AFFINRE LINKING NUMBERS AND CAUSALITY RELATIONS FOR WAVE FRONTS

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Abstract. Let M be an oriented n-dimensional manifold. We study the causal relations between the wave fronts W_1 and W_2 that originated at some points of M. We introduce a numerical topological invariant $\mathrm{CR}(W_1, W_2)$ (the so-called causality relation invariant) that, in particular, gives the algebraic number of times the wave front W_1 passed through the point that was the source of W_2 before the front W_2 originated. This invariant can be easily calculated from the current picture of wave fronts on M without the knowledge of the propagation law for the wave fronts. Moreover, in fact we even do not need to know the topology of M outside of a part $\overline{M}$ of M such that W_1 and W_2 are null-homotopic in $\overline{M}$.

Introduction

Throughout this paper M is a smooth connected oriented manifold (not necessarily compact).

Let W_1 and W_2 be two wave fronts which are propagating in M. (Generally, we assume that the fronts have different propagation laws.) We define a *dangerous intersection* between the fronts $W_1(t)$ and $W_2(t)$ at some moment of time t to be a point x where the fronts intersect and have the same direction of propagation (see Section 1 for the precise definition).

A passage of the front $W_1(t)$ through the point that is the birth point of the front W_2 at the moment of time t before the front W_2 originated is called the *baby- intersection.*

It turns out that we can associate to each dangerous intersection as well as to each baby-intersection a sign (i.e. a number ± 1). The sum of the signs up to a moment t is called a *causality relation invariant* and is denoted by $\mathrm{CR}(W_1(t), W_2(t))$.

In particular, suppose that W_1 originated before W_2 and that there are no dangerous intersections of $W_1(t)$ and $W_2(t)$ for all t, (see such examples in Section 1). Then the causality relation invariant tells us the algebraic number of times the first wave front reached the birth point of the second front before the second front originated.

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We are interested in reconstructing the value $\mathrm{CR}(W_1(t), W_2(t))$ from the current shape of the wave fronts only, without the knowledge of the propagation laws, of the birth- points of the fronts, of the topology of M etc.

It turns out that, having the current picture only, we can evaluate $\mathrm{CR}(W_1(t), W_2(t)) \in \mathbb{Z}$ modulo certain $m \in \mathbb{Z}$ that depends on M. This m is zero if $\dim M$ is even. Furthermore, for M odd- dimensional this m is divisible by the order of the fundamental group of M. In particular, if $\pi_1(M)$ is infinite then $m = 0$, i.e. we can completely evaluate CR from the current picture. The really bad case $m = 1$ (when we can not say anything about CR) appears only when M is an odd-dimensional homotopy sphere.

To evaluate CR modulo m, we introduce an affine linking number $\mathrm{AL} \in \mathbb{Z}/m$ which depends on the current picture only. Then we notice that CR and AL are congruent modulo m. Here the biggest technical difficulty appears since in order to define AL we must define the "linking number" for two spheres that are non-homologous to zero.

This theory has the following physical interpretation. Let $\overline{M}$ be the part of the manifold (the universe) M such that $\overline{M}$ contains the current picture of wave fronts W_1, W_2 and W_1, W_2 are contractible in $\overline{M}$.

We transform the wave fronts via certain allowable moves to trivial fronts, i.e. small spherical fronts with the canonical orientation and coorientation, located far away from each other. The allowable moves should be thought of as generalized Reidemeister moves: they are the passages through generic singularities (in both directions) of wave fronts and dangerous intersection moves. We count the change of the invariant CR that occurs in the process of this formal deformation, and it turns out that this change is congruent modulo m with the (unknown!) value $\mathrm{CR}(W_1, W_2)$ of the current picture. In particular, as we have already mentioned, if $\dim M$ is even or if $\pi_1(M)$ is infinite, then we can completely compute CR from the current picture, without any knowledge of the propagations, moments of birth of the fronts, and topology of M outside of $\overline{M}$.

The following observation seems to be interesting. Suppose that we have two pictures of two pairs of fronts (W_1, W_2) and (W_1', W_2') made at two unknown moments of time t_0 and t_1. (We assume that both pairs are free of dangerous intersection points.) Assume that we know that the propagation laws for the two fronts are such that the dangerous intersection points cannot appear during the propagation and that $\mathrm{CR}(W_1(t_0), W_2(t_0))$ and $\mathrm{CR}(W_1'(t_1), W_2'(t_1))$ are not comparable modulo m. Then we can conclude that the pairs $(W_1(t_0), W_2(t_0))$ and $(W_1'(t_1), W_2'(t_1))$ of wave fronts are not the pictures of the same pair of fronts taken at different moments of time.

Notice that in these calculations we disregard the dangerous self-intersections of wave fronts. (In a sense this is similar to the theory of link homotopy where different components of links are not allowed to intersect through possible deformations, but self-intersections are allowed.) The study of self-intersections of fronts on surfaces was initiated by the ground breaking work of Arnold [2], see also [4, 5, 9, 10, 14, 15, 17]. The methods developed in this paper allow us to calculate the algebraic number of dangerous self-intersection points that arise under the propagation of fronts on manifolds of arbitrary dimensions, we do it in a next paper.

The following physical speculations related to the CR invariant seem to be possible. Assume that the space-time is topologically a product $M^n \times \mathbb{R}$, and that the observable universe $\overline{M}$ is so big that we are not able to see the current picture of wave fronts (due to the finiteness of the speed of light). The propagating fronts define the mapping of the cones C_1, C_2 (over the sphere S^{n-1} parameterizing the fronts at every moment of time) into $M \times \mathbb{R}$. Let $\sec : \overline{M} \to \overline{M} \times \mathbb{R}$ be a section of the projection $p_{\overline{M}} : \overline{M} \times \mathbb{R} \to \overline{M}$, and let $V_i = C_i \cap \sec(\overline{M}), i = 1, 2$. Assume that the law of propagation is such that dangerous intersections do not occur. Then, similarly to the above, we can restore the number of baby-intersections from the picture of images of V_i under the projection $p_{\overline{M}}$. The section sec can be thought of as the picture of the universe that we see as the light from the points of $\overline{M}$ reaches the observer, and thus V_i can be regarded as the picture of fronts that we actually see.

Low [6, 7, 8] attacked a similar problem for $M = \mathbb{R}^3$, where he considered some linking invariant for linked cones C_1, C_2 as above. In this case the linking numbers can be constructed directly via the approach of Tabachnikov [13], because $\mathbb{R}^3$ has the topological end.

The paper is organized as follows. In Section 1 we discuss some preliminary information, in Section 2 we define the invariant CR, in Section 3 we prove homotopy theoretical results which we use in order to define the invariant AL, in Section 4 we define the invariant AL and fix the relation between CR and AL, in Section 5 we treat the case of propagation with respect to a certain Riemannian metric, in Section 6 we give some examples and applications.

1. Preliminaries: propagation laws, propagations and dangerous intersections

We denote by $p_T : TM \to M$ the tangent bundle over M. Let $s : M \to TM$ be the zero section of the tangent bundle. We set $\overline{TM} = TM \setminus s(M)$. The multiplicative group $\mathbb{R}^+$ of positive real numbers acts fiberwise on $TM \setminus s(M)$ by multiplication, and we set

$$STM = (TM \setminus s(M))/\mathbb{R}^+.$$

Let $\overline{p} : \overline{TM} \to STM$ be the quotient map. Clearly, the projection $p_T : TM \to M$ yields the commutative diagram

$$\begin{array}{ccccc} TM & \xrightarrow{\supset} & \overline{TM} & \xrightarrow{\overline{p}} & STM \\ {\scriptstyle p_T}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ M & = & M & = & M \end{array}$$

It is easy to see that $\mathrm{pr} : STM \to M$ is a locally trivial bundle with the fiber S^{n-1}, we call the bundle the *spherical tangent bundle*.

Given $x \in M^n$, we denote by S^{n-1}_x (or just by S_x) the fiber $\mathrm{pr}^{-1}(x)$ over x of the spherical tangent bundle and by T_xM the tangent space to M at x.

Since M is orientable, the bundle $\mathrm{pr} : STM \to M$ is also orientable and in order to orient STM it suffices to orient the fiber S^{n-1}_x. We do it as follows. Choose an orientation preserving chart for M centered at x and let S be a small $(n-1)$-sphere centered at x. We equip S with the unique orientation o by requiring that the pair (o, outer normal vector to S) gives us the orientation of M.

Given $s \in S$, the radius-vector from x to s can be regarded as a nonzero tangent vector to M at x, i.e., as a point of S^{n-1}_x. In this way we get a diffeomorphism $\psi : S \to S_x$ which gives us an orientation of S^{n-1}_x. It is easy to see that this orientation of S_x does not depend of choice of the chart. Now, the pair (the orientation of M, the orientation of S_x) gives us an orientation of STM which we fix forever.

1.1**. Definition.** We define a *propagation law* on M to be a smooth map

$$L : \overline{TM} \times \mathbb{R} \times \mathbb{R} \to \overline{TM}$$

(a time-dependent flow on $\overline{TM}$). Here $L(\mathbf{u}, s, t) \in \overline{TM}$ should be thought of as the point that corresponds to the position and the velocity vector at moment $s+t$ of a perturbation whose position and the velocity vector at moment s was $\mathbf{u}$. (We assume that a velocity of movement of a perturbation is either zero all the time or nonzero all the time.) Furthermore we assume that $L(\mathbf{u}, s, t)$ satisfies the following conditions:

a: $L(\mathbf{u}, s, 0) = \mathbf{u}$ for all $\mathbf{u} \in \overline{TM}$;
b: $\forall s, t \in \mathbb{R}$ the map $L_{s,t} : \overline{TM} \to \overline{TM}$ defined as $L_{s,t}(\mathbf{u}) = L(\mathbf{u}, s, t)$ is a diffeomorphism;
c: $L(\mathbf{u}, s, t_1 + t_2) = L\big(L(\mathbf{u}, s, t_1), s + t_1, t_2\big)$, $\forall s, t_1, t_2 \in \mathbb{R}$;
d: $\forall \mathbf{u} \in \overline{TM}$ and $\forall s_0, t_0 \in \mathbb{R}$

$$\frac{d}{dt} p_T(L(\mathbf{u}, s_0, t))\Big|_{t=t_0} = L(\mathbf{u}, s_0, t_0).$$

1.2**. Definition.** A *propagation* is a quadruple $P = (L, x, T, V)$ where L is a propagation law, $x \in M, T \in \mathbb{R}$ and $V : S^{n-1}_x \to \big(T_xM \setminus s(x)\big)$ is a smooth section of the $\mathbb{R}^+$-bundle $\big(T_xM \setminus s(x)\big) \to S^{n-1}_x$. We fix an orientation preserving diffeomorphism $S^{n-1} \to S^{n-1}_x$ and further in the text regard V as a mapping $V : S^{n-1} \to \big(T_xM \setminus s(x)\big)$.

A propagation $P = (L, x, T, V)$ produces a wave front $W(t) : S^{n-1} \to M, t \geq T$ as follows. Informally speaking, we assume that at a moment of time T something happens at a point $x \in M$ and the perturbation caused by this event starts to radiate from the point x in all the directions according to a propagation law L with the initial velocities of propagation in T_xM described by V. Formally, for $t \geq T$ we define the front $W(t)$ to be the mapping

$$W(t) := p_T(L(V, T, t - T)) : S^{n-1} \to M.$$

We put $\overline{W}(t) = L(V, T, t-T)$ and $\widetilde{W}(t) = \overline{p} \circ \overline{W}(t)$. In this case we also say that the wave front has *originated from the event* (x, T). Initially a front of an event is a smooth embedded sphere (because of 1.1(d)), but generically it soon acquires double points, folds, cusps, swallow tails, and other complicated singularities. A generic front is a singular hypersurface, whose set of singularities is a codimension two subset of M.

We denote by $\varepsilon_x : S^{n-1} \to STM$ any map of the form

$$S^{n-1} \xrightarrow{\ h\ } S^{n-1}_x \subset STM \tag{1.1}$$

where h is a map of degree 1. Clearly, the homotopy class of ε_x is well-defined and does not depend on x.

Let $\mathcal{S}$ be the space of smooth maps $S^{n-1} \to STM$ that are homotopic to a map ε_x as in (1.1). Then $\mathcal{S} \times \mathcal{S}$ is the space of ordered pairs (f_1, f_2) with $f_i \in \mathcal{S}$.

Put Σ to be the discriminant in $\mathcal{S} \times \mathcal{S}$, i.e. the subspace that consists of pairs (f_1, f_2) such that there exist $y_1, y_2 \in S^{n-1}$ with $f_1(y_1) = f_2(y_2)$.

(We do not include into Σ the maps that are singular in the common sense but do not involve double points between the two different spheres.)

1.3. **Definition.** We define Σ_0 to be a subset (stratum) of Σ consisting of all the pairs (f_1, f_2) such that there exists precisely one pair of points $y_1, y_2 \in S^{n-1}$ such that:

- **a:** $f_1(y_1) = f_2(y_2)$. And moreover this pair of points is such that:
- **b:** y_i is a regular point of $f_i, i = 1, 2$;
- **c:** $(df_1)(T_{y_1}) \cap (df_2)(T_{y_2}) = 0$. Here df_i is the differential of f_i and T_{y_i} is the tangent space to S^{n-1} at y_i.

1.4. **Construction.** Let $\rho : (a, b) \to \mathcal{S} \times \mathcal{S}$ be a path which intersects Σ_0 in a point $\rho(t_0)$. We also assume that

$$\rho(t_0 - \delta, t_0 + \delta) \cap \Sigma_0 = \rho(t_0)$$

for δ small enough. We construct a vector $\mathbf{v} = \mathbf{v}(\rho, t_0, \delta)$ as follows. We regard $\rho(t_0)$ as a pair $(f_1, f_2) \in \mathcal{S} \times \mathcal{S}$ and consider the points y_1, y_2 as in 1.3. Set $z = f_1(y_1) = f_2(y_2)$. Choose a small $\delta > 0$ and regard $\rho(t_0 + \delta)$ as a pair $(g_1, g_2) \in \mathcal{S} \times \mathcal{S}$. Set $z_i = g_i(y_i), i = 1, 2$. Take a chart for STM that contains z and z_i, $i = 1, 2$ and set

$$\mathbf{v}(\rho, t_0, \delta) := \overrightarrow{zz_1} - \overrightarrow{zz_2} \in T_z STM.$$

1.5. **Definition.** Let $\rho : (a, b) \to \mathcal{S} \times \mathcal{S}$ be a path as in 1.4. We say that ρ *intersects* Σ_0 *transversally for* $t = t_0$ if there exists $\delta_0 > 0$ such that

$$\mathbf{v}(\rho, t_0, \delta) \notin (df_1)(T_{y_1} S^{n-1}) \oplus (df_2)(T_{y_2} S^{n-1}) \subset T_z STM$$

for all $\delta \in (0, \delta_0)$.

It is easy to see that the concept of transversal intersection does not depend on the choice of the chart.

1.6. **Definition.** A path $\rho : (a, b) \to \mathcal{S} \times \mathcal{S}, -\infty \le a < b \le \infty$ is said to be *generic* if

- **a:** $\rho(a, b) \cap \Sigma = \rho(a, b) \cap \Sigma_0$;
- **b:** the set $J = \{t | \rho(t) \cap \Sigma_0 \neq \emptyset\} \subset (a, b)$ is an isolated subset of $\mathbb{R}$;
- **c:** the path ρ intersects Σ_0 transversally for all $t \in J$.

As one can expect, every path can be turned into a generic one by a small deformation. We leave a proof to the reader.

Let $P_1 = (L_1, x_1, T_1, V_1)$ and $P_2 = (L_2, x_2, T_2, V_2)$ be two propagations. They define mappings $r_i : \mathbb{R} \to \mathcal{S}$, $i = 1, 2$ as follows.

$$r_i(t) = \begin{cases} \overline{p} \circ V_i & \text{for } t \leq T_i, \\ \widetilde{W}_i(t) & \text{for } t > T_i. \end{cases}$$

1.7. **Definition.** A pair of propagations $\{P_1, P_2\}$ is said to be *generic* if the path $r = (r_1, r_2) : \mathbb{R} \to \mathcal{S} \times \mathcal{S}$ is generic and $r(T_i) \notin \Sigma$, $i = 1, 2$.

1.8. **Definition.** Let $\{P_1, P_2\}$ be a generic pair of propagations and let $r : \mathbb{R} \to \mathcal{S} \times \mathcal{S}$ be as above. Then a moment $t \in \mathbb{R}$ such that $r(t) \in \Sigma$ corresponds either to the baby-intersection or to the case where $t > \max(T_1, T_2)$ and there exists $y_1, y_2 \in S^{n-1}$ such that $W_1(t)(y_1) = W_2(t)(y_2) = z$ and $\widetilde{W}_1(t)(y_1) = \widetilde{W}_2(t)(y_2) \in S^{n-1}_z$, i.e. to the case where there is a double point of the two fronts $W_1(t)$ and $W_2(t)$ at which the directions of the propagations of the two fronts coincide. Such a double point z of two fronts is called *a point of dangerous intersection.* Notice that we do not exclude situations where the fronts are tangent at z, the so-called *dangerous tangencies*, cf. [2].

For many pairs of propagations the dangerous intersection points do not occur. Such pairs of propagations are called *dangerous intersection free.* Now we describe a source of examples of such pairs.

1.9. **Source of Examples.** Let $L : \overline{TM} \times \mathbb{R} \times R \to \overline{TM}$ be a propagation law. Suppose that there exists a section

$$\widetilde{s} : STM \times \mathbb{R} \times \mathbb{R} \to \overline{TM} \times \mathbb{R} \times \mathbb{R}$$

of the map $\overline{p} \times 1 \times 1$ such that $\mathrm{Im}(\widetilde{s})$ consists of the trajectories of L, i.e. if $(\mathbf{u}, s_0, 0) \in \mathrm{Im}(\widetilde{s})$ for some $\mathbf{u} \in \overline{T}M$ and $s_0 \in \mathbb{R}$, then $L(\mathbf{u}, s_0, t) \in \mathrm{Im}(\widetilde{s})$, for every $t \in \mathbb{R}$.

Let $P_1 = (L, x_1, T_1, V_1)$ and $P_2 = (L, x_2, T_2, V_2)$ be propagations such that $(\mathrm{Im}(V_1), T_1, 0) \subset \mathrm{Im}(\widetilde{s}|_{S_{x_1}, T_1, 0})$, $(\mathrm{Im}(V_2), T_2, 0) \subset \mathrm{Im}(\widetilde{s}|_{S_{x_2}, T_2, 0})$ and $r(T_i) \notin \Sigma$, $i = 1, 2$.

Then it is easy to see that the pair (P_1, P_2) is dangerous intersections free.

1.10. **Example.** *Propagations that are defined by a Riemannian metric.* An interesting class of examples comes from the propagation defined by the geodesics of a complete Riemannian metric g on M. In this case $L(\mathbf{u}, s, t)$ is just a point on $\overline{TM}$ that corresponds to a velocity vector at moment $s + t$ of a geodesic curve that had a velocity vector $\mathbf{u}$ at the moment s. It is easy to see that in this example at every moment of time the velocity vectors of the points on the wave front are perpendicular to the image of the front. *Thus the dangerous intersections are precisely the dangerous tangencies.*

Furthermore, if both $\operatorname{Im} V_1$ and $\operatorname{Im} V_2$ are spheres of the same radius r then the dangerous intersections (= dangerous tangencies) do not occur, since spheres of radius r in all the tangent planes produce the section $\widetilde{s}$ described above.

1.11**. Example.** *Propagation in a non-homogeneous and non- isotropic medium whose structure does not depend on time.* Assume that M is a Riemannian manifold and $\mu : STM \to \overline{TM}$ is a smooth section of the corresponding $\mathbb{R}^+$-bundle such that $\operatorname{Im}(\mu\big|_{S_x})$ bounds a strictly convex domain in T_xM for all $x \in M$. The radius vector from $s(x)$ to $\operatorname{Im}(m\big|_{S_x})$ in the given direction is the velocity vector of the distortion traveling in the direction. This information allows us to calculate for every smooth curve $\gamma : [t_1, t_2] \to M$ the total time $\tau(\gamma)$ needed for the distortion to travel along this curve.

Assume $\operatorname{Im}(V_1) \subset \operatorname{Im}\mu$ and $\operatorname{Im}(V_2) \subset \operatorname{Im}\mu$ and that propagation occurs according to the Huygens principle, i.e. distortion travels along the extremal curves of the functional τ on the space of smooth curves on M. It is clear that here we have a special case of the situation described in 1.9, and so the dangerous intersection points do not occur for such a pair of propagation. On the other hand, if the propagation happens according to the Huygens principle then at every point $W(t)(x) = z$ the normal vector to the wave front is conjugate with respect to $\mu\big|_{S_z}$ to the direction of the extremal curve along which the information travelled to this point, Arnold [3]. In particular, in this case the dangerous tangencies do not occur under the wave fronts propagation, since they are the dangerous intersections.

2. The causality relation invariant

Recall that the standard sphere S^{n-1} is assumed to be oriented. We say that a tangent frame $\mathfrak{r}$ to S^{n-1} is positive if it gives us the standard orientation of S^{n-1}.

2.1**. Definition.** Let ρ be a path in $\mathcal{S} \times \mathcal{S}$ that intersects Σ transversally in one point $\rho(t_0) \in \Sigma_0$. We associate a sign $\widetilde{\sigma}(\rho, t_0)$ to such a crossing as follows.

We regard $\rho(t_0)$ as a pair $(f_1, f_2) \in \mathcal{S} \times \mathcal{S}$ and consider the points $y_1, y_2 \in S^{n-1}$ such that $f_1(y_1) = f_2(y_2)$. Set $z = f_1(y_1) = f_2(y_2)$. Let $\mathfrak{r}_1$ and $\mathfrak{r}_2$ be frames which are tangent to S^{n-1} at y_1 and y_2, respectively, and both are assumed to be positive. Consider the frame

$$\{df_1(\mathfrak{r}_1), \mathbf{v}, df_2(\mathfrak{r}_2)\}$$

at $z \in STM$ where $\mathbf{v}$ is a vector described in 1.4. We put $\widetilde{\sigma}(\rho, t_0) = 1$ if this frame gives us the orientation of STM, otherwise we put $\widetilde{\sigma}(\rho, t_0) = -1$.

Because of the transversality and condition (c) from 1.3, the family $\{df_1(\mathfrak{r}_1), \mathbf{v}, df_2(\mathfrak{r}_2)\}$ is really a frame.

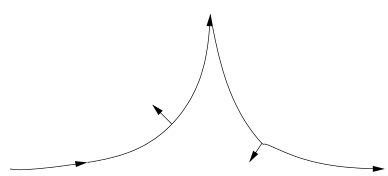
Notice also that the vector $\mathbf{v}$ is not well- defined, but the above defined sign $\widetilde{\sigma}$ is.

Clearly if we traverse the path ρ in the opposite direction then the sign of the crossing changes.

2.2. **Definition.** Suppose that a front W passes through a point $x \in M$ at the moment of time t_0 in such a way that the velocity vector $\mathbf{v}_x$ of the front $W(t_0)$ at x is transverse to $W(t_0)$, $W(t_0)$ restricted to a small neighborhood U of $W^{-1}(t_0)(x)$ is an embedding, and x has only one preimage under $W(t_0)$.

Recall that the manifold M is oriented. Let o_x be the local orientation of $W(t_0)$ at x (i.e. the orientation of the tangent plane T_x to $W(t_0)$). We say that the local orientation o_x is *positive*, and write $\sigma(W(t_0), x) = 1$ if the pair $(o_x,\, v_x)$ gives us the orientation of M; otherwise we say that the local orientation of $W(t)$ at x is *negative* and write $\sigma(W(t_0), x) = -1$.

Notice that the same wave front $W(t)$ can contain two points x and y such o_x is positive orientation while o_y is the negative one, see Figure 1.



Orientations and coorientations at the
cusp point of a one–dimensional front

FIGURE 1

Consider a generic pair (P_1, P_2) of propagations $P_1 = (L_1, x_1, V_1, T_1)$ and $P_2 = (L_2, x_2, V_2, T_2)$. In the text below we assume that $T_1 \leq T_2$. The case where $T_1 > T_2$ is treated in a similar way.

Let $t > T_2$ be a generic moment of time, i.e. the one at which dangerous intersections do not occur.

Let $c_i, i \in I \subset \mathbb{N}$ where $T_2 < c_i < t$ be moments of time when dangerous intersections did occur.

2.3. **Definition.** We define $\sigma(W_1(c_i), W_2(c_i))$ as the sign $\widetilde{\sigma}$ of the corresponding passage of Σ_0.

Notice that $\sigma(W_1(c_i), W_2(c_i))$ is symmetric if n is even and skew-symmetric if n is odd.

Let $p_j, j \in J \subset \mathbb{N}$ be the moments of time when the front W_1 passed through the point x_2 before the front W_2 originated. (Notice that $p_j < T_2$ and that $\sigma(W_1(p_j), x_2))$ is well-defined since the pair of propagations is generic.) A straightforward verification shows that

$$\sigma(W_1(p_j), x_2) = \widetilde{\sigma}(\rho, t_0).$$

where $\rho(t) = (\widetilde{W_1}(t), \varepsilon_{x_2}), t \in (p_j - \delta, p_j + \delta)$.

2.4. **Definition.** We set

$$\mathrm{CR}(W_1(t), W_2(t)) = \sum_{i \in I} \sigma(W_1(c_i), W_2(c_i)) + \sum_{j \in J} \sigma(W_1(p_j), x_2) \in \mathbb{Z}$$

and call it the *causality relation invariant* for the fronts $W_1(t)$ and $W_2(t)$ at a given moment of time t. (If in fact $T_1 > T_2$, then the second sum should be $\sum_{k \in K} (-1)^{\dim M} \sigma(W_2(q_k), x_1)$, where $q_k, k \in K \subset \mathbb{N}$ are the moments of time when the front W_2 passed through the point x_1 before the front W_1 originated. One can easily verify that $(-1)^{\dim M} \sigma(W_2(q_k), x_1)$ coincides with the sign of the corresponding crossing of Σ_0 by the path $r = (r_1, r_2)$.)

If the above pair (P_1, P_2) of propagations is dangerous intersections free, then

$$\mathrm{CR}(W_1(t), W_2(t)) = \sum_{j \in J} \sigma(W_1(p_j), x_2).$$

It is easy to see that in this case $\mathrm{CR}(W_1(t), W_2(t))$ does not depend on t provided $t > \max(T_1, T_2)$, and thus it is invariant under the propagation. In particular, if $\mathrm{CR}(W_1(t), W_2(t))$ is non-zero then we know for a fact that the perturbation caused by the first signal has reached the source point of the second signal before the second signal originated. Moreover, if $\mathrm{CR}(W_1(t), W_2(t)) = k \neq 0$, then we can say for sure that the first wave front has passed through the source point of the second front at least k times before the second signal originated. (Of course it could be that it did pass more times, because it could have passed $k + l$ times with a positive sign and l times with a negative sign.)

In case of generic propagations P_1, P_2 and $\mathrm{CR}(W_1(t), W_2(t)) = k \neq 0$ we can conclude that the sum of the number of baby-intersections and of the number of dangerous intersections is at least k. This probably could be interpreted as the quantity that measures either how much faster the first front is than the second so that they could become dangerously intersected or how many times the first front did pass through the source of the second front before the second front originated.

3. Homotopy properties of maps to STM

3.1. **Definition.** Given a map $\alpha : S^1 \times S^{n-1} \to STM$, we say that α is *special* if $\alpha\big|_{*\times S^{n-1}}$ has the form ε_x for some $x \in M$, see (1.1). Here $* \in S^1$ is the base point.

3.2. **Definition.** Given an n-dimensional manifold N and a map $\beta : N \to STM$, we define $d(\beta)$ to be the degree of the map

$$\operatorname{pr} \circ \beta : N \to M$$

3.3. **Lemma.** *Let $\alpha : S^1 \times S^{n-1} \to STM$ be a special map. Then there exists a map $\beta : S^n \to STM$ such that $d(\beta) = d(\alpha)$.*

Proof. We regard S^{n-1} as a pointed space. Consider a map $\widetilde{\alpha} : S^1 \times S^{n-1} \to STM$ such that:

- **1:** $\widetilde{\alpha}\big|_{*\times S^{n-1}} = \alpha\big|_{*\times S^{n-1}}$,
- **2:** $\widetilde{\alpha}\big|_{S^1\times *} = \alpha\big|_{S^1\times *}$,
- **3:** $\widetilde{\alpha}|_{t\times S^{n-1}} = \varepsilon\big|_{\widetilde{\alpha}(t\times *)}$.

We regard $S^1 \times S^{n-1}$ as the CW-complex with four cells e^0, e^1, e^{n-1}, e^n, $\dim e^k = k$. It is easy to see that the maps $\widetilde{\alpha}$ and α coincide on the $(n-1)$-skeleton. Thus, the maps α and $\widetilde{\alpha}$ (restricted to the n-cell) together yield a map $\beta : S^n \to STM$. Clearly $d(\widetilde{\alpha}) = 0$, and therefore $d(\beta) = d(\alpha)$. □

3.4. **Lemma.** *Suppose that there exists a map $\beta : S^n \to STM$ with $d(\beta) \neq 0$. Then the Euler class $\chi \in H^n(M)$ of the tangent bundle $TM \to M$ is zero.*

Proof. We set$f = \operatorname{pr} \circ \beta : S^n \to M$. First, notice that $H^n(M) = \mathbb{Z}$ because $d(\beta) \neq 0$. So, again since $d(\beta) \neq 0$, we conclude $f^*\chi \neq 0$ whenever $\chi \neq 0$. Now the result follows because $f^*\chi$ is the obstruction to the lifting of f to STM, while β is such a lifting of f. □

3.5. **Lemma.** *Let M be a $2k$-dimensional oriented manifold and $\beta : S^{2k} \to STM$ be a map with $d(\beta) \neq 0$. Then the Euler class of the tangent bundle $TM \to M$ is non-zero.*

Proof. We set $f = \operatorname{pr} \circ \beta : S^{2k} \to M$ and $d = d(\beta)$. Notice that M is closed because $d(\beta) \neq 0$. Let $f_! : H_*(M) \to H_*(S^{2k})$ be the transfer map, see e.g. [11, V.2.12]. Since $f_*(f^*y \cap x) = y \cap f_*x$ for all $x \in H_*(S^{2k})$ and $y \in H^*(M)$, we conclude that $f_*f_!(z) = dz$, for all $z \in H_*(M)$.

In particular, $dH_i(M) = 0$ for $0 < i < 2k$, i.e. $H_i(M;\mathbb{Q}) = 0$ for $0 < i < 2k$. So, the Euler characteristic of M is 2, and thus the Euler class of the tangent bundle is non-zero (in fact, ± 2). □

3.6. **Corollary.** *If M is an even-dimensional oriented manifold, then $d(\beta) = 0$ for every $\beta : S^n \to STM$.*

Proof. This is a direct consequence of Lemma 3.4 and 3.5. □

Let $\deg : \pi_n(M) \to \mathbb{Z}$ be the degree homomorphism, i.e., the homomorphism which assigns the degree $\deg f$ to the homotopy class of a map $f : S^n \to M$. (In fact, it coincides with the Hurewicz homomorphism $h : \pi_n(M) \to H_n(M)$ for M closed and is zero for M non-closed.)

3.7. **Definition.** Given a connected oriented manifold M^n, we define an Abelian group $\boldsymbol{A}(M)$ and a homomorphism $q = q_M : \mathbb{Z} \to \boldsymbol{A}(M)$ as follows. If n is even then $\boldsymbol{A}(M) = \mathbb{Z}$ and $q = 1_{\mathbb{Z}}$. If n is odd then $\boldsymbol{A}(M)$ is the cokernel of the degree homomorphism $\deg : \pi_n(M) \to \mathbb{Z}$ and $q : \mathbb{Z} \to \boldsymbol{A}(M)$ is the canonical epimorphism. Notice that $\boldsymbol{A}(M) = \mathbb{Z}$ for odd- dimensional non-closed manifolds.

3.8. **Proposition.** *Let M^n be a closed odd-dimensional manifold as in Definition* 3.7*. Then the following holds:*
(i) *If $\boldsymbol{A}(M) = 0$ then M is a homotopy sphere.*
(ii) *If $\pi_1(M)$ is infinite then $\boldsymbol{A}(M) = \mathbb{Z}$.*

Proof. (i) If $\boldsymbol{A}(M) = 0$ then there exists a map $S^n \to M$ of degree 1. Since every map of degree 1 induces epimorphism of fundamental groups and homology groups, we conclude that M is a homotopy sphere.

(ii) This follows because every map $S^n \to M$ passes through the universal covering, and thus the Hurewicz homomorphism is trivial. □

4. The affine linking invariant AL as a reduction of CR

4.1. **Definition.** We define Σ_1 to be the subset (stratum) of Σ consisting of all the pairs (f_1, f_2) such that there exists precisely two pairs of points $y_1, y_2 \in S^{n-1}$ as in 1.3. Here we assume that the two double points of the image are distinct.

Notice that, in a sense, Σ_i is a stratum of codimension i in Σ. In particular, a generic path in $\mathcal{S} \times \mathcal{S}$ intersects Σ_0 in a finite number of points, and a generic disk in $\mathcal{S} \times \mathcal{S}$ intersects Σ_1 in a finite number of points.

A generic path $\gamma : [0,1] \to \mathcal{S} \times \mathcal{S}$ that connects two points in $\mathcal{S} \times \mathcal{S} \setminus \Sigma$ intersects Σ_0 in finitely many points $\gamma(t_j), j \in J \subset N$ and all the intersection points are of the types described in 2.1. Put

$$\Delta_{\mathrm{AL}}(\gamma) = \sum_{j\in J} \sigma(\gamma, t_j) \in \mathbb{Z}. \tag{4.1}$$

We let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, $B_1 = \{(x, y) \in A \mid xy = 0\}$, $B_2 = \{(x, y) \in A \mid x = 0\}$, $B_3 = \{(x, y) \in A \mid y = 0\}$, $B_4 = \{(0, 0)\}$, $B_5 = \emptyset$.

We define a regular disk in $\mathcal{S} \times \mathcal{S}$ as a generically embedded disk D such that the triple $(D, D \cap \Sigma_0, D \cap \Sigma_1)$ is homeomorphic to a triple (A, B, C) where B is one of B_i's and $C \subset B_4$.

4.2**. Lemma.** *Let β be a generic loop that bounds a regular disk in $\mathcal{S} \times \mathcal{S}$. Then $\Delta_{\mathrm{AL}}(\beta) = 0$.*

Proof. Straightforward. □

4.3**. Lemma.** *Let β be a generic loop that bounds a disk in $\mathcal{S} \times \mathcal{S}$. Then $\Delta_{\mathrm{AL}}(\beta) = 0$.*

Proof. Without loss of generality we can (using a small deformation of the disk) assume the disk is the union of regular ones, cf. Arnold [2], [1]. Now the proof follows from Lemma 4.2. □

Notice that $\mathcal{S} \times \mathcal{S}$ is path connected.

4.4**. Corollary.** *The invariant Δ_{AL} induces a well-defined homomorphism $\Delta_{\mathrm{AL}} : \pi_1(\mathcal{S} \times \mathcal{S}, *) \to \mathbb{Z}$.*

Proof. Since every element of $\pi_1(\mathcal{S} \times \mathcal{S}, *)$ can be represented by a generic loop, the proof follows from Lemma 4.3. □

Let x_1, x_2 be two distinct points of M. Let $\alpha : S^1 \times S^{n-1} \to STM$ be a special map (see Definition 3.1) such that the composition

$$S^{n-1} \subset S^1 \times S^{n-1} \xrightarrow{\alpha} STM$$

has the form ε_{x_1}, and let $e : S^1 \times S^{n-1} \to STM$ be the map of the form

$$S^1 \times S^{n-1} \xrightarrow{\mathrm{proj}} S^{n-1} \xrightarrow{\varepsilon_{x_2}} STM$$

Then (α, e) is a loop in $(\mathcal{S} \times \mathcal{S}, *)$.

4.5**. Lemma.** $\Delta_{\mathrm{AL}}[(\alpha, e)] = d(\alpha)$*.*

Proof. Notice that $\Delta_{\mathrm{AL}}[(\alpha, e)]$ is the intersection index of the cycles $\alpha(S^1 \times S^{n-1})$ and $S^{n-1}_{x_2}$. This index coincides with the degree of the map $\mathrm{pr} \circ \alpha$ because the last one is equal to the algebraic number of the preimages of x_2. □

4.6**. Definition.** Choose a point $* = (f_1^0, f_2^0) \in \mathcal{S} \times \mathcal{S} \setminus \Sigma$ and put $\mathrm{AL}(*)$ to be a constant $k \in \mathbb{Z}$. (It is easy to see from the following proof that this choice is the only ambiguity in the definition of the AL-invariant.) Take an arbitrary point $f = (f_1^1, f_2^1) \in \mathcal{S} \times \mathcal{S} \setminus \Sigma$ and choose a generic path γ going from $*$ to f. We set

$$\mathrm{AL}(f) = q(k + \Delta_{\mathrm{AL}}(\gamma)) \in \boldsymbol{A}(M)$$

and call AL the *affine linking invariant.* Here q is the epimorphism from Definition 3.7.

4.7**. Theorem.** *The function* $\mathrm{AL} : \pi_0(\mathcal{S} \times \mathcal{S} \setminus \Sigma) \to \boldsymbol{A}(M)$ *is well-defined and increases by* $1 \in \boldsymbol{A}(M)$ *under the positive transverse passage through the stratum* Σ_0*.*

Furthermore, the above property determines the function AL *uniquely up to an additive constant.*

Proof. To show that AL is well-defined we must verify that the definition is independent on the choice of the path γ that goes from $*$ to f. This is the same as to show that $q(\Delta_{\mathrm{AL}}(\varphi)) = 0$ for every closed generic loop φ at $*$.

Since $\Delta_{\mathrm{AL}} : \pi_1(\mathcal{S} \times \mathcal{S}, *) \to \mathbb{Z}$ is a homomorphism, it suffices to prove that $q(\Delta_{\mathrm{AL}}(\varphi)) = 0$ for all generators φ of $\pi_1(\mathcal{S} \times \mathcal{S}, *)$.

We can and shall assume that the maps $f_1^0, f_2^0 : S^{n-1} \to STM$ are the maps $\varepsilon_{x_1}, \varepsilon_{x_2}$, respectively, where x_1, x_2 are two distinct points in M. Then the classes $[(\alpha, e)]$ and similar classes $[(e, \alpha)]$ generate the group $\pi_1(\mathcal{S} \times \mathcal{S}, *)$. By Lemma 4.5 we have

$$\Delta_{\mathrm{AL}}[(\alpha, e)] = d(\alpha).$$

Clearly, $d(\alpha) = 0$ if M is an non-closed manifold. So, we assume M to be closed. Now, for n even $d(\alpha) = 0$ by Corollary 3.6, while for n odd $q(d(\alpha)) = 0$ by Lemma 3.3. □

Let (P_1, P_2) be a generic pair of propagations, and let t be a moment of time when dangerous intersection do not occur. Let $q : \mathbb{Z} \to \boldsymbol{A}(M)$ be the epimorphism described in Definition 3.7.

4.8**. Theorem.** *The invariants* CR *and* AL *are related as follows:*

$$q\big(\mathrm{CR}(W_1(t), W_2(t)\big) = \mathrm{AL}\big(\widetilde{W}_1(t), \widetilde{W}_2(t)\big) - \mathrm{AL}(V_1, V_2).$$

Proof. The Theorem follows because the signs defined for dangerous intersections (as well as for baby-intersections) are exactly the sign of the corresponding crossings of Σ_0. □

If the pair of propagations is dangerous intersections free then the invariant $\mathrm{AL}(\widetilde{W}_1(t), \widetilde{W}_2(t)) \in \boldsymbol{A}$ gives us the number of times the wave fronts W_1 has passed through the point x_2 before the wave front W_2 has originated. It is also easy to see that if the wave front W_2 did not originated yet then $\mathrm{AL}(\widetilde{W}_1(t), \widetilde{W}_2(t))$ is the number of times the wave front W_1 has passed through x_2 at the moment of time t.

The invariant AL works especially nice for even dimensional manifolds, since in these cases $\boldsymbol{A} = \mathbb{Z}$.

5. Causality relation invariant in the case of the propagation according to Riemannian metrics.

As it was noticed in 1.10, if a propagation happens according to a complete Riemannian metric and $\operatorname{Im} V$ is a sphere then the velocities of the points of the front are always orthogonal to the front. So, if each of two propagations happens according to a complete Riemannian metric then dangerous tangency points and the dangerous intersection points are the same thing. In this section we deal only with this case and we provide an especially nice way of calculation of the CR invariant. We need some preliminaries.

Let W be a wave front, and let $x \in \operatorname{Im} W(t)$ be a non-singular point of $W(t)$. For sake of simplicity we denote $T_x \operatorname{Im} W(t)$ just by T. Let O be a small neighborhood of x in M, and let $U = O \cap \operatorname{Im} W(t)$. Without loss of generality we can and shall assume that the injectivity radius is big enough (≥ 3) for all points of O.

The Riemannian metric g on M produces a unique symmetric connection on M. So, for every $a \in O$, the parallel transport along the geodesic segment (connecting x and a) gives us an isomorphism

$$\tau_a : T_a M \to T_x M \tag{5.1}$$

Furthermore, we can regard every sphere $S_a \in STM, a \in M$ as the unit sphere in $T_a M$, and so STM can be regarded as the total space of the unit sphere subbundle of TM. Since the connection respects the Riemannian metric, we conclude that $\tau_a(S_a) = S_x$.

5.1**. Definition.** (a) We define

$$\pi : \mathrm{pr}^{-1}(O) \to S_x$$

as follows. A point $z \in \mathrm{pr}^{-1}(O)$ is a pair (a, ξ) with $a = \mathrm{pr}(z)$ and $\xi \in S_a$, and we set $\pi(z) = \tau_a(\xi)$ with τ_a as in (5.1).

(b) Given $u \in U$, let $\ell(u) \in \operatorname{Im} \widetilde{W}$ be the point with $\mathrm{pr}(\ell(u)) = u$. In this way we get a map $\ell : U \to \widetilde{W}$. We set $z = \ell(x)$. Given $\mathbf{e} \in T$, we set

$$\mathbf{e}^W := d\ell(\mathbf{e}), \quad \mathbf{e}^W \in T_z \operatorname{Im} \widetilde{W} \subset T_z STM.$$

It is clear that $d\ell : T \to T_z STM$ is an isomorphism.

(c) Let $z \in STM$ be the point described in (b). We define the horizontal section $H : O \to STM$ of pr by setting

$$H(a) = \tau_a^{-1}(z) \in S_a \subset STM.$$

Furthermore, given $\mathbf{w} \in T_a M, a \in O$, we set

$$\mathbf{w}^H = dH(\mathbf{w}) \in T_{H(a)} STM.$$

Clearly, $\mathbf{w}^H$ can be characterized by the properties

$$(d\,\mathrm{pr})(\mathbf{w}^H) = \mathbf{w}, \quad d\pi(\mathbf{w}^H) = \mathbf{0}.$$

(d) We define the Gauss map $G = G_W : U \to S_x$ by setting $G(u) = \tau_u(\mathbf{n}_u), u \in U$ where $\mathbf{n}_u$ is the unit normal vector to U at u.

(e) Let $z \in STM$ be the point described in (b). Given $\mathbf{w} \in T_x M$, we define $\mathbf{w}^S \in T_z T_x M$ as follows. We regard z as the vector $\mathbf{z} \in T_x M$. Furthermore, we regard $T_x M$ as the affine space T^{aff} over the vector space $T_x M$ and consider the parallel shift

$$P_{\mathbf{z}} : T^{\mathrm{aff}} \to T^{\mathrm{aff}}, \quad a \mapsto a + \mathbf{z}.$$

Let $o \in T^{\mathrm{aff}}$ correspond to the origin of the vector space $T_x M$. Using the obvious identification $T_x M = T_o T^{\mathrm{aff}}$, we regard $\mathbf{w}$ as the tangent vector $\mathbf{w}_o \in T_o T^{\mathrm{aff}}$, and we set

$$\mathbf{w}^S = dP_{\mathbf{z}}(\mathbf{w}_o) \in T_z T^{\mathrm{aff}} = T_z T_x M.$$

Notice that if $\mathbf{e} \in T$ then $\mathbf{e}^S \in T_z S_x$. (This is where the notation comes from: $\mathbf{e}^S$ is the spherical lifting of $\mathbf{e}$.)

5.2. **Lemma.** *For every* $\mathbf{e} \in T$ *we have* $\mathbf{e}^W - \mathbf{e}^H = dG(\mathbf{e})$.

Proof. First, notice that $G = \pi \circ \ell : U \to S_x$, because $\ell(u)$ can be regarded as the unit normal vector to U at u. So, $d\pi(\mathbf{e}^W) = dG(\mathbf{e})$. Now,

$$(d\,\mathrm{pr})(\mathbf{e}^W - dG(\mathbf{e})) = \mathbf{e} - \mathbf{0} = \mathbf{e}$$

and

$$d\pi(\mathbf{e}^W - dG(\mathbf{e})) = dG(\mathbf{e}) - dG(\mathbf{e}) = \mathbf{0}.$$

Thus, $\mathbf{e}^W - dG(\mathbf{e}) = \mathbf{e}^H$. □

5.3. **Proposition.** *Let* $\mathbf{e} \in T$*, and let* $\mathbf{n}$ *be the normal vector field to* U *in* M*. Then* $dG(\mathbf{e}) = (\nabla_{\mathbf{e}} \mathbf{n})^S$.

Here ∇ denotes the covariant differentiation operation on M.

Proof. Let $\gamma : (-\delta, \delta) \to U$ be a curve with $\dot\gamma(0) = \mathbf{e}$. We define the curve $\zeta : (-\delta, \delta) \to S_x$ by setting $\zeta(t)$ to be the end of the vector $\tau_{\gamma(t)} \mathbf{n}_{\gamma(t)}$. Since

$$\nabla_{\mathbf{e}} \mathbf{n} = \frac{d}{dt} \left(\tau_{\gamma(t)} \mathbf{n}_{\gamma(t)} - \mathbf{n}_x \right) \Big|_{t=0}$$

we conclude that $\dot{\zeta}(0) = (\nabla_{\mathbf{e}}\mathbf{n})^S$. On the other hand, $G \circ \gamma = \zeta$, and thus

$$dG(\mathbf{e}) = dG(\dot{\gamma}(0)) = (G \dot{\circ} \gamma)(0) = \dot{\zeta}(0) = (\nabla_{\mathbf{e}}\mathbf{n})^S.$$

□

5.4. **Corollary.** $\mathbf{e}^W - \mathbf{e}^H = (\nabla_{\mathbf{e}}\mathbf{n})^S$.

Proof. This is the direct consequence of 5.2 and 5.3. □

Consider the Weingarten operator

$$A = A_W : T \to T, \quad A(\mathbf{e}) = \nabla_{\mathbf{e}}\mathbf{n}. \tag{5.2}$$

The Corollary 5.4 can now be written as follows:

$$\mathbf{e}^W - \mathbf{e}^H = (A\mathbf{e})^S. \tag{5.3}$$

Now let W_1 and W_2 be two wave fronts, and let $x \in M$ be a point of dangerous tangency of $W_1(t)$ and $W_2(t)$. We assume that the corresponding pair of propagations is generic. Again, we denote by T the common tangent plane $T_x W_i(t), i = 1, 2$. Let $A_i := A_{W_i} : T \to T, i = 1, 2$ be the Weingarten operators considered in (5.2). We set $B = A_1 - A_2$. It is well known that each A_i is a self-adjoint operator, [12, Ch. 7], and therefore B is. Let $k_1, \dots, k_{n-1}$ be the eigenvalues (with multiplicities) of B.

5.5. **Proposition.** $\operatorname{Ker} B = 0$*, and so* $k_i \neq 0$ *for all* i*.*

Proof. Let $\mathbf{e} \neq \mathbf{0}$ be a vector with $B\mathbf{e} = \mathbf{0}$. Then

$$\mathbf{e}^{W_1} - \mathbf{e}^{W_2} = (\mathbf{e}^{W_1} - \mathbf{e}^H) - (\mathbf{e}^{W_2} - \mathbf{e}^H) = (A_1\mathbf{e} - A_2\mathbf{e})^S = (B\mathbf{e})^S = \mathbf{0}.$$

i.e. $T_x \operatorname{Im} W_1 \cap T_x \operatorname{Im} W_2 \neq 0$. But this is impossible because the pair of propagations is assumed to be generic (look conditions 1.6(b) and 1.3(c)). □

In particular, $\det B = k_1 \cdots k_n \neq 0$.

5.6. **Definition** (Alternative definition of the sign $\sigma(W_1(t), W_2(t))$)**.** We put $\varepsilon(W_1(t), W_2(t) = 1$ if both fronts have the same local orientations at x (as defined in 2.2) and $\varepsilon(W_1(t), W_2(t)) = -1$ if the fronts have opposite local orientations. Now we set

$$\widehat{\sigma}(W_1(t), W_2(t)) = \varepsilon(W_1(t), W_2(t)) \operatorname{sign}(\det B) \operatorname{sign}(|\mathbf{v}_1| - |\mathbf{v}_2|)$$

where $\mathbf{v}_i$ is the velocity vector of $W_i(t)$ at x.

5.7. **Theorem.** $\widehat{\sigma}(W_1(t), W_2(t)) = \sigma(W_1(t), W_2(t))$.

Proof. Given a vector $\mathbf{e}\in T$, we set $\mathbf{e}'=\mathbf{e}^{W_1}$ and $\mathbf{e}''=\mathbf{e}^{W_2}$. Choose a basis $\{\mathbf{e}_1,\dots,\mathbf{e}_{n-1}\}$ of T containing of the eigenvectors of B, i.e., $B\mathbf{e}_i=k_i\mathbf{e}_i$. Because of equality (5.3) we have

$$\mathbf{e}''_i-\mathbf{e}'_i=(B\mathbf{e}_i)^S=(k_i\mathbf{e}_i)^S=k_i\mathbf{e}_i^S. \tag{5.4}$$

We can and shall assume that the frame $\{\mathbf{e}_1,\dots,\mathbf{e}_{n-1}\}$ gives the positive (local) orientation of $W_i(t), i=1,2$ at x. Take the polyvector

$$\mathfrak{p}:=\mathbf{e}'_1\wedge\cdots\wedge\mathbf{e}'_{n-1}\wedge\mathbf{v}\wedge\mathbf{e}''_1\wedge\cdots\wedge\mathbf{e}''_{n-1}$$

where $\mathbf{v}$ is the vector defined in 1.4. Then $\mathfrak{p}\neq 0$ since the pair of propagations is assumed to be generic. Notice that $\mathfrak{p}$ gives us an orientation of STM, and we say that $\mathfrak{p}$ is positive if this orientation coincides with the original one, otherwise we say that $\mathfrak{p}$ is negative.

According to Definition 2.3, the sign of $\mathfrak{p}$ is equal to

$$\varepsilon(W_1(t),W_2(t))\sigma(W_1(t),W_2(t)).$$

So, we must prove that sign of the polyvector $\mathfrak{p}$ is equal to the sign of

$$(\det B)\operatorname{sign}(|\mathbf{v}_1|-|\mathbf{v}_2|).$$

To be definite, we assume that $|\mathbf{v}_1|>|\mathbf{v}_2|$ and prove that the sign of $\mathfrak{p}$ is equal to the sign of $\det B$.

Since $\mathbf{e}''_i=\mathbf{e}'_i+k_i\mathbf{e}_i^S$ and $\det B=k_1\cdots k_{n-1}$, we conclude that

$$\mathbf{e}'_1\wedge\cdots\wedge\mathbf{e}'_{n-1}\wedge\mathbf{v}\wedge\mathbf{e}''_1\wedge\cdots\wedge\mathbf{e}''_{n-1}=(\det B)\mathbf{e}'_1\wedge\cdots\wedge\mathbf{e}'_{n-1}\wedge\mathbf{v}\wedge\mathbf{e}^S_1\wedge\cdots\wedge\mathbf{e}^S_{n-1}.$$

So, it remains to prove that the polyvector

$$\mathbf{e}'_1\wedge\cdots\wedge\mathbf{e}'_{n-1}\wedge\mathbf{v}\wedge\mathbf{e}^S_1\wedge\cdots\wedge\mathbf{e}^S_{n-1}$$

is positive.

Since $\mathbf{e}'_1\wedge\cdots\wedge\mathbf{e}'_{n-1}\wedge\mathbf{v}\wedge\mathbf{e}^S_1\wedge\cdots\wedge\mathbf{e}^S_{n-1}\neq 0$, we conclude that the family $\{\mathbf{e}^S_1,\dots,\mathbf{e}^S_{n-1}\}$ generate T_zS_x. It is easy to see that the frame $\{\mathbf{e}^S_1,\dots,\mathbf{e}^S_{n-1}\}$ gives the original orientation of S_x, since the frame $\{\mathbf{e}_1,\dots,\mathbf{e}_{n-1}\}$ gives the positive local orientation of each of the fronts at x. So, it remains to prove that the polyvector

$$(d\operatorname{pr})(\mathbf{e}'_1\wedge\cdots\wedge\mathbf{e}'_{n-1}\wedge\mathbf{v})=\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{n-1}\wedge(d\operatorname{pr})\mathbf{v}$$

is positive, i.e. that it gives the original orientation of M. Recall that the polyvector

$$\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{n-1}\wedge(\mathbf{v}_1-\mathbf{v}_2)$$

is positive since $|\mathbf{v}_1|>|\mathbf{v}_2|$. Taking into account that the vector $\mathbf{v}_1-\mathbf{v}_2$ is orthogonal to T and points into the direction of both propagations at x, it sufices to prove that $\langle\mathbf{v}_1-\mathbf{v}_2,(d\operatorname{pr})(\mathbf{v}\rangle>0$. But this is clear because W_1 is faster then W_2 at (x,t), and so the point $\operatorname{pr}(z_1)$ (in notation of 1.4) is further then the point $\operatorname{pr}(z_2)$ from T. □

5.8**. Example.** Suppose that the fronts propagate as it is shown in Figure 2.

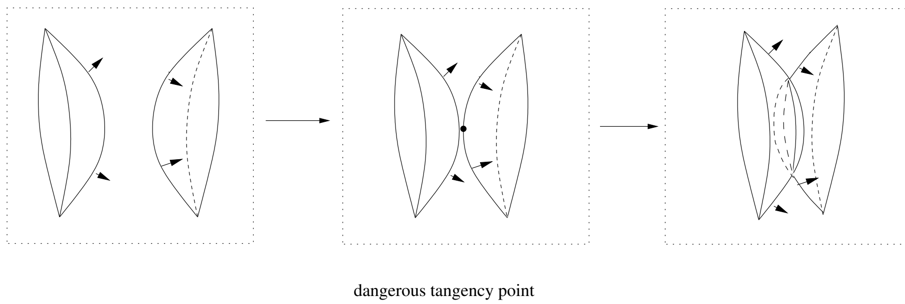


FIGURE 2

Let W_1 be the "right" front. Then, clearly,

$$\sigma(W_1(t), W_2(t)) = -\varepsilon(W_1(t), W_2(t)).$$

The negative sign appears because W_2 is faster the W_1.

6. EXAMPLES

To illustrate the usage of the affine linking invariant consider the following examples.

6.1**. Example.** Here we show how to apply AL to determining the causality relation. Let M be a smooth oriented n-dimensional manifold that is not an odd-dimensional homotopy sphere. Let W_1, W_2 be the wave fronts that originated on M long time ago and were propagating according to the dangerous intersections free pair of propagations $\{P_1, P_2\}$.

Assume that the current picture of wave fronts $W_1(t), W_2(t)$ is the one shown in Figure 3 with the velocity vectors normal to the two spheres shown in Figure 3.

Then a straightforward calculation shows that $\mathrm{AL}(\widetilde{W}_1(t), \widetilde{W}_2(t)) - \mathrm{AL}(V_1, V_2) = \pm 1 \neq 0$ (we used the notation as in Theorem 4.8), and thus the first wave front reached the birth point of the second front before the second front originated. (The sign of ± 1 in this example depends on which of the two fronts shown in Figure 3 is W_1 in the case where n is odd and is always a plus sign when n is even.)

This seems to demonstrate that AL is a very powerful invariant because in this case we know neither the propagation laws nor when and where the fronts originated. In fact, in this example we can make this conclusion even without the knowledge of the topology of M outside of the depicted part of it.

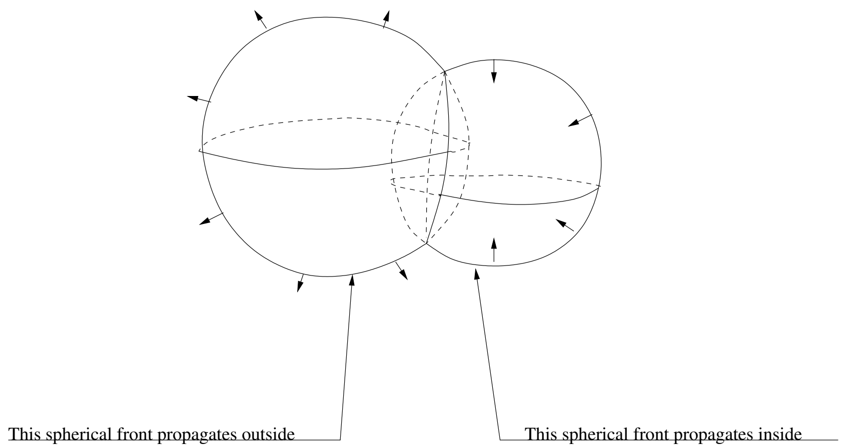


FIGURE 3

6.2. **Example.** Here we show how to apply AL to estimating of the number of times the wave front passed though a given point between the two moments of time.

Assume that we have a wave front W that propagates on M and that M is not an odd-dimensional homotopy sphere.

Assume that at a certain moment of time the picture of the wave front was the one shown in Figure 4.a and later it developed into the shape shown in Figure 4.b. (The Figure 4.b depicts a sphere that can be obtained from the trivially embedded sphere by passing three times through a point.)

a **b**

FIGURE 4

Let $\widetilde{W}(t_1), \widetilde{W}(t_2) : S^{n-1} \to STM$ be the liftings of the fronts shown in Figure 4.a and b respectively. A straightforward calculation shows that $\mathrm{AL}(\widetilde{W}(t_2), \varepsilon_x) - \mathrm{AL}(\widetilde{W}(t_1), \varepsilon_x) = 3 \in \boldsymbol{A}(M)$ for every map $\varepsilon_x : S^{n-1} \to STM$ as in (1.1).

Thus if the dimension of the ambient manifold is even, or $\pi_1(M)$ is infinite, then W has passed at least three times through the point x between the time moments shown in Figure 4.a and 4b. Once again, this conclusion does not depend on the topology of M outside of the part of it depicted in Figure 4, on the time passed between the two pictures taken, and on the propagation law.

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