Koszul duality and extensions of exponential functors

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Abstract

We study Koszul duality in the category of strict polynomial functors. We compute Koszul duals for various functors and apply these results to the problem of calculating Ext–groups between exponential functors. The main application is a full description of the Ext–groups between twisted exterior and divided powers and between twisted symmetric and divided powers.

1 Introduction

Duality between symmetric and exterior powers (called the Koszul duality) appears in many contexts in algebra. In the category of functors (or representations of general linear groups) it has suggestive combinatorial interpretation. Namely, the symmetric power functor is a Schur functor associated to a Young diagram consisting of a single column while the exterior power is that associated to a one rowed diagram. Thus it is quite natural to ask for a natural operation which would relate a Schur functor associated to a Young diagram λ to the one associated to the *conjugate* diagram $\widetilde{\lambda}$ (ie. the diagram whose rows are columns of λ). In the present article we construct such an operation. Like in other situations where Koszul duality appears, our construction has best properties when considered on the level of derived

category. More specifically, we define a self-equivalence Θ of the derived category of the category of strict polynomial functors which sends a Schur functor to a Weyl functor associated to the conjugate diagram.
I should say that such equivalence was constructed by Donkin [Do] in the category of modules over the Schur algebra which is isomorphic to the category of strict polynomial functors. Our approach is independent of that of Donkin and, I think, it deserves some attention, since it is much more elementary. But the main purpose of this paper is different. Thanks to good homological properties of the category of strict polynomial functors (which are completely invisible from the level of the Schur algebra) we are able to compute Θ for some interesting functors. Moreover, the Koszul duality turns out to be a valuable tool in homological calculations in the functor category. Using it we compute Ext–groups between various exponential functors completing calculations started in [FFSS]. Apart from these concrete applications, the Koszul duality, I believe, offers some conceptual explanation of effectiveness of the method of "untwisting functors" used so successfully in [FS], [FFSS], and implicitly in the pioneering [FLS].

The article is organized as follows. In Section 2 after giving some motivation we introduce our duality and establish its basic properties. These facts (probably with the exception of Fact 2.6 concerning twists) may be obtained by translating results of Donkin into our context. In section 3 we deal with one nontrivial example. We compute (Theorem 3.2) cohomology of the Koszul dual of the twisted divided power functor. In Section 4 we compute the Ext-groups between twisted exterior and divided powers (Corollaries 4.3, 4.4) and between twisted symmetric and divided powers (Corollary 4.5). All these calculations are surprisingly easy consequences of Theorem 3.2.

2 Duality

Our working category will be the category $\mathcal{P}_d$ of homogeneous strict polynomial functors of degree d over a fixed field $\mathbf{k}$ (see [FS, Sect. 2]). We preserve all conventions and terminology from [C1] and [C2], but for the convenience of the reader we shall recall some strict polynomial functors which will be used frequently in this article.
First of all we have a functor $jI \in \mathcal{P}_1$, the j–th direct sum: $(V \rightsquigarrow V^{\oplus j})$. We will also consider the following homogeneous strict polynomial functors

of degree d: the d–th tensor power I^d ($V \rightsquigarrow V^{\otimes d}$); the d–th symmetric power S^d ($V \rightsquigarrow (V^{\otimes d})_{\Sigma_d}$); the d–th divided power D^d ($V \rightsquigarrow (V^{\otimes d})^{\Sigma_d}$); the d–th exterior power Λ^d ($V \rightsquigarrow ((V^{\otimes d})^{alt})_{\Sigma_d} \simeq ((V^{\otimes d})^{alt})^{\Sigma_d}$). More generally, for a Young diagram $\lambda = (\lambda_1, \ldots, \lambda_k)$ of weight d (ie. $\sum_j \lambda_j = d$), we put $S^\lambda := S^{\lambda_1} \otimes \ldots \otimes S^{\lambda_k}$ and we define similarly D^λ and Λ^λ. An important fact is that the family $\{S^\lambda\}$ forms a set of injective cogenerators of $\mathcal{P}_d$ (analogously $\{D^\lambda\}$ forms a set of projective ones) [FS, p. 18]. Having these functors we are able to define a family of functors which are of great importance in representation theory. Namely, for a Young diagram λ we define the Schur functor S_λ as the image of a composition $\Lambda^\lambda \longrightarrow I^d \longrightarrow S^{\widetilde{\lambda}}$ (cf. [ABW], Sect. II.1). There is a useful contravariant duality $^\#$ in $\mathcal{P}_d$, called the Kuhn duality. We put $F^\#(V)$ to be $(F(V^*))^*$. It is easy to see that $(S^d)^\# = D^d$, whereas Λ^d is self–dual. The dual of S_λ is called the Weyl functor W_λ (it may be explicitly defined as the image of a composition $D^{\widetilde{\lambda}} \longrightarrow I^d \longrightarrow \Lambda^\lambda$).

A formal definition of dualising functor Θ, which I am going to give, may look rather unmotivated (at least for somebody not familiar with tilting modules). Thus I would like to present some elementary considerations which have led me to it. Looking at the definitions of Schur and Weyl functors one can see certain symmetry in them. Namely, the so called Straightening Rule [ABW, Sect. II.2, II. 3] provides an explicit presentation of S_λ as a kernel of certain map from $S^{\widetilde{\lambda}}$ to a sum of products of symmetric powers. But it is easy to see that the Weyl functor $W_{\widetilde{\lambda}}$ is the kernel of the "same map" from $\Lambda^{\widetilde{\lambda}}$ to a sum of products of exterior powers (I will make this statement precise in a moment). For this reason my initial approach to the Koszul duality was the following. Since any functor F may be presented as the kernel of a map between sums of S^λ, we can put $\Theta_{naive}(F)$ to be the kernel of the "same map" between sums of Λ^λ. There are many equivalent ways of explaining what the "same map" means. In fact there is an isomorphism $\mathrm{Hom}_{\mathcal{P}_d}(S^\lambda, S^{\lambda'}) \simeq \mathrm{Hom}_{\mathcal{P}_d}(\Lambda^\lambda, \Lambda^{\lambda'})$. These groups are easy to compute (see eg. [FFSS, Cor. 1.8] or [AB, Sect. 6]) and compare. For example it is easy to see that every element of $\mathrm{Hom}_{\mathcal{P}_d}(S^\lambda, S^{\lambda'})$ is a composition of transformations of three simple types (perhaps tensored with identities): the multiplication $S^a \otimes S^b \longrightarrow S^{a+b}$, the comultiplication $S^{a+b} \longrightarrow S^a \otimes S^b$ and the transposition $S^a \otimes S^b \longrightarrow S^b \otimes S^a$. An analogous fact holds for exterior powers (which also form a Hopf algebra) and the practical way of establishing an isomorphism between spaces of transformations is to say that we send the multiplication in the symmetric power to the multiplication in the exterior power etc. A more intrinsic description of this isomorphism (from which immediately follows its functoriality) may be obtained using language of Σ_d–functors and transformations developed in [C1, Sect. 3]. Namely, [C1, Lemma 3.4] says that for any $\phi \in \mathrm{Hom}_{\mathcal{P}_d}(S^\lambda, S^{\lambda'})$ there exists a unique Σ_d–transformation $\tilde{\phi} \in \mathrm{Hom}_{\mathcal{F}_{\Sigma_d}}(s^\lambda, s^{\lambda'})$ such that $\phi = \tilde{\phi}(V^{\otimes d})$. Then the "same map" between exterior powers is just $\tilde{\phi}((V^{\otimes d})^{alt})$. This "naive Koszul duality" works well when $\mathbf{k}$ has characteristic 0 but in positive characteristic it may kill objects. For example, the Frobenius twist $I^{(1)}$ (the functor which associates to a space the same space but with the action of scalars induced by the Frobenius automorphism), is the kernel of the comultiplication $S^p \longrightarrow S^{p-1} \otimes S^1$. But the corresponding comultiplication in the exterior power $\Lambda^p \longrightarrow \Lambda^{p-1} \otimes \Lambda^1$ is a monomorphism. More careful analysis of this example reveals that when we extend our duality in an obvious way to complexes of symmetric powers then we will find our lost object: it was not killed but shifted (see Fact 2.6). This suggests that if we want our duality to be an isomorphism we should extend it to the derived category. The definition just given (we apply degreewise the naive duality) is perfectly correct, but it is not convenient in practice since it is usually impossible to describe explicitly resolution of a functor by symmetric powers (for example it is an open problem even for Schur functors). For this reason we will give a more sophisticated definition. It relies on the observation ([C1, Th. 5.1] or rather its additive counterpart) that $\mathrm{Ext}^*(\Lambda^d \circ jI, S^\lambda) = \mathrm{Hom}(\Lambda^d \circ jI, S^\lambda) = \Lambda^d(A'_j)$, where $A'_j = \mathrm{Hom}(jI, I) = \mathbf{k}^j$. Of course this construction is functorial in S^λ and it follows from [C1, Th. 5.1] that it takes $\phi \in \mathrm{Hom}_{\mathcal{P}_d}(S^\lambda, S^{\lambda'})$ to $\tilde{\phi}((V^{\otimes d})^{alt})$. Thus we can say that the value of Θ on $\mathbf{k}^j$ is just $\mathrm{RHom}(\Lambda^d \circ jI, F)$. Since it is easy to generalize the formula from [C1, Th. 5.1] to the form $\mathrm{Ext}^*(\Lambda^d \circ P_V, S^\lambda) = \mathrm{Hom}(\Lambda^d \circ P_V, S^\lambda) = \Lambda^d(V)$, where $P_V \in \mathcal{P}_1$ is given by the formula $P_V(W) := \mathrm{Hom}_{\mathbf{k}}(V, W)$, we can give elegant purely algebro–homological definition of Θ.

Definition 2.1 *We define a functor $\Theta : \mathbf{D}\mathcal{P}_d \longrightarrow \mathbf{D}\mathcal{P}_d$ by the formula*

$$\Theta(F)(V) := \mathrm{RHom}(\Lambda^d \circ P_V, F),$$

where $\mathbf{D}\mathcal{P}_d$ denotes the category of finite cohomological complexes of objects $\mathcal{P}_d$ modulo quasiisomorphisms (we recall that $\mathcal{P}_d$ has finite homological di-

mension and that every object has a finite injective resolution by (sums of products of) symmetric powers (cf. [To]).

Of course, this invariant definition is much more convenient than that referring to resolutions, but the former one is sometimes useful in concrete computations.

Fact 2.2 *For every Young diagram λ of weight d,*

$$\Theta(S_\lambda) = W_{\widetilde{\lambda}}.$$

Proof: Since $\mathrm{H}^0\Theta = \Theta_{naive}$ and, as we have explained earlier, $\Theta_{naive}(S_\lambda) = W_{\widetilde{\lambda}}$, it remains to show that $\mathrm{H}^j\Theta(S_\lambda) = 0$ for $j > 0$. But since $\Lambda^d \circ P_V$ is a direct sum of functors Λ^μ, it suffices to show that $\mathrm{Ext}^j(\Lambda^\mu, S_\lambda) = 0$ for every μ, λ. The last problem, by the Decomposition Spectral Sequence [C1, Cor. 2.5], may be reduced to showing that $\mathrm{Ext}^j(\Lambda^{d'}, S_{\lambda'}) = 0$. But this follows from the general fact that $\mathrm{Ext}^j(W_{\mu'}, S_{\lambda'}) = 0$ for every μ', λ' and $j > 0$ [CPS, proof of Th. 3.11]. ■

In order to show that Θ is an isomorphism we describe explicitly its inverse $\widetilde{\Theta}$, which may be thought of as projective version of duality.

Definition 2.3 *A functor $\widetilde{\Theta} : \mathbf{D}\mathcal{P}_d \longrightarrow \mathbf{D}\mathcal{P}_d$ is defined by the formula*

$$\widetilde{\Theta}(F)(V) := (\mathrm{RHom}(F, \Lambda^d \circ P_V))^{\#}.$$

It is easy to see that $\widetilde{\Theta}$ takes divided powers to exterior ones, hence it satisfies the dual version of Fact 2.2: $\widetilde{\Theta}(W_\lambda) = S_{\widetilde{\lambda}}$.

Corollary 2.4

$$\widetilde{\Theta} \circ \Theta = \mathrm{Id} = \Theta \circ \widetilde{\Theta}.$$

Proof: Since $\widetilde{\Theta} \circ \Theta(S^\lambda) = \widetilde{\Theta}(\Lambda^\lambda) = S^\lambda$, this composition is an identity on any complex of symmetric powers. But every complex is quasiisomorphic to a complex of symmetric powers. For the second composition we do the same with divided powers. ■

Thus we have shown that Θ is a self–equivalence of $\mathbf{D}\mathcal{P}_d$. Of course for $\mathbf{k}$ of characteristic 0 $\mathcal{P}$ is semisimple and $\Theta = \Theta_{naive}$ is just a self-equivalence of $\mathcal{P}_d$. Therefore from now on we will assume that our ground field $\mathbf{k}$ is of positive characteristic p.

Corollary 2.5 *For every pair of diagrams λ, λ'*

$$\mathrm{Ext}^*_{\mathcal{P}_d}(S_\lambda, S_{\lambda'}) = \mathrm{Ext}^*_{\mathcal{P}_d}(W_{\widetilde{\lambda}}, W_{\widetilde{\lambda'}}) = \mathrm{Ext}^*_{\mathcal{P}_d}(S_{\widetilde{\lambda'}}, S_{\widetilde{\lambda}}).$$

Proof: The first equality follows from the fact that Θ is an isomorphism. The second equality is just the Kuhn duality. I have used it in order to come back to Schur functors. ■

This fact corresponds to Corollary 3.9 in [Do], and as Donkin pointed out was known even earlier [AB, Th. 7.7].
The last fact in this section allows us to extend Corollary 2.5 to twisted Schur functors, which seems to be a new result.

Fact 2.6 *For any $F \in \mathcal{P}_d$:*

$$\Theta(F^{(i)}) = (\Theta(F))^{(i)}[d(p^i - 1)],$$

$$\Theta(F \otimes G) = \Theta(F) \otimes \Theta(G),$$

(we take convention $(\mathbf{C}[l])^k := \mathbf{C}^{k+l}$*, hence* $\mathrm{Hom}^*(\mathbf{C}[l], \mathbf{C}'[l']) = \mathrm{Hom}^*(\mathbf{C}, \mathbf{C}')[l' - l]$.

Proof: As we know from [FFSS, Th. 5.8], $\mathrm{Ext}^*(\Lambda^{dp^i}, S^{d(i)}) = \mathbf{k}[d(p^i - 1)]$. Thus, arguing like in untwisted case, we obtain $\mathrm{Ext}^*(\Lambda^{dp^i} \circ P_V, S^{\lambda(i)}) = \Lambda^{\lambda(i)}(V)[d(p^i - 1)]$. Hence $\mathrm{RHom}(\Lambda^{dp^i} \circ P_V, S^{\lambda(i)})$ is formal and we get $\Theta(S^{d(i)}) := \mathrm{RHom}(\Lambda^{dp^i} \circ P, S^{\lambda(i)}) \simeq \mathrm{Ext}^*(\Lambda^{dp^i} \circ P, S^{\lambda(i)}) = \Lambda^{\lambda(i)}[d(p^i - 1)] = (\Theta(S^\lambda))^{(i)}[d(p^i - 1)]$. In order to get our assertion for general F we take a resolution of F by symmetric powers $0 \longrightarrow F \longrightarrow S^{\lambda^0} \longrightarrow \ldots$ After i–times twisting and applying $\mathrm{RHom}(\Lambda^d \circ P_V, -)$ we get an exact sequence of complexes

$$0 \longrightarrow \mathrm{RHom}(\Lambda^d \circ P_V, F^{(i)}) \longrightarrow \mathrm{RHom}(\Lambda^d \circ P_V, S^{\lambda^0(i)}) \longrightarrow \ldots$$

Since starting from the second spot we have formal complexes with cohomology concentrated in degree $d(p^i - 1)$, we get a quasiisomorphism

$$\mathrm{RHom}(\Lambda^d \circ P_V, F^{(i)}) \simeq \Lambda^{\lambda^0(i)}(V)[d(p^i - 1)] \longrightarrow \ldots$$

functorial in V. But the right–hand side is just $(\Theta(F))^{(i)}[d(p^i - 1)]$.
The proof for tensor product is straightforward. ■

3 Calculation of $\mathrm{H}\Theta(D^{d(i)})$

There is the fundamental asymmetry in the duality Θ coming from the fact that we define it using (at least implicitly) injective resolutions. Thus one should not expect as easy description of values of Θ on projective objects as we have got for injective or Schur objects. In the present section we deal with the problem of computing $\Theta(D^d)$. This complex seems to be far from being formal, but its cohomology still has a reasonable description.
Since $\Lambda \circ P_V$ has decomposition (not functorial in V) into a sum of Λ^λ, then, by the Decomposition Spectral Sequence, the main ingredient in computation of $\mathrm{H}\Theta(D^d)$ will be the calculation of $\mathrm{Ext}^*(\Lambda^d, D^d)$. Already this computation is nontrivial in contrast to the situation considered in [FFSS] where the starting point ie. determination of Ext-groups between untwisted functors was tautological. Since we are going to proceed by induction using Koszul and De-Rham complexes (like in [FLS], [FS], [FFSS]), we are forced to consider the four-graded object $\mathrm{Ext}^k(\Lambda^{dp^i}, \Lambda^{l(i)} \otimes D^{d-l(i)})$. Luckily, since Λ^* and $\Lambda^{*(i)} \otimes D^{*(i)}$ are Hopf algebras so are Ext–groups between them (cf. ([Ku, Sect. 5], [FFSS, Lemma 1.10]), which greatly helps to organize computations. Since structural arrows in the Hopf algebra $\mathrm{Ext}^*(\Lambda^{*p^*}, \Lambda^{*(*)} \otimes D^{*-*(*)})$ preserve index i we may fix it and describe the trigraded algebras $\mathrm{Ext}^*(\Lambda^{*p^i}, \Lambda^{*(i)} \otimes D^{*-*(i)})$ for each i separately. Our terminology slightly differs here from that of [FFSS, Sect. 1]. We just call a Hopf algebra n–graded when our object has n indices which can vary independently. The advantage of convention taken in [FFSS] is that the commutativity relations [FFSS, Lemma 1.11] take more elegant form with it, but since we do not use them explicitly, I decided not to increase the number of indices artificially.

Fact 3.1 *The trigraded Hopf algebra* $\mathrm{Ext}^*(\Lambda^{*p^i}, \Lambda^{*(i)} \otimes D^{*-*(i)})$ *is isomorphic to*

$$D^*(x^{(i)}) \otimes D^*(\alpha_s^{(i)}) \otimes \Lambda^*(\beta_s^{(i)}),$$

for primitive generators:

- $x^{(i)} \in \mathrm{Ext}^{p^i-1}(\Lambda^{p^i}, \Lambda^{1(i)})$,
- $\alpha_s^{(i)} \in \mathrm{Ext}^{p^{s+i}-2}(\Lambda^{p^{s+i}}, D^{p^s(i)})$, *for* $s = 1, 2 \ldots$
- $\beta_s^{(i)} \in \mathrm{Ext}^{p^{s+i}-1}(\Lambda^{p^{s+i}}, D^{p^s(i)})$, *for* $s = 0, 1 \ldots$

Proof: Let $\mathbf{K}^d$ and $\mathbf{R}^d$ be the Kuhn duals of respectively Koszul and De–Rham complexes (we recall that $(\mathbf{K}^d)_l = (\mathbf{R}^d)^l = \Lambda^{d-l} \otimes D^l$ but the Koszul complex is equipped with a homological differential whereas De-Rham complex with a cohomological one (cf. [FS, Sect. 4])). We will consider spectral sequences converging to $\mathrm{hExt}^*(\Lambda^{dp^i}, \mathbf{K}^{d(i)})$ and $\mathrm{hExt}^*(\Lambda^{dp^i}, \mathbf{R}^{d(i)})$ calling them (twisted) Koszul and De–Rham spectral sequences (the first one, of course, converges to 0). Observe that since the structural maps in the Hopf algebra $\Lambda^* \overset{\circ}{\otimes} D^*$ ($\overset{\circ}{\otimes}$ indicates that we take graded tensor product of Hopf algebras (cf. [ABW, Sect. V.1]) commute with the Koszul and De–Rham differentials [C2, Sect. 3], the Koszul and De–Rham spectral sequences are sequences of Hopf algebras ie. the structural maps in the Hopf algebra $\mathrm{Ext}^*(\Lambda^{*p^i}, \Lambda^{*(i)} \otimes D^{*-*(i)})$ commute with differentials in spectral sequences.
We will compute $\mathrm{Ext}^*(\Lambda^{dp^i}, \Lambda^{d-*(i)} \otimes D^{*(i)})$ by induction on d. For needs of induction we should also understand Koszul and De–Rham spectral sequences. Since our sequences are sequences of Hopf algebras, they are determined by the action of differentials on primitive generators $\alpha_s^{(i)}, \beta_s^{(i)}, x^{(i)}$. In the course of induction we will show that

- All differentials in the second De-Rham spectral sequence converging to $\mathrm{hExt}^*(\Lambda^{dp^i}, \mathbf{R}^{d(i)})$ are trivial.
- In the first De–Rham spectral sequence, we have $\delta(\alpha_s^{(i)}) = \delta(\beta_s^{(i)}) = 0, \delta(x^{(i)}) = \beta_0^{(i)}$.
- In the first Koszul spectral sequence converging to $\mathrm{hExt}^*(\Lambda^{p^{s+i}}, \mathbf{K}^{p^s(i)})$, we have $\partial(\alpha_s^{(i)}) = x^{p^s - p^{s-1}(i)} \otimes \beta_{s-1}^{(i)}, \partial(\beta_s^{(i)}) = x^{p^s(i)}, \partial(x) = 0$.

We start with $d = 1$. Then, by [FFSS, Th. 5.8], $\mathrm{Ext}^*(\Lambda^{p^i}, \Lambda^{1(i)}) = \mathrm{Ext}^*(\Lambda^{p^i}, D^{1(i)}) = \mathbf{k}[p^i - 1]$ and we choose our generators $x^{(i)}, \beta_0^{(i)}$ in such a way that $\partial(\beta_0^{(i)}) = x^{(i)}$ (of course, here $\delta = \partial^{-1}$).
Now we turn to the induction step. Let $p > 2$. Let us first state explicitly, which conditions we should check (in addition to counting dimensions of course) to control the structure of Hopf algebra and differentials in the spectral sequences. In fact, there is anything to do only for $d = p^s$. Here we should show that two new indecomposable primitive elements appear (these are our candidates for $\alpha_s^{(i)}$ and $\beta_s^{(i)}$) and compute their differentials. Fact that $\beta_s^{(i)}$ generates an exterior algebra follows from the parity of its multigrading (see [FFSS, Lemma 1.11]) but the structure of a summand generated by $\alpha_s^{(i)}$

depends on components of higher degrees. Thus we need not to care about $\alpha_s^{(i)}$ at the moment but we should check some facts concerning the previous generators $\alpha_{s'}^{(i)}$. Namely, we should show that for $s' < s$, $(\alpha_{s'}^{(i)})^{p^{s-s'}} = 0$ (in a divided power algebra we should carefully differ between the kth power of generator $\alpha_{s'}^{(i)}$ which we denote by $(\alpha_{s'}^{(i)})^k$ and the canonical nonzero element $\alpha_{s'}^{k(i)}$ (in fact $(\alpha_{s'}^{(i)})^k = k! \cdot \alpha_{s'}^{k(i)}$)), and that we can choose an element $\alpha_{s'}^{p^{s-s'}(i)}$ in such a way that $\Delta(\alpha_{s'}^{p^{s-s'}(i)}) = (\alpha_{s'}^{(i)})^{\otimes p^{s-s'}}$ where Δ is a component in comultiplication on Ext–groups induced by $(\Lambda^{p^{s'+i}})^{\otimes p^{s-s'}} \longrightarrow \Lambda^{p^{s+i}}$.
Let us start with the case $d \neq 0, 1 (\mathrm{mod}\ p)$. Here we need only to compute dimensions. The only nontrivial thing to show is that $\mathrm{Ext}^*(\Lambda^{dp^i}, D^{d(i)}) = 0$. Look at the first Koszul spectral sequence converging to $\mathrm{hExt}^*(\Lambda^{dp^i}, \mathbf{K}^{d(i)})$. It follows from $\delta_1(x^{(i)}) = \beta_0^{(i)}$ and the induction assumption that δ_1 yields an isomorphism between the $(d'p)$th and $(d'p+1)$th column in the first term of the sequence. This shows that the last column must be trivial. Let now d be divisible by p. The same analysis of the first De–Rham spectral sequence reveals that only the last column survives. An obvious dimension counting shows then, that this group (ie. $\mathrm{Ext}^*(\Lambda^{dp^i}, D^{d(i)})$) has the desired (graded) dimension if and only if all differentials in the second De–Rham spectral sequence are trivial. But the second term of this sequence is known by the induction assumption (for $i+1$). Thus the triviality of differentials follows from the induction assumption unless $d = p^s$ where we a priori do not know this for generators $\alpha_{s-1}^{(i+1)}, \beta_{s-1}^{(i+1)}$. But these elements lie in the last column and the triviality of differentials on them follows immediately from dimension argument. It remains to check facts concerning Hopf structure and differentials which, again, are nontrivial only for $d = p^s$. Observe that $\mathrm{Ext}^{p^{s+i}-2}(\Lambda^{p^{s+i}}, D^{p^s(i)})$, and $\mathrm{Ext}^{p^{s+i}-1}(\Lambda^{p^{s+i}}, D^{p^s(i)})$ are one dimensional and that the Koszul differential sends $\mathrm{Ext}^{p^{s+i}-1}(\Lambda^{p^{s+i}}, D^{p^s(i)})$ to $\mathrm{Ext}^{p^{s+i}-p^s}(\Lambda^{p^{s+i}}, \Lambda^{p^s(i)})$ where $x^{p^s(i)}$ belongs to. Hence we put $\beta_s^{(i)}$ to the preimage of $x^{p^s(i)}$ under this differential. In order to choose $\alpha_s^{(i)}$ in a similar manner we should show that the Koszul differential sends $\mathrm{Ext}^{p^{s+i}-2}(\Lambda^{p^{s+i}}, D^{p^s(i)})$ to $\mathrm{Ext}^{p^{s+i}-p^s+p^{s-1}-1}(\Lambda^{p^{s+i}}, \Lambda^{p^s-p^{s-1}(i)} \otimes D^{p^{s-1}(i)})$ where $\beta_{s-1}^{(i)} \otimes x^{p^s-p^{s-1}(i)}$ lives. To this end, observe that $\beta_{s-1}^{(i)} \otimes x^{p^s-p^{s-1}(i)}$ is a cycle with respect to the Koszul differential, and that the element which kills it must lie in $\mathrm{Ext}^{p^{s+i}-2}(\Lambda^{p^{s+i}}, D^{p^s(i)})$ by dimension argument. In this way we have defined $\alpha_s^{(i)}$ and $\beta_s^{(i)}$ with the expected action of the Koszul differential. Fact that they are primitive in-

decomposable, and that both De-Rham differentials act on them trivially follows from dimension argument. Then we turn to the analysis of generators $\alpha_{s'}^{(i)}$ for $s' < s$. Let us start with the observation that $(\alpha_{s'}^{(i)})^{p^{s-s'}} = 0$ since the Koszul differential on a $p^{s-s'}$th power must be zero and our element lies in the last column, so there is nothing to kill it. Hence, there exists a nonzero indecomposable element in $\mathrm{Ext}^{p^{s-s'}(p^{s'+i}-2)}(\Lambda^{p^{s+i}}, D^{p^s(i)})$ which will be our candidate for $\alpha_{s'}^{p^{s-s'}(i)}$. But to finish the proof, we should also show that $\Delta(\alpha_{s'}^{p^{s-s'}(i)}) = (\alpha_{s'}^{(i)})^{\otimes p^{s-s'}}$ To this end, it suffices to show that $\partial(\alpha_{s'}^{p^{s-s'}(i)}) = \alpha_{s'}^{(i)} \otimes x^{p^s - p^{s'}(i)}$ (up to scalar by which we can always modify our definition of $\alpha_{s'}^{p^{s-s'}(i)}$). But this follows from the fact that the latter element is a Koszul cycle and cannot be killed by any decomposable element by the induction assumption. This finishes the proof for d divisible by p. The remaining case $d = 1(\mathrm{mod}\ p)$ is easy. We need only to compute dimensions. The right answer follows from the fact that the De-Rham differential gives an isomorphism between the $(d-1)$-th and dth column.

For $p = 2$ only minor modifications of the proof are needed. The only problem is that our Hopf algebra is now just commutative and hence, we cannot a priori rule out the possibility that $\alpha_s^{(i)}$ generates exterior algebra or $\beta_s^{(i)}$ generates divided powers or eg. (truncated) polynomials. But in our situation we still know that the pth (ie. second) powers of our candidates for generators vanish and our method for deciding whether given element is primitive also works. ■

Now we turn to computing $\mathrm{H}\Theta(D^{d(i)})$. In fact, thanks to Fact 2.6, it would suffice to compute $\mathrm{H}\Theta(D^d)$, but since we were already forced to deal with twists in Fact 3.1, computation of $\mathrm{H}\Theta(D^{d(i)})$ will take no additional work. Thus our task is to compute $\mathrm{Ext}^*(\Lambda^{dp^i} \circ P_V, D^{d(i)})$. Since $\Lambda^{dp^i} \circ P_V$ is a direct sum of Λ^λ and, by the Exponential Formula [C1, Sect. 2], $\mathrm{Ext}^*(\Lambda^\lambda, D^{d(i)}) = \bigotimes_k \mathrm{Ext}^*(\Lambda^{\lambda_k}, D^{\lambda_k/p^i(i)})$, Fact 3.1 provides us all needed computational input. The only problem is to organize results in a functorial way (we recall that the decomposition of $\Lambda^{dp^i} \circ P_V$ into a sum of Λ^λ is not functorial in V). Again it will be easier to describe the entire exponential functor $\mathrm{H}\Theta(D^{*(i)})$. Let us rewrite the computation achieved in Fact 3.1 in a suggestive form:

$$\mathrm{Ext}^*(\Lambda^*, D^*) = \Lambda^*(\beta_0^{(i)}) \otimes \bigotimes_{s\geq 1} \Lambda^*(\beta_s^{(i)}) \otimes D^*(\alpha_s^{(i)}),$$

(observe that all products are finite in any multidegree). Roughly speaking, in order to get the functor $\mathrm{H}\Theta(D^{*(i)})$ we should replace in the above formula generators $\alpha_s^{(i)}$ and $\beta_s^{(i)}$ by a space $V^{(s+i)}$.

Theorem 3.2 *There is an isomorphism of exponential functors*

$$\Phi : F_{i,0} \otimes \bigotimes_{s\geq 1} F_{i,s} \otimes G_{i,s} \longrightarrow \mathrm{H}\Theta(D^{*(i)})$$

where $F_{i,s}(V) := \Lambda^{*(i+s)}(V)$ *for* $V^{(i+s)}$ *placed in* $\mathrm{H}^{p^{i+s}-1}\Theta(D^{p^s(i)})$*, and* $G_{i,s}(V) := D^{*(i+s)}(V)$ *for* $V^{(i+s)}$ *placed in* $\mathrm{H}^{p^{i+s}-2}\Theta(D^{p^s(i)})$.

Proof: We shall describe Φ quite explicitly. Let $v \in V$ and $\gamma \in \mathrm{Ext}(\Lambda^{dp^i}, D^{d(i)})$. We define the element $\gamma(v) \in \mathrm{Ext}(\Lambda^{dp^i} \circ P_V, D^{d(i)})$ in the following way. The element v determines the transformation $\phi_v : P_V \longrightarrow I$ in an obvious way (we send $f : V \longrightarrow W$ to $f(v)$). Hence $\Lambda^{dp^i}(\phi_v)$ is a transformation from $\Lambda^{dp^i} \circ P_V$ to Λ^{dp^i} and we put $\gamma(v)$ to be $(\Lambda^{dp^i}(\phi_v))^*(\gamma)$. This construction is clearly functorial in v and it is easy to see that the assignment $v \rightsquigarrow \alpha_s^{(i)}(v)$ produces a transformation from $I^{(s+i)}$ to $\mathrm{Ext}^{p^{s+i}-2}(\Lambda^{p^{s+i}} \circ P, D^{p^s(i)})$ and analogously $v \rightsquigarrow \beta_s^{(i)}(v)$ determines a transformation from $I^{(s+i)}$ to $\mathrm{Ext}^{p^{s+i}-1}(\Lambda^{p^{s+i}} \circ P, D^{p^s(i)})$. Then we define Φ on the factor $G_{i,s}$ by the formula $v_1^{d_1} \cdot \ldots \cdot v_k^{d_k} \mapsto \alpha_s^{d_1(i)}(v_1) \cdot \ldots \cdot \alpha_s^{d_k(i)}(v_k)$ and analogously on $F_{i,s}$ by the formula $v_1 \wedge \ldots \wedge v_d \mapsto \beta_s^{(i)}(v_1) \wedge \ldots \wedge \beta_s^{(i)}(v_d)$. Thanks to Fact 3.1 the transformation Φ is well defined. To show that it is an isomorphism we observe that Φ is an exponential transformation (ie. it is compatible with decompositions of functors applied to direct sums). But an exponential transformation is essentially determined by its action on a one dimensional space.

Lemma 3.3 *Let* $\Psi : A^* \longrightarrow B^*$ *be an exponential transformation. If* $\Psi(\mathbf{k})$ *is an isomorphism, so is the entire transformation* Ψ *.*

Proof: is an obvious double induction on dimension and degree. ■

Thus it suffices to show that $\Phi(\mathbf{k})$ is an isomorphism. Let us take $1^d \in D^d(\mathbf{k}) = G_{i,s}(\mathbf{k})$. Then $\Phi(1^d) = \alpha_s^{d(i)}(1) = \alpha_s^{(i)}$. Analogously (even easier) we show an isomorphism on factors $F_{i,s}$. This completes the proof of Theorem 3.2. ■

Although computing $\mathrm{H}\Theta(D^{d(i)})$ is sufficient for applications, it would be interesting to describe $\Theta(D^{d(i)})$ (of course, in a way from which it would be clear how to obtain cohomology, which is not the case for the resolution of D^d constructed in [To]). Let us consider the simplest nontrivial example D^p. Then $\mathbf{K}^p$ provides a Λ^p–acyclic resolution of D^p and it is easy to derive from it that $\Theta(D^p)$ is $\mathbf{R}^p$ with removed the degree 0 component. This example suggests that $\Theta(D^d)$ is built out of Λ^d and some De-Rham complexes. Since the same example shows that $\Theta(D^d)$ is not formal we can only hope that

Conjecture 3.4 $\mathrm{H}\Theta(D^d)$ *can be realized as a complex which is isomorphic up to filtration to the complex*

$$\Lambda^* \otimes \bigotimes_{s\geq 0} \mathbf{R}^{*(s)}.$$

In fact this conjecture suggested me the way of organizing results in Theorem 3.2. It seems that $\Theta(D^d)$ consists of "naive part" which is Λ^d (warning: this is not exactly $\Theta_{naive}(D^d)$ when $p = 2$) and some additional "homological part" in which De–Rham complexes are involved.

4 Applications to exponential functors

As we know from [C1, Th. 4.3], the groups $\mathrm{Ext}^*(D^{d(j)}, F^{(j)})$ are computable for any F. The Koszul duality will allow us to extend this class of calculations significantly. For example: $\mathrm{Ext}^*(\Lambda^{d(j)}, F^{(j)}) = \mathrm{Hom}^*(\Theta(\Lambda^{d(j)}), \Theta(F^{(j)})) = \mathrm{hExt}^*(D^{d(j)}, \Theta(F)^{(j)})$. By this method we will compute the Ext–groups between twisted exterior and divided powers and between twisted symmetric and divided powers.

In order to make our considerations precise, we start with stating some immediate consequences of [C1, Th. 4.3] which were not formulated explicitly there. Let $\mathcal{P}_d^{(j)}$ denote the full subcategory of $\mathcal{P}_{dp^j}$ consisting of j–times twisted functors. Since the embedding $\mathcal{P}_d \longrightarrow \mathcal{P}_{dp^j}$ is faithfully full, $\mathcal{P}_d^{(j)}$ is an abelian category which may be identified with the image of this embedding.

Fact 4.1 *For any $F \in \mathcal{P}_d$ there is an isomorphism*

$$\mathrm{Ext}^*(D^{d(j)}, F^{(j)}) \simeq F(A_j),$$

where $A_j = \mathrm{Ext}^*(I^{(j)}, I^{(j)})$ (it is a one dimensional space in degrees $2k$ for $k = 0, \ldots, p^j - 1$ and trivial elsewhere), which is functorial in F ie. for any

transformation $\phi : F \longrightarrow G$ *the induced map* $(\phi^{(j)})_* : \mathrm{Ext}^*(D^{d(j)}, F^{(j)}) \longrightarrow \mathrm{Ext}^*(D^{d(j)}, G^{(j)})$ *corresponds to* $\phi(A_j)$.
Moreover, for any bounded above complex $\mathbf{C}$ *of objects of* $\mathcal{P}_d^{(j)}$, *we have*

$$\mathrm{hExt}^*(D^{d(j)}, \mathbf{C}) = \mathrm{H}\mathbf{C}(A_j),$$

(all Ext–groups are taken in $\mathcal{P}_{dp^j}$*).*

Proof: Description of Ext–groups is just [C1, Th. 4.3] for $\mu = (d)$ (I discuss this example also at the end of [C1, Sect. 5]). In [C1] I examined functoriality of this description with respect to the first variable, which is rather delicate question. Functoriality in F follows immediately from the machinery developed in [C1, Sect. 4]. There is only one subtlety, which was also discussed more thoroughly in [C1]. Strictly speaking we not just apply F to A_j but its injective symmetrization f_{in} to $A_j^{\otimes d}$. The difference is that f_{in} carries information about grading. In other words, choosing f_{in} allows to extend F to a functor on graded vector spaces which a priori may be done in many ways. In practice, in order to find f_{in} one should present F as a kernel of a map between symmetric powers which we regard as functors on graded spaces in an obvious way. All this has a practical consequence which otherwise could be easily overlooked. Namely, it is easy to check that an injective symmetrization of $I^{(i)}$ is not just the identity but the Σ_{p^i}–functor $i^{(i)}$ which multiplies degrees of components by p^i(ie. $(i^{(i)}(V))^{p^i s} := V^s$ and is trivial elsewhere). Thus $I^{(i)}(A_j)$ has nontrivial components in degrees $2kp^i$ for $k = 0, \ldots, p^j - 1$.
For the second part of Fact 4.1 we observe that the functor $\mathrm{Ext}^*(D^{d(j)}, -)$ is exact on sequences of objects of $\mathcal{P}_d^{(j)}$. Thus, since all cycles, boundaries and cohomology spaces in $\mathbf{C}$ belong to $\mathcal{P}_d^{(j)}$ (because it is a full subcategory), all connecting homomorphisms which could produce nontrivial higher differentials in the hyperExt spectral sequences converging to $\mathrm{hExt}(D^{d(j)}, \mathbf{C})$ are trivial, hence these spectral sequences collapse. ■

This fact shows that in order to compute $\mathrm{Ext}^*(\Lambda^{d(j)}, F^{(j)})$ we only need to know cohomology of $\Theta(F)$.

Corollary 4.2 *For any* $F \in \mathcal{P}_d$

$$\mathrm{Ext}^*(\Lambda^{d(j)}, F^{(j)}) = \mathrm{H}\Theta(F)(A_j).$$

Proof: As we have already observed, $\mathrm{Ext}^*(\Lambda^{d(j)}, F^{(j)}) = \mathrm{hExt}^*(D^{d(j)}, \Theta(F)^{(j)})$, and then we apply Fact 4.1 (in fact it is crucial that Θ preserves the family of complexes of objects of $\mathcal{P}_d^{(j)}$ which is a consequence of Fact 2.6). ■

In the remainder of this section we will apply Corollary 4.2 to various exponential functors. We start with computing $\mathrm{Ext}^*(\Lambda^{d'p^i(j)}, D^{d'(i+j)})$. For this we take Corollary 4.2 for $d = d'p^i$ and $F = D^{d'(i)}$. Gathering up results for various d' we get

Corollary 4.3 *There is an isomorphism of bigraded Hopf algebras*

$$\mathrm{Ext}^*(\Lambda^{*p^i(j)}, D^{*(i+j)}) \simeq \Lambda^*(B_{i,j,0}) \otimes \bigotimes_{s\geq 1} \Lambda^*(B_{i,j,s}) \otimes D^*(B'_{i,j,s}).$$

The space $B_{i,j,s} \subset \mathrm{Ext}^*(\Lambda^{p^{i+s}(j)}, D^{p^s(i+j)})$ *is one dimensional in Ext–degrees* $p^{i+s} - 1 + 2kp^{i+s}$ *for* $k = 0, \ldots, p^j - 1$ *and trivial elsewhere.*
The space $B'_{i,j,s} \subset \mathrm{Ext}^*(\Lambda^{p^{i+s}(j)}, D^{p^s(i+j)})$ *is one dimensional in Ext–degrees* $p^{i+s} - 2 + 2kp^{i+s}$ *for* $k = 0, \ldots, p^j - 1$ *and trivial elsewhere.*

Thus we have computed Ext–groups between larger exterior and smaller divided powers. Trying to compute $\mathrm{Ext}^*(\Lambda^{*(i+j)}, D^{*p^i(j)})$ we face the problem that $\Theta(D^{*p^i(j)})$ need not to be $(i+j)$–times twisted, hence we cannot apply Corollary 4.2 and it may be not that easy to compute $\mathrm{Ext}^*(D^{*(i+j)}, \Theta(D^{*p^i(j)}))$. We bypass this difficulty with the aid of projective version of the Koszul duality. Applying it to the groups under consideration we get $\mathrm{Ext}^*(\Lambda^{d(i+j)}, D^{dp^i(j)}) = \mathrm{Hom}^*(\tilde{\Theta}(\Lambda^{d(i+j)}), \tilde{\Theta}(D^{dp^i(j)})) = \mathrm{Ext}^*(S^{d(i+j)}, \Lambda^{dp^i(j)})[d(p^i-1)] = \mathrm{Ext}^*(\Lambda^{dp^i(j)}, D^{d(i+j)})[d(p^i-1)]$ reducing the problem to that we have already solved. Therefore, the result is

Corollary 4.4 *There is an isomorphism of bigraded Hopf algebras*

$$\mathrm{Ext}^*(\Lambda^{*(i+j)}, D^{*p^i(j)}) \simeq \Lambda^*(C_{i,j,0}) \otimes \bigotimes_{s\geq 1} \Lambda^*(C_{i,j,s}) \otimes D^*(C'_{i,j,s}).$$

The space $C'_{i,j,s} \subset \mathrm{Ext}^*(\Lambda^{p^s(i+j)}, D^{p^{i+s}(j)})$ *is one dimensional in Ext–degrees* $p^{i+s} + p^s(p^i - 1) - 1 + 2kp^{i+s}$ *for* $k = 0, \ldots, p^j - 1$ *and trivial elsewhere.*
The space $C'_{i,j,s} \subset \mathrm{Ext}^*(\Lambda^{p^s(i+j)}, D^{p^{i+s}(j)})$ *is one dimensional in Ext–degrees* $p^{i+s} + p^s(p^i - 1) - 2 + 2kp^{i+s}$ *for* $k = 0, \ldots, p^j - 1$ *and trivial elsewhere.*

Our last task will be computation of $\mathrm{Ext}^*(S^{dp^i(j)}, D^{d(i+j)})$. Applying Θ twice we replace these groups by $\mathrm{Ext}^*(D^{dp^i(j)}, \Theta^2(D^{d(i+j)}))$. Thus we should compute $\mathrm{H}\Theta^2(D^*)$. This would be easy if we knew that $\mathrm{H}\Theta^2(D^*) = \mathrm{H}\Theta(\mathrm{H}\Theta(D^*))$. For this we should show that all differentials in the second spectral sequence converging to $\mathrm{hExt}^*(\Lambda^{dp^i}, \Theta(D^{d(i)}))$ are trivial. Since $\Theta(D^{*(i)})$ is an exponential functor (in the category of complexes) it suffices to show the triviality of differentials on primitive generators of the Hopf algebra $\mathrm{Ext}^*(\Lambda^{*p^i}, \mathrm{H}\Theta(D^{*(i)}))$. But this is obvious by dimension argument. Therefore we get $\mathrm{H}\Theta^2(D^{*(i)}) = \mathrm{H}\Theta(\mathrm{H}\Theta(D^{*(i)}))$ which is by Theorem 3.2 equal to

$$G_{i,0,0} \otimes \bigotimes_{s\geq 1} G_{i,s,0} \otimes \bigotimes_{s\geq 1, t\geq 1} F_{i,s,t} \otimes G_{i,s,t}.$$

In the above formula $G_{i,s,t}(V) := D^{*(i+s+t)}(V)$ for $V^{(i+s+t)}$ placed in $\mathrm{H}^{p^{i+s+t}-1+p^t(p^{i+s}-2)}\Theta^2(D^{p^{s+t}(i)})$ for $t \geq 1$ and in $\mathrm{H}^{2(p^{i+s}-1)}(\Theta^2(D^{p^s(i)})$ for $t = 0$; and $F_{i,s,t}(V) := \Lambda^{*(i+s+t)}(V)$ for $V^{(i+s+t)}$ placed in $\mathrm{H}^{p^{i+s+t}-2+p^t(p^{i+s}-2)}\Theta^2(D^{p^{s+t}(i)})$. This leads to the following description of the Ext–groups.

Corollary 4.5 *There is an isomorphism of bigraded Hopf algebras*

$$\mathrm{Ext}^*(S^{*p^i(j)}, D^{*(i+j)}) \simeq D^*(C'_{i,j,0,0}) \otimes \bigotimes_{s\geq 1} D^*(C'_{i,j,s,0}) \otimes \bigotimes_{s\geq 1, t\geq 1} \Lambda^*(C_{i,j,s,t}) \otimes D^*(C'_{i,j,s,t}).$$

The space $C'_{i,j,s,t} \subset \mathrm{Ext}^*(S^{p^{i+s+t}(j)}, D^{p^{s+t}(i+j)})$ *is one dimensional in Ext–degrees* $p^{i+s+t} - 1 + p^t(p^{i+s} - 2) + 2kp^{i+s+t}$ *for* $k = 0, \dots, p^j - 1$ *and trivial elsewhere, for* $t \geq 1$*;*
and is one dimensional in Ext–degrees $2(p^{i+s}-1)+2kp^{i+s}$ *for* $k = 0, \dots, p^j-1$ *and trivial elsewhere, for* $t = 0$.
The space $C'_{i,j,s,t} \subset \mathrm{Ext}^*(S^{p^{i+s+t}(j)}, D^{p^{s+t}(i+j)})$ *is one dimensional in Ext–degrees* $p^{i+s+t} - 2 + p^t(p^{i+s} - 2) + 2kp^{i+s+t}$ *for* $k = 0, \dots, p^j - 1$ *and trivial elsewhere.*

The Ext–groups between smaller symmetric and larger divided powers are exactly the same, since by the Kuhn duality

$$\mathrm{Ext}^*(S^{*p^i(j)}, D^{*(i+j)}) = \mathrm{Ext}^*(S^{*(i+j)}, D^{*p^i(j)}).$$

Computation of Ext–groups between exponential functors in the category $\mathcal{P}$ immediately leads to a paralel calculation in the category $\mathcal{F}$ of functors in a naive sense over a finite field $\mathbf{k}$, by methods of [FFSS, Sect. 6]. We recall that by [FFSS, Cor. 6.2] it suffices to compute the groups $\mathrm{Ext}^*_{\mathcal{P}\to\mathcal{F}}(F,G) := \mathrm{colim}_j \mathrm{Ext}^*_{\mathcal{P}}(F^{(j)}, G^{(j)})$ for certain functors F, G. It is easy to see that in our context we obtain the groups $\mathrm{Ext}^*_{\mathcal{P}\to\mathcal{F}}$ just putting in formulae in Cor. 4.3, 4.4, 4.5 instead of spaces $B_{i,j,s}$ etc. the limit spaces (with respect to j) $B_{i,s}$ etc. in which we drop the condition $k \leq p^j - 1$. Hence, by [FFSS, Cor. 6.2], we get complete calculations in $\mathcal{F}$ (to make formulae simpler we restrict ourselves to the case of prime field $\mathbf{k}$, leaving the general case to the interested reader).

Corollary 4.6 *Let $|\mathbf{k}| = p$. Then we have isomorphisms of trigraded Hopf algebras:*

$$\mathrm{Ext}^*_{\mathcal{F}}(\Lambda^*, D^*) \simeq$$
$$\bigotimes_{i\geq 0}((\Lambda^*(B_{i,0}) \otimes \bigotimes_{s\geq 1} \Lambda^*(B_{i,s}) \otimes D^*(B'_{i,s})))\otimes$$
$$\bigotimes_{i>0}(\Lambda^*(C_{i,0}) \otimes \bigotimes_{s\geq 1} \Lambda^*(C_{i,s}) \otimes D^*(C'_{i,s})),$$

and

$$\mathrm{Ext}^*_{\mathcal{F}}(S^*, D^*) \simeq$$
$$\bigotimes_{i\geq 0}(D^*(C'_{i,0,0}) \otimes \bigotimes_{s\geq 1} D^*(C'_{i,s,0}) \otimes \bigotimes_{s\geq 1, t\geq 1} \Lambda^*(C_{i,s,t}) \otimes D^*(C'_{i,s,t}))\otimes$$
$$\bigotimes_{i>0}(D^*(C'_{i,0,0}) \otimes \bigotimes_{s\geq 1} D^*(C'_{i,s,0}) \otimes \bigotimes_{s\geq 1, t\geq 1} \Lambda^*(C_{i,s,t}) \otimes D^*(C'_{i,s,t})),$$

where multidegrees of generators are as in Cor. 4.3, 4.4, 4.5.

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