EQUIVARIANT CELLULAR HOMOLOGY AND ITS APPLICATIONS

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ABSTRACT. In this work we develop a cellular equivariant homology functor and apply it to prove an equivariant Euler-Poincaré formula and an equivariant Lefschetz theorem.

1. INTRODUCTION

Let D be an arbitrary small topologically enriched category. In this paper we develop a D-CW-homology functor which allows for easy computation of the *ordinary* D-equivariant homology defined by E. Dror Farjoun in [1]. Our approach is a generalization of the G-CW-(co)homology functor constructed by S.J. Willson in [13] for the case of G being compact Lie group.

Then we apply the D-CW-homology functor to obtain:

(i) Equivariant Euler-Poincaré formula:

$$\chi^D(\underset{\sim}{X}) = \sum_{n=0}^{\infty}(-1)^n \widetilde{rk}_{\mathrm{HS}}(H_n^D(\underset{\sim}{X};\mathcal{I})) \tag{1}$$

This formula establishes a connection between the equivariant homology and an equivariant Euler characteristic; $\widetilde{rk}_{\mathrm{HS}}(\cdot)$ is a slight modification of Hattori-Stallings rank (originally defined in [8],[11]).

(ii) Equivariant Lefschetz theorem: Let $\underset{\sim}{X}$ be a triangulated D-space, $f : \underset{\sim}{X} \to \underset{\sim}{X}$ an equivariant map. If the equivariant Lefschetz number

$$\Lambda_D(f) = \sum_{n=0}^{\infty}(-1)^n \widetilde{tr}_{\mathrm{HS}}(H_n^D(f;\mathcal{I})) \tag{2}$$

is not equal to zero, then there are f-invariant orbits in $\underset{\sim}{X}$, moreover the orbit types of the invariant orbits may be recovered from $\Lambda_D(f)$.

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2. PRELIMINARIES

2.1. ***D*-spaces.** Let $\mathcal{T}\mathrm{op}$ denote the category of the compactly generated Hausdorff topological spaces. Fix an arbitrary small category D enriched over $\mathcal{T}\mathrm{op}$. We work in the category $\mathcal{T}\mathrm{op}^D$ of functors from D to $\mathcal{T}\mathrm{op}$. The objects of this category

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are called topological diagrams or just D-spaces. The arrows in $\mathcal{T}\mathrm{op}^D$ are natural transformations of functors or *equivariant maps.*

2.2. **D-homotopy.** An equivariant homotopy between two D-maps $f, g : \underset{\sim}{X} \to \underset{\sim}{Y}$, where $\underset{\sim}{X}, \underset{\sim}{Y}$ are D-diagrams, is a D-map $H : \underset{\sim}{X} \times I \to \underset{\sim}{Y}$, where I denotes the constant D-space $I(d) = [0, 1]$. A *homotopy equivalence* $f : \underset{\sim}{X} \to \underset{\sim}{Y}$ is a map with a (two sided) D-homotopy inverse.

2.3. **D-orbits.** We recall now the central concept of the D-homotopy theory (introduced in [1],[3]) – that of D-orbit. A D-orbit is a D-space $T : D \to \mathcal{T}\mathrm{op}$, such that $\mathrm{colim}_D T = \{*\}$. A **free** D-orbit generated in $d \in ob(D)$ is $\mathcal{T}\mathrm{op}^D \ni F^d = \mathrm{hom}_D(d, \cdot)$, i.e. $F^d(d') = \mathrm{hom}_D(d, d')$ and $F^d(d' \to d'')$ is given by the composition. Clearly F^d is a D-orbit. A D-space $\underset{\sim}{X}$ is called free iff for any $s \in \mathrm{colim}_{\mathcal{C}} \underset{\sim}{X}$ the full orbit T_s lying over s is free.

2.4. **D-CW-complexes.** A *D-cell* is a D-space of the form $T \times e^n$, where T is a D-orbit and e^n is the standard n-cell. An *attaching map* of this D-cell to some D-space $\underset{\sim}{X}$ is a map $\phi : T \times \partial e^n \to \underset{\sim}{X}$.

A (*relative*) D-CW-complex $(\underset{\sim}{X}, \underset{\sim}{X}_{-1})$ is a D-space $\underset{\sim}{X}$ together with a filtration $\underset{\sim}{X}_{-1} \subset \underset{\sim}{X}_0 \subset \ldots \subset \underset{\sim}{X}_n \subset \underset{\sim}{X}_{n+1} \subset \ldots \subset \underset{\sim}{X} = \mathrm{colim}_n \underset{\sim}{X}_n$, such that $\underset{\sim}{X}_{n+1}$ is obtained from $\underset{\sim}{X}_n$ by attaching a set of n-dimensional D-cells. Namely one has a push-out diagram of D-spaces:

$$\begin{array}{ccc} \coprod_i (T_i \times \partial e^n) & \xrightarrow{\phi} & \underset{\sim}{X}_{n-1} \\ \downarrow & & \downarrow \\ \coprod_i (T_i \times e^n) & \xrightarrow{\Phi} & \underset{\sim}{X}_n \end{array}$$

If $\underset{\sim}{X}_{-1} = \varnothing$ we call the D-CW-complex *absolute*.

Let $\underset{\sim}{X}$ be a D-CW-complex. A D-subspace $\underset{\sim}{Y} \subset \underset{\sim}{X}$ is called the *cellular subspace* if $\underset{\sim}{Y}$ has a D-CW-structure such that each cell of $\underset{\sim}{Y}$ is also a cell of $\underset{\sim}{X}$.

2.5. **The category of orbits.** The *category of orbits* $\mathbb{O}$ is a full topological subcategory of $\mathcal{T}\mathrm{op}^D$ generated by all D-orbits.

Usually $\mathbb{O}$ is not a small category. For example for $D = J = (\bullet \longrightarrow \bullet)$, then $\mathbb{O} \cong \mathcal{T}\mathrm{op}$. A model category has been constructed for the D-spaces of arbitrary orbit type in [2]. We will be interested in the diagrams which are homotopy equivalent to the finite D-CW-complexes, i.e. only finite number of orbit types appear in such diagram. We collect those orbits into the a full subcategory $\mathcal{O}'$ of $\mathbb{O}$ with a finite amount of objects.

2.6. **Orbit point $(\cdot)^{\mathcal{O}}$ and realization $|\cdot|_D$ functors.** Suppose $\mathcal{O}$ is a *small* category of D-orbits. An orbit point functor $(\cdot)^{\mathcal{O}} : \mathcal{T}\mathrm{op}^D \to \mathcal{T}\mathrm{op}^{\mathcal{O}^{\mathrm{op}}}$ is a generalization to the diagram case of Bredon's fixed point functor. For any D-space $\underset{\sim}{X}$ (usually of type $\mathcal{O}$) $(\underset{\sim}{X})^{\mathcal{O}}$ is a $\mathcal{O}^{\mathrm{op}}$ diagram such that $(\underset{\sim}{X})^{\mathcal{O}}(T) = \mathrm{hom}_D(T, \underset{\sim}{X})$ for all $T \in ob(\mathcal{O})$ and the arrows of the diagram are induced by composition with the maps between orbits.

If $f : \underset{\sim}{X} \to \underset{\sim}{Y}$ is an equivariant map between two D-spaces, then there exist an $\mathcal{O}^{\mathrm{op}}$-equivariant map $f^{\mathcal{O}} : \underset{\sim}{X}^{\mathcal{O}} \to \underset{\sim}{Y}^{\mathcal{O}}$, which is obtained from f by composition:

$$\underset{\sim}{X}^{\mathcal{O}}(T) = \mathrm{hom}_D(T, \underset{\sim}{X}) \ni g \overset{f^{\mathcal{O}}}{\longmapsto} f \circ g \in \mathrm{hom}_D(T, \underset{\sim}{Y}) = \underset{\sim}{Y}^{\mathcal{O}}(T)$$

The fundamental property of $(\cdot)^{\mathcal{O}}$ functor is that for any D-space $\underset{\sim}{X}$ the $\mathcal{O}^{\mathrm{op}}$-space $(\underset{\sim}{X})^{\mathcal{O}}$ is $\mathcal{O}^{\mathrm{op}}$-free [1, 3.7].

Example 2.1. Consider a free D-space $\underset{\sim}{X}$. And let the orbit category $\mathcal{O}$ consists of all the free orbits. Then $\mathcal{O}$ is isomorphic to D^{op} as a category and $(\underset{\sim}{X})^{\mathcal{O}} \cong \underset{\sim}{X}$ (Yoneda's lemma).

Another easy case occurs then $\underset{\sim}{X}$ is a D-CW-space. We shall discuss it in the next section.

There exist a left adjoint to $(\cdot)^{\mathcal{O}}$. It is called *realization functor* $|\cdot|_D$, since it takes an $\mathcal{O}^{\mathrm{op}}$-space and produce a D-space with the prescribed orbit point data (up to local weak equivalence). Realization functor in the group case has been constructed by A.D.Elmendorf in [7] and has been generalized to the arbitrary diagram case by W.Dwyer and D.Kan in [6]. Compare also [1].

2.7. **Equivariant Euler characteristic.** Let $\underset{\sim}{X}$ be a finite D-CW-complex, then $\underset{\sim}{X}$ is of type $\mathcal{O}$ for some category of orbits with finite amount of objects. We define the Equivariant Euler characteristic to be the Universal Additive Invariant [10, I.5] $(U(D), \chi^D)$. Or, equivalently, we say that $\chi^D(\underset{\sim}{X}) \in U(D)$ is equal to the alternating sum of the orbit types over the dimensions of the cells of $\underset{\sim}{X}$ in the free abelian group $U(D) = \bigoplus_{T \in Iso(h\mathcal{O})} \mathbb{Z}$ generated by the homotopy types of orbits.

3. Equivariant cellular homology

3.1. **$\mathcal{O}^{\mathrm{op}}$-$CW$ structure on the orbit point space of a D-CW-complex.** Let $\mathcal{C}$ denote a subcategory of $\mathcal{T}\mathrm{op}^{\mathcal{O}^{\mathrm{op}}}$ which is obtained as the image of $\mathcal{T}\mathrm{op}^D$ under the functor $(\cdot)^{\mathcal{O}}$. Recall that there is the inclusion of the categories $\imath : D \hookrightarrow \mathcal{O}^{\mathrm{op}}$, where $\imath(d) = F^d$, for each $d \in ob(D)$. Hence there is a functor $\mathrm{Res} : \mathcal{T}\mathrm{op}^{\mathcal{O}^{\mathrm{op}}} \to \mathcal{T}\mathrm{op}^D$. By abuse of notation we denote by Res also $\mathrm{Res}|_{\mathcal{C}}$.

Lemma 3.1. *The functor $(\cdot)^{\mathcal{O}}$ is fully faithful.*

Proof. The faithfulness is clear. We have to show only that for any map $f_0 : \underset{\sim}{X}^{\mathcal{O}} \to \underset{\sim}{Y}^{\mathcal{O}}$ there exists a map $f : \underset{\sim}{X} \to \underset{\sim}{Y}$ s.t. $f_0 = f^{\mathcal{O}}$. Take $f = \mathrm{Res}(f_0)$, then if $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ were orbits the result follows from the bijective correspondence induced by the Yoneda's lemma: $\mathrm{h}om_D(\underset{\sim}{X}, \underset{\sim}{Y}) = \mathrm{hom}_{\mathcal{O}^{\mathrm{op}}}(\underset{\sim}{X}^{\mathcal{O}}, \underset{\sim}{Y}^{\mathcal{O}})$. The general claim will follow from the comparison of f_0 and $f^{\mathcal{O}}$ orbitwise, i.e. by their action on each full orbit. Fortunately the functor $(\cdot)^{\mathcal{O}}$, being right adjoint, commutes with taking full orbit (pullback). □

Lemma 3.2. *The pair of functors* $\mathrm{Res}(\cdot) : \mathcal{C} \leftrightarrow \mathcal{T}\mathrm{op}^D : (\cdot)^{\mathcal{O}}$ *induce the equivalence of the categories* $\mathcal{C}$ *and* $\mathcal{T}\mathrm{op}^D$.

Proof. We need to construct the natural isomorphisms of the functors $id_{\mathcal{T}\mathrm{op}^D} \cong \mathrm{Res}((\cdot)^{\mathcal{O}})$ and $id_{\mathcal{C}} \cong (Res(\cdot))^{\mathcal{O}}$.

Let $\underset{\sim}{X} \in \mathcal{T}\mathrm{op}^D$, then $\mathrm{Res}(\underset{\sim}{X}^{\mathcal{O}}) \cong \underset{\sim}{X}$ because the generalized lemma of Yoneda [9] induces the objectwise homeomorphisms and the equivariance is preserved by the naturality of the Yoneda's isomorphism. But an equivariant map which is the objectwise homeomorphism is an isomorphism of D-spaces, hence the first isomorphism of functors.

Let $\underset{\sim}{X}^{\mathcal{O}} \in \mathcal{C}$, then $\mathrm{Res}(\underset{\sim}{X}^{\mathcal{O}}) \cong \underset{\sim}{X}$ by the first homeomorphism, then $(\mathrm{Res}(\underset{\sim}{X}^{\mathcal{O}}))^{\mathcal{O}} \cong \underset{\sim}{X}^{\mathcal{O}}$. Hence the second isomorphism.

□

Proposition 3.3. *Let $\underset{\sim}{X}$ be a (pointed) D-CW–space of orbit type $\mathcal{O}$,where $\mathcal{O}$is a small category of orbits. Consider the $\mathcal{O}^{\rm op}$–space $\underset{\sim}{X}^{\mathcal{O}}$ to be the orbit point space of $\underset{\sim}{X}$.*

Then $\underset{\sim}{X}^{\mathcal{O}}$ has $\mathcal{O}^{\rm op}$-CW structure which corresponds to the D-CW structure of $\underset{\sim}{X}$ in the following sense: let $pt_D \subseteq \underset{\sim}{X}_0 \subseteq \underset{\sim}{X}_1 \subseteq \cdots \subseteq \underset{\sim}{X}_n \subseteq \cdots \subseteq \underset{\sim}{X} = \mathrm{colim}_n \underset{\sim}{X}_n$ is a D-CW filtration of $\underset{\sim}{X}$, such that each $\underset{\sim}{X}_n$ is a push-out:

$$\begin{array}{ccc} \coprod_i T_i \times S^{n-1} & \xrightarrow{\phi} & \underset{\sim}{X}_{n-1} \\ \downarrow & & \downarrow{\scriptstyle i_n} \\ \coprod_i T_i \times D^n & \xrightarrow{\Phi} & \underset{\sim}{X}_n \end{array}$$

then there exist a $\mathcal{O}^{\rm op}$-CW–filtration: $pt_{\mathcal{O}^{\rm op}} \subseteq \underset{\sim}{X}^{\mathcal{O}}_0 \subseteq \underset{\sim}{X}^{\mathcal{O}}_1 \subseteq \cdots \subseteq \underset{\sim}{X}^{\mathcal{O}}_n \subseteq \cdots \subseteq \underset{\sim}{X}^{\mathcal{O}} = \mathrm{colim}_n \underset{\sim}{X}^{\mathcal{O}}_n$, such that $\underset{\sim}{X}^{\mathcal{O}}_n = (\underset{\sim}{X}_n)^{\mathcal{O}}$, and

$$\begin{array}{ccc} \coprod_i F^{T_i} \times S^{n-1} & \xrightarrow{\phi^{\mathcal{O}}} & \underset{\sim}{X}^{\mathcal{O}}_{n-1} \\ \downarrow & & \downarrow{\scriptstyle i_n^{\mathcal{O}}} \\ \coprod_i F^{T_i} \times D^n & \xrightarrow{\Phi^{\mathcal{O}}} & \underset{\sim}{X}^{\mathcal{O}}_n \end{array}$$

is a push-out square.

Proof. We proceed by the induction on the skeleton of $\underset{\sim}{X}$.

$$\underset{\sim}{X}^{\mathcal{O}}_0(T) = (\coprod T_i)^{\mathcal{O}}(T) = \coprod((T_i)^{\mathcal{O}}(T)) = \coprod \hom_D(T, T_i) = \coprod \hom_{\mathcal{O}^{\rm op}}(F^T, F^{T_i}) = \coprod F^{T_i}(T).$$

Hence the base of the induction.

Suppose we know the claim for $\underset{\sim}{X}_n$. Then it follows for $\underset{\sim}{X}_{n+1}$ since $(\cdot)^{\mathcal{O}}$ is both left and right adjoint, so commutes both with push-outs and products. □

3.2. ***D-CW*-homology functor.** The construction of the (co)homology functor in [1, 4.16] depends on the specific *D-CW*-decomposition of $\underset{\sim}{X}^{\mathcal{O}}$. We apply this construction to the cellular structure of $\underset{\sim}{X}^{\mathcal{O}}$, which was constructed in 3.3 and obtain the required *D-CW*-homology functor.

3.3. **Isotropy ring $\mathcal{I}$.** In [13] a universal coefficient system for the G-equivariant homology have been developed(where G is a compact Lie group). Let us generalize this approach to the coefficient systems for the classical D-homology theory. Suppose $\mathcal{O}'$ is a small, full subcategory of the orbit category $\mathcal{O}$. Let $\underset{\sim}{X}$ be a D-space of orbit type $\mathcal{O}'$. Then a coefficient system for the ordinary (co)homology is a homotopy (co)functor $M : \mathcal{O}' \to (R - \mathrm{mod})$.

Definition 3.4. Let R be a commutative ring. An *isotropy ring* $\mathcal{I} = I_{\rm D}^{{\rm R},\mathcal{O}'}$ is generated by $mor(h\mathcal{O}')$ as a free $R - \mathrm{mod}$. Define the multiplication on the generators by

$$fg = \begin{cases} f \circ g, & \text{if codom}(g) = \text{dom}(f) \\ 0, & \text{otherwise} \end{cases}$$

and extend the definition to the rest of the elements of $\mathcal{I}$ by linearity.

Proposition 3.5. *The category $\mathcal{M}$ of the left $\mathcal{I}$-modules which satisfy:*

$$\forall M \in ob(\mathcal{M}),\ M = \bigoplus_{T \in ob(h\mathcal{O}')} 1_T M \tag{3}$$

(where $\{1_T M\}_{T \in ob(h\mathcal{O}')}$ are left R-modules) and the category of $R(h\mathcal{O}')$-mod of functors from $h\mathcal{O}'$ to the category of left R-modules are equivalent.

Proof. Let us define a pair of functors which induce the required equivalence:

$$\zeta : \mathcal{M} \leftrightarrows R(h\mathcal{O}')\text{-}mod : \xi.$$

Let $M \in ob(\mathcal{M})$, $T \in ob(h\mathcal{O}')$, then define

$$\zeta M(T) = 1_T M.$$

If $\mathrm{mor}(h\mathcal{O}') \ni f : T_1 \to T_2$, then define

$$\zeta M(f)(1_{T_1} m) = f 1_{T_1} m = (1_{T_2} f) 1_{T_1} m \in 1_{T_2} M.$$

Obviously the morphisms of the left $\mathcal{I}$ modules correspond to the natural transformations of the functors.

Given a $R(h\mathcal{O}')$-module N, then

$$\xi N = \bigoplus_{T \in ob(h\mathcal{O}')} N(T), \text{ as a left } R\text{-module}.$$

Define the left $\mathcal{I}$-module structure on ξN by $f(\dots, n, \dots) = (\dots, fn, \dots)$, where

$$N(\mathrm{codom}(f)) \ni fn = \begin{cases} f(n), & \text{if } n \in N(\mathrm{dom}(f)) \\ 0, & \text{otherwise} \end{cases}$$

Now it is clear that the defined functors provide the equivalence of the categories. □

Remark 3.6. The ring $\mathcal{I}$ considered as a left $\mathcal{I}$-module is an object of $\mathcal{M}$, because $\mathcal{I} \cong \bigoplus_{T \in ob(h\mathcal{O}')} 1_T \mathcal{I}$ (as left R-modules) by the construction. But it also carries an obvious structure of the right $\mathcal{I}$ module, so the $\zeta\mathcal{I}(T)$.

Remark 3.7. If $ob(h\mathcal{O}')$ is finite then the ring $\mathcal{I}$ has a two-sided identity element $1 = \sum_{T \in ob(h\mathcal{O}')} 1_T$ together with its decomposition into the sum of the orthogonal idempotents and the condition (3) is redundant.

Definition 3.8. The augmentation $\varphi : \mathcal{I} \to \bigoplus_{T \in Iso(ob(h\mathcal{O}'))} R$ is defined for any

$$\mathcal{I} \ni g = \sum_{T \in ob(h\mathcal{O}')} \sum_{f \in \mathrm{mor}(T,T)} r_f f + \sum_{h \in \mathrm{mor}(T_1, T_2), T_1 \neq T_2} s_h h$$

(only a finite number of $r_f, s_h \in R$ is non equal to zero) to be

$$\varphi(g) = (\dots, \sum_{f \in \mathrm{mor}(T,T)} r_f, \dots) \in \bigoplus_{T \in Iso(ob(h\mathcal{O}'))} R$$

Remark 3.9. The idempotents in $\mathcal{I}$ which correspond to the D-homotopy equivalent orbits are identified under φ. Apparently, φ is an epimorphism of rings. Consider the abelinization functor $Ab : (Rings) \to \mathcal{A}b$ which corresponds to a ring its additive group divided by the commutator subgroup. Then $Ab(\varphi) : Ab(\mathcal{I}) \to \bigoplus_{T \in Iso(ob(h\mathcal{O}'))} R$. The last map will be used to obtain a generalization of the Euler-Poincaré formula.

4. Applications

Let $\underset{\sim}{X}$ be a finite D-CW-complex of type $\mathcal{O}'$ for some orbit category $\mathcal{O}'$ with $\mathrm{obj}(\mathcal{O}')$ a *finite* set.

4.1. **Equivariant Euler-Poincaré formula.** We remind that the equivariant Euler characteristic lies in the abelian group $U(D) \cong \bigoplus_{Iso(obj(h\mathcal{O}'))} \mathbb{Z}$, so in order to apply Hattori–Stallings machinery we need to choose a coefficient system for the equivariant homology such that the resulting chain complex and homology groups will be endowed with the module structure over some ring S which allows an epimorphism $\varepsilon : Ab(S) \longrightarrow U(D)$.

Our choice of the coefficient system for the equivariant homology will be the isotropy ring $\mathcal{I} = I_{\mathrm{D}}^{\mathbb{Z},O'}$ taken over itself as a left module.

Lemma 4.1. *Let $\underset{\sim}{X}$ be a finite D-CW-complex. Suppose $\underset{\sim}{X}$ has n_q q-dimensional cells and $t_1 + \cdots + t_s = n_q$, t_i is the number of q-dimensional cells of the same homotopy type $T_i \in Iso(obj(h\mathcal{O}'))$. Then $\mathcal{C}_q(\underset{\sim}{X}) \otimes_{\mathcal{O}'} \zeta\mathcal{I} \cong \zeta\mathcal{I}(T_1)^{t_1} \oplus \cdots \oplus \zeta\mathcal{I}(T_n)^{t_s}$ as a left $\mathbb{Z}$-module.*

Proof. Let $t_i = r_{i1} + \cdots + r_{ik}$, where r_{ij} is the number of q-dimensional cells of type $T_{ij} \in ob(\mathcal{O})$ of homotopy type T_i. By the construction of the equivariant homology $\mathcal{C}_q(\underset{\sim}{X}) = \bigoplus_{i=1}^{s}(\oplus_{j=1}^{k}\mathbb{Z}(\hom_{\mathcal{O}'}(?, T_{ij})^{r_{ij}}))$. The dual Yoneda isomorphism [9, p.74] implies:

$$\mathcal{C}_q(\underset{\sim}{X}) \otimes_{\mathcal{O}'} \zeta\mathcal{I} \cong \bigoplus_{i=1}^{s}(\oplus_{j=1}^{k}\zeta\mathcal{I}(T_{ij})^{r_{ij}}) \cong \bigoplus_{i=1}^{s} \oplus_{j=1}^{k}(1_{T_{ij}}\mathcal{I})^{r_{ij}},$$

If T_{ij_1} is isomorphic to T_{ij_2} in $h\mathcal{O}'$ then there is an obvious isomorphism of the left $\mathbb{Z}$-modules and right $\mathcal{I}$-modules $1_{T_{ij_1}}\mathcal{I} \cong 1_{T_{ij_2}}\mathcal{I}$. Let us choose a representative T_i of each isomorphism class of objects in $h\mathcal{O}'$, then

$$\mathcal{C}_q(\underset{\sim}{X}) \otimes_{\mathcal{O}'} \zeta\mathcal{I} \cong \bigoplus_{i=1}^{s}(1_{T_i}\mathcal{I})^{(\sum_{j=1}^{k} r_{ij})} \cong \bigoplus_{i=1}^{s}(1_{T_i}\mathcal{I})^{t_i} \cong \bigoplus_{i=1}^{s}(\zeta\mathcal{I}(T_i))^{t_i}$$

□

Because of 3.6 the equivariant chain complex $\{\mathcal{C}_q(\underset{\sim}{X}) \otimes_{\mathcal{O}'} \zeta\mathcal{I}\}_{q=0}^{\dim \underset{\sim}{X}}$ is a complex of *projective* right $\mathcal{I}$-modules and the equivariant homology is endowed with the right $\mathcal{I}$-module structure.
Notation: $\chi_{\mathrm{HS}}(\cdot)$ means Euler characteristic of a $\mathcal{I}$ differential complex with respect to $rk_{\mathrm{HS}}(\cdot)$.

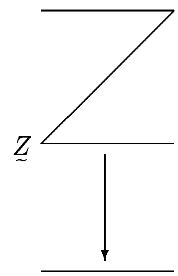
Proposition 4.2. *Let $K_* = \mathcal{C}_*(\underset{\sim}{X}) \otimes_{\mathcal{O}'} \zeta\mathcal{I}$ be a right $\mathcal{I}$-complex, then $\chi^D(\underset{\sim}{X}) = Ab(\varphi)(\chi_{\mathrm{HS}}(K_*))$ whenever left side is defined.*

Proof. It is easy to see that $rk_{\mathrm{HS}}(1_T\mathcal{I}) = 1_T \in Ab(\mathcal{I})$. Lemma 4.1 together with 3.9 completes the proof. □

Now we combine 4.2 with the additivity properties of the Hattory-Stallings rank and obtain the following

Theorem 4.3. $\chi^D(\underset{\sim}{X}) = Ab(\varphi)(\sum_{n=0}^{\infty}(-1)^n rk_{\mathrm{HS}} H_n^D(\underset{\sim}{X};\zeta\mathcal{I}))$, *whenever the left side is defined.*

Example 4.4. Consider the J-diagram:



$\underset{\sim}{Z}$ has two 0-cells of type $T_2 = [\underset{\cdot}{\overset{\cdot\cdot}{\downarrow}}]$ and one 1-cell of type $T_3 = [\underset{\cdot}{\overset{\cdot\cdot\cdot}{\downarrow}}]$, hence $\chi^J(\underset{\sim}{Z}) = 2[\underset{\cdot}{\overset{\cdot\cdot}{\downarrow}}] - [\underset{\cdot}{\overset{\cdot\cdot\cdot}{\downarrow}}]$.

The category $\mathcal{O}'$ of orbits contains two objects: T_2, T_3. The cellular chain complex tensored with the coefficients $\mathcal{I} = \mathcal{I}_J^{\mathbb{Z},\{T_2,T_3\}}$ becomes:

$$\cdots \to 0 \to 1_{T_3}\mathcal{I} \xrightarrow{\partial_1} (1_{T_2}\mathcal{I})^2$$

and $\partial_1 = 0$ from the orbit type considerations.

$U(J) = \mathbb{Z} \oplus \mathbb{Z}$ in that case. And $H_0^J(\underset{\sim}{Z}, \mathcal{I}) = (1_{T_2}\mathcal{I})^2$, $H_1^J(\underset{\sim}{Z}, \mathcal{I}) = 1_{T_3}\mathcal{I}$ are right $\mathcal{I}$ - modules. Hence, $\chi^J(\underset{\sim}{Z}) = (2,0) - (0,1) = (2,-1)$.

Let us, for comparison, calculate the J-equivariant homology of $\underset{\sim}{Z}$ with $\mathbb{Z}^{\mathcal{O}'}$ coefficients: $H_i^J(\underset{\sim}{Z}, \mathbb{Z}^{\mathcal{O}'}) = H_i(\operatorname{colim}_J \underset{\sim}{Z}, \mathbb{Z})$ (see [1, 5.2]). Then $\operatorname{colim}_J \underset{\sim}{Z} = I = [0,1]$ and

$$H_i^J(\underset{\sim}{Z}, \mathbb{Z}^{\mathcal{O}'}) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

We can see that $\mathbb{Z}^{\mathcal{O}'}$ coefficients are inappropriate to the Euler-Poincaré formula.

4.2. **Equivariant Lefschetz theorem.** Using cellular equivariant homology functor we are able now to proof a version of the equivariant Lefschetz theorem.

Some result of the Lefschetz type in the equivariant setting may be obtained already by applying the ordinary Lefschetz theorem: consider an equivariant map $f : \underset{\sim}{X} \to \underset{\sim}{X}$, where $\underset{\sim}{X}$ is a diagram over small category D, then if the Lefschetz number $\Lambda(\operatorname{colim}_D \underset{\sim}{X}) \neq 0$ there are f- invariant D-orbits in $\underset{\sim}{X}$. However the advantage of using the equivariant homology and equivariant Lefschetz number $\Lambda_D(\underset{\sim}{X}) \in U(D)$ is that we obtain the specific information about orbit type of the invariant orbit.

First we give a technical

Definition 4.5. A D-CW-complex $\underset{\sim}{X}$ will be called the triangulated D-space if the natural CW-structure of $\operatorname{colim} \underset{\sim}{X}$ also triangulates $\operatorname{colim} \underset{\sim}{X}$.

The following lemma will be used in the proof of the equivariant Lefschetz theorem.

Lemma 4.6. *Let $\underset{\sim}{X}$ be a triangulated diagram, then for any refinement Y of the triangulation of* $\operatorname{colim} \underset{\sim}{X}$*, there exists a D-CW-complex $\underset{\sim}{X}'$, such that $\underset{\sim}{X}'$ is D-homeomorphic to $\underset{\sim}{X}$ and* $\operatorname{colim} \underset{\sim}{X}' = Y$ *(as the triangulated spaces). $\underset{\sim}{X}'$ will be called the refinement of $\underset{\sim}{X}$.*

Proof. Consider a new simplex Δ in the triangulation of Y. It lies in some old simplex of $\operatorname{colim} \underset{\sim}{X} : \Delta \in \Delta'$. Then consider the pull-back:

$$\lim \left(\begin{array}{ccc} & & \underset{\sim}{X} \\ & & \downarrow \\ \Delta & \hookrightarrow & \operatorname{colim}_D \underset{\sim}{X} \end{array} \right) = T \times \Delta,$$

where T is the orbit which lies over Δ'.

We've obtained the cell of the new D-$CW-$complex $\underset{\sim}{X}'$. Continuing in the same way for the rest of the simplices of Y completes the construction of $\underset{\sim}{X}'$. Hence D-CW-complex $\underset{\sim}{X}'$ has the same underlying topological diagram as $\underset{\sim}{X}$, therefore they are D-homeomorphic. □

Definition 4.7. Let $f : \underset{\sim}{X} \to \underset{\sim}{X}$ be a map of the finite triangulated D-space $\underset{\sim}{X}$ of orbit type $\mathcal{O}'$, where $\mathcal{O}'$ is an orbit category with the finite number n of objects. Let $\mathcal{I} = \mathcal{I}_D^{\mathbb{Z},\mathcal{O}'}$. Then the equivariant Lefschetz number of f:

$$U(D) \ni (\lambda_1, \ldots, \lambda_n) = \Lambda_D(f) = Ab(\varphi)(\sum_{k=0}^{\infty}(-1)^k \mathrm{tr}_{\mathrm{HS}}(H_k(f;\mathcal{I})))$$

Theorem 4.8. *Let $\underset{\sim}{X}$ be a finite triangulated diagram over D. $f : \underset{\sim}{X} \to \underset{\sim}{X}$ be a D-map. $\Lambda_D(f) = (\lambda_1, \ldots, \lambda_n) \in U(D)$ - Lefschetz number of f. Then if there is no f-invariant orbit of type T_m, $\lambda_m = 0$.*

Proof. A simplex in $\operatorname{colim} \underset{\sim}{X}$ will be called of type T if the overlaying orbit is of type T in $\underset{\sim}{X}$. Then the condition that there are no invariant orbits of type T_m is equivalent to the condition that there are no fixed points in the simplices of type T_m.

Since $\underset{\sim}{X}$ is a finite triangulated diagram, $\operatorname{colim} \underset{\sim}{X}$ is a finite triangulated space, hence it is a compact metric space. If there are no fixed points of type T_m, then there exists a refinement Y of the triangulation such that if Δ is a simplex of type T_m in Y, $\Delta \cap (\operatorname{colim} f)(\Delta) = \varnothing$.

Consider the refinement $\underset{\sim}{X}'$ of $\underset{\sim}{X}$, which exists by lemma 4.6. Since $\underset{\sim}{X}' \cong \underset{\sim}{X}$, $H_*^D(\underset{\sim}{X}';\mathcal{I}) = H_*^D(\underset{\sim}{X};\mathcal{I})$, $\Lambda(\tilde{f'}) = \Lambda(\tilde{f})$, where $f' : \underset{\sim}{X}' \to \underset{\sim}{X}'$ is equal to f, $\tilde{\lambda'_m} = \tilde{\lambda}_m$. Therefore, it is enough to show that $\lambda'_m = 0$.

Now,

$$\Lambda_D(f') = Ab(\varphi)(\sum_{k=0}^{\infty}(-1)^k \mathrm{tr}_{\mathrm{HS}}(H_k(f';\mathcal{I}))) = Ab(\varphi)(\sum_{k=0}^{\infty}(-1)^k \mathrm{tr}_{\mathrm{HS}}(\mathcal{C}_k(f';\mathcal{I}))),$$

where $\mathcal{C}_k(f';\mathcal{I})$ is the map induced by f on the chains $\mathcal{C}_k(\underset{\sim}{X};\mathcal{I}) = \mathcal{C}_k(\underset{\sim}{X}) \otimes_{\mathcal{O}'} \mathcal{I} = (1_{T_1}\mathcal{I})^{t_1} \oplus \ldots \oplus (1_{T_n}\mathcal{I})^{t_n}$ as $\mathcal{I}$-module. Because of the property: $\Delta \cap (\operatorname{colim} f)(\Delta) = \varnothing$ for *any* simplex Δ of type T_m, the induced map on $\mathcal{C}_k(\underset{\sim}{X};\mathcal{I})$ will take the generator 1_{T_m} corresponding to Δ outside the submodule $1_{T_m}\mathcal{I}$, that it generates. Then the m-th entry of $Ab(\varphi)(\sum_{k=0}^{\infty}(-1)^k \mathrm{tr}_{\mathrm{HS}}(\mathcal{C}_k(f';\mathcal{I})))$ will be zero. This is true for all k, hence $\lambda_m = 0$. □

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