(R)$–modules (at least for $R$ being a field). This connection suggests the possibility of generalization of computations of Ext–groups begun in [FLS] by using ideas from representation theory. I started this program in [C1] where I extended calculations of Ext–groups to a large class of functors important in representation theory. But to push the computations further we need appropriate generalizations of the Koszul and De-Rham complexes. Although it turns out that the Schur complex is exactly this generalization of the Koszul complex we need, it seems, surprisingly enough, that nobody has tried to generalize the De–Rham complex in a similar manner. This generalization, which I call the Schur–De-Rham complex is the objective of the present paper.

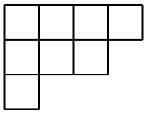
I focus on the most fundamental case of the complex associated to the identity map (this is the case important for the applications I have in mind). Thus our complex is associated to an $R$–module $V$ and a Young diagram $\lambda$ (we denote it by $\mathbf{S}_\lambda(V)$). In the parallel case, the "classical" Schur complex is exact but the Schur–De-Rham complex is not. Its cohomology reflects deep properties of the Young diagram (if $R$ is a field of positive characteristic). It turns out that the cohomology of $\mathbf{S}_\lambda$ may be nonzero if $\lambda$ has a trivial $p$–core (cf. Fact 4.3). The main result of this paper (Theorem 5.3) describes $H^*(\mathbf{S}_\lambda)$ for $\lambda$ with a trivial $p$–core and $p$–quotient consisting of a single diagram. This connection between the cohomology of the Schur–De-Rham complex and combinatorics introduced for description of blocks of modular representations was quite surprising for me. I hope it will be developed and better understood in the future.

I introduced the Schur–De-Rham complex in order to generalize computations of Ext–groups initiated in [FLS] where the De-Rham complex played an important role. But this complex turned out to be interesting for its own. I hope it will also be useful in other branches of representation theory. For this reason I decided to devote a separate paper to the study of the Schur–De-Rham complex and I tried to make this paper independent of the other parts of my work on the functor category ([C1],[C2]). The applications of the results of the present paper to computations of Ext–groups will appear in [C2].

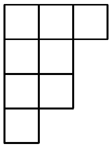
## 2 Basic combinatorics

We recall here some standard facts concerning Young diagrams mainly in order to fix notation and terminology. We will usually think of partitions of $d$ (ie. weakly decreasing sequences of positive integers with sum $d$) as Young diagrams of weight $d$. The Young diagram of weight $d$ is an arrangement of $d$ squares (boxes) associated to a given partition in the obvious way. Given a diagram $\lambda$ we may form its *conjugate* $\widetilde{\lambda}$ by reflecting the diagram $\lambda$ in its diagonal:

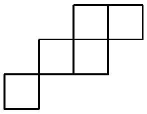
$\lambda = (4, 3, 1)$



$\widetilde{\lambda} = (3, 2, 2, 1)$

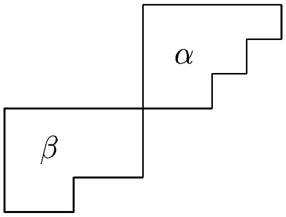


We will also consider *skew partitions* $\lambda/\mu$ for $\mu \subseteq \lambda$ (ie. $\mu_i \leq \lambda_i$ for all $i$). We get the Young diagram for a skew partition $\lambda/\mu$ by removing the boxes belonging to $\mu$ from the diagram for $\lambda$. For example, for $(4, 3, 1)/(2, 1)$ we get



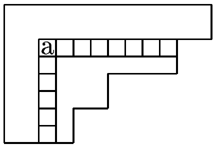
Sometimes to emphasize the difference between diagrams nad skew diagrams we will call the former *solid diagrams.*

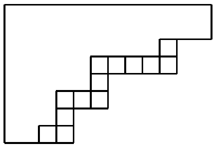
We say that a skew diagram $\lambda/\mu$ is a *disconnected sum* of diagrams $\alpha/\alpha'$ and $\beta/\beta'$ if the set of boxes of $\lambda/\mu$ is a disjoint sum of the sets of boxes of $\alpha/\alpha'$ and $\beta/\beta'$, and each box in $\alpha/\alpha'$ lies above and to the right of each box in $\beta/\beta'$. For example, for diagrams $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_l)$, the (skew) diagram $\lambda/\mu = (\beta_1+\alpha_1,\ldots,\beta_1+\alpha_k,\beta_1,\ldots,\beta_l)/(\beta_1^k)$ is their disconnected sum

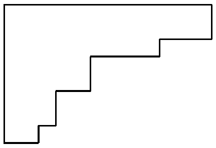


There is a natural total ordering of the set of solid diagrams called the *lexicographic order*. We say that $\mu$ is lexicographically smaller than $\lambda$ (notation: $\mu<\lambda$) if for the smallest $i$ such that $\mu_i \neq \lambda_i$ we have $\mu_i > \lambda_i$. The last inequality looks like a misprint. I explained the reason for taking this strange convention in ([C1], sect. 2). The best way to memorize it is to remember that the diagram $(d)$ is the smallest among diagrams of weight $d$, while $(1,\ldots,1)$ (we will denote this diagram by $(1^d)$) is the largest.

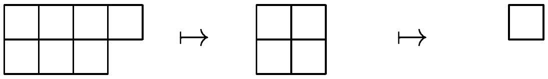
At last, I would like to distinguish certain types of diagrams which will be important later. We call a diagram of shape $(n,1^m)$ a *hook*. We call its first row the *arm* of a hook, its first column — the *leg* of a hook. The last box in the arm is called the *hand*, and the lowest box in the leg is called the *foot*. The box in which the leg and the arm intersect is called the *corner*. I would like to recall a combinatorial algorithm which is important in block theory (see [JK] pp. 75–76). Let $\lambda$ be a diagram and $a$ be its box lying in the $i$th row and the $j$th column. We consider a set consisting of boxes in the $i$th row lying to the right of $a$ and boxes in the $j$th column lying below $a$. Of course, such an arrangement of boxes looks like a hook with the corner $a$ and the hand and foot lying on the rim of $\lambda$. We call such an arrangement of boxes an *$n$–hook* in $\lambda$, where $n$ is the number of its boxes. Observe that if we remove from $\lambda$ a part of rim contained between the hand and foot of some $n$–hook in $\lambda$, we get a new diagram, say $\lambda'$ (of course, $|\lambda'| = |\lambda| - n$, where "$|\ |$" means the weight of a diagram). We call this removed part of a diagram a *rim $n$–hook*, and we say that $\lambda'$ was obtained from $\lambda$ by removing the rim $n$–hook. The following picture shows: a diagram $\lambda$ with some $n$–hook marked off, $\lambda$ with the corresponding rim $n$–hook, the diagram $\lambda'$







We call a diagram an $n$–*core* if it does not contain any $n$–hook. For an arbitrary diagram $\lambda$ we may perform the following procedure. We find an $n$–hook in it and get rid off its rim $n$–hook. In the obtained diagram we again find some $n$–hook and remove its rim $n$–hook etc. We continue the procedure until we get an $n$–core (perhaps being the empty diagram). It is a highly nontrivial fact that this $n$–core does not depend on the order of removals of rim $n$–hooks ([JK], Th. 2.7.16). We call this diagram the $n$–core of $\lambda$ and denote it by $c(\lambda)$. Here are presented two ways of reaching the 3–core of $(4, 3)$ which is $(1)$:



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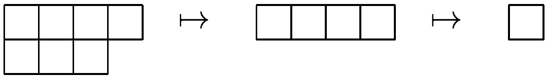

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# 3 The Schur–De-Rham complex

Although all definitions given in this article work over any ground commutative ring $R$, some theorems hold (or are interesting) for fields of positive characteristic only. Therefore from now on, the term *space* will mean a finite dimensional vector space over a fixed field $\mathbf{k}$ of positive characteristic $p$.

The Schur–De-Rham complex $\mathbf{S}_\lambda$ is a complex of functors. By this we mean that for any space $V$ we have a complex $\mathbf{S}_\lambda(V)$ and the construction is natural with respect to linear maps $V \longrightarrow W$. In fact it is a complex of "homogeneous strict polynomial functors of degree $|\lambda|$" in the sense of [FS]. I will use the language of strict polynomial functors (SP–functors for short) throughout this article because it is best adapted for the applications in [C2]. But all results of the present work could be, with only minor changes, stated and proved without appealing to this concept. The reader not familiar with strict polynomial functors may think of them just as functors in a usual sense. In a few places where the difference between these notions is substantial I will explain what is really going on.

We now turn to defining the Schur–De-Rham complex. Our construction may be thought of as an analogue of the construction of the Schur functor in the category of complexes. To make this analogy clear I shall briefly recall the construction of the Schur functor $S_{\lambda/\mu}$ (cf. [ABW], II.1). Given a skew diagram $\lambda/\mu$ of weight $d$ (ie. $d = |\lambda| - |\mu|$), we set $\Lambda^{\lambda/\mu}(V) := \Lambda^{\lambda_1-\mu_1}(V) \otimes \ldots \otimes \Lambda^{\lambda_l-\mu_l}(V)$. Then the natural embedding $\Lambda^{\lambda/\mu}(V) \longrightarrow V^{\otimes d}$ may be seen as an inclusion of the invariants of the alternating action of the horizontal Young subgroup $\Sigma_{\lambda/\mu}$ of $\Sigma_d$ (see [JK] p. 29) on the $d$–th tensor power of $V$ (the alternating action is given by the formula: $\sigma.(v_1 \otimes \ldots \otimes v_d) = (-1)^{|\sigma|} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$). Similarly, we define $S^{\widetilde{\lambda/\mu}}$ and we have an epimorphism $V^{\otimes d} \longrightarrow S^{\widetilde{\lambda/\mu}}$ which is a projection to the coinvariants of the permutative action of the vertical Young subgroup $\Sigma_{\lambda/\mu}$ of $\Sigma_d$ on the $d$–th tensor power of $V$. The Schur functor $S_{\lambda/\mu}(V)$ is defined to be the image of the composition $\Lambda^{\lambda/\mu}(V) \longrightarrow V^{\otimes d} \longrightarrow S^{\widetilde{\lambda/\mu}}$ (this definition needs a modification for $p = 2$, I discuss this point in length in ([C1], sect. 3)). It comes with two natural transformations which we call the structural arrows: the epimorphism $\phi_{\lambda/\mu} : \Lambda^{\lambda/\mu} \longrightarrow S_{\lambda/\mu}$ (which for $\lambda/\mu = (d)$ yields an isomorphism $\Lambda^d \simeq S_{(d)}$) and the monomorphism $\psi_{\lambda/\mu} : S_{\lambda/\mu} \longrightarrow S^{\widetilde{\lambda/\mu}}$ (which for $\lambda/\mu = (1^d)$ gives $S_{(1^d)} \simeq S^d$).

To repeat this construction in the category of complexes we take the sequence $\mathbf{K}(V) = \{0 \longrightarrow V = V \longrightarrow 0\}$ regarded as a cohomological complex concentrated in degrees 0 and 1 and consider its $d$–th tensor power $(\mathbf{K}(V))^{\otimes d}$. There is a natural "permutative" action of $\Sigma_d$ on this complex (the transposition $(j, j+1)$ sends an element $v_1 \otimes \ldots \otimes v_j \otimes v_{j+1} \otimes \ldots \otimes v_d$ to $(-1)^{|v_j||v_{j+1}|} v_1 \otimes \ldots \otimes v_{j+1} \otimes v_j \otimes \ldots \otimes v_d$, ($\pm 1$ is necessary to obtain an action in the category of complexes)). The Schur–De-Rham complex $\mathbf{S}_{\lambda/\mu}(V)$ is simply the image of the composition of the maps of complexes $(((\mathbf{K}(V))^{\otimes d})^{alt})^{\Sigma_{\lambda/\mu}} \longrightarrow (\mathbf{K}(V))^{\otimes d} \longrightarrow ((\mathbf{K}(V))^{\otimes d})_{\Sigma_{\widetilde{\lambda/\mu}}}$ (for any $\Sigma_d$–module $M$, $M^{alt}$ stands for $M \otimes \mathrm{sgn}$).

In fact, this construction is only a slight modification of the one which was known earlier. Namely, it is easy to see that if we apply the above construction to the $\mathbf{K}(V)$ considered as a homological complex we get nothing but the Schur complex introduced in ([ABW], sect. V.1). This purely formal difference in definition, which looks quite innocent, affects the homology dramatically. While the Schur complex is exact (([ABW], Cor. V.1.5), we recall that in the terminology of that paper we deal with a Schur complex associated to the identity map $V = V$), the cohomology of the Schur–De-Rham complex is complicated and interesting.

We now list some basic properties of the Schur–De-Rham complex. Let us first look at the simplest cases. For $\lambda/\mu = (1)^d$ we have $\mathbf{S}_{\lambda/\mu} = (\mathbf{K}^{\otimes d})_{\Sigma_d}$, ie. the complex of coinvariants of the action. We will denote this complex by $\mathbf{S}^d(V)$ for it is an analogue of the symmetric power of a space. We can identify the $i$th degree component of $\mathbf{K}^{\otimes d}(V)$ with the $\Sigma_d$–module $(V^{\otimes d-i} \otimes (V^{\otimes i})^{alt}) \otimes_{\mathbf{k}[\Sigma_{d-i}\times\Sigma_i]} \mathbf{k}[\Sigma_d]$. Hence $(\mathbf{S}^d)^i(V) = S^{d-i}(V) \otimes \Lambda^i(V)$. It is easy to write down the differential explicitly. We get the formula $\delta(x_1 \dots x_{d-i} \otimes y_1 \wedge \dots \wedge y_i) = \sum_{j=1}^{d-i} x_1 \dots x_{j-1}x_{j+1} \dots x_{d-i} \otimes y_j \wedge y_1 \wedge \dots \wedge y_i$, which is nothing but the differential in the algebraic De–Rham complex (cf. [FLS], sect. 3):

$$0 \longrightarrow S^d(V) \xrightarrow{\delta} S^{d-1}(V) \otimes \Lambda^1(V) \xrightarrow{\delta} \dots \xrightarrow{\delta} \Lambda^d(V) \longrightarrow 0.$$

Let us compare this complex with a corresponding Schur complex. Of course, we have the same graded space but the differential changes direction. It is given by the formula $\partial(x_1 \dots x_{d-i} \otimes y_1 \wedge \dots \wedge y_i) = \sum_{j=1}^{i} (-1)^{j-1} x_1 \dots x_{d-i} x_j \otimes y_1 \wedge \dots \wedge y_{j-1} \wedge y_{j+1} \wedge \dots \wedge y_i$, hence this time we obtained the Koszul complex (see again [FLS], sect. 3).

Already at this stage we can see the difference between the homological and cohomological complex. $\mathbf{S}^d(V)$ equipped with the Koszul differential is acyclic, but equipped with the De-Rham has nontrivial cohomology for $d$ divisible by $p$. Namely, Cartier's theorem ([Ca]; [FS], Th. 4.1) says that there is an isomorphism of graded functors $H^*(\mathbf{S}^{pd}) \simeq (\mathbf{S}^d)^{(1)}$, where $(-)^{(1)}$ means the Frobenius twist of a functor. Here is the first place where we use the assumption on $\mathbf{k}$ and also the language of strict polynomial functors plays some (positive) role. I shall describe this point in some detail. Perhaps I should begin with explaining what the "Frobenius twist" means in this context. The Frobenius twist $V^{(1)}$ of a space $V$ is the same space but with the action of scalars induced by the Frobenius automorphism ($c.v := c^p \cdot v$). The assignment $V \longrightarrow V^{(1)}$ is (a strict polynomial) functor and, strictly speaking, $F^{(1)}$ means the composition of functors $(-)^{(1)} \circ F$ (cf. [FS], sect. 1). Thus of course, this concept makes sense for fields of positive characteristic only (in fact, the De–Rham complex over a field of characteristic 0 is acyclic (=exact)). Viewing $F^{(1)}$ as a SP–functor may be sometimes profitable. To see this, let us assume for the moment that $\mathbf{k}$ has cardinality $p$. Then the functors $F$ and $F^{(1)}$ are isomorphic, but they are not as SP–functors. Thus the language of SP–functors provides some additional information about functors. For example, there is a notion of degree of an SP–functor ([FS], Lemma 2.2) refining the notion of the Eilenberg–LacLane degree of a functor. The principal advantage of the SP–degree is that there are no transformations between SP–functors of different degree (to be precise: between homogeneous functors in the terminology of [FS]). It will be important for us that $\mathbf{S}_{\lambda/\mu}$ consists of (homogeneous) SP–functors of degree $|\lambda| - |\mu|$ and that the Frobenius twist of a functor of degree $d$ has degree $pd$, which enables us to compute the SP–degree of functors appearing in Cartier's Theorem. This information, for example, allows to exclude the possibility of existence of some maps as we will see in Section 6.

Now I would like to discuss the case $\lambda/\mu = (d)$ which will be even more important than $(1^d)$. We will denote the corresponding Schur–De-Rham complex by $\mathbf{\Lambda}^d(V)$. This time in the $i$th degree component we have $\Lambda^{d-i}(V) \otimes D^i(V)$, where $D^i(V)$ stands for the $i$–th divided power of a space ie. $D^i(V) = (V^{\otimes i})^{\Sigma_d}$ for the permutative action. In order to describe the differential it is convenient to use a multiplicative structure on $D^*(V)$. Since this structure is less popular than that on the symmetric power, I shall recall it briefly (cf. [ABW], I.4). Let $Y = \{y_1, \ldots, y_n\}$ be a basis of $V$. Then, like in the symmetric power, the set of weakly increasing sequences of elements of Y of length $n$ forms a basis of $D^n(V)$. There exists a multiplication $D^n(V) \otimes D^m(V) \longrightarrow D^{n+m}(V)$. The point is, that it does not act on basis sequences just as concatenation, but some multiplicities occur when we take a power of an element. Namely: $y^n \cdot y^m = \left(\begin{array}{c} n+m \\ n \end{array}\right) y^{n+m}$. Thus for example: $y_1 y_2^2 y_4 \cdot y_2^3 y_3 y_4 = 20 y_1 y_2^5 y_3 y_4^2$. The cohomological differential is given by the formula: $\delta(x_1 \wedge \ldots \wedge x_{d-i} \otimes y_1^{k_1} \ldots y_{i'}^{k_{i'}}) = \sum_{j=0}^{d-i-1} (-1)^{d-i-j} x_1 \wedge \ldots \wedge x_{j-1} \wedge x_{j+1} \wedge \ldots \wedge x_{d-i} \otimes y_j \cdot y_1^{k_1} \ldots y_{i'}^{k_{i'}}$. It turns out that $\mathbf{\Lambda}^d$ is a dual complex to $\mathbf{S}^d$ in the sense of the Kuhn duality. This duality (known in representation theory as "contravariant duality") is a contravariant involution on the category of functors which sends a functor $F$ to $F^{\#}$ which is defined by the formula $F^{\#}(V) := (F(V^*))^*$. Then it is easy to see that $(\mathbf{S}^d(V))^i = ((\mathbf{\Lambda}^d(V))^{d-i})^{\#}$ and that the differentials are mutually dual. Thus the "dual Cartier's theorem" holds: $H^*(\mathbf{\Lambda}^{pd}) = \mathbf{\Lambda}^{d(1)}[(p-1)d]$ (the shift of grading is caused by the fact that the duality "turns the complex upside down"). Similarly, $\mathbf{\Lambda}^d$ equipped with a homological differential is dual to the Koszul complex.

Luckily, many basic properties of the Schur–De-Rham complex may be obtained by repeating word for word proofs of the corresponding properties of the Schur complex. The most important for us will be a description of elements of $\mathbf{S}_{\lambda/\mu}$ in terms of tableaux, which we recall in the moment, but in the forthcoming sections we will also use the Decomposition Formula ([ABW], Th. V.1.13), the Littlewood–Richardson rule [Bo] etc.

We now give the promised description of a basis of $\mathbf{S}_{\lambda/\mu}(V)$. Let $X = \{x_1, \ldots, x_n\}$ be a basis of a copy of $V$ placed in degree 0 of $\mathbf{K}$ and $Y = \{y_1, \ldots, y_n\}$ a basis of that in degree 1. We order totally the set $X \cup Y$ putting $x_1 < y_1 < x_2 < \ldots < y_n$ and assume that the differential in $\mathbf{K}$ takes $x_i$ to $y_i$. By a *tableau of shape* $\lambda/\mu$ *with values in* $X \cup Y$ we mean a function from the set of boxes of $\lambda/\mu$ to $X \cup Y$. We say that a tableau is *row* $Y$*–standard* if we have in each row a nondecreasing sequence and repetitions occur only among elements of $Y$, we say it is *column* $Y$*–standard* if we have in each column a nondecreasing sequence and repetitions occur only among elements of $X$. A tableau which is both row and column $Y$–standard is called $Y$*–standard.* Since $\mathbf{\Lambda}^d(V) = \bigoplus_{i=0}^d \Lambda^{d-i}(V) \otimes D^i(V)$, the set of row $Y$–standard tableaux of shape $\lambda/\mu$ with values in $X \cup Y$ forms its basis. The main result of [ABW] (Th. V.1.10) says that the the structural epimorphism $\phi_{\lambda/\mu} : \mathbf{\Lambda}^{\lambda/\mu}(V) \longrightarrow \mathbf{S}_{\lambda/\mu}(V)$ takes the set of $Y$–standard tableaux of shape $\lambda/\mu$ with values $X \cup Y$ to a basis of $\mathbf{S}_{\lambda/\mu}(V)$. This theorem was proved by establishing an effective algorithm called the Straightening Formula allowing to express any tableau (regarded as an element of $\mathbf{S}_{\lambda/\mu}(V)$) as a linear combination of $Y$–standard tableaux. We will use this algorithm in the next section.

# 4 Cohomology: the first steps

We start with the simplest diagrams. We say that a skew diagram $\lambda/\mu$ is a *skew hook* if it is connected (ie. it cannot be presented as a disconnected sum of nontrivial diagrams) and it does not contain a subdiagram isomorphic to $(2, 2)$. The set of boxes of a skew hook may be totally ordered in an obvious manner: we say that a box $(\lambda/\mu)_{ij}$ (ie. belonging to the $i$th row and $j$th column) is greater than $(\lambda/\mu)_{i'j'}$ if $i < i'$ or $j > j'$. The smallest box of a skew hook will be called the foot while the largest — the hand. Of course, our terminology agrees with that concerning hooks and rim hooks, which are also skew hooks. The following easy observation explains the importance of this class of diagrams.

**Fact 4.1** *If* $\dim(V) = 1$*, then* $\mathbf{S}_{\lambda/\mu}(V) \neq 0$ *if and only if* $\lambda/\mu$ *is a disconnected sum of skew hooks.*

**Proof:** It is easy to see that there are no $Y$–standard tableaux of shape $(2,2)$ with values in the set $X \cup Y = \{x, y\}$. Thus if $\lambda/\mu$ contains $(2,2)$ then $\mathbf{S}_\lambda(V) = 0$ (we recall that $x$ spans $\Lambda^1(V)$ and $y$ spans $D^1(V)$). But a diagram does not contain $(2,2)$ if and only if it is a disconnected sum of skew hooks.

To conclude the proof we should construct a $Y$–standard tableau for any disconnected sum of skew hooks. Since a Schur–De-Rham complex associated to a disconnected sum is a tensor product of Schur–De-Rham complexes associated to factors, it suffices to consider the case when $\lambda/\mu$ is a skew hook. In order to construct a tableau we will fill subsequent boxes of $\lambda/\mu$ with $x$ or $y$ going from the foot to the hand according to the total order. We start by putting $x$ or $y$ to the foot. Observe that in the next box the condition of $Y$–standardness already determines an element we should put. Namely if this box lies above the foot then we must put $x$ while if it lies to the right of the foot then we put $y$. Turning to the next box we must use this rule again etc. Thus we see that there exist exactly two $Y$–standard tableaux of shape $\lambda/\mu$ with values in $\{x, y\}$ which only differ by an element put to the foot. We denote them by $\Psi(x)$ and $\Psi(y)$ respectively:

$$\Psi(x) = \begin{ytableau} \none & \none & \mathrm{x} \\ \none & \none & \mathrm{x} \\ \mathrm{x} & \mathrm{y} & \mathrm{y} \\ \mathrm{x} \\ \mathrm{x} \end{ytableau} \qquad\qquad \Psi(y) = \begin{ytableau} \none & \none & \mathrm{x} \\ \none & \none & \mathrm{x} \\ \mathrm{x} & \mathrm{y} & \mathrm{y} \\ \mathrm{x} \\ \mathrm{y} \end{ytableau}$$

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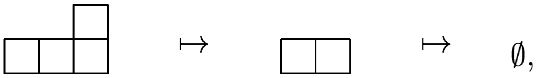
The next natural question concerns the behaviour of a differential for a one–dimensional space. The following fact is responsible for the complexity of the cohomology of the Schur–De-Rham complex.

**Fact 4.2** *For any skew hook* $\lambda/\mu$ *of weight* $n$*, we have* $\delta(\Psi(x)) = \pm n\Psi(y)$.

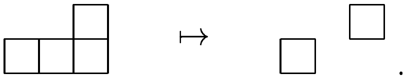
**Proof:** Let us first recall the action of $\delta$ in terms of tableaux. Writing down a tableau is just taking the preimage of an element in $\mathbf{\Lambda}^{\lambda/\mu} = \mathbf{\Lambda}^{\lambda_1-\mu_1} \otimes \ldots \otimes \mathbf{\Lambda}^{\lambda_l-\mu_l}$.Thus it suffices to remember the formula for the differential in

$\mathbf{\Lambda}^k$, which is $\delta(x \otimes y^{k-1}) = ky^k$. Throughout the proof we will identify a tableau $t$ of shape $\lambda/\mu$ with the sequence $(t^1, \ldots, t^l)$ consisting of tableaux filling successive rows of $\lambda/\mu$ ($l$ is the number of rows of $\lambda/\mu$). It is easy to see that we associate to $\Psi(x)$ the sequence $(\Psi(x), \ldots, \Psi(x))$. Therefore $\delta(\Psi) = \sum_{j=1}^{l}(-1)^{h(j)}(\lambda_j - \mu_j)s^j$, where $s^j$ stands for the tableau $(\Psi(x), \ldots, \Psi(y), \ldots, \Psi(x))$ in which $\Psi(y)$ is placed in the $j$th row and $h(j)$ is a sum of degrees of elements standing in the earlier rows. This sum is equal to the number of appearings of $y$ in the earlier rows or to the number of columns lying to the left of the hand of the $j$th row. Thus we got a sum of exactly $n$ tableaux (with different signs). The proof will be finished if we show that all factors in this sum are in fact equal in $\mathbf{S}_{\lambda/\mu}(V)$. But observe that if we apply the Straightening Formula ([ABW], p. 264) to the $j$th and $(j+1)$th row of $\lambda/\mu$, we get the ralation $s^{j+1} = (-1)^{\lambda_j - \mu_j - 1}s^j$. Therefore $(-1)^{h(j+1)}s^{j+1} = (-1)^{h(j)}s^j$, which completes the proof. ■

These elementary observations lead to a general necessary condition for the cohomological nontriviality of the Schur complex. To express it we should slightly generalize the notion of the $n$–core of a diagram. Namely, we call a $n$–hook in a skew diagram $\lambda/\mu$ any $n$–hook in $\lambda$ whose rim $n$–hook is contained in $\lambda \setminus \mu$ (we do not demand the original hook to be contained in $\lambda \setminus \mu$). We alert the reader that the notion of the $n$–core of a skew diagram is not well defined. for example, by removing rim 2–hooks from the digram $(3,3)/(2)$ we may get the empty diagram



or the diagram $(3,1)/(2)$



Nevertheless, we say that $\lambda/\mu$ has a trivial $n$–core if it is possible to reach the empty diagram by the removing rim $n$–hooks (hence, according to our definition, the diagram $(3,3)/(2)$ has a trivial 2–core).

**Fact 4.3** *If $H^*(\mathbf{S}_{\lambda/\mu})$ is nontrivial then $\lambda/\mu$ has a trivial $p$–core (we recall* *that $p$ is the characteristic of our ground field).*

**Proof:** We proceed by induction on $\dim(V)$. If $\dim(V) = 1$ then, according to Fact 4.1, the only diagrams for which our claim is nontrivial are disconnected sums of skew hooks. Therefore, thanks to Fact 4.2, we should only show that a skew hook of weight divisible by $p$ has a trivial $p$–core. Thus let us take such a skew hook and try to find a rim $p$–hook in it. When we take the first $p$ boxes (counting from the foot), then they form a rim $p$–hook if and only if the $(p+1)$th box lies above the $p$th. But if the $(p+1)$th box lies to the right of the $p$th, then it can be a foot of a hook in $\lambda/\mu$. Thus, the boxes from the $(p+1)$th to the $(2p)$th form a rim $p$–hook if and only if the $(2p+1)$th lies above the $(2p)$th. If we are still unlucky then we can try the next $p$ boxes etc. In the worst case it will turn out that the last $p$ boxes form a rim $p$–hook. Thus we see that any skew hook of weight divisible by $p$ contains a rim $p$–hook. Moreover, it follows from the construction that this rim $p$–hook may be chosen in such a way that after removing it our skew hook breaks up into two skew hooks of weights divisible by $p$. This finishes the proof in the case $\dim(V) = 1$.

We now assume our assertion for all spaces of dimension smaller than $\dim(V)$. We take $V = W \oplus L$ where $\dim(L) = 1$. By the Decomposition Formula ([ABW], Th. V.1.10) we have a filtration on $\mathbf{S}_{\lambda/\mu}(W \oplus L)$ with the associated graded object

$$\bigoplus_{\mu \subseteq \alpha \subseteq \lambda} \mathbf{S}_{\alpha/\mu}(W) \otimes \mathbf{S}_{\lambda/\alpha}(L).$$

It suffices to show that all factors in the sum are acyclic. We may restrict our attention to the factors where $\lambda/\alpha$ is a disconnected sum of skew hooks of weights divisible by $p$, since otherwise $\mathbf{S}_{\lambda/\alpha}(L)$ is acyclic. We will show that in such a situation $\mathbf{S}_{\alpha/\mu}(W)$ is acyclic. Thanks to the induction assumption it suffices to show that $\alpha/\mu$ has a nontrivial $p$–core. But observe that $\alpha/\mu$ may be obtained from $\lambda/\mu$ by removing rim $p$–hooks, since $\lambda/\alpha$ has a trivial $p$–core (we use here the obvious fact that a $p$–hook in $\lambda/\alpha$ is also a $p$–hook in $\lambda/\mu$). Thus if $\alpha/\mu$ would have a trivial $p$–core then the same would hold for $\lambda/\mu$. This contradiction completes the proof. ■

We finish this section by examining the simplest situation where the cohomology is nontrivial.

**Fact 4.4** $H^*(\mathbf{S}_{(k+1,1^{p-k-1})} = \mathbf{S}^{(1)}_{(1)}[k]$.

**Proof:** This fact may be easily derived from Fact 4.2 and the Decomposition Formula. The proof which will be presented is slightly more complicated but gives some additional information which will be useful later. We proceed by induction on $k$. For $k = 0$ we get Cartier's Theorem. In order to make the induction step we apply the Littlewood-Richardson rule (see [Bo]) to the complex $\mathbf{\Lambda}^k \otimes \mathbf{S}^{p-k}$. We get the short exact sequence

$$0 \longrightarrow \mathbf{S}_{(k+1,1^{p-k-1})} \longrightarrow \mathbf{\Lambda}^k \otimes \mathbf{S}^{p-k} \longrightarrow \mathbf{S}_{(k,1^{p-k})} \longrightarrow 0,$$

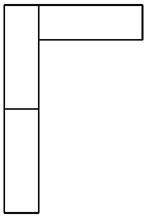
in which the middle term is acyclic by Cartier's Theorem. Thus we get the required shift of the cohomological grading. ■

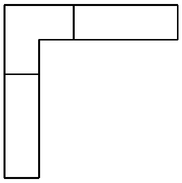
We now derive some consequences of the proof. Namely we would like to obtain a more explicit description of the cohomology. We start with the case $k = 0$. Then the isomorphism $\mathbf{I}^{(1)} \simeq H^*(\mathbf{S}^p)$ is realized by the formulae $x \mapsto x^p$, $y \mapsto x^{p-1} \otimes y$ (see [FLS], p. 520), where the second formula becomes linear only after taking cohomology. Observe that in the language of tableaux this is the map $\Psi$ which we defined at the beginning of the section. To describe explicitly the cohomology of other hooks we recall that the connecting homomorphism in the sequence which we used in the proof $\delta_{k,k-1} : H^*(\mathbf{S}_{(k,1^{p-k})}) \longrightarrow H^{*+1}(\mathbf{S}_{(k+1,1^{p-k-1})})$ is an isomorphism. Computing its values directly we get the following relations: $\delta_{k,k-1}(\Psi_{k-1}(x)) = k\Psi_k(x)$, and $\delta_{k,k-1}(\Psi_{k-1}(y)) = (-1)^k k\Psi_k(x)$, where $\Psi_k$ is a map which sends an element to a tableau of shape $(k+1, 1^{p-k-1})$. Thus the composition $\delta_{k,k-1} \circ \ldots \circ \delta_{1,0} \circ \Psi_0$ yields the isomorphism $\Phi_k : \mathbf{I}^{(1)}[k] \simeq H^*(\mathbf{S}_{(k+1,1^{p-k-1})})$, which, up to a nonzero scalar factor (depending on $k$ and cohomological degree), is equal to $\Psi_k$.

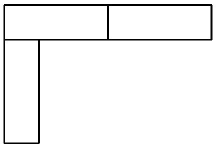
# 5 Cores, quotients and enlargement of a diagram

We now should look closer at diagrams with a trivial $p$–core. As we saw in Facts 4.2 and 4.3, in a sense, cohomology is concentrated in rim $p$–hooks. Thus we should understand how a diagram may be divided into rim $p$–hooks. A combinatorial structure which controls this process is $p$–quotient. A Young diagram $\lambda$ is determined by its $p$–core $c(\lambda)$ and the ordered family od diagrams $\{q^0(\lambda), \ldots, q^{p-1}(\lambda)\}$ called the *p-quotient of* $\lambda$. When we remove from a diagram a rim $p$–hook then we do not change its core but we remove one box from its quotient. The precise algorithm may be found in ([JK], sect. 2.7). For our purposes the most important thing will be to recognize from which diagram in the quotient we should remove a box. The answer is that if the hand of a hook belongs to the $i$th row and $j$th column then we choose the diagram $q^k(\lambda)$, for $k = (j-i)(\mathrm{mod}\ p)$ (this number is called the *$p$–residue* of a box). Let us illustrate this algorithm by a simple example. We consider the diagram $\lambda = (2,2)$ for $p = 2$. Then $c(\lambda) = \emptyset, q^0(\lambda) = q^1(\lambda) = (1)$. According to what was just said, there should be two ways of reaching the empty diagram from $(2,2)$. The first when we start with removing the (only) box from $q^0(\lambda)$ and the second when we start with $q^1(\lambda)$. Indeed: in the first case we remove from $(2,2)$ the second row (whose hand has residue 0) and we are left with the diagram $(2)$ which is a 2–hook with a hand of residue 1. In the second case we do the same with the columns of $\lambda$. Already this example suggests that the cohomology of the Schur–De-Rham complex should be particularly simple for diagrams with a quotient consisting of only one nonempty diagram. From now on we focus on this particular situation.

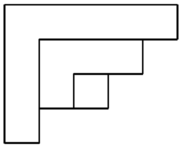
We start with introducing another bit of notation. For a given diagram $\lambda$ we will denote by $F_k(\lambda)$ the diagram with $q^k(F_k(\lambda)) = \lambda$ and $c(F_k(\lambda) = q^0(F_k(\lambda)) = \ldots = q^{k-1}(F_k(\lambda)) = q^{k+1}(F_k(\lambda)) = q^{p-1}(F_k(\lambda)) = \emptyset$. Our first task will be to understand how $F_k(\lambda)$ looks like, which is not at all clear relying on its definition from [JK] only. We first consider the case of a hook $\lambda = (n, 1^m)$. Then it is easy to check (eg. with the aid of a "star diagram" ([JK], p. 85)) that $F_k(\lambda) = (p(n-1)+k+1, 1^{p(m+1)-k-1})$. Moreover, in the case of hooks, the process of removing of rims $p$–hooks is, in a sense, uniquely determined. Assume first that $\lambda$ is not of the form $(n)$ or $(1^m)$. Then there are two $p$–hooks in $F_k(\lambda)$: the first consists of the first $p$ boxes (counting from the foot) in the leg of $F_k(\lambda)$ and corresponds to the foot of $\lambda$, the second consists of the last $p$ boxes in the arm of $F_k(\lambda)$ and corresponds to the hand. Thus of course, we may remove rim $p$–hooks in many different ways (until a $p$–quotient becomes one–rowed or one–columned). Uniqueness means the following: if we divide the set of boxes of $F_k(\lambda)$ into $p$–elemented subsets consisting of successive rim $p$–hooks, then we always get the same family of subsets (no matter in which order we removed boxes from the $p$–quotient). Moreover, to a given box in $\lambda$ there corresponds the same set of boxes in $F_k(\lambda)$. Thus we get a bijection between the set of boxes in $\lambda$ and some family of disjoint $p$–elemented subsets of the set of boxes of $F_k(\lambda)$. It is also easy to see that in this bijection we assign to the boxes from the leg of $F_k(\lambda)$ (except the corner) the subsets in $F_k(\lambda)$ of shape $(1)^p$, to the boxes from the arm (except the corner again) the subsets of shape $(p)$, and finally we assign to the corner of $\lambda$ the subset of shape $(k+1, 1^{p-k-1}) = F_k((1))$. Here we draw $F_0((2,1))$, $F_1((2,1))$, $F_2((2,1))$ for $p=3$, dividing diagrams into appropriate arrangements of boxes







Let us turn to the case of an arbitrary diagram $\lambda$. We say that $\lambda$ has the *principal diagonal of length* $e$ if $\lambda_e \geq e$ but $\lambda_{e+1} \leq e$. We assign to $\lambda$ a sequence of hooks $(\chi_1, \dots, \chi_e)$ in the following manner. We take as $\chi_1$ a hook consisting of the first row and the first column of $\lambda$, as $\chi_2$ a hook consisting of the first row and the first column of $\lambda \setminus \chi_1$ etc. It is clear that after $e$ steps we are left with the empty diagram. We call this procedure the *decomposition of a diagram into hooks.* The picture presents the decomposition of $(5,4,3,1)$ into hooks

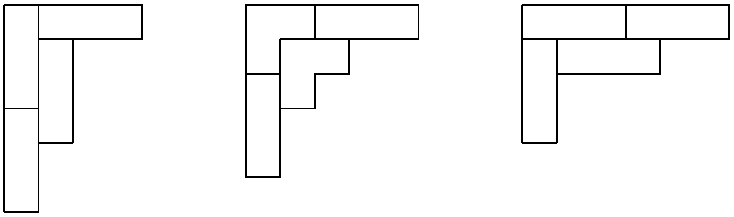


This point of view allows to describe $F_k(\lambda)$ in a very convenient way

**Fact 5.1** *Let* $(\chi_1, \dots, \chi_e)$ *be the decomposition of* $\lambda$ *into hooks. Then the sequence* $(F_k(\chi_1), \dots, F_k(\chi_e))$ *is the decomposition into hooks of* $F_k(\lambda)$.

To prove this claim it suffices to draw a "star diagram" ([JK] p. 85) for a diagram with decomposition into hooks $(F_k(\chi_1), \dots, F_k(\chi_e))$. ■

Moreover, we observe that we still have a bijection between the set of boxes in $\lambda$ and certain dissection of the set of boxes in $F_k(\lambda)$ into $p$–elemented subsets. We call this dissection the *slicing* of $F_k(\lambda)$, and we call these $p$–elemented subsets the *slices.* We obtain this bijection just by assembling the bijections for all hooks in the decomposition of $\lambda$ into hooks. Again the best way to understand how it works is to draw a picture. Here we have the slicings of the diagrams $F_0((2,2))$, $F_1((2,2))$, $F_2((2,2))$ for $p=3$



Similarly to the case of hooks, the set of slices (and also the correspondence between the slices in $F_k(\lambda)$ and the boxes in $\lambda$) does not depend on the order of removals of rim $p$–hooks. This time however, to avoid any confusion, I make this statement precise:

**Fact 5.2** *Let $F_k(\lambda) = \alpha^0 \supset \alpha^1 \supset \ldots \supset \alpha^d = \emptyset$ be a sequence of diagrams in which each diagram $\alpha^j$ is obtained from $\alpha^{j-1}$ by removing a rim $p$–hook. Let $\lambda = \beta^0 \supset \beta^1 \supset \ldots \supset \beta^d = \emptyset$ be the corresponding sequence of $p$–quotients. Then for every $0 \le j < d$ the set of boxes $\alpha^j \setminus \alpha^{j+1}$ is the slice associated to the box $\beta^j \setminus \beta^{j+1}$ in the bijection we just described.*

**Proof:** In order to understand that this fact requires any proof one should recall the diagram $(2,2)$ which was not of the form $F_k(\lambda)$. For that diagram, as we remember, a different order of removing boxes from the $p$–quotient led to different "sets of slices" (which consisted either of rows or of columns of the diagram).

But in our situation, thanks to detailed knowledge about the structure of a diagram $F_k(\lambda)$, the proof is easy. In fact, the only nontrivial thing is to check that each rim $p$–hook in $F_k(\lambda)$ is a slice. But this follows immediately from the fact that the lengths of two consecutive rows whose ends lie above the principal diagonal of $F_k(\lambda)$ differ by $-1(\text{mod } p)$, and the analogous fact for columns with ends above the diagonal. ■

Thus the operation $F_k$ may be thought of as a kind of $p$–times enlargement of a diagram, for we replace each box of $\lambda$ by a $p$–hook. Taking into account Fact 4.4 it is quite reasonable to expect the following description of the cohomology of the Schur–De-Rham complex

**Theorem 5.3** *For any skew diagram $\lambda/\mu$ and $0 \le k < p$*

$$H^*(\mathbf{S}_{F_k(\lambda)/F_k(\mu)}) = \mathbf{S}^{(1)}_{\lambda/\mu}[h_k(\lambda/\mu)],$$

*where the shift of grading is given by the formula*

$$h_k(\lambda/\mu) = (p-1)f_{\lambda/\mu} + ke_{\lambda/\mu},$$

*where $e_{\lambda/\mu}$ is the number of boxes of $\lambda/\mu$ lying on the principal diagonal and $f_{\lambda/\mu}$ is the number of boxes of $\lambda/\mu$ lying above it.*

Unfortunately the proof of this theorem is quite involved and undirect. We must start by better understanding relationship between enlargement of a diagram and the Decomposition Formula.

# 6 Homological Decomposition Formula and compatible families of transformations

Since some formulae in this section will be quite complicated, I would like to simplify notation. Namely from now on, we will denote skew diagrams just by $\lambda$ etc. not specifying, if not necessary, a subdiagram we divide through. For example, the expression $\alpha \subset \lambda$ is an abbreviation for saying that we have skew diagrams $\lambda/\mu$, $\alpha/\mu$ for which $\alpha \subset \lambda$. Also eg. $\lambda_j$ means in fact the skew diagram $(\lambda_1, \dots, \lambda_j)/(\lambda_1, \dots, \lambda_{j-1}, \mu_j)$. To get used to this convention we recall the Decomposition Formula ([ABW], sect. II.4) for a skew diagram $\lambda$. It is an SP–filtration $\{M_\alpha\}_{\alpha\subseteq\lambda}$ of a functor in two variables $\mathbf{S}_\lambda(V \oplus W)$ with the graded object

$$\bigoplus_\alpha \mathbf{S}_\alpha(V) \otimes \mathbf{S}_{\lambda/\alpha}(W).$$

Here we made further simplification of the notation not writing that the sum is taken over $\alpha$ contained in $\lambda$. We will assume it tacitly whenever in our formulae skew diagrams $\lambda/\alpha$ appear. The ordering in the filtration is the lexicographic order.

In general this filtration does not split. Nevertheless, dividing it into parts of different SP–degree we get a splitting in the category of bifunctors

$$\mathbf{S}_\lambda(V \oplus W) = \bigoplus_{0\le j\le|\lambda|} {}_j\mathbf{S}_\lambda(V \oplus W),$$

(of course, $|\lambda|$ means the weight of a skew diagram) and each ${}_j\mathbf{S}_\lambda(V \oplus W)$ has a filtration with the graded object

$$\bigoplus_{|\alpha|=j} \mathbf{S}_\alpha(V) \otimes \mathbf{S}_{\lambda/\alpha}(W).$$

Here we benefit from the fact that there are no transformations between homogeneous SP–functors of different degrees. To prove the existence of this splitting without appealing to SP–functors one should observe that the filtration is defined over the integers and investigate the effect of extension of scalars. I leave the details to the (interested) reader.

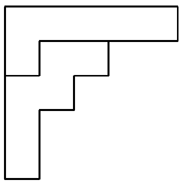
From this splitting we derive useful consequences: if $\alpha$ is the lexicographically smallest subdiagram of a given weight in $\lambda$, then we have a transformation

$$\mathbf{S}_{\alpha}(V) \otimes \mathbf{S}_{\lambda/\alpha}(W) \longrightarrow \mathbf{S}_{\lambda}(V \oplus W),$$

and similarly, if $\beta$ is the largest subdiagram of a given weight in $\lambda$, then we have

$$\mathbf{S}_{\lambda}(V \oplus W) \longrightarrow \mathbf{S}_{\beta}(V) \otimes \mathbf{S}_{\lambda/\beta}(W).$$

We now turn to diagrams enlarged by the operation $F_k$. Also here we will use a simplified notation. First, recall that we have not considered the notion of the $p$–quotient of a skew diagram. Thus when $\lambda$ is skew (ie. "$\lambda = \lambda/\mu$") then $F_k(\lambda)$ stands for the diagram $F_k(\lambda)/F_k(\mu)$. Next, by $\mathbf{\Lambda}^{F_k(\lambda)}$ we do not mean $\mathbf{\Lambda}^{(F_k(\lambda))_1} \otimes \ldots \otimes \mathbf{\Lambda}^{(F_k(\lambda))_l}$ which complex never appears in our considerations but rather $\mathbf{S}_{F_k((\lambda_1))} \otimes \ldots \otimes \mathbf{S}_{F_k((\lambda_l))}$. I should alert the reader that even if $\lambda$ is solid, the diagram $F_k((\lambda_1))$ need not to be a horizontal hook (for $k < p-1$), and that $F_k((\lambda_j))$ for $j > 1$ is just a skew hook. The picture presents $F_1((2,2))$ for $p = 3$ divided into skew hooks corresponding to the rows of $(2,2)$



We take a similar convention for $\mathbf{S}^{F_k(\widetilde{\lambda})}$ (one should also remember that $\widetilde{F_k(\lambda)} = F_{p-1-k}(\widetilde{\lambda})$).

Let us look at the Decomposition Formula for $\mathbf{S}_{F_k(\lambda)}$, but we will be interested only in factors cohomologically nontrivial. Notice a simple but important combinatorial fact.

**Fact 6.1** *If $\beta \subseteq F_k(\lambda)$ is such that $F_k(\lambda)/\beta$ has a trivial $p$–core then $\beta = F_k(\alpha)$ for some $\alpha \subseteq \lambda$.*

**Proof:** Readily, it suffice to consider the situation when $\lambda$ is solid. Then, since $F_k(\lambda)/\beta$ has a trivial $p$–core, we may obtain $\beta$ from $F_k(\lambda)$ by removing rim $p$–hooks. Therefore $\beta$ must, like $F_k(\lambda)$, have a trivial $p$–core and the $p$–quotient contained in the $p$–quotient of $F_k(\lambda)$. This completes the proof. ■

Thanks to this fact and Fact 4.3, if we neglect acyclic factors in the Decomposition Formula for $\mathbf{S}_{F_k(\lambda)}$, we get the graded object labeled by the same set of diagrams as in the formula for $\mathbf{S}_\lambda$ ($F_k$ preserves the lexicographic order):

$$\bigoplus_{0\leq j\leq|\lambda|}\ \bigoplus_{|\alpha|=j} \mathbf{S}_{F_k(\alpha)}(V)\otimes\mathbf{S}_{F_k(\lambda/\alpha)}(W),$$

but one should remember that the cohomology of the total object need not to have a filtration with the quotients being the cohomlogy of the quotients of the filtration, because the spectral sequence of the filtration may have nontrivial differentials. Next, we observe that if $\alpha$ is the (lexicographically) smallest subdiagram of a given weight in $\lambda$, then we still have a well defined transformation

$$H^*(\mathbf{S}_{F_k(\alpha)})(V)\otimes H^*(\mathbf{S}_{F_k(\lambda/\alpha)})(W)\longrightarrow H^*(\mathbf{S}_{F_k(\lambda)})(V\oplus W),$$

and an analogous map exists for the largest subdiagram.

The main difficulty in the proof of Theorem 5.3 is to construct a transformation $\Phi_\lambda : \mathbf{S}^{(1)}_\lambda[h_k(\lambda)] \longrightarrow H^*(\mathbf{S}_{F_k(\lambda)})$ with some good properties . Once we have it at our disposal we will prove in rather formal manner that it is an isomorphism. We postpone its construction to the next section. Now, we make it precise what we mean by “good properties” of a transformation and how to use them to prove Theorem 5.3.

We will prove that the transformation is an isomorphism inductively using the Decomposition Formula. Therefore we should, together with $\Phi_\lambda$, define transformations $\Phi_\alpha, \Phi_{\lambda/\alpha}$ for all $\alpha\subset\lambda$ in a way compatible with the Decomposition Formula. The following definition extracts properties of transformations we need for a proof.

**Definition 6.2** *A family of transformations*

$$\Phi_{\alpha/\alpha'} : \mathbf{S}^{(1)}_{\alpha/\alpha'}[h_k(\alpha/\alpha')]\longrightarrow H^*(\mathbf{S}_{F_k(\alpha/\alpha')}),$$

*defined for all pairs $\alpha, \alpha'$ such that $\alpha'\subseteq\alpha\subseteq\lambda$ is called a compatible family of transformation for $\lambda$, if it satisfies for all $\alpha'\subseteq\alpha''\subseteq\alpha$ the following conditions:*

- *There exists a morphism* $\widetilde{\Phi}_{\alpha/\alpha',\alpha''} : M^{*(1)}_{\alpha''}(V \oplus W)[h_k(\alpha/\alpha')] \longrightarrow H^*(M_{F_k(\alpha'')})(V \oplus W)$ *making the diagram*

$$\begin{tikzcd}
M^{*(1)}_{\alpha''}(V \oplus W)[h_k(\alpha/\alpha')] \arrow[r, "\widetilde{\Phi}_{\alpha/\alpha',\alpha''}"] \arrow[dr, "\Phi_{\alpha/\alpha'}\big|_{M^{*(1)}_{\alpha''}}"'] & H^*(M_{F_k(\alpha'')})(V \oplus W) \arrow[d] \\
 & H^*(\mathbf{S}_{F_k(\alpha/\alpha')})(V \oplus W)
\end{tikzcd}$$

  *commutative.*

- *Therefore if $\alpha'''$ is the lexicographic predecessor of $\alpha''$, then we obtain the commutative diagram*

$$\begin{tikzcd}
M^{*(1)}_{\alpha'''}(V \oplus W)[h_k(\alpha/\alpha')] \arrow[r, "\widetilde{\Phi}_{\alpha/\alpha',\alpha'''}"] \arrow[d] & H^*(M_{F_k(\alpha''')})(V \oplus W) \arrow[d] \\
M^{*(1)}_{\alpha''}(V \oplus W)[h_k(\alpha/\alpha')] \arrow[r, "\widetilde{\Phi}_{\alpha/\alpha',\alpha''}"] \arrow[d] & H^*(M_{F_k(\alpha'')})(V \oplus W) \arrow[d] \\
\mathbf{S}^{(1)}_{\alpha''/\alpha'}(V) \otimes \mathbf{S}^{(1)}_{\alpha/\alpha''}(W)[h_k(\alpha/\alpha')] & H^*(\mathbf{S}_{F_k(\alpha''/\alpha')})(V) \otimes H^*(\mathbf{S}_{F_k(\alpha/\alpha'')})(W),
\end{tikzcd}$$

  *which may be completed to the commutative diagram by exactly one bottom arrow. We requires this arrow to be* $\Phi_{\alpha''/\alpha'}(V) \otimes \Phi_{\alpha/\alpha''}(W)$.

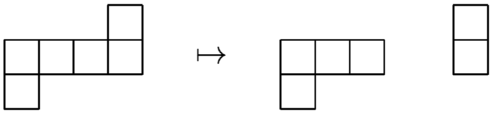
The meaning of the first condition is that a compatible family takes the filtration $M^{*(1)}_{\alpha''}[h_k(\alpha/\alpha')]$ to $H^*(M_{F_k(\alpha'')})$. But since we do not know a priori whether the map induced on the cohomology by the embedding $M^*_{F_k(\alpha'')} \longrightarrow \mathbf{S}_{F_k(\alpha/\alpha')}$ is a monomorphism we were forced to express this condition in such an awkward way. We should also remember that the second column in the diagram in the second condition need not to be a short exact sequence. But the existence and uniqueness of the bottom arrow follows easily by a diagram chasing. Also, perhaps at first glance it is not clear why we introduced yet another diagram $\alpha''$. The reason is that it makes our definition hereditary ie. a compatible family for $\lambda$ restricts to a compatible family for any $\alpha/\alpha'$ such that $\alpha' \subseteq \alpha \subseteq \lambda$.

Now we show, as we have promised, that the very existence of a compatible family is almost sufficient to prove Theorem 5.3.

**Lemma 6.3** *If in a compatible family for $\lambda$ all transformations $\Phi_{\alpha/\alpha'}$ for $\alpha' \subset \alpha \subseteq \lambda$ such that $\alpha/\alpha'$ consists of a single box are isomorphisms, then $\Phi_\lambda$ is an isomorphism.*

**Proof:** We proceed by induction on $\dim(V)$. Let us first assume that $\dim(V) = 1$. Since $F_k(\lambda)$ is a skew hook if and only if so is $\lambda$, we may assume that $\lambda$ is a skew hook. We start another induction, this time with respect to the weight of diagrams. More precisely: we are going to show that $\Phi_{\alpha/\alpha'}(V)$ is an isomorphism inductively on the weight of $\alpha/\alpha'$. Thanks to the assumption of the lemma we may begin our induction. We consider an arbitrary $\alpha/\alpha'$ assuming an isomorphism for all smaller diagrams. Since the definition of a compatible family is hereditary, we may assume that $\alpha/\alpha' = \lambda$ which simplifies notation.

Now, I shall describe some general construction which will be used repeatedly. We say that $\lambda = \alpha|_v\beta$ if the diagram $\alpha$ consists exactly of these boxes of $\lambda$ which belong to at most $j$th column for some number $j$, while $\beta$ consists of these which belong to at least $(j+1)$th. For example $(4,4,1)/(3) = (3,1)|_v(1^2)$:



In such a situation there exists an embedding $\mathbf{S}_\lambda \longrightarrow \mathbf{S}_\alpha \otimes \mathbf{S}_\beta$. To see this, consider a diagram

$$\begin{tikzcd}
\mathbf{\Lambda}^\lambda \arrow[r] \arrow[d] & \mathbf{S}^{\widetilde{\lambda}} \arrow[d, equal] \\
\mathbf{\Lambda}^\alpha \otimes \mathbf{\Lambda}^\beta \arrow[r] & \mathbf{S}^{\widetilde{\alpha}} \otimes \mathbf{S}^{\widetilde{\beta}},
\end{tikzcd}$$

in which the horizontal arrows are compositions of structural arrows in respective Schur complexes, while the left vertical arrow is a comultiplication in the Hopf algebra $\mathbf{\Lambda}^*$ (see [ABW], sect. V.1). Thus we see that the image of the top arrow (ie. $\mathbf{S}_\lambda$) is contained in the image of the bottom one (ie. $\mathbf{S}_\alpha \otimes \mathbf{S}_\beta$), which yields the required embedding. Analogously, we write

$\lambda = \alpha|_h\beta$ if $\alpha$ consists of the first $j$ rows of $\lambda$, and $\beta$ of the rest. In this situation there exists an epimorphism $\mathbf{S}_\alpha \otimes \mathbf{S}_\beta \longrightarrow \mathbf{S}_\lambda$.

It will turn out to be plausible to describe these transformations in terms of the Decomposition Formula. Observe that if $\lambda = \alpha|_v\beta$, then $\alpha$ is the largest subdiagram of a given weight in $\lambda$. Since $\lambda/\alpha = \beta$, we have a morphism $\mathbf{S}_\lambda(V \oplus W) \longrightarrow \mathbf{S}_\alpha(V) \otimes \mathbf{S}_\beta(W)$. It is easy to check on tableaux that if we compose this morphism with a map $\mathbf{S}_\lambda(V) \longrightarrow \mathbf{S}_\lambda(V \oplus V)$ induced by the diagonal we get the map $\mathbf{S}_\lambda \longrightarrow \mathbf{S}_\alpha \otimes \mathbf{S}_\beta$ which we defined earlier in terms of structural maps. The drawback of this new description is that it is not clear from it that the map is a monomorphism. In a similar manner we can interpret the epimorphism $\mathbf{S}_\alpha \otimes \mathbf{S}_\beta \longrightarrow \mathbf{S}_\lambda$ when $\lambda = \alpha|_h\beta$. The only differences are that this time we have the smallest diagram and we compose with a map induced by an addition $V \oplus V \longrightarrow V$.

We come back to the proof of the lemma. Let us first assume that $\lambda$ consists of at least two columns. If so, we can write it as $\lambda = \alpha|_v\beta$ where $\beta$ is the last column of $\lambda$. Thus we have a monomorphism

$$\mathbf{S}_\lambda(V) \longrightarrow \mathbf{S}_\lambda(V \oplus V) \longrightarrow \mathbf{S}_\alpha(V) \otimes \mathbf{S}_\beta(V).$$

Now we will try to lift this morphism to the level of $F_k$. Observe that $F_k(\lambda) = F_k(\alpha)|_v F_k(\beta)$. It is because the foot of a slice corresponding to the first box in the last column of $\lambda$ lies to the right of the hand of a slice corresponding to the last box in the last but one column (we count boxes from the foot to the hand). Hence we get the composition

$$\mathbf{S}_{F_k(\lambda)}(V) \longrightarrow \mathbf{S}_{F_k(\lambda)}(V \oplus V) \longrightarrow \mathbf{S}_{F_k((\alpha)}(V) \otimes \mathbf{S}_{F_k(\beta)}(V).$$

We apply cohomology to this sequence and consider the diagram

$$\begin{tikzcd}
\mathbf{S}^{(1)}_\lambda(V)[h_k(\lambda)] \arrow[r, "\Phi_\lambda"] \arrow[d] & H^*(\mathbf{S}_{F_k(\lambda)})(V) \arrow[d] \\
\mathbf{S}^{(1)}_\lambda(V \oplus V)[h_k(\lambda)] \arrow[r, "\Phi_\lambda"] \arrow[d] & H^*(\mathbf{S}_{F_k(\lambda)})(V \oplus V) \arrow[d] \\
\mathbf{S}^{(1)}_\alpha(V) \otimes \mathbf{S}^{(1)}_\beta(V)[h_k(\lambda)] \arrow[r, "\Phi_\alpha \otimes \Phi_\beta"] & H^*(\mathbf{S}_{F_k(\alpha)})(V) \otimes H^*(\mathbf{S}_{F_k(\beta)})(V).
\end{tikzcd}$$

This diagram commutes: the commutativity of the first square follows from the naturality of $\Phi_\lambda$, and the commutativity of the second from the compatibility of a family. Since the bottom arrow is an isomorphism by the induction assumption and the composition of the left vertical arrows is monic, $\Phi_\lambda(V)$ is monic. But by Fact 4.2, the dimensions of the source and target are equal. Hence $\Phi_\lambda(V)$ is an isomorphism. Thus we finished the proof in the case $\dim(V) = 1$ for a diagram consisting of at least two columns.

To cover all diagrams of weight greater than 1 it suffices to consider the symmetric case of a diagram consisting of at least two rows. We argue in a similar manner. This time we write $\lambda$ in the form $\alpha|_h\beta$, where $\beta$ is the last row of $\lambda$. Then we consider the commutative diagram

$$\begin{tikzcd}
\mathbf{S}_\alpha^{(1)}(V) \otimes \mathbf{S}_\beta^{(1)}(V)[h_k(\lambda)] \arrow[r, "\Phi_\alpha\otimes\Phi_\beta"] \arrow[d] & H^*(\mathbf{S}_{F_k(\alpha)})(V) \otimes H^*(\mathbf{S}_{F_k(\beta)})(V) \arrow[d] \\
\mathbf{S}_\lambda^{(1)}(V)[h_k(\lambda)] \arrow[r, "\Phi_\lambda"] & H^*(\mathbf{S}_{F_k(\lambda)})(V).
\end{tikzcd}$$

This time the situation is slightly more complicated, because in order to conclude that the bottom arrow is onto (hence iso) we should show that the right vertical arrow is epic. But according to Fact 4.2, the complexes under consideration have trivial differentials. Therefore the right vertical arrow is just the epimorphism

$$\mathbf{S}_{F_k(\alpha)}(V) \otimes \mathbf{S}_{F_k(\beta)} \longrightarrow \mathbf{S}_{F_k(\lambda)}(V),$$

provided by the decomposition $F_k(\lambda) = F_k(\alpha)|_h F_k(\beta)$. This completes the proof for a one–dimensional space $V$.

We now turn to the induction step (with respect to $\dim(V)$). Let $V = W \oplus L$. We shall show by induction on the lexicographic order among $\alpha$ contained in $\lambda$ that $\Phi_\lambda|M_\alpha^{(1)}$ induces an isomorphism: $M_\alpha^{*(1)}(W \oplus L) \simeq H^*(M_{F_k(\alpha)})(W \oplus L)$. We can start this induction thanks to the assumption of the external induction (with respect to dimension). Let $\alpha'$ be the lexicographic predecessor of $\alpha$. We consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_{\alpha'}^{*(1)}(W \oplus L)[h_k(\lambda)] \arrow[r] \arrow[d, "\Phi_\lambda|_{M_{\alpha'}^{*(1)}}"] & M_\alpha^{*(1)}(W \oplus L)[h_k(\lambda)] \arrow[r] \arrow[d, "\Phi_\lambda|_{M_\alpha^{*(1)}}"] & \mathbf{S}_\alpha^{*(1)}(W) \otimes \mathbf{S}_{\lambda/\alpha}^{*(1)}(W)[h_k(\lambda)] \arrow[r] \arrow[d, "\Phi_\alpha \otimes \Phi_{\lambda/\alpha}"] & 0 \\
& H^*(M_{F_k(\alpha')})(W \oplus L) \arrow[r] & H^*(M_{F_k(\alpha)})(W \oplus L) \arrow[r] & H^*(\mathbf{S}_{F_k(\alpha)})(W) \otimes H^*(\mathbf{S}_{F_k(\lambda/\alpha)})(W). &
\end{tikzcd}$$

The top row in this diagram is exact and the left and right vertical arrows are isomorphisms by the induction assumption. Therefore the right bottom arrow is epic. But we have in the bottom row a long exact sequence. Thus we have just shown that this long exact sequence splits into short exact sequences. Hence by the Five–Lemma, the middle vertical arrow is an isomorphism. This completes the proof of Lemma 6.3. ■

# 7 Construction of a compatible family

Thus our task is to construct a compatible family satisfying the assumptions of Lemma 6.3. We start with a special case when $\lambda$ consists of a single row. Let $\chi^1, \ldots, \chi^d$ be the set of slices of $F_k(\lambda)$ ordered from the one corresponding to the foot of $\lambda$ to the one corresponding to the hand (I recall that $F_k(\lambda)$ may be a skew diagram). Since $F_k(\lambda) = \chi^1|_v\chi^2|_v\ldots|_v\chi^d$, we have the embedding $\mathbf{S}_{F_k(\lambda)} \longrightarrow \mathbf{S}_{\chi^1} \otimes \ldots \otimes \mathbf{S}_{\chi^d}$. The required transformation $\Phi_\lambda$ will close (ie. make commutative) a diagram

$$\begin{tikzcd}
\mathbf{\Lambda}^d[h_k(\lambda)] \arrow[r] & \mathbf{I}^{(1)}[h(\chi^1)] \otimes \ldots \otimes \mathbf{I}^{(1)}[h(\chi^d)] \arrow[d] \\
H^*(\mathbf{S}_{F_k(\lambda)}) \arrow[r] & H^*(\mathbf{S}_{\chi^1} \otimes \ldots \mathbf{S}_{\chi^d}).
\end{tikzcd}$$

in which the bottom arrow is the tensor product $\Phi_{h(\chi^1)} \otimes \ldots \otimes \Phi_{h(\chi^d)}$ of isomorphisms $\Phi_{h(\chi^s)} : \mathbf{I}^{(1)}[h(\chi^s)] \longrightarrow H^*(\mathbf{S}_{\chi^s})$ described in remarks after the proof of Fact 4.4 ($h(\chi_s)$ is equal to the number of columns of $\chi_s$ minus 1). To close this diagram we should understand the behaviour of its bottom arrow $H^*(\mathbf{S}_{F_k(\lambda)}) \longrightarrow H^*(\mathbf{S}_{\chi^1} \otimes \ldots \mathbf{S}_{\chi^d})$. To describe the image of this map we endow $H^*(\mathbf{S}_{\chi^1} \otimes \ldots \mathbf{S}_{\chi^d})$ with a structure of permutative $\Sigma_d$–module pulling it by the isomorphisms

$$\mathbf{I}^{\otimes d(1)} \simeq \mathbf{I}^{(1)}[h(\chi^1)] \otimes \ldots \otimes \mathbf{I}^{(1)}[h(\chi^d)] \xrightarrow{\Phi_{h(\chi^1)} \otimes \ldots \Phi_{h(\chi^d)}} H^*(\mathbf{S}_{\chi^1} \otimes \ldots \mathbf{S}_{\chi^d}).$$

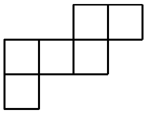
These isomorphisms shift the grading but we regard them as morphisms in the ungraded category. Thus when we consider the alternating action on $H^*(\mathbf{S}_{\chi^1} \otimes \ldots \mathbf{S}_{\chi^d})$, then, independently of the parity of $h(\chi_i)$, we always get $(H^*(\mathbf{S}_{\chi^1} \otimes \ldots \mathbf{S}_{\chi^d}))^{\Sigma_d} = \mathbf{\Lambda}^{d(1)}[h_k(\lambda)]$. This subcomplex turns out to be the desired image. We show this even in a slightly greater generality.

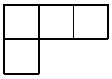
**Lemma 7.1** *Let* $\alpha = \chi^1|_v\ldots|_v\chi^d$ *for* $p$*–hooks* $\chi^1, \ldots, \chi^d$ *placed in such a way that the foot of the next hook lies to the right of the hand of the previous* *one (in such a situation we say that the hooks are placed horizontally). Then the map* $\mathbf{S}_\alpha \longrightarrow \mathbf{S}_{\chi^1} \otimes \ldots \otimes \mathbf{S}_{\chi^d}$ *induces an isomorphism*

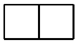
$$H^*(\mathbf{S}_\alpha) \simeq (H^*(\mathbf{S}_{\chi^1} \otimes \ldots \otimes \mathbf{S}_{\chi^d}))^{\Sigma_d},$$

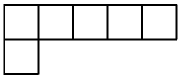
*with respect to the alternating action on* $H^*(\mathbf{S}_{\chi^1} \otimes \ldots \mathbf{S}_{\chi^d})$ *which we just described.*

**Proof:** We proceed by induction on the number of rows of $\alpha$. When $\alpha = (pd)$, then $\mathbf{S}_\alpha = \mathbf{\Lambda}^{pd}$ is the Kuhn dual of the De–Rham complex. Thus our description follows from Cartier's Theorem. Now we turn to the induction step. Let $\alpha^1$ be the first row of $\alpha$ and let $\alpha^2 = \alpha/\alpha^1$. Then $\alpha = \alpha^1|_h\alpha^2$. We form a new diagram: $\alpha' = \alpha^2|_v\alpha^1$ in such a way that the foot of $\alpha^1$ lies to the right of the hand of $\alpha^2$:


$\alpha$


$\alpha^2$


$\alpha^1$


$\alpha'$

We consider the sequence

$$0 \longrightarrow \mathbf{S}_{\alpha'} \longrightarrow \mathbf{S}_{\alpha^1} \otimes \mathbf{S}_{\alpha^2} \longrightarrow \mathbf{S}_\alpha \longrightarrow 0.$$

We shall show that this sequence is exact. The fact that the composition of arrows is trivial may be checked directly on tableaux. Therefore it suffices to check that the sum of dimensions of the first and the last term is equal to the dimension of the middle term. To see this, we use the Littlewood–Richardson rule (cf. [Bo]). According to it, we should check that the number of Yamanouchi's words of shape $\beta$ being a disconnected sum of $\alpha^1$ and $\alpha^2$ of a given content is equal to the sum of the number of Yamanouchi's words of shape $\alpha$ and the number of Yamanouchi's words of shape $\alpha'$ (of this given content).

To this end, let us take a word of shape $\beta$. If the number placed in the hand of $\alpha^2$ is greater than or equal to the number placed in the foot of $\alpha^1$, then we can associate to it in the obvious manner a word of shape $\alpha$ of the same content. Otherwise (if this number is smaller), we can form a word of shape $\alpha'$. It is easy to see that in this way we get a bijection between considered sets of Yamanouchi's words. This finishes the proof of the exactness of the sequence.

We now observe that the weight of $\alpha^1$ cannot be divisible by $p$, because the slices of $\alpha$ are placed horizontally. Hence the complex $\mathbf{S}_{\alpha^1}$ is acyclic. Let $|\alpha^1| = sp + j$ for some $0 < j < p$. We have a morphism of short exact sequences with acyclic middle terms

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathbf{S}_{\alpha'} \arrow[r] \arrow[d] & \mathbf{S}_{\chi^1} \otimes \ldots \otimes \mathbf{S}_{(\chi^{d-s})'} \otimes \ldots \otimes \mathbf{S}_{\chi^d} \arrow[d] \\
\mathbf{S}_{\alpha^1} \otimes \mathbf{S}_{\alpha^2} \arrow[r] \arrow[d] & \mathbf{S}_{\chi^1} \otimes \ldots \otimes \mathbf{\Lambda}^j \otimes \mathbf{S}^{p-j} \otimes \ldots \otimes \mathbf{S}_{\chi^d} \arrow[d] \\
\mathbf{S}_{\alpha} \arrow[r] \arrow[d] & \mathbf{S}_{\chi^1} \otimes \ldots \otimes \mathbf{S}_{\chi^{d-s}} \otimes \ldots \otimes \mathbf{S}_{\chi^d} \arrow[d] \\
0 & 0,
\end{tikzcd}$$

where arrows in the right column come from the sequence

$$0 \longrightarrow \mathbf{S}_{(\chi^{d-s})'} \longrightarrow \mathbf{\Lambda}^j \otimes \mathbf{S}^{p-j} \longrightarrow \mathbf{S}_{\chi^{d-s}} \longrightarrow 0$$

inducing shift in grading of the cohomology of hooks (see remark after Fact 4.4), tensored with identities. Taking the connecting homomorphisms in the long sequences of cohomology we get the commutative diagram

$$\begin{tikzcd}
H^*(\mathbf{S}_\alpha) \arrow[r] \arrow[d, "\partial"'] & \mathbf{I}[h(\chi^1)] \otimes \ldots \otimes \mathbf{I}[h(\chi^{d-s})] \otimes \ldots \otimes \mathbf{I}[h(\chi^d)] \arrow[d, "\partial'"] \\
H^*(\mathbf{S}_{\alpha'})[1] \arrow[r] & \mathbf{I}[h(\chi^1)] \otimes \ldots \otimes \mathbf{I}[h(\chi'^{d-s})] \otimes \ldots \otimes \mathbf{I}[h(\chi^d)],
\end{tikzcd}$$

in which the bottom row, by the induction assumption, has the desired description. The proof is concluded by the observation that the right vertical arrow is, up to sign, an isomorphism of $\Sigma_d$–modules (for $\partial' = \pm id \otimes \ldots \otimes \delta_{j+1,j} \otimes \ldots \otimes id$ and $\Phi_{j+1} = \delta_{j+1,j} \circ \Phi_j$). ■

Let us observe that Lemma 7.1 enables us to close (in the unique way) the diagram defining $\Phi_\lambda$. Moreover, the proof of Lemma 7.1 provides an explicit description of $\Phi_\lambda$ in terms of tableaux. A $Y$–standard tableau of shape $\lambda$ is sent (up to a nonzero scalar factor depending on cohomological degree and shapes of slices) to a $Y$–standard tableau of shape $F_k(\lambda)$ constructed in a following way: if in a given box in $\lambda$ there is an element $z$, then in a slice corresponding to this box we put tableau $\Psi(z)$. It follows immediately from this description that $\{\Phi_\lambda\}$ where $\lambda$ ranges over the set of all one–rowed diagrams form a compatible family of transformations for any such a diagram. In fact, we proved in Lemma 7.1 the assertion which is slightly more general than Theorem 5.3 for one–rowed diagrams.

We now turn to the general case. For an arbitrary diagram $\lambda$ we consider a diagram

$$\begin{tikzcd}
\mathbf{\Lambda}^{\lambda(1)}[h_k(\lambda)] \arrow[r] \arrow[d] & \mathbf{S}^{(1)}_\lambda[h_k(\lambda)] \\
H^*(\mathbf{\Lambda}^{F_k(\lambda)}) \arrow[r] & H^*(\mathbf{S}_{F_k(\lambda)}).
\end{tikzcd}$$

The top arrow in this diagram is the structural arrow in the Schur complex, the left vertical arrow is the tensor product of transformations $\Phi_{(\lambda_s)}$ for one–rowed diagrams which we constructed in the previous paragraph. But the bottom arrows needs some explanation, for it is the first arrow which exists only on the level of cohomology. We recall that since $\lambda = (\lambda_1)|_h \ldots |_h(\lambda_l)$, the structural arrow may be described as a composition of the map

$$\mathbf{S}_{(\lambda_1)}(V) \otimes \ldots \otimes \mathbf{S}_{(\lambda_l)}(V) \longrightarrow \mathbf{S}_\lambda(V \oplus \ldots \oplus V)$$

obtained by $l$–times taking the smallest diagram (of appropriate weight) in the Decomposition Formula for $\mathbf{S}_\lambda(V \oplus \ldots \oplus V)$, with the map induced by addition. We perform an analogous construction in the bottom row of our diagram but on the level of cohomology. In order to construct a map

$$H^*(\mathbf{S}_{F_k((\lambda_1))})(V) \otimes \ldots \otimes H^*(\mathbf{S}_{F_k((\lambda_l))})(V) \longrightarrow H^*(F_k(\mathbf{S}_\lambda))(V \oplus \ldots \oplus V)$$

we observe that (we start with detaching the last row) the diagram $F_k((\lambda_1, \ldots, \lambda_{l-1}))$ is the smallest subdiagram in $F_k(\lambda)$ among these which give in the Decomposition Formula for $\mathbf{S}_{F_k(\lambda)}(V \oplus V)$ a cohomologically nontrivial factor. As we noticed in Section 6, it enables to construct the map

$$H^*(\mathbf{S}_{F_k((\lambda_1,\ldots,\lambda_{l-1}))})(V^{\oplus(l-1)}) \otimes H^*(\mathbf{S}_{F_k((\lambda_l))})(V) \longrightarrow H^*(F_k(\mathbf{S}_\lambda))(V^{\oplus l}).$$

Iterating this procedure we obtain the bottom arrow in the diagram.

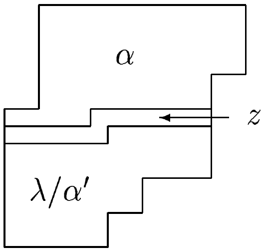
Now we would like to show that it is possible to close this diagram by the right vertical arrow (thanks to the epimorphicity of the top arrow it will be unique). This closure will be our transformation $\Phi_\lambda$. Our task will be completed if we show that the left arrow sends the kernel of the top arrow to the kernel of the bottom one. We start by recalling (and adapting to our context) a known description of the kernel of the structural arrow $\phi_\lambda$ (cf. [ABW], p. 264).

Let us introduce the temporary notation: we call a diagram $(n+m-1,m)/(m-1)$ a *zigzag* and denote it by $z(n,m)$. Since $\mathbf{S}_{z(n,m)}$ is very close to $\mathbf{\Lambda}^n \otimes \mathbf{\Lambda}^m$, the description of the kernel of the structural map is quite simple in this case. We just have an exact sequence

$$0 \longrightarrow \mathbf{\Lambda}^{n+m} \longrightarrow \mathbf{\Lambda}^n \otimes \mathbf{\Lambda}^m \xrightarrow{\phi_{z(n,m)}} \mathbf{S}_{z(n,m)} \longrightarrow 0,$$

where the first arrow is a comultiplication in the Hopf algebra $\mathbf{\Lambda}^*$ (cf. [ABW], sect. V.1).

For a general diagram $\lambda$ we say that $\alpha \subset \lambda$ generates zigzag if there exists the diagram $\alpha' \subset \lambda$ (it is unique, if exists) such that $\alpha'/\alpha = z$ is a zigzag



To any $\alpha$ generating zigzag we shall associate a space $K_\alpha$ contained in $\ker(\mathbf{\Lambda}^\lambda \longrightarrow \mathbf{S}_\lambda)$. To do this we observe that both $\alpha$ and $\alpha'$ are the smallest subdiagrams of their weights. Hence we have the commutative diagram

$$\begin{tikzcd}
\mathbf{\Lambda}^\alpha \otimes \mathbf{\Lambda}^{n+m} \otimes \mathbf{\Lambda}^{\lambda/\alpha'} \arrow[r] & \mathbf{\Lambda}^\alpha \otimes \mathbf{\Lambda}^n \otimes \mathbf{\Lambda}^m \otimes \mathbf{\Lambda}^{\lambda/\alpha'} \arrow[r] \arrow[d] & \mathbf{S}_\alpha \otimes \mathbf{S}_w \otimes \mathbf{S}_{\lambda/\alpha'} \arrow[d] \\
 & \mathbf{\Lambda}^\lambda \arrow[r] & \mathbf{S}_\lambda,
\end{tikzcd}$$

in which the composition of the top arrows is trivial. Therefore the space $\mathbf{K}_\alpha = \operatorname{im}(\mathbf{\Lambda}^\alpha \otimes \mathbf{\Lambda}^{n+m} \otimes \mathbf{\Lambda}^{\lambda/\alpha'}) \longrightarrow \mathbf{\Lambda}^\lambda)$ is contained in the kernel of the structural arrow. The main point of ([ABW], Th. V.1.10) is that the spaces $\mathbf{K}_\alpha$ for all $\alpha$ generating zigzags span this kernel. Thus it suffices to show that

$\Phi_{(\lambda_1)} \otimes \ldots \otimes \Phi_{(\lambda_l)}(\mathbf{K}_\alpha)$ is contained in the kernel of a homological analogue of the structural arrow, for all $\alpha$ generating zigzag. To see this, let us consider the commutative diagram

$$\begin{tikzcd}
\mathbf{\Lambda}^{\alpha(1)} \otimes \mathbf{\Lambda}^{n+m(1)} \otimes \mathbf{\Lambda}^{\lambda/\alpha'(1)}[h_k(\lambda)] \arrow[r] \arrow[d] & \mathbf{\Lambda}^{\lambda(1)}[h_k(\lambda)] \arrow[d] & \\
H^*(\mathbf{\Lambda}^{F_k(\alpha)}) \otimes H^*(\mathbf{\Lambda}^{F_k((n+m))}) \otimes H^*(\mathbf{\Lambda}^{F_k(\lambda/\alpha')}) \arrow[r] & H^*(\mathbf{\Lambda}^{F_k(\lambda)}) \arrow[r] & H^*(\mathbf{S}_\lambda),
\end{tikzcd}$$

in which the bottom arrows exist only on the level of cohomology, by lexicographic properties of $\alpha$ and $\alpha'$. It will be sufficient if we show that the composition of the bottom arrows is trivial. Moreover, it suffices to show it for $\lambda$ being a zigzag, because tensoring with a trivial map remains trivial. But in this special case, the considered sequence of cohomology groups

$$H^*(\mathbf{S}_{F_k((n+m))}) \longrightarrow H^*(\mathbf{S}_{F_k((n))} \otimes \mathbf{S}_{F_k((m))}) \longrightarrow H^*(\mathbf{S}_{F_k(W(n,m))})$$

is induced by the "real" sequence of complexes

$$\mathbf{S}_{F_k((n+m))} \longrightarrow \mathbf{S}_{F_k((m))} \otimes \mathbf{S}_{F_k((n))}) \longrightarrow \mathbf{S}_{F_k(W(n,m))},$$

because $F_k((n))$ is the smallest (not only among nonacyclic!) subdiagram of a given weight in $F_k(w(n,m))$. Now it is easy to check on tableaux that the composition of the arrows in this sequence of complexes is trivial. This finishes the construction of $\Phi_\lambda$.

What remains is to show that the family $\Phi_\lambda$ satisfies the conditions of Definition 6.2. Since our construction was uniform for all diagrams, it suffices to check the conditions for $\alpha/\alpha' = \lambda$. In this situation, to shorten notation, we will denote the diagram $\alpha''$ just by $\alpha$.

We take an arbitrary $\alpha \subseteq \lambda$ and consider the diagram

$$\begin{tikzcd}
\bigoplus_{\beta \leq \alpha} \mathbf{\Lambda}^{\beta(1)}(V) \otimes \mathbf{\Lambda}^{\lambda/\beta(1)}(W)[h_k(\lambda)] \arrow[r, "\Phi_{\mathbf{\Lambda}^\beta} \otimes \Phi_{\mathbf{\Lambda}^{\lambda/\beta}}"] \arrow[d] & \bigoplus_{\beta \leq \alpha} H^*(\mathbf{\Lambda}^{F_k(\beta)})(V) \otimes H^*(\mathbf{\Lambda}^{F_k(\lambda/\beta)})(W) \arrow[d] \\
\mathbf{\Lambda}^{\lambda(1)}(V \oplus W)[h_k(\lambda)] \arrow[r, "\Phi_{\mathbf{\Lambda}^\lambda}(V \oplus W)"] \arrow[d] & H^*(\mathbf{\Lambda}^{F_k(\lambda)})(V \oplus W) \arrow[d] \\
\mathbf{S}_\lambda^{(1)}(V \oplus W)[h_k(\lambda)] \arrow[r, "\Phi_\lambda(V \oplus W)"] & H^*(\mathbf{S}_{F_k(\lambda)})(V \oplus W),
\end{tikzcd}$$

in which $\Phi_{\mathbf{\Lambda}^\beta} : \mathbf{\Lambda}^\beta[h_k(\lambda)] \longrightarrow H^*(\mathbf{\Lambda}^{F_k(\beta)})$ stands for the product $\Phi_{(\beta_1)} \otimes \ldots \otimes \Phi_{(\beta_l)}$. By the definition of filtration yielding the Decomposition Formula ([ABW], Def. V.1.11), $M_\alpha^{(1)}$ is the image of the composition of the arrows in the first column, while $M_{F_k(\alpha)}^{(1)}$ — in the second. Hence to show that $\Phi_\lambda$ preserves the filtration it suffices to show that this diagram commutes. But the commutativity of the upper square follows from the compatibility of the family $\Phi$ for one–rowed diagrams, and the commutativity of the lower one, from the diagram defining $\Phi_\lambda$.

It remains to identify the quotient map. We consider a diagram which is best seen as a cube (in order not to make it even larger I neglect shifts of grading)

$$\begin{tikzcd}
\bigoplus_{\beta\leq\alpha} H^*(\mathbf{\Lambda}^{F_k(\beta)})(V) \otimes H^*(\mathbf{\Lambda}^{F_k(\lambda/\beta)})(W) \arrow[rrr] \arrow[ddd] & & & H^*(\mathbf{\Lambda}^{F_k(\alpha)})(V) \otimes H^*(\mathbf{\Lambda}^{F_k(\lambda/\alpha)})(W) \arrow[ddd] \\
& \bigoplus_{\beta\leq\alpha} \mathbf{\Lambda}^{\beta(1)}(V) \otimes \mathbf{\Lambda}^{\lambda/\beta(1)}(W) \arrow[r] \arrow[ul, "\Phi_{\mathbf{\Lambda}^\beta} \otimes \Phi_{\mathbf{\Lambda}^{\lambda/\beta}}"'] \arrow[d, "\rho"'] & \mathbf{\Lambda}^{\alpha(1)}(V) \otimes \mathbf{\Lambda}^{\lambda/\alpha(1)}(W) \arrow[ur, "\Phi_{\mathbf{\Lambda}^\alpha} \otimes \Phi_{\mathbf{\Lambda}^{\lambda/\alpha}}"] \arrow[d] & \\
& M_\alpha^{*(1)}(V \oplus W) \arrow[r] \arrow[dl, "\Phi_\lambda | C_\alpha^{*(1)}"] \arrow[dr, phantom, "?"] & \mathbf{S}_{\alpha(1)}(V) \otimes \mathbf{S}_{\lambda/\alpha(1)}(W) \arrow[dr, "\Phi_\alpha \otimes \Phi_{\lambda/\alpha}"'] & \\
H^*(M_{F_k(\alpha)})(V \oplus W) \arrow[rrr] & & & H^*(\mathbf{S}_{F_k(\alpha)})(V) \otimes H^*(\mathbf{S}_{F_k(\lambda/\alpha)})(W)
\end{tikzcd}$$

We have to show the commutativity of a square with question mark inserted. I claim that the commutativity of all other squares follows from the earlier considerations. Indeed: the commutativity of the external and internal squares follows from the definition of the filtration. The commutativity of the side squares follows from the second condition in the definition of a compatible family. The commutativity of the top square is obvious. Now, to obtain the commutativity of the square we are interested in, it suffices to observe that the arrow $\rho$ is epic and perform a standard diagram chasing. This completes the proof of compatibility of the family $\Phi_\lambda$ and hence, of Theorem 5.3. ■

## 8 Remarks on multiple quotients

A situation when a diagram has a trivial $p$–core but its $p$–quotient consists of a few diagrams is much more complicated and I have not succeed yet in understanding the cohomology of the Schur–De-Rham complex in this case. In this section I would like to discuss some conjectures suggested by numerical computations and support them by a simple example where a complete calculation of the cohomology is possible.

Both numerical computations and common sense suggest the existence of some relation between the cohomology of $\mathbf{S}_\lambda$ and the tensor product $\mathbf{S}^{(1)}_{q^0(\lambda)} \otimes \ldots \otimes \mathbf{S}^{(1)}_{q^{p-1}(\lambda)}$ for diagrams appearing in the $p$–quotient of $\lambda$. Alas, the simplest possible example shows that one cannot hope for an isomorphism

**Fact 8.1**

$$H^n(\mathbf{S}_{(p,p)}) = \begin{cases} \Lambda^{2(1)} & \text{for } n = 2p-4, 2p-3, 2p-1, 2p \\ 0 & \text{otherwise.} \end{cases}$$

**Proof:** Applying the Littlewood–Richardson rule to the skew diagram $(2p-1,p)/(p)$ we get the following decomposition (up to filtration) $\mathbf{S}_{(2p-1,p)/(p-1)} = \bigoplus_{a=1}^{p} \mathbf{S}(2p-a,a)$. But among the diagrams at the right–hand side only $(2p-1,1)$ and $(p,p)$ have trivial $p$–cores. Hence in the spectral sequence of filtration converging to $H^*(\mathbf{S}_{(2p-1,p)/(p)})$ there are only two nontrivial columns. Thus this spectral sequence degenerates to the long exact sequence

$$\ldots \longrightarrow H^*(\mathbf{S}_{(2p-1,1)}) \longrightarrow H^*(\mathbf{S}_{(2p-1,p)/(p-1)}) \longrightarrow H^*(\mathbf{S}_{(p,p)}) \longrightarrow H^{*+1}(\mathbf{S}_{(2p-1,1)}) \longrightarrow \ldots$$

We recall that according to Lemma 7.1 (and its Kuhn dual), we have $H^*(\mathbf{S}_{(2p-1,p)/(p)}) = \mathbf{S}^{2(1)}[2p-2]$, and $H^*(\mathbf{S}_{(2p-1,1)}) = \mathbf{\Lambda}^{2(1)}[2p-3]$. Therefore, a nontrivial part of this sequence looks like this

$$\begin{aligned} &0 \longrightarrow H^{2p-4}(\mathbf{S}_{(p,p)}) \longrightarrow \Lambda^{2(1)} \longrightarrow 0 \longrightarrow H^{2p-3}(\mathbf{S}_{(p,p)}) \longrightarrow I^{2(1)} \longrightarrow \\ &\longrightarrow S^{2(1)} \longrightarrow H^{2p-2}(\mathbf{S}_{(p,p)}) \longrightarrow D^{2(1)} \longrightarrow I^{2(1)} \longrightarrow H^{2p-1}(\mathbf{S}_{(p,p)}) \longrightarrow 0 \longrightarrow \\ &\longrightarrow \Lambda^{2(1)} \longrightarrow H^{2p}(\mathbf{S}_{(p,p)}) \longrightarrow 0. \end{aligned}$$

From this we immediately get that $H^n(\mathbf{S}_{(p,p)}) = 0$ for $n < 2p-4$, and for $n > 2p$, and also that $H^{2p-4}(\mathbf{S}_{(p,p)}) = \Lambda^{2(1)} = H^{2p}(\mathbf{S}_{(p,p)})$. Moreover, we observe that since $H^{2p-3}(\mathbf{S}_{(p,p)})$ is a subobject in $I^{2(1)}$, it must be equal to $\Lambda^{2(1)}$ or to $D^{2(1)}$. To rule out the second possibility it suffices to notice that $H^*(\mathbf{S}_{(p,p)})$ evaluated on a one–dimensional space is trivial since the underlying complex is trivial. Thus we conclude that $H^{2p-3}(\mathbf{S}_{(p,p)}) = \Lambda^{2(1)}$. By a similar reasoning $H^{2p-1}(\mathbf{S}_{(p,p)}) = \Lambda^{2(1)}$. Now, the exactness of the sequence forces that $H^{2p-2}(\mathbf{S}_{(p,p)}) = 0$ which completes the proof. ■

Thus we may suppose that $H^*(\mathbf{S}_{(p,p)})$ comes from (suitably shifted) $\mathbf{\Lambda}^{2(1)} \oplus \mathbf{S}^{2(1)}$ divided thorough some differential acting between the factors. To connect this observation with remarks made at the beginning of this section we recall that $q^{p-2}((p,p)) = q^{p-1}((p,p)) = (1)$. Therefore, the Littlewood–Richardson rule applied to the tensor product of diagrams from $p$–quotient gives $\mathbf{S}_{(1)} \otimes \mathbf{S}_{(1)} = \mathbf{S}_{(2)} + \mathbf{S}_{(1^2)}$, that is, exactly these Schur complexes whose remains we discovered in $H^*(\mathbf{S}_{(p,p)})$. Also some other partial calculations which I have performed with the aid of *Mathematica* support the following conjecture. It seems, there exists a spectral sequence converging to $H^*(\mathbf{S}_\lambda)$, whose columns in $E_1$–term are Schur complexes for diagrams appearing in the Littlewood–Richardson decomposition of a tensor product of diagrams from the $p$–quotient. I constructed such a spectral spectral sequence only for diagrams consisting of at most two rows. The construction is quite artificial and it is difficult to generalize it to an arbitrary diagram.

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