

KRULL-SCHMIDT DECOMPOSITIONS FOR THICK SUBCATEGORIES

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ABSTRACT. Following Krause [Kra99], we prove Krull-Schmidt theorems for thick subcategories of various triangulated categories: derived categories of rings, noetherian stable homotopy categories, stable module categories over Hopf algebras, and the stable homotopy category of spectra. In all these categories, it is shown that the thick ideals of small objects decompose uniquely into indecomposable thick ideals. Some consequences of these decomposition results are also discussed. In particular, it is shown that all these decompositions respect K -theory.

1. INTRODUCTION

Using ideas from modular representation theory of finite groups, Krause [Kra99] proved a Krull-Schmidt theorem for thick subcategories of the stable module category. More precisely, he showed that the thick ideals of $\text{stmod}(KG)$ decompose uniquely into indecomposable thick ideals. In this paper, we show that such decompositions exist in various other stable homotopy categories like the derived categories of commutative rings and the stable homotopy category of spectra. To this end, we first generalise Krause's definition to arbitrary triangulated categories.

Definition 1.1. [Kra99] Let \mathcal{T} denote a tensor triangulated category. A thick subcategory \mathcal{A} of \mathcal{T} is a *thick ideal* if $X \wedge Y$ belongs to \mathcal{A} for all $X \in \mathcal{A}$ and all $Y \in \mathcal{T}$. If \mathcal{A} is a thick (ideal) subcategory of \mathcal{T} , a family of thick (ideal) subcategories $(\mathcal{A}_i)_{i \in I}$ is a *decomposition* of \mathcal{A} if

- (1) the objects of \mathcal{A} are the finite coproducts of objects from the \mathcal{A}_i , and
- (2) $\mathcal{A}_i \cap \mathcal{A}_j = 0$ for all $i \neq j$.

A decomposition $(\mathcal{A}_i)_{i \in I}$ of \mathcal{A} is denoted by $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$, and we say that \mathcal{A} is *indecomposable* if $\mathcal{A} \neq 0$ and any decomposition $\mathcal{A} = \mathcal{C} \amalg \mathcal{D}$ implies that $\mathcal{C} = 0$ or $\mathcal{D} = 0$. A *Krull-Schmidt decomposition* of \mathcal{A} is a unique decomposition $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$ where all the \mathcal{A}_i are indecomposable.

The above definition reminds one of the classical Krull-Schmidt theorem which states that any finite length module over an Artin ring admits a unique direct sum decomposition into indecomposable modules. Since all thick subcategories studied in this paper consist of finite objects in some triangulated category, it is reasonable to call these decompositions as Krull-Schmidt decompositions.

Date: July 8, 2005.

2000 Mathematics Subject Classification. Primary:55p42; Secondary:18E30.

Key words and phrases. Krull-Schmidt, tensor product property, stable homotopy theory, stable module category, thick subcategory, derived category, Grothendieck groups.

We now discuss the main results in this paper. The first one is about the effect of K -theory of a thick subcategory under a Krull-Schmidt decomposition.

Theorem 1.2. *Let \mathcal{T} denote a triangulated category and let \mathcal{C} be a thick subcategory of \mathcal{T} . If $\coprod_{i \in I} \mathcal{C}_i$ is a Krull-Schmidt decomposition of \mathcal{C} , then under some mild conditions (see theorem 2.4), there is a natural isomorphism in K -theory,*

$$\bigoplus_{i \in I} K_0(\mathcal{C}_i) \cong K_0(\mathcal{C}).$$

Our motivation for studying K -theory for thick subcategories comes from the problem of classifying the triangulated subcategories in a given triangulated category. Thomason [Tho97] proved that if \mathcal{T} is any triangulated category, then the subgroups of \mathcal{T} are in bijection with the dense triangulated subcategories of \mathcal{T} . (A triangulated subcategory \mathcal{A} is *dense* in \mathcal{T} if \mathcal{T} is the smallest thick subcategory that contains \mathcal{A} .) Since every triangulated subcategory \mathcal{A} of \mathcal{T} is dense in a unique thick subcategory (namely, the one obtained by taking the intersection of all the thick subcategories of \mathcal{T} that contain \mathcal{A}), one can classify all the triangulated subcategories of \mathcal{T} by computing the Grothendieck groups of all the thick subcategories of \mathcal{T} . Towards this, the above theorem tells us that we can restrict ourselves to indecomposable thick subcategories.

We use Landsburg's criterion (see lemma 2.1) and some general lemmas about triangulated categories which are developed in the Section 2 to prove the above theorem. It applies to all the decompositions we study in this paper.

The next theorem deals with Krull-Schmidt decompositions in the aforementioned categories.

Theorem 1.3. *The thick subcategories of finite spectra and those of perfect complexes in the derived category of a noetherian ring admit Krull-Schmidt decompositions. Conversely, in both these cases, given any collection of thick subcategories $(\mathcal{C}_i)_{i \in I}$ such that $\mathcal{C}_i \cap \mathcal{C}_j = 0$ for all $i \neq j$, there exists a unique thick subcategory \mathcal{C} such that $\mathcal{C} = \coprod_{i \in I} \mathcal{C}_i$.*

In proving these decomposition theorems, we make good use of the thick subcategory theorems of Hopkins-Smith [HS98] and Hopkins-Neeman [Nee92]. In the derived category of a noetherian ring R , Hopkins-Neeman result states that the thick subcategories of perfect complexes are in bijection with the *thick supports* (subset of $\text{Spec}(R)$ that are a union of closed sets). Given a thick subcategory \mathcal{T}_S (corresponding to the thick support S under this bijection), we define a graph G_S using the minimal prime ideals of S . It is shown that the indecomposable pieces that constitute \mathcal{T}_S in a Krull-Schmidt decomposition correspond precisely to the connected components of this graph. This decomposition theorem gives the following interesting algebraic result; see corollary 4.15 for a stronger result.

Corollary 1.4. *Let X be a perfect complex over a noetherian ring R . Then there exists a unique decomposition*

$$X \cong \bigoplus_{i \in I} X_i$$

such that the supports of the X_i are pairwise disjoint and indecomposable.

We also generalise the above results on Krull-Schmidt decomposition to more general noetherian stable homotopy categories. We use the language and results

from [HPS97] to achieve this generalisation. In particular, this generalisation gives Krull-Schmidt decompositions in some categories arising in modular representation theory. We summarise these results in the next theorem.

Theorem 1.5. *Let B denote a finite dimensional graded co-commutative Hopf algebra satisfying the tensor product property (Examples: group algebras of finite groups and the finite dimensional sub-Hopf algebras of the mod 2 Steenrod algebra). Then,*

- (1) *Every thick ideal of small objects in $K(\text{Proj } B)$ (the chain-homotopy category of graded projective B -modules) is indecomposable.*
- (2) *Every thick ideal of $\text{stmod}(B)$ admits a Krull-Schmidt decomposition.*

As mentioned above, Krause [Kra99] proved that thick ideals of $\text{stmod}(KG)$ (G a finite group) admit Krull-Schmidt decompositions. It is known [BCR96] that group algebras of finite groups satisfy the tensor product property and therefore part 2 of the above theorem is a generalisation of Krause's result.

The paper is organised as follows. In section 2, after developing necessary lemmas about triangulated categories and thick subcategories, we show that Krull-Schmidt decompositions respect K -theory under some mild hypothesis. In the later sections we study Krull-Schmidt decompositions in various stable homotopy categories: We review Krause's result for the stable module categories of group algebras in section 3, derived categories of rings in section 4, noetherian stable homotopy categories in section 5, stable module categories over Hopf algebras in section 6, and finally the stable homotopy category of spectra in section 7. We end the paper with a few questions which ask for further extensions of these decompositions.

Acknowledgements: I would like to thank Henning Krause, Srikanth Iyengar, and John Palmieri for the many interesting and fruitful discussions I had with them on this subject.

2. K -THEORY FOR THICK SUBCATEGORIES

In this section we study how the K -theory of triangulated categories behaves under a Krull-Schmidt decomposition. We begin with some preliminaries on triangulated categories. Let \mathcal{T} denote a triangulated category that is essentially small (i.e., a category that has only a set of isomorphism classes of objects). The *Grothendieck group* $K_0(\mathcal{T})$ is defined to be the free abelian group on the isomorphism classes of \mathcal{T} modulo the Euler relations: $[B] = [A] + [C]$, whenever $A \rightarrow B \rightarrow C \rightarrow \Sigma A$ is an exact triangle in \mathcal{T} (here $[X]$ denotes the element in the Grothendieck group that is represented by the isomorphism class of the object X). This is clearly an Abelian group with $[0]$ as the identity element and $[\Sigma X]$ as the inverse of $[X]$. We always have the identity $[A] + [B] = [A \amalg B]$ in the Grothendieck group. Also note that any element of $K_0(\mathcal{T})$ is of the form $[X]$ for some X in \mathcal{T} .

Having defined K -theory, we now study how it behaves under a Krull-Schmidt decomposition. We begin with some lemmas that will be needed in studying K -theory. We will start with the following extremely useful Lemma due to Landsburg [Lan91]. This is a nice criterion for the equality of two classes in the Grothendieck group.

Lemma 2.1. [Lan91] *Let \mathcal{T} be an essentially small triangulated category. If X and Y are objects in \mathcal{T} , then $[X] = [Y]$ in $K_0(\mathcal{T})$ if and only if there are objects A, B ,*

\mathcal{C} in \mathcal{T} and maps such that there are exact triangles

$$\begin{aligned} A &\xrightarrow{f} B \amalg X \xrightarrow{g} C \xrightarrow{h} \Sigma A, \\ A &\xrightarrow{f'} B \amalg Y \xrightarrow{g'} C \xrightarrow{h'} \Sigma A. \end{aligned}$$

It is well-known that coproducts of exact triangles are exact in any triangulated category; see [Mar83, Appendix 2, Prop. 10]. Under some additional hypothesis, the next lemma gives a converse to this well-known fact.

Lemma 2.2. *Let \mathcal{T} denote a triangulated category and let*

$$A \xrightarrow{f} B \rightarrow C \rightarrow \Sigma A \quad \text{and} \quad A' \xrightarrow{f'} B' \rightarrow C' \rightarrow \Sigma A'$$

be two sequences of maps in \mathcal{T} such that their sum $A \amalg A' \xrightarrow{f \amalg f'} B \amalg B' \rightarrow C \amalg C' \rightarrow \Sigma(A \oplus A')$ is an exact triangle. If we also have

- (1) $\text{Hom}(\text{Cone}(f), C') = 0 = \text{Hom}(\text{Cone}(f'), C)$ and
- (2) $\text{Hom}(C', \text{Cone}(f)) = 0 = \text{Hom}(C, \text{Cone}(f'))$,

then the given sequences of maps are exact triangles.

Proof. Complete the maps f and f' to exact triangles in \mathcal{T} :

$$A \xrightarrow{f} B \rightarrow \text{Cone}(f) \rightarrow \Sigma A \quad \text{and} \quad A' \xrightarrow{f'} B' \rightarrow \text{Cone}(f') \rightarrow \Sigma A'.$$

Since the coproduct of exact triangles is exact, adding these two triangles gives another triangle,

$$A \amalg A' \xrightarrow{f \amalg f'} B \amalg B' \rightarrow \text{Cone}(f) \amalg \text{Cone}(f') \rightarrow \Sigma(A \oplus A').$$

We know from the axioms for a triangulated category that there is a fill-in map H in the diagram below.

$$\begin{array}{ccccccc} A \amalg A' & \xrightarrow{f \amalg f'} & B \amalg B' & \longrightarrow & \text{Cone}(f) \amalg \text{Cone}(f') & \longrightarrow & \Sigma(A \amalg A') \\ \downarrow = & & \downarrow = & & \downarrow H & & \downarrow = \\ A \amalg A' & \xrightarrow{f \amalg f'} & B \amalg B' & \longrightarrow & C \amalg C' & \longrightarrow & \Sigma(A \amalg A') \end{array}$$

Note that three out of the four vertical maps in the above diagram are isomorphisms and therefore so is H ; see [Mar83, Appendix 2, Prop. 6]. Now the hypothesis

$$\text{Hom}(\text{Cone}(f), C') = \text{Hom}(\text{Cone}(f'), C) = 0$$

implies that $H = h \amalg h'$. So we have $h \amalg h' : \text{Cone}(f) \amalg \text{Cone}(f') \rightarrow C \amalg C'$ is an isomorphism. The hypothesis

$$\text{Hom}(C', \text{Cone}(f)) = \text{Hom}(C, \text{Cone}(f')) = 0$$

implies that the inverse G of this isomorphism is of the form $g \amalg g'$. This forces both h and h' to be isomorphisms and hence $C \cong \text{Cone}(f)$ and $C' \cong \text{Cone}(f')$. Since exact triangles in \mathcal{T} form a replete class, the two sequences of maps under consideration are exact triangles. \square

Lemma 2.3. *Let \mathcal{C} and \mathcal{D} be thick ideals in \mathcal{T} . If $\mathcal{C} \cap \mathcal{D} = 0$, then $\mathcal{C} \wedge \mathcal{D} = 0$. ($\mathcal{C} \wedge \mathcal{D}$ is the full subcategory of objects of the form $X \wedge Y$ where $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.)*

Proof. Clear, since $\mathcal{C} \wedge \mathcal{D} \subseteq \mathcal{C} \cap \mathcal{D} = 0$. \square

We are now ready to state and prove our main theorem on K -theory for thick subcategories.

Theorem 2.4. *Let $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$ be a Krull-Schmidt decomposition of a thick (ideal) subcategory \mathcal{A} in a triangulated category \mathcal{T} . If \mathcal{A} is thick, assume that (1) holds; if \mathcal{A} is a thick ideal, assume any one of the following three conditions.*

- (1) $\text{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for all $i \neq j$.
- (2) *There exist an object S_i in \mathcal{A}_i such that $S_i \wedge X = X$ for all $X \in \mathcal{A}_i$.*
- (3) *There exists an object S_i in \mathcal{T} such that $S_i \wedge X = X$ for all $X \in \mathcal{A}_i$, and whenever $j \neq i$, $S_i \wedge X = 0$ for all $X \in \mathcal{A}_j$.*

Then the inclusion functors $\mathcal{A}_i \hookrightarrow \mathcal{A}$ give rise to an isomorphism

$$\Xi : \bigoplus_{i \in I} K_0(\mathcal{A}_i) \cong K_0(\mathcal{A}).$$

Proof. The inclusion functors $\mathcal{A}_i \hookrightarrow \mathcal{A}$ induce maps on the Grothendieck groups which can be assembled to obtain the map $\Xi : \bigoplus_{i \in I} K_0(\mathcal{A}_i) \rightarrow K_0(\mathcal{A})$ that we want to show is an isomorphism. Showing that Ξ is surjective is easy: note that every element in $K_0(\mathcal{A})$ is of the form $[X]$ for some $X \in \mathcal{A}$. Since the family of thick ideals $(\mathcal{A}_i)_{i \in I}$ is a decomposition for \mathcal{A} , we can express X as a finite coproduct; $X = \coprod X_i$ with $X_i \in \mathcal{A}_i$. This gives $[X] = [\coprod X_i] = \sum_{i \in I} [X_i]$. The last quantity is clearly the image of $\bigoplus [X_i]$ under the map Ξ and therefore Ξ is surjective.

To show that Ξ is injective, we use Landsburg's criterion 2.1. Suppose $\Xi(\bigoplus [X_i]) = 0$. Then, as noted above, $\Xi(\bigoplus [X_i]) = \sum [X_i] = [\coprod X_i]$ and hence $[\coprod X_i] = [0]$. This now gives, by Landsburg's criterion, two exact triangles in \mathcal{A} :

$$(3a) \quad A \rightarrow B \amalg \left(\coprod_{j \in I} X_j \right) \rightarrow C \rightarrow \Sigma A$$

$$(3b) \quad A \rightarrow B \rightarrow C \rightarrow \Sigma A$$

We want to show that $\bigoplus [X_i] = 0$ or equivalently $[X_i] = 0$ for all $i \in I$. First assume that condition (1) holds. Then consider the decompositions $A = \coprod \mathcal{A}_i$, $B = \coprod \mathcal{B}_i$, and $C = \coprod \mathcal{C}_i$ (which exist because $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$). Substituting these decompositions in the above triangles (3a) (3b) gives

$$\coprod \mathcal{A}_i \rightarrow \coprod (\mathcal{B}_i \amalg \mathcal{X}_i) \rightarrow \coprod \mathcal{C}_i \rightarrow \Sigma \coprod \mathcal{A}_i,$$

$$\coprod \mathcal{A}_i \rightarrow \coprod \mathcal{B}_i \rightarrow \coprod \mathcal{C}_i \rightarrow \Sigma \coprod \mathcal{A}_i.$$

Now our assumption (1) together with lemma 2.2 will enable us to split these two triangles into exact triangles in \mathcal{A}_i . So for each $i \in I$, we get exact triangles

$$A_i \rightarrow B_i \amalg X_i \rightarrow C_i \rightarrow \Sigma A_i,$$

$$A_i \rightarrow B_i \rightarrow C_i \rightarrow \Sigma A_i.$$

in \mathcal{A}_i . This implies (by Landsburg's criterion) that $[X_i] = 0$ in $K_0(\mathcal{A}_i)$. So Ξ is injective if condition (1) holds.

Now assume that \mathcal{A} is a thick ideal and that for each fixed $i \in I$ either (2) or (3) holds. Smash the above triangles (3a) (3b) with S_i to get two exact triangles in \mathcal{T} :

$$A \wedge S_i \rightarrow (B \wedge S_i) \amalg \left(\coprod_{j \in I} (X_j \wedge S_i) \right) \rightarrow C \wedge S_i \rightarrow \Sigma(A \wedge S_i)$$

$$A \wedge S_i \rightarrow B \wedge S_i \rightarrow C \wedge S_i \rightarrow \Sigma(A \wedge S_i)$$

It is easily seen that these triangles are in fact triangles in \mathcal{A}_i : This is trivial if condition (2) holds (because \mathcal{A}_i are ideals and $S_i \in \mathcal{A}_i$). If condition (3) holds, write $A = \coprod A_i$ with $A_i \in \mathcal{A}_i$, then $A \wedge S_i = \coprod (A_i \wedge S_i) = A_i \in \mathcal{A}_i$. Similarly $B \wedge S_i$ and $C \wedge S_i$ also belong to \mathcal{A}_i .

Now we claim that $\coprod_{j \in I} (X_j \wedge S_i) = X_i$. If (2) holds, then by the lemma 2.3, we get $X_j \wedge S_i = 0$ whenever $i \neq j$, and since S_i is a unit for \mathcal{A}_i , $X_i \wedge S_i = X_i$. If (3) holds, this is obviously true. So in both cases (conditions (2) and (3)), the above triangles can be simplified to obtain the following triangles in \mathcal{A}_i .

$$A \wedge S_i \rightarrow (B \wedge S_i) \amalg X_i \rightarrow C \wedge S_i \rightarrow \Sigma(A \wedge S_i),$$

$$A \wedge S_i \rightarrow B \wedge S_i \rightarrow C \wedge S_i \rightarrow \Sigma(A \wedge S_i).$$

This implies that $[X_i] = 0$ in $K_0(\mathcal{A}_i)$. This show that Ξ is injective, completing the proof of the theorem. \square

Here is another crucial lemma for studying Krull-Schmidt decompositions.

Lemma 2.5. *Let \mathcal{T} be a triangulated category and let \mathcal{A} and \mathcal{B} be two thick (ideal) subcategories of \mathcal{T} . If $\text{Hom}(\mathcal{A}, \mathcal{B}) = 0 = \text{Hom}(\mathcal{B}, \mathcal{A})$, then the full subcategory $\mathcal{A} \amalg \mathcal{B}$, consisting of objects of the form $A \amalg B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, is a thick (ideal) subcategory of \mathcal{T} .*

Proof. The key observation here is that every map $H : A \amalg B \rightarrow A' \amalg B'$ in $\mathcal{A} \amalg \mathcal{B}$ is forced by the given hypothesis to be of the form $f \amalg g$. It is very clear that $\mathcal{A} \amalg \mathcal{B}$ satisfies the ideal condition. To see that $\mathcal{A} \amalg \mathcal{B}$ satisfies the 2 out of 3 condition, start with a map H as above and complete it to a triangle. Then we have the following diagram where the rows are triangles in \mathcal{T} . (The bottom row is the coproduct of two triangles in \mathcal{T} .)

$$\begin{array}{ccccccc} A \amalg B & \xrightarrow{f \amalg g} & A' \amalg B' & \longrightarrow & \text{Cone}(f \amalg g) & \longrightarrow & \Sigma(A \amalg B) \\ \downarrow = & & \downarrow = & & \downarrow H & & \downarrow = \\ A \amalg B & \xrightarrow{f \amalg g} & A' \amalg B' & \longrightarrow & \text{Cone}(f) \amalg \text{Cone}(g) & \longrightarrow & \Sigma(A \amalg A') \end{array}$$

There exists a fill-in map H which turns out to be an isomorphism as before. Therefore $\mathcal{A} \amalg \mathcal{B}$ is a triangulated subcategory.

It remains to show thickness, i.e., $\mathcal{A} \amalg \mathcal{B}$ is closed under retractions. Consider a retraction map $e : A \amalg B \rightarrow A \amalg B$ (so $e^2 = e$). Since $e = a \amalg b$, the equation $e^2 = e$ implies $(a \amalg b)^2 = a^2 \amalg b^2 = a \amalg b$. This shows both a and b are retractions. So we are done. \square

Having developed all the necessary tools, we now turn our attention to Krull-Schmidt decompositions for thick subcategories. We begin with Krause's decomposition result [Kra99] for the stable module category in the next section.

3. STABLE MODULE CATEGORIES OVER GROUP ALGEBRAS

Consider the stable module category $\text{StMod}(KG)$, where G is a finite group and K is some field. The objects of this category are the (right) KG -modules and morphisms are equivalence classes of KG -module homomorphisms where two homomorphisms are equivalent if and only if their difference factors through a projective module. This category is well-known to be a triangulated category and has a well-defined tensor product (ordinary tensor product of K -vector spaces with the diagonal G -action) which makes it into a tensor triangulated category. The full subcategory of small objects in $\text{StMod}(KG)$ is equivalent to the full subcategory consisting of finitely generated KG -modules and is denoted by $\text{stmod}(KG)$. The main theorem of [Kra99] then states:

Theorem 3.1. [Kra99] *Every thick ideal \mathcal{A} in $\text{stmod}(KG)$ decomposes uniquely into indecomposable thick ideals $(\mathcal{A}_i)_{i \in I}$; $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$. Conversely given thick ideals $(\mathcal{A}_i)_{i \in I}$ such that $\mathcal{A}_i \cap \mathcal{A}_j = 0$ for all $i \neq j \in I$, there exists a thick ideal \mathcal{A} such that $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$.*

We will briefly outline how Krause arrives at this decomposition. The key idea is to consider the Bousfield localisation with respect to the localising subcategory \mathcal{A}^Π generated by \mathcal{A} . The inclusion $\mathcal{A}^\Pi \hookrightarrow \text{StMod}(KG)$ has a right adjoint $e : \text{StMod}(KG) \rightarrow \mathcal{A}^\Pi$. In fact, $e(M)$ is just the fibre of the Bousfield localisation map $M \rightarrow M_{\mathcal{A}^\Pi}$. Thus for each KG -module M , there is a natural triangle in $\text{StMod}(KG)$ given by $e(M) \xrightarrow{\epsilon} M \rightarrow M_{\mathcal{A}^\Pi} \rightarrow \Sigma e(M)$. Applying this right adjoint to the trivial module K gives a module $e(K)$, which is denoted by $E_{\mathcal{A}}$. The KG -module $E_{\mathcal{A}}$ associated to the thick ideal \mathcal{A} in this way is an idempotent module (i.e., $E_{\mathcal{A}} \otimes E_{\mathcal{A}} = E_{\mathcal{A}}$). (These modules were introduced by Rickard and they proved to be very useful objects in modular representation theory.) Krause then shows that this idempotent module is an endofinite object (see [Kra99, Definition 1.1]) in \mathcal{A}^Π and hence admits a splitting into indecomposable modules: $E_{\mathcal{A}} = \bigoplus_{i \in I} E_i$ with $E_i \otimes E_j = 0$ for $i \neq j$. Let $(\epsilon_i) : \coprod_i E_i \rightarrow K$ be the decomposition of $\epsilon : E_{\mathcal{A}} \rightarrow K$ and define \mathcal{A}_i to be the collection of all modules X in $\text{stmod}(KG)$ such that $\epsilon_i \otimes X$ is an isomorphism. It then follows that $\mathcal{A} = \coprod_i \mathcal{A}_i$ is the Krull-Schmidt decomposition for \mathcal{A} . See [Kra99] for more details.

We draw the following corollary by applying theorem 2.4 to the above decomposition.

Proposition 3.2. *Let \mathcal{A} be a thick ideal of $\text{stmod}(KG)$ and let $\coprod \mathcal{A}_i$ be the Krull-Schmidt decomposition of \mathcal{A} . Then,*

$$K_0(\mathcal{A}) \cong \bigoplus_{i \in I} K_0(\mathcal{A}_i).$$

Proof. We show that condition (1) of our main theorem is satisfied. This is done in [Kra99, Lemma 2.4] but we include it here for the reader's convenience. For any two thick ideals \mathcal{A}_1 and \mathcal{A}_2 that occur in the decomposition for \mathcal{A} , we want to show that $\text{Hom}(\mathcal{A}_1, \mathcal{A}_2) = 0$. Towards this, consider objects $X \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$ and note that every map $X \rightarrow Y$ in $\text{stmod}(KG)$ factors through $X \otimes \text{Hom}_K(X, Y)$ in the obvious way. So we will be done if we can show that $\text{Hom}_K(X, Y)$ is a projective KG -module or equivalently a trivial module in $\text{stmod}(KG)$. To see this, first note that $\text{Hom}_K(X, Y) \cong X^* \otimes Y$, where X^* denotes the K -dual of X . But X^* is a retract of $X^* \otimes X \otimes X^*$ (see [HPS97, Lemma A.2.6]) and therefore belongs to \mathcal{A}_1 .

This implies that $X^* \otimes Y \in \mathcal{A}_1 \otimes \mathcal{A}_2$. The latter is zero by lemma 2.3 and therefore $\text{Hom}_K(X, Y) = X^* \otimes Y$ is zero in $\text{stmod}(KG)$. \square

4. DERIVED CATEGORIES OF RINGS

In this section we will prove a Krull-Schmidt theorem for the thick subcategories of perfect complexes over a noetherian ring. We work in some level of formality here so that we can generalise our results easily to noetherian stable homotopy categories in the sense of Hovey-Palmieri-Strickland [HPS97].

Recall that a subset S of $\text{Spec}(R)$ is a *thick support* if it is a union of (Zariski) closed sets S_α such that $\text{Spec}(R) - S_\alpha$ is a quasi-compact set. It is an exercise [AM69, Page 12, Exc.17(vii)] to show that $\text{Spec}(R) - S_\alpha$ is quasi-compact if and only if $S_\alpha = V(I_\alpha)$ for some finitely generated ideal I_α . ($V(I)$ denotes the collection of all primes that contain I .) Since all ideals in a noetherian ring are finitely generated, it follows that the thick supports for noetherian rings are precisely subsets of $\text{Spec}(R)$ that are a union of closed sets (also known as *specialisation closed subsets*). It is a celebrated theorem of Thomason that the thick subcategories of perfect complexes are completely determined by these thick supports. More precisely:

Theorem 4.1. [Tho97] *The lattice of thick subcategories in $D^b(\text{proj } R)$ is isomorphic to the lattice of thick supports of $\text{Spec}(R)$. Under this isomorphism, a thick support S corresponds to the thick subcategory \mathcal{T}_S consisting of all complexes X such that $\text{Supp}(X) \subseteq S$, and a thick subcategory \mathcal{C} corresponds to the thick support $\bigcup_{X \in \mathcal{C}} \text{Supp}(X)$.*

Using this theorem, we now work our way to the Krull-Schmidt theorem for thick subcategories of perfect complexes. We begin with some lemmas.

Lemma 4.2. *For X in $D^b(\text{proj } R)$, let $DX = \text{RHom}(X, R)$ denote its Spanier-Whitehead dual. Then $\text{Supp}(X) = \text{Supp}(DX)$.*

Proof. X is a retract of $X \otimes DX \otimes X$ [HPS97, Lemma A.2.6], and therefore DX is a retract of $DX \otimes X \otimes DX$. Now it is clear that both X and DX have the same support. \square

Following [HPS97], we will denote $\text{Hom}_{D^b(\text{proj } R)}(\Sigma^* X, Y)$ by $[X, Y]_*$ and the internal function spectrum $\text{RHom}(X, Y)$ by $F(X, Y)$. With these notations, we have the following natural isomorphism [HPS97]

$$[X \otimes A, B]_* \cong [X, F(A, B)]_*$$

This isomorphism gives the following useful lemma.

Lemma 4.3. *If X and Y are perfect complexes such that $\text{Supp}(X) \cap \text{Supp}(Y) = \emptyset$, then $[X, Y]_* = 0$. In particular if A and B are any two disjoint thick supports of $\text{Spec}(R)$, then $[\mathcal{T}_A, \mathcal{T}_B]_* = 0$.*

Proof. By Spanier-Whitehead duality, $[X, Y]_* = [R \otimes X, Y_*] = [R, F(X, Y)]_*$. Therefore $[X, Y]_* = 0$ if and only if $F(X, Y) = 0$. But since X is a small object in $D(R)$, $F(X, Y) = DX \otimes Y$ [HPS97, Appendix A.2]. So we have to show that $DX \otimes Y = 0$. Since $\text{Supp}(DX) = \text{Supp}(X)$ is disjoint with $\text{Supp}(Y)$, given any prime p , either p is not in $\text{Supp}(DX)$ or it is not in $\text{Supp}(Y)$. In the former case, $R_p \otimes DX = 0$, and in the latter, $R_p \otimes Y = 0$. In either case, we get $DX \otimes Y \otimes R_p = 0$. Since p is an arbitrary prime, we get $DX \otimes Y = 0$. \square

By a *thick decomposition* of a thick support S , we will mean a decomposition $S = \bigcup S_i$ into non-empty thick supports, where $S_i \cap S_j = \emptyset$ if $i \neq j$. A thick decomposition is *Krull-Schmidt* if the S_i do not admit nontrivial thick decompositions. Here are a few examples.

Example 4.4. All rings considered are commutative.

- (1) If R is a PID, then every non-zero prime ideal is a maximal ideal and so the thick supports are: $\text{Spec}(R)$, and all subsets of the set of maximal ideals. In particular, $\text{Spec}(R)$ is an indecomposable thick support.
- (2) If R is an Artinian ring, it has only finitely many prime ideals and every prime ideal is also a maximal ideal. So every non-empty subset S of $\text{Spec}(R)$ is a thick support. It is then clear that $\bigcup_{p \in S} \{p\}$ is the Krull-Schmidt decomposition for S .

Note: The components of the Krull-Schmidt decomposition of a thick support S are not (in general) the connected components of S . For example, $\text{Spec}(\mathbb{Z}) - \{0\}$ is a connected subset of $\text{Spec}(\mathbb{Z})$; however, it is not indecomposable as a thick support. In fact,

$$\text{Spec}(\mathbb{Z}) - \{0\} = \bigcup_{p \text{ prime}} \{(p)\}$$

is its Krull-Schmidt decomposition.

Now we establish the strong connection between the decompositions of thick supports and the decompositions of thick subcategories.

Proposition 4.5. *Let $C = A \cup B$ be a thick decomposition of a thick support. Then this induces a decomposition of the associated thick subcategories:*

$$\mathcal{T}_C = \mathcal{T}_A \amalg \mathcal{T}_B.$$

Proof. Clearly $\mathcal{T}_A \cap \mathcal{T}_B = 0$ because A and B are disjoint by definition. We have to show that the objects of \mathcal{T}_C are coproducts of objects in \mathcal{T}_A and \mathcal{T}_B . We give an indirect proof of this statement using the thick subcategory theorem. Lemma 4.3 and lemma 2.5 tell us that $\mathcal{T}_A \amalg \mathcal{T}_B$ is a thick subcategory. So we will be done if we can show that the thick support corresponding to $\mathcal{T}_A \amalg \mathcal{T}_B$ is C . The thick support corresponding to $\mathcal{T}_A \amalg \mathcal{T}_B$ is given by $\bigcup_{x \in \mathcal{T}_A \amalg \mathcal{T}_B} \text{Supp}(x)$. This clearly contains both A and B and hence their union ($= C$). To see the other inclusion, just note that $\text{Supp}(a \oplus b) = \text{Supp}(a) \cup \text{Supp}(b) \subseteq A \cup B = C$. \square

Now we show that decompositions of thick subcategories gives rise to thick decompositions of the corresponding thick supports.

Proposition 4.6. *Let $\mathcal{C} = \mathcal{A} \amalg \mathcal{B}$ be a decomposition of a thick subcategory \mathcal{C} of $D^b(\text{proj } R)$ and let A , B , and C denote the corresponding thick supports of these thick subcategories. Then $C = A \cup B$ is a thick decomposition of C .*

Proof. We first observe that the intersection of thick supports is again a thick support: Let S and T be two thick supports of $\text{Spec}(R)$. Then $S = \bigcup_{\alpha} V(I_{\alpha})$ and $T = \bigcup_{\beta} V(J_{\beta})$ for some finitely generated ideals I_{α} and J_{β} in R . Now,

$$S \cap T = \bigcup_{\alpha, \beta} (V(I_{\alpha}) \cap V(J_{\beta})) = \bigcup_{\alpha, \beta} V(I_{\alpha} + J_{\beta}).$$

Since the sum of two finitely generated ideals is finitely generated, we conclude that $S \cap T$ is a thick support.

We now argue that A and B are disjoint. If A and B are not disjoint, then their intersection being a thick support corresponds to a non-zero thick subcategory that is contained in both \mathcal{A} and \mathcal{B} . This contradicts the fact that $\mathcal{A} \cap \mathcal{B} = 0$, so A and B have to be disjoint. It remains to show that $A \cup B$ is C . For this, note that every complex $c \in \mathcal{C}$ splits as $c = a \amalg b$ and recall that $\text{Supp}(c) = \text{Supp}(a) \cup \text{Supp}(b)$. It follows that $C = A \cup B$. \square

Remark 4.7. It can be easily seen that the last proposition, together with lemma 4.3, implies that the cancellation property holds for thick subcategories of $D^b(\text{proj } R)$.

Combining proposition 4.5 and 4.6 we get the following decomposition result.

Theorem 4.8. *Let R be a commutative ring and let \mathcal{T}_S be the thick subcategory of $D^b(\text{proj } R)$ corresponding to a thick support S . Then \mathcal{T}_S admits a Krull-Schmidt decomposition if and only if S admits one.*

Applying our main theorem 2.4 to the above decomposition gives,

Corollary 4.9. *Let $\mathcal{T}_S = \coprod_{i \in I} \mathcal{T}_{S_i}$ be a decomposition corresponding to a thick decomposition $S = \bigcup_{i \in I} S_i$. Then,*

$$K_0(\mathcal{T}_S) \cong \bigoplus_{i \in I} K_0(\mathcal{T}_{S_i}).$$

Proof. Condition (1) of our main theorem holds here by lemma 4.3. \square

The question that remains to be addressed is the following. When do thick supports of $\text{Spec}(R)$ admit Krull-Schmidt decompositions? We show that this is always possible if R is noetherian.

Proposition 4.10. *Let R be a noetherian ring and let S be a thick support of $\text{Spec}(R)$. Then there exists a unique Krull-Schmidt decomposition $\bigcup S_i$ for S .*

Proof. It is well-known that the set of prime ideals in a noetherian ring satisfies the descending chain condition [AM69, Corollary 11.12]. (This condition is equivalent to saying that any non-empty collection of primes ideals has a minimal element.) To start, let S be a thick support in $\text{Spec}(R)$ and let $(p_i)_{i \in I}$ be the collection of all minimal elements in S – i.e., primes $p \in S$ which do not contain any other prime in S . It is now clear (using the above fact about noetherian rings) that every prime $p \in S$ contains a minimal element $p_i \in S$, therefore $S = \bigcup_{i \in I} V(p_i)$. (Also note that each $V(p_i)$ is a closed subset of $\text{Spec}(R)$ and hence a thick support.) Now define the *thick graph* G_S of S as follows: The vertices are the minimal primes $(p_i)_{i \in I}$ in S , and two vertices p_i and p_j are adjacent if and only if $V(p_i) \cap V(p_j) \neq \emptyset$. Let $(C_k)_{k \in K}$ be the connected components of this graph and for each C_k define a thick support

$$S_k := \bigcup_{p_i \in C_k} V(p_i).$$

By construction it is clear that $\bigcup S_k$ is a thick decomposition of S . It is not hard to see that each S_k is indecomposable. This is done by showing that any thick decomposition of S_k disconnects the connected component C_k of the thick graph of S . Finally the uniqueness part: let $\bigcup T_k$ be another Krull-Schmidt decomposition

of S . It can be easily verified that the minimal primes in T_k are precisely the minimal primes of S that are contained in T_k . Thus the Krull-Schmidt decomposition $\bigcup T_k$ gives a partition of the set of minimal primes in S . This partition induces a decomposition of the thick graph of S into its connected (since each T_k is indecomposable) components. Since the decomposition of a graph into its connected components is unique, the uniqueness of Krull-Schmidt decomposition follows. \square

Remark 4.11. The careful reader will perhaps note that the proof of proposition 4.10 makes use of *only* the following two conditions on R .

- (1) Every open subset of $\text{Spec}(R)$ is compact.
- (2) $\text{Spec}(R)$ satisfies the descending chain condition.

Therefore the proof generalises to any R which satisfies these two properties. As mentioned above, it is well-known that noetherian rings satisfy these two properties. Moreover, any ring which has finitely many prime ideals automatically satisfies these properties. Note that there are non-noetherian rings which have finitely many primes. For example, the ring

$$R = \mathbb{C}[x_2, x_3, x_4, \dots]/(x_2^2, x_3^3, x_4^4, \dots)$$

is a non-noetherian ring with only one prime ideal. So everything that we are going to state for the remainder of this section will work for rings which have these two properties, but we state our results only for noetherian rings for obvious cognitive reasons.

After all that work, the following theorem is now obvious.

Theorem 4.12. *If R is a noetherian ring, then every thick subcategory of $D^b(\text{proj } R)$ admits a unique Krull-Schmidt decomposition. Conversely, given any collection of thick subcategories $(\mathcal{C}_i)_{i \in I}$ in $D^b(\text{proj } R)$ such that $\mathcal{C}_i \cap \mathcal{C}_j = 0$ for all $i \neq j$, there exists a thick subcategory \mathcal{C} such that $\mathcal{C} = \coprod_{i \in I} \mathcal{C}_i$.*

Proof. The first part follows from theorem 4.8 and proposition 4.10. For the second part, define \mathcal{C} be the full subcategory of all finite coproducts of objects from the \mathcal{C}_i . Thickness of \mathcal{C} follows by combining theorem 4.8, lemma 4.3, and lemma 2.5. \square

Corollary 4.13. *A noetherian ring R is local if and only if every thick subcategory of $D^b(\text{proj } R)$ is indecomposable.*

Proof. If R is local, then it is clear that every thick support of $\text{Spec}(R)$ contains the unique maximal ideal. In particular, we cannot have two non-empty disjoint thick supports and therefore every thick subcategory of $D^b(\text{proj } R)$ is indecomposable. Conversely, if R is not local, it is clear that the thick subcategory supported on the set of maximal ideals is decomposable. \square

Theorem 4.12 also gives the following interesting algebraic results.

Corollary 4.14. *Let X be a perfect complex over a noetherian ring R . Then X admits a unique splitting into perfect complexes,*

$$X \cong \bigoplus_{i \in I} X_i$$

such that the supports of the X_i are pairwise disjoint and indecomposable.

Proof. Let $\bigcup S_i$ be the Krull-Schmidt decomposition of $\text{Supp}(X)$ (which exists by proposition 4.10). It is clear from theorem 4.12 that X admits a splitting ($X \cong \bigoplus X_i$) where $\text{Supp}(X_i) \subseteq S_i$. To see that we have equality in this inclusion, observe that

$$\text{Supp}(X) = \text{Supp}\left(\bigoplus_{i \in I} X_i\right) = \bigcup_{i \in I} \text{Supp}(X_i) \subseteq \bigcup_{i \in I} S_i = \text{Supp}(X).$$

Uniqueness: If X admits another decomposition $\bigoplus Y_i$ as above, then by proposition 4.10 we know that both $\bigcup \text{Supp}(X_i)$ and $\bigcup \text{Supp}(Y_i)$ are the same decompositions of $\text{Supp}(X)$. Lemma 4.3 then implies that $X_i \cong Y_i$ for all i . So the decomposition is unique. \square

Srikanth Iyengar pointed out that this splitting holds in a much more generality. In fact it holds for all complexes in the derived category which have bounded and finite homology (i.e. complexes whose homology groups $H_i(-)$ are finitely generated and are zero for all but finitely many i .) The full subcategory of such complexes will be denoted by $D^f(R)$.

Corollary 4.15. *Let R be a noetherian ring. Every complex X in $D^f(R)$ admits a splitting*

$$X \cong \bigoplus_{i \in I} X_i$$

such that the supports of the X_i are pairwise disjoint and indecomposable.

Proof. Since X has bounded and finite homology, $\text{Supp}(X)$ is a closed set and therefore proposition 4.10 applies and we get a decomposition

$$\text{Supp}(X) = \bigcup_{i \in I} S_i.$$

We now use a result of Neeman [Nee92] which states that the lattice of localising subcategories (thick subcategories that are closed under arbitrary coproducts) is isomorphic to the lattice of all subsets of $\text{Spec}(R)$. If \mathcal{L} denotes the localising subcategory generated by X , then we get a Krull-Schmidt decomposition $\mathcal{L} \cong \coprod \mathcal{L}_i$ that corresponds to the above decomposition of the support of X . (This follows exactly as in the case of thick subcategories.) The splitting of X as stated in the corollary is now clear. \square

Example 4.16. We illustrate the last two corollaries with some examples.

- (1) For every integer $n > 1$, let $M(n)$ denote the Moore complex $0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow 0$ and let $p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$ be the unique prime factorisation of n . It is easy to see that $\bigcup_{t=1}^k \{(p_t)\}$ is the Krull-Schmidt decomposition of $\text{Supp}(M(n))$. Then it follows that

$$M(n) \cong \bigoplus_{t=1}^k M(p_t^{i_t})$$

is the splitting of $M(n)$ corresponding to this Krull-Schmidt decomposition.

- (2) Let R be a self-injective noetherian ring (hence also an Artinian ring). Then it follows (because every prime ideal is also maximal) that $\bigcup_{p \in \text{Spec}(R)} \{p\}$

is the Krull-Schmidt decomposition for $\text{Supp}(R)$. It can be shown that

$$R \cong \bigoplus_{p \in \text{Spec}(R)} E(R/p)$$

is the splitting of R corresponding to this Krull-Schmidt decomposition. (Here $E(R/p)$ denotes the injective hull of R/p .)

5. NOETHERIAN STABLE HOMOTOPY CATEGORIES.

Motivated by the work in the previous section, we now state a Krull-Schmidt theorem for noetherian stable homotopy categories. We use the language and results of [HPS97] freely. We explain how the proofs of the previous section generalise to give us a Krull-Schmidt theorem in this more general setting by invoking the appropriate results from [HPS97]. We begin with some definitions and preliminaries from Axiomatic Stable Homotopy Theory [HPS97].

Definition 5.1. [HP01] A *unital algebraic stable homotopy category* is a tensor triangulated category \mathcal{C} with the following properties.

- (1) Arbitrary products and coproducts of objects in \mathcal{C} exist.
- (2) \mathcal{C} has a finite set \mathcal{G} of weak generators: i.e., $X \cong 0$ if and only if $[A, X]_* = 0$ for all $A \in \mathcal{G}$.
- (3) The unit object S and the objects of \mathcal{G} are small.

\mathcal{C} is a *noetherian stable homotopy category* if in addition the following conditions are satisfied,

- $\pi_*(S) := [S, S]_*$ is commutative and noetherian as a bigraded ring.
- For small objects Y and Z of \mathcal{C} , $[Y, Z]_*$ is a finitely generated module over $[S, S]_*$.

The stable homotopy category of spectra is unital and algebraic but not noetherian. The derived category $D(R)$ of a commutative ring R is a noetherian stable homotopy category if and only if R is noetherian (since $[R, R]_* = R$). $\text{Spec}(\pi_*S)$ will stand for the collection of all homogeneous prime ideals of π_*S with the Zariski topology.

Henceforth \mathcal{C} will denote a bigraded noetherian stable homotopy category. For each thick ideal \mathcal{A} of finite objects in \mathcal{C} , there is a *finite localisation functor* $L_{\mathcal{A}}$ (also denoted $L_{\mathcal{A}}^f$) on \mathcal{C} whose finite acyclics are precisely the objects of \mathcal{A} ; see [HP01, Theorem 2.3]. For each bihomogeneous prime ideal \mathfrak{p} in $\pi_*(S)$, there are finite localisation functors $L_{\mathfrak{p}}$ and $L_{<\mathfrak{p}}$ whose finite acyclics are $\{X \text{ finite} \mid X_{\mathfrak{q}} = 0 \forall \mathfrak{q} \subseteq \mathfrak{p}\}$ and $\{X \text{ finite} \mid X_{\mathfrak{q}} = 0 \forall \mathfrak{q} \subsetneq \mathfrak{p}\}$ respectively. These functors define, for each $X \in \mathcal{C}$, a natural exact triangle

$$M_{\mathfrak{p}}X \rightarrow L_{\mathfrak{p}}X \rightarrow L_{<\mathfrak{p}}X \rightarrow \Sigma M_{\mathfrak{p}}X.$$

We set $M_{\mathfrak{p}} := M_{\mathfrak{p}}S$. See [HP01] for more details.

We say that \mathcal{C} has the *tensor product property* if for all bihomogeneous prime ideals \mathfrak{p} in $\pi_*(S)$ and all objects $X, Y \in \mathcal{C}$, $M_{\mathfrak{p}} \wedge X \wedge Y = 0$ if and only if either $M_{\mathfrak{p}} \wedge X = 0$ or $M_{\mathfrak{p}} \wedge Y = 0$.

We now mimic the setup for the derived category of a ring. For X in \mathcal{C} , define its *cohomological support variety* as

$$\text{Supp}(X) = \{\mathfrak{p} \mid M_{\mathfrak{p}} \wedge X \neq 0\} \subseteq \text{Spec}(\pi_*S),$$

and if \mathcal{D} is any thick subcategory of \mathcal{C} , define

$$\text{Supp}(\mathcal{D}) = \bigcup_{X \in \mathcal{D}} \text{Supp}(X).$$

It is a fact [HPS97, Theorem 6.1.7] that for X a small object in \mathcal{C} , $\text{Supp}(X)$ is a Zariski closed subset and hence also a thick support of $\text{Spec } \pi_*(S)$.

We are now ready to state the thick subcategory theorem for \mathcal{C} .

Theorem 5.2. [HP01] *Let \mathcal{C} be a noetherian stable homotopy category satisfying the tensor product property. Then the lattice of thick ideals of small objects in \mathcal{C} is isomorphic to the lattice of thick supports of $\text{Spec}(\pi_*S)$. Under this isomorphism, a thick ideal \mathcal{A} corresponds to $\text{Supp}(\mathcal{A})$, and a thick support T of $\text{Spec}(\pi_*S)$ corresponds to the thick ideal $\{Z \in \mathcal{C} \mid \text{Supp}(Z) \subseteq T\}$.*

Now using this thick subcategory theorem, we can run our proofs from the previous section which generalise to prove theorem 5.4 below. We will also mention the minor changes that are necessary for this generalisation. We begin with an algebraic lemma.

Lemma 5.3. *Let R be a graded commutative noetherian K -algebra. Then R satisfies the descending chain condition on the set $\text{Spec}(R)$ of homogeneous prime ideals of R .*

Proof. First note that if $\text{char}(K) = 2$, then R is a strictly commutative graded ring and the result for such R is well-known; see [AM69, Corollary 11.12]. So assume that $\text{char}(K) \neq 2$. Since every homogeneous prime ideal contains the nilradical, the ring map

$$R \rightarrow R/\text{nilrad } R$$

induces an order-preserving bijection between $\text{Spec}(R)$ and $\text{Spec}(R/\text{nilrad } R)$. So in view of this remark, we can assume without any loss of generality that $\text{nilrad } R = 0$. But if $\text{nilrad } R = 0$, then it is concentrated in even degrees: If x is an odd degree element, then $x^2 = -x^2$. This implies $x^2 = 0$ (since $\text{char}(K) \neq 2$), or equivalently $x \in \text{nilrad } R = 0$. This shows that R is concentrated in even degrees. In particular it is a strictly commutative graded ring, so we are done. \square

Theorem 5.4. *Let \mathcal{C} be a noetherian stable homotopy category satisfying the tensor product property. Then every thick ideal \mathcal{A} of small objects in \mathcal{C} admits a Krull-Schmidt decomposition: $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$. Consequently,*

$$K_0(\mathcal{A}) \cong \bigoplus_{i \in I} K_0(\mathcal{A}_i).$$

Conversely, if \mathcal{A}_i are thick ideals of \mathcal{C} such that $\mathcal{A}_i \cap \mathcal{A}_j = 0$ for all $i \neq j$, then there exists a thick ideal \mathcal{A} such that $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$.

Proof. We will explain why all the lemmas and propositions leading up to theorem 4.12 hold in this generality. We proceed in order starting from lemma 4.2.

Lemma 4.2: This holds for any strongly dualisable object in a closed symmetric monoidal category [HPS97, Lemma A.2.6]. By [HPS97, theorem 2.1.3 (d)], every small object is strongly dualisable in a unital algebraic stable homotopy category and therefore lemma 4.2 holds for small objects of \mathcal{C} .

Lemma 4.3: Proceeding in the same way, it boils down to showing that $DX \wedge Y$ is zero. Since DX and Y have disjoint supports, for each prime \mathfrak{p} in $\text{Spec}(\pi_*S)$,

either $Y \wedge M_{\mathfrak{p}} = 0$ or $DX \wedge M_{\mathfrak{p}} = 0$. This implies that $DX \wedge Y \wedge M_{\mathfrak{p}} = 0$ for all \mathfrak{p} . By theorem [HPS97, Theorem 6.1.9], we have the following equality of Bousfield classes:

$$\langle S \rangle = \coprod_{\mathfrak{p} \in \text{Spec}(\pi_* S)} \langle M_{\mathfrak{p}} \rangle.$$

This implies $(DX \wedge Y) \wedge S = DX \wedge Y = 0$.

Proposition 4.5: This goes through verbatim without any changes.

Proposition 4.6: This is actually much easier. Thick supports of $\text{Spec}(\pi_* S)$ are just unions of Zariski-closed sets and therefore it is obvious that intersection of thick supports is thick. The rest of the proof of this proposition follows exactly the same way.

Using these lemmas and proposition, theorem 4.8 and corollary 4.9 follow. Proposition 4.10 also holds in the graded noetherian case; see lemma 5.3. Thus the first part of theorem follows by combining all these lemmas and propositions.

The converse follows as before by combining theorem 4.8, lemma 4.3, and lemma 2.5. \square

Corollary 5.5. *Let \mathcal{C} be a noetherian stable homotopy category satisfying the tensor product property. Then every object X in \mathcal{C} admits a unique decomposition*

$$X \cong \coprod_{i \in I} X_i,$$

such that the support varieties of the X_i are pairwise disjoint and indecomposable.

Remark 5.6. The derived category $D(R)$ of a noetherian ring R is a noetherian stable homotopy category which satisfies the tensor product property. (Note that $\pi_*(S) = [R, R]_* = R$.) So theorem 5.2 recovers our Krull-Schmidt theorem for derived categories of noetherian rings.

We now give another example of a noetherian stable homotopy category for which theorem 5.4 applies.

Example 5.7. [HP01, HP00] Let B denote a finite dimensional graded co-commutative Hopf algebra over a field k ($\text{char}(k) > 0$). Then the category $K(\text{Proj } B)$, whose objects are unbounded chain complexes of projective B -modules and whose morphisms are chain homotopy classes of chain maps, is a noetherian stable homotopy category. In this category the unit object is a projective resolution of k by B -modules and therefore $\pi_*(S)$ is isomorphic to $\text{Ext}_B^{*,*}(k, k)$. It is well-known that this ext algebra is a bigraded noetherian algebra; see [FS97] and [Wil81]. In particular, if B is a finite dimensional sub-Hopf algebra of the mod 2 Steenrod algebra, it is a deep result [HP01, Corollary 8.6] that the category $K(\text{Proj } B)$ satisfies the tensor product property. So theorem 5.4 applies and we get the following result.

Theorem 5.8. *Let B denote a finite dimensional graded co-commutative Hopf algebra satisfying the tensor product property. Then every thick subcategory of small objects in $K(\text{Proj } B)$ is indecomposable.*

Proof. The augmentation ideal of $\text{Ext}_B^{*,*}(k, k)$ which consists of all elements in positive degrees is the unique maximal homogeneous ideal and therefore is contained in every thick support of $\text{Spec } \text{Ext}_B^{*,*}(k, k)$. Now by theorem 5.4 it follows that every thick subcategory of small objects in this category is indecomposable. \square

6. STABLE MODULE CATEGORIES OVER HOPF ALGEBRAS

In this section we give a generalised version of Krause's result [Kra99, Theorem A]. To this end, let B denote a finite dimensional graded co-commutative Hopf algebra over a field k . Following [HP01], we say that B satisfies the tensor product if the category $K(\text{Proj } B)$ does. We now prove a Krull-Schmidt theorem for $\text{stmod}(B)$, whenever B satisfies the tensor product property. The category $\text{StMod}(B)$ fails to be a noetherian stable homotopy category in general. (In fact, $\pi_*(S) = \text{Hom}_{\text{stmod}(B)}(k, k)$ is isomorphic to the Tate cohomology of B which is not noetherian in general.) So we take the standard route of using a finite localisation functor [HPS97, Section 9] to get around this problem.

We now review some preliminaries from [HPS97, Section 9]. There is a finite localisation functor $L_B^f : K(\text{Proj } B) \rightarrow K(\text{Proj } B)$, whose finite acyclics are precisely the objects in the thick subcategory generated by B . It has been shown [HPS97] that the category $\text{StMod}(B)$ is equivalent to the full subcategory of L_B^f -local objects. This finite localisation functor also establishes a bijection between the nonzero thick ideals of small objects in $K(\text{Proj } B)$ and the thick ideals of $\text{stmod}(B)$. In fact, the thick ideals of $\text{stmod}(B)$ are in poset bijection with the non-empty specialisation closed subsets of $\text{Spec Ext}_B^{*,*}(k, k)$. (Note that for the category $K(\text{Proj } B)$, the empty set of $\text{Spec Ext}_B^{*,*}(k, k)$ corresponds to the thick subcategory consisting of the zero chain complex.) Finally let \mathfrak{m} denote the unique bihomogeneous maximal ideal of $\text{Ext}_B^{*,*}(k, k)$ which corresponds to the trivial thick subcategory in $\text{stmod}(B)$. Then here is the main result of this section.

Theorem 6.1. *Let B denote a finite dimensional graded co-commutative Hopf algebra satisfying the tensor product property. Then every thick ideal of $\text{stmod}(B)$ admits a unique Krull-Schmidt decomposition. Conversely, given thick ideals \mathcal{T}_i in $\text{stmod}(B)$ satisfying $\mathcal{T}_i \cap \mathcal{T}_j = 0$ for $i \neq j$, there exists a thick ideal \mathcal{T} such that $\mathcal{T} \cong \coprod \mathcal{T}_i$. Moreover,*

$$K_0(\mathcal{T}) \cong \bigoplus_{i \in I} K_0(\mathcal{T}_i).$$

Proof. Recall once again that the thick ideals of the stable module category are in poset bijection with the non-empty subsets of $\text{Spec Ext}_B^{*,*}(k, k)$ that are closed under specialisation. To begin, let \mathcal{T} be a thick ideal of $\text{stmod}(B)$ and let S denote the corresponding specialisation closed subset. Since $\text{Ext}_B^{*,*}(k, k)$ is a graded noetherian algebra, it follows that S admits a unique decomposition $S = \bigcup_i S_i$ such that $S_i \cap S_j = \mathfrak{m}$ for all $i \neq j$ and S_i indecomposable (the proof of Proposition 4.10 can be adapted to the set $S - \mathfrak{m}$ by making use of lemma 5.3). We now claim that $\mathcal{T} = \coprod \mathcal{T}_{S_i}$, where \mathcal{T}_{S_i} denotes the thick ideal of $\text{stmod}(B)$ corresponding to S_i .

The first thing to note is that for $i \neq j$, $\mathcal{T}_{S_i} \cap \mathcal{T}_{S_j} = 0$: Suppose not, then the non-zero thick ideal $\mathcal{T}_{S_i} \cap \mathcal{T}_{S_j}$ corresponds to a non-empty subset of $S_i \cap S_j - \mathfrak{m}$ which is impossible. Therefore it follows that $\mathcal{T}_{S_i} \cap \mathcal{T}_{S_j} = 0$.

The next step is to show that the objects in \mathcal{T} are the finite coproducts of the objects in \mathcal{T}_{S_i} . We give an indirect proof (as before) by showing that $\coprod \mathcal{T}_{S_i}$ is a thick ideal. Then the theorem follows because the specialisation closed subset corresponding to $\coprod \mathcal{T}_{S_i}$ is clearly $\bigcup S_i = S$. To show that $\coprod \mathcal{T}_{S_i}$ is thick (ideal condition holds trivially), by lemma 2.5, all we need to show is that $\text{Hom}_{\text{stmod}(A, B)} = 0$, whenever $A \in \mathcal{T}_{S_i}$ and $B \in \mathcal{T}_{S_j}$ ($i \neq j$). This was already done for group algebras

in proposition 3.2. The proof given there can be easily generalised to finite dimensional co-commutative Hopf algebras. So that completes the proof of the existence of Krull-Schmidt decompositions.

For the converse, define \mathcal{T} be the full subcategory of all finite coproducts of objects in $(\mathcal{T}_i)_{i \in I}$ and use the fact that $\text{Hom}_{\text{stmod}}(\mathcal{T}_i, \mathcal{T}_j) = 0$ for $i \neq j$, to complete the argument.

The statement about K_0 groups follows from theorem 2.4. \square

Remark 6.2. It is worth pointing out the difference between the decompositions in theorem 5.8 and those in theorem 6.1: Every thick ideal of small objects in $K(\text{Proj } B)$ is indecomposable; on the other hand, thick ideals of small objects in $\text{StMod}(B)$ admit Krull-Schmidt decompositions. This striking difference is really the effect of Bousfield localisation ($\text{StMod}(B)$ is a Bousfield localisation of $\text{Stable}(B)$).

6.1. Sub-Hopf algebras of the mod-2 Steenrod algebra. It has been shown in [HP01] that every finite dimensional sub-Hopf algebra of A (the mod 2 Steenrod algebra) satisfies the tensor product property. So theorem 6.1 applies and we get interesting Krull-Schmidt decompositions in the stable module categories over these sub-Hopf algebras. We now look at some standard sub-Hopf algebras of A and analyse decompositions in their stable module categories. For every non-negative integer n , denote by $A(n)$ the finite dimensional sub-Hopf algebra of A generated by $\{\text{Sq}^k \mid 1 \leq k \leq 2^n\}$.

$A(0)$ - This is the sub-Hopf algebra of A generated by Sq^1 and therefore it is isomorphic to the exterior algebra $\mathbb{F}_2[\text{Sq}^1]/(\text{Sq}^1)^2$ on Sq^1 . It can be easily shown that $\text{Ext}_{A(0)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_0]$ with $|h_0| = (1, 1)$. Therefore the bihomogeneous prime spectrum of $\text{Ext}_{A(0)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ is $\{(0), (h_0)\}$. It is now clear from the thick subcategory theorem that there are only two thick subcategories of $\text{stmod}(A(0))$; the trivial category which corresponds to maximal prime (h_0) and the entire category $\text{stmod}(A(0))$ which corresponds to $\{(0), (h_0)\}$. These subsets of prime ideals are clearly indecomposable closed subsets and therefore we conclude by theorem 6.1 that all thick subcategories of $\text{stmod}(A(0))$ are indecomposable.

$A(1)$ - This is the sub-Hopf algebra of A generated by Sq^1 and Sq^2 . This algebra can be shown (using the Adem relations) to be isomorphic to the free non-commutative algebra (over \mathbb{F}_2) generated by symbols Sq^1 and Sq^2 modulo the two sided ideal

$$(\text{Sq}^1 \text{Sq}^1, \text{Sq}^2 \text{Sq}^2 + \text{Sq}^1 \text{Sq}^2 \text{Sq}^1).$$

The cohomology algebra of $A(1)$ has been computed in [Wil81]:

$$\text{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_0, h_1, w, v]/(h_0 h_1, h_1^3, h_1 w, w^2 - h_0^2 v),$$

where the bi-degrees are: $|h_0| = (1, 1)$, $|h_1| = (1, 2)$, $|w| = (3, 7)$, and $|v| = (4, 12)$. The bihomogeneous prime lattice of the above cohomology ring consists of the minimal prime (h_1) ; primes $p_0 = (h_1, w, h_0)$ and $p_2 = (h_1, w, v)$ of height one; and the maximal prime $\mathfrak{m} = (h_1, w, h_0, v)$.

The thick subcategory theorem now tells us that there are five thick subcategories in $\text{stmod}(A(1))$ which correspond to the thick supports $\{\mathfrak{m}\}$, $\{p_0, \mathfrak{m}\}$, $\{p_1, \mathfrak{m}\}$,

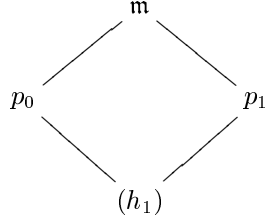


FIGURE 1. Bihomogeneous prime lattice of $\text{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$

$\{p_0, p_1, \mathfrak{m}\}$, and $\{(h_1), p_0, p_1, \mathfrak{m}\}$. It is clear that the only decomposable thick support in this list is $\{p_0, p_1, \mathfrak{m}\}$; in fact,

$$\{p_0, p_1, \mathfrak{m}\} = \{p_0, \mathfrak{m}\} \cup \{p_1, \mathfrak{m}\}$$

is the desired decomposition. This induces a Krull-Schmidt decomposition of thick subcategories supported on these thick supports:

$$\mathcal{T}_{\{p_0, p_1, \mathfrak{m}\}} \cong \mathcal{T}_{\{p_0, \mathfrak{m}\}} \amalg \mathcal{T}_{\{p_1, \mathfrak{m}\}}.$$

Note that the remaining four thick subcategories in $\text{stmod}(A(1))$ are indecomposable.

Decompositions in stable module categories over $A(2)$ and higher are much more complicated.

7. FINITE SPECTRA

In this section we work in the triangulated category \mathcal{F} of finite spectra. Recall that a spectrum X is torsion if $\pi_*(X)$ is a torsion group. In particular, a finite spectrum X is torsion (p -torsion) if $\pi_*(X)$ consists of finite abelian groups (p -groups) in every degree. We will now show that the thick subcategories of finite spectra admit Krull-Schmidt decompositions.

\mathcal{F}_{tor} will denote the category of finite torsion spectra, and $\mathcal{C}_{n,p}$ will denote the thick subcategory of finite p -torsion spectra consisting of all $K(n-1)_*$ acyclics

Lemma 7.1. *If \mathcal{A} is a thick subcategory of \mathcal{F} , then either $\mathcal{A} = \mathcal{F}$ or $\mathcal{A} \subseteq \mathcal{F}_{\text{tor}}$.*

Proof. It suffices to show that if X is any non-torsion finite spectrum, then the integral sphere spectrum S belongs to the thick subcategory generated by X . Consider the cofibre sequence

$$X \xrightarrow{p} X \rightarrow X \wedge M(p) \rightarrow \Sigma X.$$

It is easy to see that $X \wedge M(p)$ is a type-1 spectrum, therefore the thick subcategory generated by X contains all finite torsion spectra. The next step is to show that there is a cofibre sequence

$$\coprod S^r \rightarrow X \rightarrow T \rightarrow \coprod S^{r+1},$$

where T is a finite torsion spectrum; see [Mar83, Proposition 8.9]. It is clear from these exact triangles that S belongs to the thick subcategory generated by X . \square

We now start analysing decompositions for thick subcategories of finite spectra.

Proposition 7.2. *The category \mathcal{F} is indecomposable.*

Proof. If $\mathcal{F} = \mathcal{A} \amalg \mathcal{B}$ is any decomposition of \mathcal{F} , then that would give a splitting $A \vee B$ of the sphere spectrum (S). Since S is an indecomposable spectrum, either A or B must be the sphere spectrum, which implies that either \mathcal{A} or \mathcal{B} is equal to \mathcal{F} . \square

We now show that the thick subcategories of finite *torsion* spectra admit non-trivial Krull-Schmidt decompositions.

Theorem 7.3. *Every thick subcategory \mathcal{D} of \mathcal{F}_{tor} admits a unique Krull-Schmidt decomposition. Conversely, given any collection of thick subcategories \mathcal{D}_i in \mathcal{F}_{tor} such that $\mathcal{D}_i \cap \mathcal{D}_j = 0$ for all $i \neq j$, there exists a thick subcategory \mathcal{D} such that $\coprod_i \mathcal{D}_i$ is a Krull-Schmidt decomposition for \mathcal{D} . Moreover $K_0(\mathcal{D}) \cong \bigoplus_i K_0(\mathcal{D}_i)$.*

Proof. For every prime number p , define $\mathcal{D}_p := \mathcal{D} \cap \mathcal{C}_{1,p}$. By [Mar83, 8.2, Theorem 20] every torsion spectrum X can be written as $X = \bigvee_p X_{(p)}$ where $X_{(p)}$ is the p -localisation of X . If X is a finite torsion spectrum, it is easy to see that $X_{(p)}$ is a finite p -torsion spectrum and therefore belongs to $\mathcal{C}_{1,p} (\subseteq \mathcal{F}_{tor})$. Since \mathcal{D} is a thick subcategory of \mathcal{F}_{tor} , all the summands $X_{(p)}$ also belong to \mathcal{D} . Therefore $X_{(p)}$ belongs to $\mathcal{C}_{1,p} \cap \mathcal{D} = \mathcal{D}_p$. Also since X is a finite spectrum, the above wedge runs over only a finite set of primes. This implies that the spectra in \mathcal{D} are the finite coproducts of spectra in (\mathcal{D}_p) .

\mathcal{D}_p is a thick subcategory in \mathcal{F}_{tor} : Showing the $\mathcal{C}_{1,p}$ is a thick subcategory of \mathcal{F}_{tor} is a routine verification. Since intersection of thick subcategories is thick, $\mathcal{D} \cap \mathcal{C}_{1,p}$ is thick in \mathcal{F}_{tor} .

$\mathcal{D}_p \cap \mathcal{D}_q = 0$: It is clear that if a finite spectrum is both p -torsion and q -torsion, for primes $p \neq q$, then it is trivial – i.e., $\mathcal{C}_{1,p} \cap \mathcal{C}_{1,q} = 0$. But $\mathcal{D}_p \cap \mathcal{D}_q \subseteq \mathcal{C}_{1,p} \cap \mathcal{C}_{1,q}$, therefore $\mathcal{D}_p \cap \mathcal{D}_q = 0$.

\mathcal{D}_p is indecomposable: \mathcal{D}_p is by definition a thick subcategory of $\mathcal{C}_{1,p}$. The thick subcategories of $\mathcal{C}_{1,p}$ are well-known to be nested [HS98]. In particular they are all indecomposable.

This completes the proof of the existence of a Krull-Schmidt decomposition for any thick subcategory \mathcal{D} in \mathcal{F}_{tor} .

Uniqueness: Let $\coprod \mathcal{E}_k$ be another Krull-Schmidt decomposition for \mathcal{D} . Then since each \mathcal{E}_k is an indecomposable thick subcategory of \mathcal{F}_{tor} , we conclude that $\mathcal{E}_k = \mathcal{C}_{n_k, p_k}$ for some integer n_k and some prime p_k . Thus the decomposition is unique.

For the converse, define \mathcal{D} be the full subcategory of all finite coproducts of objects from \mathcal{D}_p . We just have to show that \mathcal{D} is a thick subcategory and then the rest follows. Note that $[X, Y] = 0$ whenever X is a p -torsion spectrum and Y is a q -torsion spectrum [Mar83, 8.2, Lemma 21]. Therefore thickness of \mathcal{D} follows from lemma 2.5. \square

The following corollary is clear from these decompositions.

Corollary 7.4. *Let \mathcal{A} be a thick subcategory of \mathcal{F} . Then either $\mathcal{A} = \mathcal{F}$, or $\mathcal{A} = \coprod \mathcal{C}_{i,p}$, where i ranges over some subset of positive integers and p over some subset of primes.*

8. QUESTIONS

8.1. Krull-Schmidt decompositions in $D^b(\text{proj } R)$. We have shown that the thick subcategories of perfect complexes over a *noetherian* ring admit Krull-Schmidt decompositions. Note that proposition 4.10 was crucial for this decomposition. Now the question that remains to be answered is whether this proposition holds for arbitrary commutative rings. In other words, if R is any commutative ring, is it true that every thick support of $\text{Spec}(R)$ admits a unique Krull-Schmidt decomposition? An affirmative answer to this question would imply, by theorem 4.8, that thick subcategories of perfect complexes over commutative rings (not necessarily noetherian) admit Krull-Schmidt decompositions.

8.2. The tensor product property. Recall that our decompositions for the thick subcategories of small objects in $K(\text{Proj } B)$ and $\text{StMod}(B)$ assumed that the finite dimensional co-commutative Hopf algebra B satisfies the tensor product property. Now the question that arises is which finite dimensional Hopf algebras satisfy the tensor product property? This seems to be a question of independent interest; see [HP01, Corollary 3.7]. Friedlander and Pevtsova [FP04] have shown recently that the finite dimensional co-commutative *ungraded* Hopf algebras satisfy the tensor product property. Using their ideas, or otherwise, we would like to know if this property also holds for finite dimensional graded co-commutative Hopf algebras. Also we do not know if the tensor product property in [FP04] is equivalent to the one given in [HP01].

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