ON THE HOMOTOPY TYPE OF LIE GROUPOIDS

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Abstract. We propose a notion of groupoid homotopy for generalized maps. This notion of groupoid homotopy generalizes the notions of natural transformation and strict homotopy for functors. The groupoid homotopy type of a Lie groupoid is shown to be invariant under Morita equivalence. As an application we consider orbifolds as groupoids and study the orbifold homotopy between orbifold maps induced by the groupoid homotopy.

1. Introduction

We develop a notion of homotopy between groupoid maps which is suitable for applications to orbifolds. There are notions of homotopy for functors in general categories that can be applied to groupoids. But they fail to be invariant of Morita equivalence when considering Lie groupoids. Since two Lie groupoids define the same orbifold if they are Morita equivalent, we need a notion of homotopy between Lie groupoid maps to be invariant of Morita equivalence.

For general categories (no topology nor smooth structure involved) natural transformations play the role of homotopy for functors [13, 4]. Two functors are homotopic if there is a natural transformation between them. We call this notion of homotopy a *natural transformation*.

For topological categories $\mathcal{T}$ and $\mathcal{T}'$ the usual notion of homotopy is just a functor which is an ordinary homotopy on objects and on arrows. We say that two continuous functors $f, g \colon \mathcal{T} \to \mathcal{T}'$ are homotopic if there is a continuous functor $H \colon \mathcal{T} \times I \to \mathcal{T}'$ such that $H_0 = f$ and $H_1 = g$. We call this notion of homotopy a *strict homotopy*.

Both notions of natural transformation and strict homotopy can be adapted to Lie groupoids by requiring all the maps involved to be smooth. None of the two notions is invariant of Morita equivalence.

For an example of Morita equivalent Lie groupoids which are not equivalent by a natural transformation nor a strict homotopy, consider the holonomy groupoid associated to a Seifert fibration $\mathcal{F}_S$ on a Möbius band M, $\mathcal{G} = \mathrm{Hol}(M, \mathcal{F}_S)$ and its reduced holonomy groupoid $\mathcal{K} = \mathrm{Hol}_T(M, \mathcal{F}_S)$ to a transversal interval T. Since the double covering of the Möbius band by the annulus has no global section, these two groupoids cannot be equivalent by a natural transformation (Section 3.10). These groupoids are clearly not

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equivalent by a strict homotopy since the space of objects of $\mathcal{G} = \mathrm{Hol}(M, \mathcal{F}_S)$ is a Möbius band and the space of objects of $\mathcal{K} = \mathrm{Hol}_T(M, \mathcal{F}_S)$ is an interval.

We look for a notion of homotopy which is invariant of Morita equivalence and generalizes the notions of natural transformation and strict homotopy. To achieve this, we follow a bicategorical approach as emphasized by Landsman in [11]. We introduce a *homotopy bicategory* $\mathfrak{H}$ of Lie groupoids and invert the essential homotopy equivalences W. Our notion of *weak homotopy equivalence* for Lie groupoids amounts to isomorphism of objects in $\mathfrak{H}[W^{-1}]$.

The bicategory $\mathfrak{G}$ of Lie groupoids, functors and natural transformations is a subbicategory of $\mathfrak{H}$ and when inverting the essential equivalences E, the following diagram of bicategories commutes:

$$\begin{tikzcd}
\mathfrak{G} \arrow[r, "U|_{\mathfrak{G}}"] \arrow[d, "i_{\mathfrak{G}}"'] & \mathfrak{G}[E^{-1}] \arrow[d, "i_{\mathfrak{G}[E^{-1}]}"] \\
\mathfrak{H} \arrow[r, "U"] & \mathfrak{H}[W^{-1}]
\end{tikzcd}$$

where U is the universal homomorphism as defined in [22].

As a main application we study the homotopy type of orbifolds as groupoids by considering the induced homotopy between orbifold morphisms.

The organization of the paper is as follows. In Section 1 we introduce some basic definitions and constructions for Lie groupoids. We set the notion of Morita equivalence and give examples. We give some background on bicategories in Section 2 and emphasize the bicategorical viewpoint by presenting two equivalent constructions of the Morita bicategory $\mathfrak{M}$ of Lie groupoids and generalized maps. In the last two sections we present our notion of homotopy. First we recall in Section 3 the construction of the Haefliger's $\mathcal{G}$-paths and the fundamental groupoid of a groupoid. In Section 4 we introduce the homotopy bicategory $\mathfrak{H}$ of Lie groupoids and prove that it admits a right calculus of fractions that inverts the essential homotopy equivalences W. Our notion of homotopy between groupoid maps corresponds to 2-morphisms in the bicategory $\mathfrak{H}(W^{-1})$. We call *groupoid homotopy* to this notion of homotopy. We prove that the notion of groupoid homotopy equivalence obtained in this bicategory is invariant of Morita equivalence and generalizes the notions of natural transformation and strict homotopy. In Section 5 we study the notion of homotopy for orbifolds induced by a groupoid homotopy.

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2. Lie Groupoids

2.1. **Preliminaries.** A *groupoid* $\mathcal{G}$ is a small category in which each arrow is invertible [15]. Thus a groupoid consist of a set of arrows G_1 and a set of objects G_0, together with five structure maps:

1. The maps $s, t : G_1 \rightrightarrows G_0$ called source and target maps. An element $g \in G_1$ with $s(g) = x$ and $t(g) = y$ is an arrow from x to y and will be denoted by $g : x \to y$.
2. The composition map $m : G_1 \times_{G_0} G_1 \to G_1$ mapping a pair (h, g) with $s(h) = t(g)$ to the composition $m(g, h) = hg$.
3. A unit map $u : G_0 \to G_1$ mapping each $x \in G_0$ to a two-sided unit $u(x) = 1_x$ for the composition.
4. An inverse map $i : G_1 \to G_1$ mapping each arrow $g : x \to y$ in G to a two-sided inverse $i(g) = g^{-1}$ for the composition.

The set of arrows from x to y is denoted $G(x, y) = \{g \in G_1 | s(g) = x \text{ and } t(g) = y\}$. The set of arrows from x to itself is a group called the *isotropy* group of G at x and denoted by $G_x = G(x, x)$. The *orbit* of x is the set $ts^{-1}(x)$. The orbit space $|\mathcal{G}|$ of $\mathcal{G}$ is the quotient of G_0 under the equivalence relation: $x \sim y$ iff x and y are in the same orbit.

A *Lie groupoid* $\mathcal{G}$ is a groupoid such that G_1 and G_0 are manifolds, the structure maps are smooth and s and t are submersions [14].

Example 2.1. *Unit groupoid.* Let M be a smooth manifold. Consider the groupoid $\mathcal{G}$ with $G_0 = G_1 = M$. This is a Lie groupoid whose arrows are all units, called the unit groupoid and denoted $\mathcal{G} = u(M)$. The orbit space $|\mathcal{G}|$ is again the manifold M.

Example 2.2. *Pair groupoid.* Let M be a smooth manifold. Consider the groupoid $\mathcal{G}$ with $G_0 = M$ and $G_1 = M \times M$. This is a Lie groupoid with exactly one arrow from any object x to any object y, called the pair groupoid and denoted $\mathcal{G} = \mathrm{Pair}(M)$ or $M \times M$. The orbit space $|\mathcal{G}|$ has only one point.

Example 2.3. *Point groupoid.* Let G be a Lie group. Let $\bullet$ be a point. Consider the groupoid $\mathcal{G}$ with $G_0 = \bullet$ and $G_1 = G$. This is a Lie groupoid with exactly one object $\bullet$ and G is the manifold of arrows in which the maps s and t coincide. We denote the point groupoid by $\bullet^G$. If G_1 is also a point, then the point groupoid is called the *trivial* groupoid and denoted by $\mathbf{1}$.

Example 2.4. *Translation groupoid.* Let K be a Lie group acting (on the left) on a smooth manifold M. Consider the groupoid $\mathcal{G}$ with $G_0 = M$ and $G_1 = K \times M$. This is a Lie groupoid with arrows (k, x) from any object x to $y = kx$, called the translation or action groupoid and denoted $\mathcal{G} = K \ltimes M$. The orbit space $|\mathcal{G}|$ is the orbit space of the action M/K which is not always a manifold.

Example 2.5. *Holonomy groupoid.* Let $(M, \mathcal{F})$ be a foliated manifold M. Consider the groupoid $\mathcal{G}$ with $G_0 = M$ and whose arrows from x to y on the same leaf $L \in \mathcal{F}$ are the holonomy classes of paths in L from x to y. There are no arrows between points in different leaves. This is a Lie groupoid called the holonomy groupoid and denoted $\mathcal{G} = \mathrm{Hol}(M, \mathcal{F})$. The orbit space $|\mathcal{G}|$ is the leaf space of the foliation.

2.2. **Morphisms and equivalences.** From now on all groupoids will be assumed Lie groupoids.

A *morphism* $\phi : \mathcal{K} \to \mathcal{G}$ of groupoids is a functor given by two smooth maps $\phi : K_1 \to G_1$ and $\phi : K_0 \to G_0$ that together commute with all the structure maps of the groupoids $\mathcal{K}$ and $\mathcal{G}$.

A *natural transformation* T between two homomorphisms $\phi, \psi : \mathcal{K} \to \mathcal{G}$ is a smooth map $T : K_0 \to G_1$ with $T(x) : \phi(x) \to \psi(x)$ such that for any arrow $h : x \to y$ in K_1, the identity $\psi(h)T(x) = T(y)\phi(h)$ holds. We write $\phi \sim_T \psi$.

A morphism $\phi : \mathcal{K} \to \mathcal{G}$ of groupoids is an *equivalence* of groupoids if there exists a morphism $\psi : \mathcal{G} \to \mathcal{K}$ of groupoids and natural transformations T and T' such that $\psi\phi \sim_T id_{\mathcal{K}}$ and $\phi\psi \sim_{T'} id_{\mathcal{G}}$.

A morphism $\epsilon : \mathcal{K} \to \mathcal{G}$ of groupoids is an *essential equivalence* of groupoids if

(i) ϵ is essentially surjective in the sense that
$$t\pi_1 : G_1 \times_{G_0} K_0 \to G_0$$
is a surjective submersion where $G_1 \times_{G_0} K_0$ is the pullback along the source $s : G_1 \to G_0$;

(ii) ϵ is fully faithful in the sense that K_1 is given by the following pullback of manifolds:
$$\begin{tikzcd}
K_1 \arrow[r, "\epsilon"] \arrow[d, "{(s,t)}"'] & G_1 \arrow[d, "{(s,t)}"] \\
K_0 \times K_0 \arrow[r, "\epsilon \times \epsilon"] & G_0 \times G_0
\end{tikzcd}$$

The first condition implies that for any object $y \in G_0$, there exists an object $x \in K_0$ whose image $\epsilon(x)$ can be connected to y by an arrow $g \in G_1$. The second condition implies that for all $x, z \in K_0$, ϵ induces a diffeomorphism $K(x, z) \to G(\epsilon(x), \epsilon(z))$ between the submanifolds of arrows.

For general categories the notions of equivalence and essential equivalence coincide. This applies to the particular case in which the categories are groupoids. But when some extra structure is involved (continuity or differentiability) these two notions are not the same anymore. An essential equivalence implies the existence of the inverse functor using the axiom of choice but not the existence of a *smooth* functor. The impossibility of somehow invert essential equivalences is what will lead to the definition of generalized maps, a category where essential equivalences can be inverted.

Proposition 2.6. [18] *Every equivalence of Lie groupoids is an essential equivalence.*

Remark 2.7. The converse does not hold for Lie groupoids.

Morita equivalence is the smallest equivalence relation between Lie groupoids such that they are equivalent whenever there exists an essential equivalence between them.

Definition 2.8. Two Lie groupoids $\mathcal{K}$ and $\mathcal{G}$ are *Morita equivalent* if there exists a Lie groupoid $\mathcal{J}$ and essential equivalences

$$\mathcal{K} \overset{\epsilon}{\leftarrow} \mathcal{J} \overset{\sigma}{\rightarrow} \mathcal{G}.$$

This defines an equivalence relation that we denote $\mathcal{K} \sim_M \mathcal{G}$. In this case, it is always possible to chose the equivalences ϵ and σ being surjective submersions on objects.

Remark 2.9. An essential equivalence of Lie groupoids $\epsilon : \mathcal{K} \to \mathcal{G}$ induces a homeomorphism $|\epsilon| : |\mathcal{K}| \to |\mathcal{G}|$ between quotient spaces.

Example 2.10. *Reduced holonomy groupoid.* Let $\mathcal{G} = \mathrm{Hol}(M, \mathcal{F})$ be the holonomy groupoid of a foliation $\mathcal{F}$. Given a complete transversal $T \subset M$, consider the arrows whose source and target are in T. The reduced holonomy groupoid to T, $\mathcal{K} = \mathrm{Hol}_T(M, \mathcal{F})$, is defined as the groupoid whose manifold of objects is $K_0 = T$ and the manifold of arrows K_1 is given by the following pullback of manifolds:

$$\begin{tikzcd} K_1 \arrow[r] \arrow[d, "{(s,t)}"'] & G_1 \arrow[d, "{(s,t)}"] \\ T \times T \arrow[r, "i"] & M \times M \end{tikzcd}$$

The inclusion functor $\mathrm{Hol}_T(M, \mathcal{F}) \to \mathrm{Hol}(M, \mathcal{F})$ is an essential equivalence. For a given foliation, all the reduced holonomy groupoids to complete transversals are Morita equivalent.

The notion of groupoid homotopy to be invariant of Morita equivalence will be introduced in a more general context of Hilsum-Skandalis or generalized maps. We review some general constructions for Lie groupoids to be used in the definition of these maps.

2.3. **Weak pullbacks.** Given the morphisms of groupoids $\psi : \mathcal{K} \to \mathcal{G}$ and $\phi : \mathcal{L} \to \mathcal{G}$, the *weak pullback* $\mathcal{K} \times_{\mathcal{G}} \mathcal{L}$ is a groupoid whose space of objects is

$$(\mathcal{K} \times_{\mathcal{G}} \mathcal{L})_0 = K_0 \times_{G_0} G_1 \times_{G_0} L_0$$

consisting of triples (x, g, y) with $x \in K_0$, $y \in L_0$ and g an arrow in G_1 from $\psi(x)$ to $\phi(y)$. An arrow between (x, g, y) and (x', g', y') is a pair of arrows (k, l) with $k \in K_1$, $l \in L_1$ such that $g'\psi(k) = \phi(l)g$. The space of arrows can be identify with $(\mathcal{K} \times_{\mathcal{G}} \mathcal{L})_1 = K_1 \times_{G_0} G_1 \times_{G_0} L_1$.

If at least one of the two maps is a submersion on objects, then the weak pullback $\mathcal{K} \times_{\mathcal{G}} \mathcal{L}$ is a Lie groupoid. In this case, the diagram of Lie groupoids

$$\begin{tikzcd}
\mathcal{K} \times_{\mathcal{G}} \mathcal{L} \arrow[r, "p_1"] \arrow[d, "p_3"'] & \mathcal{K} \arrow[d, "\psi"] \\
\mathcal{L} \arrow[r, "\phi"] & \mathcal{G}
\end{tikzcd}$$

commutes up to a natural transformation and it is universal with this property.

Remark 2.11. If ϕ is an essential equivalence, then $\mathcal{K} \times_{\mathcal{G}} \mathcal{L}$ is a Lie groupoid and p_1 is an essential equivalence too.

2.4. **Actions of groupoids on manifolds.** Let M be a manifold, $\mathcal{G}$ a groupoid and $\mu : M \to G_0$ a smooth map. A *right action* of $\mathcal{G}$ on M is a map

$$M \times^t_{G_0} G_1 \to M, \qquad (x, g) \mapsto xg$$

defined on $M \times^t_{G_0} G_1$ given by the following pullback of manifolds along the target map:

$$\begin{tikzcd}
M \times^t_{G_0} G_1 \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[d, "\mu"] \\
G_1 \arrow[r, "t"] & G_0
\end{tikzcd}$$

such that $\mu(xg) = t(g)$, $x1 = x$, $(xg)h = x(gh)$.

Analogously, we have a *left action* by considering the pullback $G_1 \times^s_{G_0} M$ along the source map.

The *translation groupoid* $M \rtimes \mathcal{G}$ associated to a right action of $\mathcal{G}$ on M is given by $(M \rtimes \mathcal{G})_0 = M$ and $(M \rtimes \mathcal{G})_1 = M \times^t_{G_0} G_1$ where the source map is given by the action $s(x, g) = xg$ and the target map is just the projection $t(x, g) = x$.

Remark 2.12. There is a natural right action of $\mathcal{G}$ on the space of objects $(M \rtimes \mathcal{G})_0 = M$ given by the original action.

The *double translation groupoid* $\mathcal{K} \ltimes M \rtimes \mathcal{G}$ associated to a left action of $\mathcal{K}$ on M and a right action of $\mathcal{G}$ on M is given by $(\mathcal{K} \ltimes M \rtimes \mathcal{G})_0 = M$ and $(\mathcal{K} \ltimes M \rtimes \mathcal{G})_1 = K_1 \times^s_{K_0} M \times^t_{G_0} G_1$ where the space of arrows is obtained by the following pullbacks of manifolds:

$$\begin{tikzcd}
K_1 \times^s_{K_0} M \times^t_{G_0} G_1 \arrow[rr] \arrow[d] & & K_1 \arrow[d, "s"] \\
M \times^t_{G_0} G_1 \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[r, "\tau"] \arrow[d, "\rho"] & K_0 \\
G_1 \arrow[r, "t"] & G_0 &
\end{tikzcd}$$

then $K_1 \times^s_{K_0} M \times^t_{G_0} G_1 = \{(h, x, g) \mid s(h) = \tau(x) \text{ and } t(g) = \rho(x)\}$

Remark 2.13. There is a natural left action of $\mathcal{K}$ and right action of $\mathcal{G}$ on the space of objects $(\mathcal{K} \ltimes M \rtimes \mathcal{G})_0 = M$.

3. The bicategory $\mathfrak{M}$

3.1. **Bicategories.** A bicategory $\mathfrak{B}$ consists of a class of objects, morphisms between objects and 2-morphisms between morphisms together with various ways of composing them. We will picture the objects as points:

$$\bullet\mathcal{G}$$

the morphisms between objects as arrows:

$$\begin{tikzcd} \mathcal{K}\bullet \arrow[r, "\phi"] & \bullet\mathcal{G} \end{tikzcd}$$

and the 2-morphisms between morphisms as double arrows:

$$\begin{tikzcd} \mathcal{K}\bullet \arrow[r, bend left=60, "\phi"{name=U}] \arrow[r, bend right=60, "\psi"'{name=D}] & \bullet\mathcal{G} \arrow[Rightarrow, from=U, to=D, "a"'] \end{tikzcd}$$

Definition 3.1. A 2-morphism $a : \phi \Rightarrow \psi$ is a *2-equivalence* if it is invertible: i.e. if there exists a 2-morphism $b : \psi \Rightarrow \phi$ such that $ab = id_\psi$ and $ba = id_\phi$. In this case we will say that the morphisms ϕ and ψ are equivalent and write $\phi \sim \psi$.

Definition 3.2. A morphism $\varphi : \mathcal{K} \to \mathcal{G}$ is an *equivalence* if it is invertible up to a 2-equivalence: i.e. if there exists a morphism $\xi : \mathcal{G} \to \mathcal{K}$ such that $\varphi\xi \sim id_{\mathcal{G}}$ and $\xi\varphi \sim id_{\mathcal{K}}$. In this case we will say that the objects $\mathcal{K}$ and $\mathcal{G}$ are equivalent and write $\mathcal{K} \sim \mathcal{G}$.

The composition $\phi\varphi$ of morphisms $\varphi : \mathcal{L} \to \mathcal{K}$ and $\phi : \mathcal{K} \to \mathcal{G}$ is denoted by:

$$\begin{tikzcd} \mathcal{L}\bullet \arrow[r, "\varphi"] & \bullet \arrow[r, "\phi"] & \bullet\mathcal{G}. \end{tikzcd}$$

We can compose 2-morphisms in two ways called horizontal and vertical composition. The horizontal composition ab of 2-morphisms $b : \varphi \Rightarrow \varphi'$ and $a : \phi \Rightarrow \phi'$ is denoted by:

$$\begin{tikzcd} \mathcal{L}\bullet \arrow[r, bend left=60, "\varphi"{name=U1}] \arrow[r, bend right=60, "\varphi'"'{name=D1}] & \bullet \arrow[r, bend left=60, "\phi"{name=U2}] \arrow[r, bend right=60, "\phi'"'{name=D2}] & \bullet\mathcal{G}. \arrow[Rightarrow, from=U1, to=D1, "b"'] \arrow[Rightarrow, from=U2, to=D2, "a"'] \end{tikzcd}$$

The vertical composition $a \cdot b$ of 2-morphisms $b : \phi \Rightarrow \varphi$ and $a : \varphi \Rightarrow \psi$ is denoted by:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend left=60, "\phi"{name=U}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, "\psi"'{name=D}] & \mathcal{G}.
\arrow[Rightarrow, from=U, to=M, "b"'] \arrow[Rightarrow, from=M, to=D, "a"']
\end{tikzcd}$$

The following interchange law relates horizontal and vertical compositions: $(a \cdot b)(c \cdot d) = (ac) \cdot (bd)$

$$\begin{tikzcd}
\mathcal{L} \arrow[r, bend left=60, "\varphi"{name=U1}] \arrow[r, ""{name=M1}] \arrow[r, bend right=60, "\varphi''"'{name=D1}] & \bullet \arrow[r, bend left=60, "\phi"{name=U2}] \arrow[r, ""{name=M2}] \arrow[r, bend right=60, "\phi''"'{name=D2}] & \mathcal{G}.
\arrow[Rightarrow, from=U1, to=M1, "d"'] \arrow[Rightarrow, from=M1, to=D1, "c"'] \arrow[Rightarrow, from=U2, to=M2, "b"'] \arrow[Rightarrow, from=M2, to=D2, "a"']
\end{tikzcd}$$

Vertical composition is strictly associative whereas horizontal composition is only associative up to 2-equivalence (associator). The unit laws for morphisms hold up to 2-equivalences (left and right unit constrains). Associator and unit constrains are required to be natural with respect to their arguments and verify certain axioms [2].

A 2-category is a bicategory in which the natural 2-equivalences above are the identity. The category of Lie groupoids and functors can be seen as a 2-category $\mathfrak{G}$ with natural transformations as 2-morphisms. We will show two different constructions of a bicategory $\mathfrak{M} \supset \mathfrak{G}$ in which the essential equivalences are invertible.

3.2. **Hilsum-Skandalis maps.** These maps were introduced by Skandalis for holonomy groupoids [10] and by Moerdijk in the context of topos theory [16]. Mrčun studied these maps for general groupoids in his 1996 thesis [21], on which work this section is largely based.

A *right $\mathcal{G}$-bundle M* over B is a surjective submersion $\pi : M \to B$ with a right action of $\mathcal{G}$ on M preserving the fibers. A right $\mathcal{G}$-bundle is *principal* if the map

$$M \times^t_{G_0} G_1 \xrightarrow{\alpha} M \times_B M, \qquad \alpha(x, g) = (xg, x)$$

is a diffeomorphism.

A *$\mathcal{K}$-$\mathcal{G}$ bibundle M* is a left $\mathcal{K}$-bundle over G_0 as well as a right $\mathcal{G}$-bundle over K_0. We represent a $\mathcal{K}$-$\mathcal{G}$ bibundle by the following diagram:

$$\begin{tikzcd}
& M \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\
G_0 & & K_0
\end{tikzcd}$$

where ρ is a left $\mathcal{K}$-bundle and τ is a right $\mathcal{G}$-bundle. We denote a $\mathcal{K}$-$\mathcal{G}$ bibundle M as $(\mathcal{K}, M, \mathcal{G})$ and we write $(\mathcal{K}, \rho, M, \mathcal{G}, \tau)$ if we need to specify the submersions.

A $\mathcal{K}$-$\mathcal{G}$ bibundle M is *right principal* if the right $\mathcal{G}$-bundle $\tau : M \to K_0$ is principal. In this case, $M \times^t_{G_0} G_1$ is diffeomorphic to $M \times_{K_0} M$ and $M/\mathcal{G}$ is diffeomorphic to K_0. Analogously, left principal.

A $\mathcal{K}$-$\mathcal{G}$ bibundle M is *biprincipal* if both left and right bundles ρ and τ are principal.

Two $\mathcal{K}$-$\mathcal{G}$ bibundles M and N are *isomorphic* if there is a diffeomorphism $f : M \to N$ that intertwines the maps $M \to G_0$, $M \to K_0$ with the maps $N \to G_0$, $N \to K_0$ and also intertwines the $\mathcal{K}$ and $\mathcal{G}$ actions. In other words, $f(hxg) = hf(x)g$ and $\tau = \tau' f$, $\rho = \rho' f$. We write $(\mathcal{K}, M, \mathcal{G}) \sim (\mathcal{K}, N, \mathcal{G})$.

Definition 3.3. A Hilsum-Skandalis map $|(\mathcal{K}, M, \mathcal{G})|$ is an isomorphism class of right principal $\mathcal{K}$-$\mathcal{G}$ bibundles.

Proposition 3.4. [11] *The collection of all Lie groupoids as objects, right principal $\mathcal{K}$-$\mathcal{G}$ bibundles as morphisms and isomorphisms f as 2-morphisms is a bicategory.*

We will denote this bicategory by $\mathfrak{M}'$. All 2-morphisms in $\mathfrak{M}'$ are isomorphisms. For each groupoid $\mathcal{G}$ the unit arrow is defined as the $\mathcal{G}$-$\mathcal{G}$ bibundle

$$\begin{tikzcd}
 & G_1 \arrow[dl, "t"'] \arrow[dr, "s"] & \\
G_0 & & G_0
\end{tikzcd}$$

The left and right actions of $\mathcal{G}$ on G_1 are given by the multiplication in the groupoid $\mathcal{G}$.

The composition of arrows $(\mathcal{K}, M, \mathcal{G})$ and $(\mathcal{G}, N, \mathcal{L})$ is given by the bibundle $(\mathcal{K}, (M \times_{G_0} N)/\mathcal{G}, \mathcal{L})$ where $M \times_{G_0} N$ is given by the following pullback of manifolds:

$$\begin{tikzcd}
M \times_{G_0} N \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[d, "\rho_M"] \\
N \arrow[r, "\tau_N"] & G_0
\end{tikzcd}$$

In addition, $\mathcal{G}$ acts on the manifold $M \times_{G_0} N$ by the right: $(x, y)g = (xg, g^{-1}y)$. The orbit space is a $\mathcal{K}$-$\mathcal{L}$ bibundle:

$$\begin{tikzcd}
 & (M \times_{G_0} N)/\mathcal{G} \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\
L_0 & & K_0
\end{tikzcd}$$

where $\tau([x, y]) = \tau_M(x)$ and $\rho([x, y]) = \rho_N(y)$. The left $\mathcal{K}$-action is given by $k[x, y] = [kx, y]$ and the right $\mathcal{L}$-action by $[x, y]l = [x, yl]$. This bibundle is right principal.

This multiplication is associative up to isomorphism.

The unit arrow $(\mathcal{G}, G_1, \mathcal{G})$ is a left and right unit for this multiplication of arrows up to isomorphism. We have that the bibundle $(\mathcal{K}, (M \times^t_{G_0} G_1)/\mathcal{G}, \mathcal{G})$ is isomorphic to $(\mathcal{K}, M, \mathcal{G})$ since the map

$$(M \times^t_{G_0} G_1)/\mathcal{G} \xrightarrow{f} M, \qquad f([x,y]) = xy$$

is a diffeomorphism verifying $f(h[x,y]g) = hf[x,y]g$. Hence there is a 2-morphism f from $(\mathcal{K}, M, \mathcal{G})$ to the composition $(\mathcal{G}, G_1, \mathcal{G}) \circ (\mathcal{K}, M, \mathcal{G}) = (\mathcal{K}, (M \times^t_{G_0} G_1)/\mathcal{G}, \mathcal{G})$.

3.3. **Generalized maps.** The bicategory $\mathfrak{M}'$ can be obtained also as a bicategory of fractions [22, 6]. We describe in this section the generalized maps obtained by this approach.

A *generalized map* from $\mathcal{K}$ to $\mathcal{G}$ is a pair of morphisms

$$\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$$

such that ϵ is an essential equivalence. We denote a generalized map by (ϵ, ϕ). Roughly speaking, a generalized map from $\mathcal{K}$ to $\mathcal{G}$ is obtained by first replacing $\mathcal{K}$ by another groupoid $\mathcal{J}$ essentially equivalent to it and then mapping $\mathcal{J}$ into $\mathcal{G}$ by an ordinary morphism.

Two generalized maps from $\mathcal{K}$ to $\mathcal{G}$, $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{G}$, are *isomorphic* if there exists a groupoid $\mathcal{L}$ and essential equivalences

$$\mathcal{J} \xleftarrow{\alpha} \mathcal{L} \xrightarrow{\beta} \mathcal{J}'$$

such that the diagram

$$\begin{tikzcd} & \mathcal{J} \arrow[dl, "\epsilon"'] \arrow[dr, "\phi"] & \\ \mathcal{K} & \mathcal{L} \arrow[u, "\alpha"'] \arrow[d, "\beta"] & \mathcal{G} \\ & \mathcal{J}' \arrow[ul, "\epsilon'"] \arrow[ur, "\phi'"'] & \end{tikzcd}$$

commutes up to natural transformations. We write $(\epsilon, \phi) \sim (\epsilon', \phi')$.

In other words, there are natural transformations T and T' such that the generalized maps

$$\mathcal{K} \xleftarrow{\epsilon\alpha} \mathcal{L} \xrightarrow{\phi\alpha} \mathcal{G}$$

$$\mathcal{K} \xleftarrow{\epsilon'\beta} \mathcal{L} \xrightarrow{\phi'\beta} \mathcal{G}$$

verify $\epsilon\alpha \sim_T \epsilon'\beta$ and $\phi\alpha \sim_{T'} \phi'\beta$.

Remark 3.5. (1) If $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi'} \mathcal{G}$ are two generalized maps with $\phi \sim_T \phi'$ then $(\epsilon, \phi) \sim (\epsilon, \phi')$.

(2) If $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{G}$ are two generalized maps and $\delta : \mathcal{J}' \to \mathcal{J}$ an essential equivalence with $\phi' = \phi\delta$ and $\epsilon' = \epsilon\delta$ then $(\epsilon, \phi) \sim (\epsilon', \phi')$.

Proposition 3.6. *The collection of Lie groupoids as objects, generalized maps as morphisms and diagrams as 2-morphisms is a bicategory.*

We will denote this bicategory by $\mathfrak{M}$. All the 2-morphisms in $\mathfrak{M}$ are isomorphisms.

For each groupoid $\mathcal{G}$ the unit arrow (id, id) is defined as the generalized map $\mathcal{G} \xleftarrow{id} \mathcal{G} \xrightarrow{id} \mathcal{G}$. The composition of two arrows $(\mathcal{G} \xleftarrow{\delta} \mathcal{J}' \xrightarrow{\varphi} \mathcal{K}) \circ (\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G})$ is given by the generalized map:

$$\mathcal{K} \xleftarrow{\epsilon p_1} \mathcal{J} \times_{\mathcal{G}} \mathcal{J}' \xrightarrow{\varphi p_3} \mathcal{K}$$

where p_1 and p_3 are the projections in the following pullback of groupoids:

$$\begin{tikzcd}
\mathcal{J} \times_{\mathcal{G}} \mathcal{J}' \arrow[r, "p_1"] \arrow[d, "p_3"'] & \mathcal{J} \arrow[r, "\epsilon"] \arrow[d, "\phi"] & \mathcal{K} \\
\mathcal{J}' \arrow[r, "\delta"] \arrow[d, "\varphi"] & \mathcal{G} & \\
\mathcal{K} & &
\end{tikzcd}$$

The morphism p_1 is an essential equivalence since it is the pullback of the essential equivalence δ. Then ϵp_1 is an essential equivalence. This composition is associative up to isomorphism.

The unit arrow is a left and right unit for this multiplication of arrows up to isomorphism. The composition $(\mathcal{G} \xleftarrow{id} \mathcal{G} \xrightarrow{id} \mathcal{G}) \circ (\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G})$ is the generalized map $\mathcal{K} \xleftarrow{\epsilon p_1} \mathcal{J} \times_{\mathcal{G}} \mathcal{G} \xrightarrow{\varphi p_3} \mathcal{G}$. Since $\varphi = \delta = id$ implies that $\varphi p_3 = \phi p_1$ and p_1 is an essential equivalence. We have that $(\epsilon p_1, \phi p_1) \sim (\epsilon, \phi)$ by Remark 3.5 (2).

Considering the bicategory $\mathfrak{G}$ of Lie groupoids, functors and natural transformations, the bicategory $\mathfrak{M}$ is obtained as the bicategory of fractions of $\mathfrak{G}$ when inverting the essential equivalences E, $\mathfrak{M} = \mathfrak{G}[E^{-1}]$.

3.4. **The isomorphism between bicategories $\mathfrak{M}'$ and $\mathfrak{M}$.** We will show an explicit construction of a bijective correspondence between generalized maps and bibundles which will allow us to switch from one formulation to the other when needed. In addition $\mathfrak{M}'$ is biequivalent to $\mathfrak{M}$. This completes the construction given in [11], [12] and [19].

Recall that a homomorphism of bicategories $\mathfrak{B} \to \mathfrak{B}'$ is a generalization of the notion of a functor sending objects, morphisms and 2-morphisms of $\mathfrak{B}$ to items of the same types in $\mathfrak{B}'$ preserving compositions and units up to 2-equivalence [2].

Given a right principal $\mathcal{K}$-$\mathcal{G}$ bibundle M:

$$\begin{tikzcd}
 & M \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\
G_0 & & K_0
\end{tikzcd}$$

where ρ is a left $\mathcal{K}$-bundle and τ is a right $\mathcal{G}$-bundle we construct a generalized map

$$\mathcal{K} \overset{\epsilon}{\leftarrow} \mathcal{J} \overset{\phi}{\rightarrow} \mathcal{G}$$

by taking $\mathcal{J} = \mathcal{K} \ltimes M \rtimes \mathcal{G}$ and the following morphisms ϵ and ϕ:

$$M \overset{\epsilon_0}{\rightarrow} K_0, \qquad \epsilon_0 = \tau \quad \text{and} \quad K_1 \times^s_{K_0} M \times_{G_0} G_1 \overset{\epsilon_1}{\rightarrow} K_1, \qquad \epsilon_1 = p_1$$

$$M \overset{\phi_0}{\rightarrow} G_0, \qquad \phi_0 = \rho \quad \text{and} \quad K_1 \times^s_{K_0} M \times_{G_0} G_1 \overset{\phi_1}{\rightarrow} G_1, \qquad \phi_1 = p_3$$

since τ is a principal bundle, we have that ϵ is an essential equivalence.

We will show that if $(\mathcal{K}, M, \mathcal{G}) \sim (\mathcal{K}, N, \mathcal{G})$ then the associated generalized maps (ϵ, ϕ) and (ϵ', ϕ') are isomorphic. Let $f : M \to N$ be the equivariant diffeomorphism that intertwines the bundles. Define

$$\bar{f} : \mathcal{K} \ltimes M \rtimes \mathcal{G} \to \mathcal{K} \ltimes N \rtimes \mathcal{G}$$

by $\bar{f}_0 = f$ on objects and $\bar{f}_1(h, x, g) = hxg$ on arrows. These maps commute with all the structural maps by the equivariance of f. Since $\bar{f}_0$ is a diffeomorphism, it is in particular a surjective submersion and the manifold of arrows $K_1 \times^s_{K_0} M \times_{G_0} G_1$ is obtained from the following pullback of manifolds:

$$\begin{tikzcd}
K_1 \times^s_{K_0} M \times_{G_0} G_1 \arrow[r, "\bar{f}_1"] \arrow[d, "(s,t)"'] & K_1 \times^s_{K_0} N \times_{G_0} G_1 \arrow[d, "(s,t)"] \\
M \times M \arrow[r, "\bar{f}_0 \times \bar{f}_0"] & N \times N
\end{tikzcd}$$

Then $\bar{f}$ is an essential equivalence. Also, as f intertwines the bundles, we have that $\phi' = \phi\bar{f}$ and $\epsilon' = \epsilon\bar{f}$ and by Remark 3.5 (2) follows that $(\epsilon, \phi) \sim (\epsilon', \phi')$.

We can define a morphism of bicategories by

$$\Gamma : \mathfrak{M}' \to \mathfrak{M}, \qquad \Gamma((\mathcal{K}, M, \mathcal{G})) = (\epsilon, \phi)$$

as constructed above on morphisms and being the identity on objects.

For 2-morphisms $f : M \to N$, we define $\Gamma(f)$ as the following diagram:

$$\begin{tikzcd}
& \mathcal{K} \ltimes M \rtimes \mathcal{G} \arrow[dl, "\epsilon"'] \arrow[dr, "\phi"] & \\
\mathcal{K} & \mathcal{K} \ltimes M \rtimes \mathcal{G} \arrow[u, "id"'] \arrow[d, "\beta"] & \mathcal{G} \\
& \mathcal{K} \ltimes N \rtimes \mathcal{G} \arrow[ul, "\epsilon'"] \arrow[ur, "\phi'"'] &
\end{tikzcd}$$

where $\beta : \mathcal{K} \ltimes M \rtimes \mathcal{G} \to \mathcal{K} \ltimes N \rtimes \mathcal{G}$ is defined by $\beta(x) = f(x)$ on objects and $\beta(h, x, g) = (h, f(x), g)$ on arrows. Since $\tau' f = \tau$ and $\rho' f = \rho$ we have that $s(h) = \tau'(f(x))$ and $t(g) = \rho'(f(x))$.

Conversely, given a generalized map from $\mathcal{K}$ to $\mathcal{G}$

$$\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$$

we construct an associated right principal $\mathcal{K}$-$\mathcal{G}$ bibundle M

$$\begin{tikzcd}
& M \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\
G_0 & & K_0
\end{tikzcd}$$

where M is obtained as the quotient by the action of $\mathcal{J}$ on $\tilde{M} = K_0 \times^t_{K_0} K_1 \times^t_{G_0} G_1$ given by the following pullbacks of manifolds:

$$\begin{tikzcd}
K_0 \times^t_{K_0} K_1 \times_{K_0} K_0 \times^t_{G_0} G_1 \arrow[r] \arrow[d] \arrow[rr, bend left, "p_2"] \arrow[dd, bend right, "p_4"'] & K_0 \times_{K_0} K_1 \arrow[r] \arrow[d] & K_1 \arrow[r, "s"] \arrow[d, "t"] & K_0 \\
K_0 \times_{G_0} G_1 \arrow[r] \arrow[d] & K_0 \arrow[r, "\epsilon"] \arrow[d, "\phi"] & K_0 & \\
G_1 \arrow[r, "t"] \arrow[d, "s"] & G_0 & & \\
G_0 & & &
\end{tikzcd}$$

The maps ρ and τ are induced in the quotient by $\tilde{\rho} = sp_4$ and $\tilde{\tau} = sp_2$. The action of $\mathcal{J}$ on $\tilde{M}$ is given by

$$\tilde{M} \times^t_{K_0} K_1 \to \tilde{M}, \qquad ((a, b, c, d), k) \mapsto (s(k), b\epsilon(k^{-1}), s(k), d\phi(k^{-1})).$$

The left action of $\mathcal{K}$ on $M = \tilde{M}/\mathcal{J}$ is given by

$$K_1 \times^s_{K_0} M \to M, \qquad (h, [a,b,c,d]) \mapsto [a, b^{-1}, c, d]$$

and the right action of $\mathcal{G}$ by

$$M \times^t_{G_0} G_1 \to M, \qquad ([a,b,c,d]), g) \mapsto [a,b,c,dg].$$

If $(\epsilon, \phi) \sim (\epsilon', \phi')$ then the associated bibundles $(\mathcal{K}, M, \mathcal{G})$ and $(\mathcal{K}, N, \mathcal{G})$ are isomorphic.

The isomorphism Γ preserves compositions and units up to 2-equivalence.

3.5. **Strict maps.** We will use now this isomorphism to characterize the generalized maps that come from an strict map.

Any strict morphism $\phi : \mathcal{K} \to \mathcal{G}$ can be viewed as a generalized map by

$$\mathcal{K} \xleftarrow{id} \mathcal{K} \xrightarrow{\phi} \mathcal{G}.$$

The corresponding bibundle is constructed by taking

$$M = (K_1 \times^t_{K_0} K_1 \times^t_{G_0} G_1)/\mathcal{K}.$$

Considering the following identifications:

$$(K_0 \times^t_{K_0} K_1 \times_{K_0} K_0 \times^t_{G_0} G_1)/\mathcal{K} = K_0 \times_{K_0} K_0 \times^t_{G_0} G_1 = K_0 \times^t_{G_0} G_1$$

$$[a,b,c,d] \mapsto (b,c,d) \mapsto (c,d)b = (s(b), d\phi(b^{-1}))$$

this bibundle is isomorphic to

$$\begin{tikzcd} & K_0 \times^t_{G_0} G_1 \arrow[dl, "sp_2"'] \arrow[dr, "p_1"] & \\ G_0 & & K_0 \end{tikzcd}$$

Proposition 3.7. [12] *Let $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ be a generalized map and $(\mathcal{K}, \rho, M, \mathcal{G}, \tau)$ the associated right principal bibundle. Then ϕ is an essential equivalence iff ρ is principal.*

Remark 3.8. Essential equivalences correspond to *biprincipal* bibundles.

In both bicategories $\mathfrak{M}$ and $\mathfrak{M}'$, Morita equivalences are the invertible morphisms (equivalences in Definition 3.2). If $\mathcal{K} \sim_M \mathcal{G}$, let $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\delta} \mathcal{G}$ be the associated generalized map in $\mathfrak{M}$ with ϵ and δ essential equivalences, then the inverse generalized map is $\mathcal{G} \xleftarrow{\delta} \mathcal{J} \xrightarrow{\epsilon} \mathcal{K}$.

In the category $\mathfrak{M}'$, let

$$\begin{tikzcd} & M \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\ G_0 & & K_0 \end{tikzcd}$$

be the biprincipal $\mathcal{K}$-$\mathcal{G}$ bibundle representing the Morita equivalence, then the inverse biprincipal $\mathcal{G}$-$\mathcal{K}$ bibundle is

$$\begin{tikzcd}
& M \arrow[dl, "\tau"'] \arrow[dr, "\rho"] & \\
K_0 & & G_0
\end{tikzcd}$$

where the new actions are obtained from the original ones composing with the inverse: the left action of $\mathcal{G}$ on M is given by $g * x = xg^{-1}$ induced by the right action of $\mathcal{G}$ on the original bibundle. Similarly, the left action of $\mathcal{K}$ induces a right action in the inverse bundle.

Proposition 3.9. *Let* $(\mathcal{K}, \rho, M, \mathcal{G}, \tau)$ *be a right principal* $\mathcal{K}$-$\mathcal{G}$ *bibundle and* $(\epsilon, \varphi) = \Gamma((\mathcal{K}, M, \mathcal{G}))$ *its associated generalized map. Then* $(\epsilon, \varphi) \sim (\mathit{id}, \phi)$ *if and only if* τ *has a section.*

In other words, a generalized map comes from a strict map iff when seen it as a bibundle, the right principal $\mathcal{G}$-bundle has a section.

If a strict map $\epsilon : \mathcal{K} \to \mathcal{G}$ is an essential equivalence, regarded as a generalized map it will be invertible. The inverse of a generalized (strict) map

$$\mathcal{K} \overset{id}{\leftarrow} \mathcal{J} \overset{\epsilon}{\rightarrow} \mathcal{G}$$

is the generalized map $\mathcal{G} \overset{\epsilon}{\leftarrow} \mathcal{J} \overset{id}{\rightarrow} \mathcal{K}$ which will not always come from a strict map.

Example 3.10. Consider the holonomy groupoid associated to the Seifert fibration on the Möbius band, $\mathcal{G} = \mathrm{Hol}(M, \mathcal{F}_S)$, and the reduced holonomy groupoid $\mathcal{K} = \mathrm{Hol}_T(M, \mathcal{F}_S)$ to a transversal T given by an interval I transversal to the leaves. The inclusion functor $i_{\mathcal{K}} : \mathcal{K} \hookrightarrow \mathcal{G}$ is an essential equivalence. The associated $\mathcal{K}$-$\mathcal{G}$ right principal bibundle is

$$\begin{tikzcd}
& I \times^t_M (M \times S^1) = I \times S^1 \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\
M & & I \arrow[ul, bend right]
\end{tikzcd}$$

which has a section since it comes from the strict map $i_{\mathcal{K}}$.

The inverse $\mathcal{G}$-$\mathcal{K}$ bibundle is

$$\begin{tikzcd}
& I \times^t_M (M \times S^1) = I \times S^1 \arrow[dl, "\tau"'] \arrow[dr, "\rho"] & \\
I & & M
\end{tikzcd}$$

where the map $\rho : I \times S^1 \to M$ is the double covering of the Möbius band by an annulus, which does not have a section. So the corresponding generalized map is not isomorphic to a strict map. This example shows that the strict map $i_{\mathcal{K}}$ has an inverse which is not an strict map. The groupoids $\mathcal{K}$ and $\mathcal{G}$ are Morita equivalent but they are not equivalent by a natural transformation.

4. The fundamental groupoid of a groupoid

We first recall the notions of equivalence and homotopy of $\mathcal{G}$-paths due to Haefliger [7, 9]. Let $\mathcal{G}$ be a Lie groupoid and x, y objects in G_0. A *$\mathcal{G}$-path* from x to y over a subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$ is a sequence:

$$(g_0, \alpha_1, g_1, \ldots, \alpha_n, g_n)$$

where

(1) $\alpha_i : [t_{i-1}, t_i] \to G_0$ is a path for all $1 \leq i \leq n$ and
(2) $g_i \in G_1$ is an arrow such that
  $s(g_0) = x$ and $t(g_n) = y$
  $s(g_i) = \alpha_i(t_i)$ for all $0 < i \leq n$
  $t(g_i) = \alpha_{i+1}(t_i)$ for all $0 \leq i < n$

$$\begin{tikzcd}
x\,\bullet \arrow[r, "g_0"] & \bullet \arrow[r, squiggly, "\alpha_1"] & \bullet \arrow[r, "g_1"] & \bullet & \cdots & \bullet \arrow[r, squiggly, "\alpha_n"] & \bullet \arrow[r, "g_n"] & \bullet\, y
\end{tikzcd}$$

We define an *equivalence* relation $\sim$ among $\mathcal{G}$-paths generated by the following operations:

(1) Add a new point $s \in [t_{i-1}, t_i]$ to the subdivision, take the restrictions α_i' and α_i'' of the corresponding path α_i to the new intervals $[t_{i-1}, s]$ and $[s, t_i]$ and add the identity arrow $1_{\alpha(s)}$

$$\begin{tikzcd}
\bullet \arrow[r, squiggly, "\alpha_i"] & \bullet & \sim & \bullet \arrow[r, squiggly, "\alpha_i'"] & \bullet \arrow[loop above] \arrow[r, squiggly, "\alpha_i''"] & \bullet
\end{tikzcd}$$

(2) Given a map $h_i : [t_{i-1}, t_i] \to G_1$ with $s \circ h_i = \alpha_i$, replace:
  α_i by $t \circ h_i$
  g_{i-1} by $h_i(t_{i-1})g_{i-1}$ and
  g_i by $g_i(h_i(t_i))^{-1}$

$$\begin{tikzcd}
\bullet \arrow[r, "g_{i-1}"] & \bullet \arrow[r, squiggly, "\alpha_i"] & \bullet \arrow[r, "g_i"] & \bullet & \sim & \bullet \arrow[r, "g_{i-1}"] & \bullet \arrow[r, squiggly, "\alpha_i"] \arrow[d, "h_i(t_{i-1})"'] & \bullet \arrow[r, "g_i"] \arrow[d, "h_i(t_i)"] & \bullet \\
 & & & & & & \bullet \arrow[r, squiggly, "t\circ h_i"] & \bullet &
\end{tikzcd}$$

A *deformation* between the $\mathcal{G}$-paths $(g_0, \alpha_1, g_1, \ldots, \alpha_n, g_n)$ and $(g_0', \alpha_1', g_1', \ldots, \alpha_n', g_n')$ from x to y is given by homotopies

$$H_i : [t_{i-1}, t_i] \times I \to G_0 \text{ with } (H_i)_0 = \alpha_i \text{ and } (H_i)_1 = \alpha_i'$$

for $i = 1, \ldots, n$ and

$$\gamma_i : I \to G_1 \text{ with } (\gamma_i)_0 = g_i \text{ and } (\gamma_i)_1 = g_i'$$

for $i = 1, \dots, n-1$, such that $(g_0, (H_1)_s, (\gamma_1)_s, \dots, (\gamma_{n-1})_s, (H_n)_s, g_n)$ is a $\mathcal{G}$-path for each $s \in I$

$$\begin{tikzcd}
\bullet \arrow[r, "g_{i-1}"] & \bullet \arrow[r, squiggly, "\alpha_i"] & \bullet \arrow[r, "g_i"] & \bullet \\
 & \vdots & & \\
\bullet \arrow[r, "(\gamma_{i-1})_s"] & \bullet \arrow[r, squiggly, "(H_i)_s"] & \bullet \arrow[r, "(\gamma_i)_s"] & \bullet
\end{tikzcd}$$

Definition 4.1. Two $\mathcal{G}$-paths between x and y are *homotopic* if one can be obtained from the other by a sequence of equivalences and deformations.

We define a multiplication of the homotopy classes of the $\mathcal{G}$-paths by

$$[(g'_0, \alpha'_1, \dots, \alpha'_n, g'_n)]\,[(g_0, \alpha_1, \dots, \alpha_n, g_n)] = [g_0, \alpha_1, \dots, \alpha_n, g'_0 g_n, \alpha'_1, \dots, \alpha'_n, g'_n]$$

where $g'_0 g_n$ is the multiplication of two composable arrows in $\mathcal{G}$ and the paths α_i are reparametrized to the new subdivision.

The inverse of the homotopy class of the $\mathcal{G}$-path $[(g_0, \alpha_1, \dots, \alpha_n, g_n)]$ from x to y is the class of the $\mathcal{G}$-path from y to x

$$[(g_k^{-1}, \alpha'_1, \dots, g_1^{-1}, \alpha'_k, g_0^{-1})]$$

over the same subdivision where $\alpha'_i : [t_{i-1}, t_i] \to G_0$ is given by

$$\alpha'_i(t) = \alpha_{k-i+1}\left(t_{k-i+1} + \left(\frac{t_{k-i} - t_{k-i+1}}{t_{i-1} - t_i}\right)(t_{i-1} - t)\right)$$

Definition 4.2. [19] The fundamental groupoid $\pi_1(\mathcal{G})$ of the Lie groupoid $\mathcal{G}$ is a groupoid over G_0 whose arrows are the homotopy classes of $\mathcal{G}$-paths with the multiplication defined above.

We will denote the fundamental groupoid $\pi_1(\mathcal{G})$ also by $\mathcal{G}_*$.

Proposition 4.3. [19] *The fundamental groupoid $\mathcal{G}_*$ of a Lie groupoid $\mathcal{G}$ is a Lie groupoid.*

A morphism $\phi : \mathcal{K} \to \mathcal{G}$ of Lie groupoids induces a morphism $\phi_* : \mathcal{K}_* \to \mathcal{G}_*$ between the fundamental groupoids given by $\phi_* = \phi$ on objects and

$$\phi_*([g_0, \alpha_1, g_1, \dots, \alpha_n, g_n]) = [\phi(g_0), \phi \circ \alpha_1, \phi(g_1), \dots, \phi \circ \alpha_n, \phi(g_n)]$$

on arrows.

Let $i_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}_*$ be the identity on objects and $i_{\mathcal{K}}(g) = [g]$ on arrows. We have the following commutative diagram of morphisms of Lie groupoids:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "\phi"] \arrow[d, "i_{\mathcal{K}}"] & \mathcal{G} \arrow[d, "i_{\mathcal{G}}"] \\
\mathcal{K}_* \arrow[r, "\phi_*"] & \mathcal{G}_*
\end{tikzcd}$$

The Haefliger's fundamental group of a groupoid [3, 8, 7, 9] at x_0 , $\pi_1(\mathcal{G}, x_0)$, coincides with the isotropy group $(\mathcal{G}_*)_{x_0}$ at x_0 of the fundamental groupoid $\mathcal{G}_*$.

Proposition 4.4. [19]

(1) *If $\epsilon : \mathcal{K} \to \mathcal{G}$ is an essential equivalence, then $\epsilon_* : \mathcal{K}_* \to \mathcal{G}_*$ is an essential equivalence as well.*
(2) *If $\mathcal{K} \sim_M \mathcal{G}$ then $\mathcal{K}_* \sim_M \mathcal{G}_*$ and the fundamental groups are isomorphic.*
(3) *The fundamental groupoid $\mathcal{G}_{**}$ of $\mathcal{G}_*$ is isomorphic to $\mathcal{G}_*$.*

5. The groupoid homotopy

5.1. **The bicategory $\mathfrak{H}$.** Consider the category of Lie groupoids and functors. We are now ready to introduce a notion of homotopy as an arrow between functors.

Definition 5.1. The morphisms $\phi : \mathcal{K} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{G}$ are *homotopic* if their induced morphisms ϕ_* and ψ_* are equivalent by a natural transformation in the fundamental groupoids.

Since a natural transformation of ϕ_* in ψ_* associates to each object x in $(G_*)_0 = G_0$ an arrow $g_x = [g_0, \alpha_1, g_1, \ldots, \alpha_n, g_n]$ in $(G_*)_1$ from $\phi(x)$ to $\psi(x)$, this notion of homotopy corresponds to the intuitive idea of continuously deforming ϕ into ψ by morphisms from $\mathcal{K}$ to $\mathcal{G}$ along $\mathcal{G}$-paths.

We have that $\phi \simeq_H \psi$ if $\phi_* \sim_a \psi_*$ where a is a natural transformation:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend left=60, "\phi"{name=U}] \arrow[r, bend right=60, "\psi"'{name=D}] \arrow[d] & \mathcal{G} \arrow[d] \\
\mathcal{K}_* \arrow[r, bend left=60, "\phi_*"{name=U2}] \arrow[r, bend right=60, "\psi_*"'{name=D2}] & \mathcal{G}_*
\arrow[Rightarrow, from=U, to=D, "H"']
\arrow[Rightarrow, from=U2, to=D2, "a"']
\end{tikzcd}$$

Lie groupoids, functors and homotopies as 2-morphisms form a bicategory $\mathfrak{H}$. All the 2-morphisms in $\mathfrak{H}$ are isomorphisms.

Horizontal and vertical composition of 2-morphisms are given by the horizontal and vertical composition of natural transformations, $a_* b_*$ and $a_* \cdot b_*$ respectively.

This notion of homotopy generalizes the concepts of natural transformation and strict homotopy, we have the following

Proposition 5.2. *Let $\phi : \mathcal{K} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{G}$ be morphisms of Lie groupoids.*

(1) *If $\phi \sim_T \psi$ where T is a natural transformation, then there is a 2-morphism $H : \phi \Rightarrow \psi$.*
(2) *If $\phi \simeq_F \psi$ where F is a strict homotopy, then there is a 2-morphism $H : \phi \Rightarrow \psi$.*

Proof. We will construct in each case a natural transformations $a : K_0 \to G_{1*}$ verifying $\psi([g_0, \alpha_1, g_1, \dots, \alpha_n, g_n])a(x) = a(y)\phi([g_0, \alpha_1, g_1, \dots, \alpha_n, g_n]$.

(1) If $T : K_0 \to G_1$ is a natural transformation with $T(x) : \phi(x) \to \psi(x)$ an arrow in G_1, define a natural transformation $a : K_0 \to G_{1*}$ by $a(x) = [T(x)]$. We have that $a(x)$ is an arrow in G_{1*} from $s(T(x)) = \phi(x)$ to $t(T(x)) = \psi(x)$ verifying the required equality.

(2) A strict homotopy $F : \mathcal{K} \times I \to \mathcal{G}$ with $F_0 = \phi$ and $F_1 = \psi$ determines for each $x \in K_0$ a path $F_x : I \to G_0$ from $\phi(x)$ to $\psi(x)$. Define $a : K_0 \to G_{1*}$ by $a(x) = [1_{\phi(x)}, F_x, 1_{\psi(x)}]$

$$\begin{tikzcd}
\bullet \arrow[loop above] \arrow[r, squiggly, "F_x"] & \bullet \arrow[loop above]
\end{tikzcd}$$

□

A *homotopy equivalence* is a morphism $\phi : \mathcal{K} \to \mathcal{G}$ such that there exists another morphism $\varphi : \mathcal{G} \to \mathcal{K}$ and 2-morphisms $\phi\varphi \Rightarrow \mathit{id}_{\mathcal{G}}$ and $\varphi\phi \Rightarrow \mathit{id}_{\mathcal{K}}$. We will say that two groupoids $\mathcal{K}$ and $\mathcal{G}$ have the same *homotopy type* if they are equivalent in the bicategory $\mathfrak{H}$. But this notion of homotopy *is not* invariant of Morita equivalence. We need to add more morphisms and 2-morphisms to the bicategory $\mathfrak{H}$ and define our notion of homotopy in an extended bicategory.

We can characterize the homotopy equivalences in terms of the morphisms that induce a (strong) equivalence between the fundamental groupoids. Recall that $\mathfrak{G}$ is the bicategory of Lie groupoids, functors and natural transformations.

Proposition 5.3. *If $\phi : \mathcal{K} \to \mathcal{G}$ is a homotopy equivalence in $\mathfrak{H}$, then $\phi_* : \mathcal{K}_* \to \mathcal{G}_*$ is an equivalence in $\mathfrak{G}$.*

Proof. We have the following diagram:

$$\begin{tikzcd}
& \bullet \arrow[dr, bend left, "\varphi"] \arrow[d, Rightarrow, "H"'] & \\
\mathcal{K} \arrow[ur, bend left, "\phi"] \arrow[rr, "\mathit{id}_{\mathcal{K}}"'] & {} & \mathcal{K}.
\end{tikzcd}$$

then $(\varphi\phi)_* \sim_a (\mathit{id}_{\mathcal{K}})_*$ where $a : K_0 \to (K_*)_1$ is a natural transformation. Since $(\varphi\phi)_* = \varphi_*\phi_*$ and $(\mathit{id}_{\mathcal{K}})_* = \mathit{id}_{\mathcal{K}_*}$, we have that $\varphi_*\phi_* \sim_a \mathit{id}_{\mathcal{K}_*}$. In the same way $\phi_*\varphi_* \sim_b \mathit{id}_{\mathcal{G}_*}$ and ϕ_* is an equivalence. □

Consider a 2-functor $\pi : \mathfrak{G} \to \mathfrak{G}$ between bicategories given by $\pi(\mathcal{G}) = \mathcal{G}_*$, $\pi(\phi) = \phi_*$ and $\pi(T) = T_*$, where $T_* : \phi_* \Rightarrow \psi_*$ is a natural transformation defined in the following way. For each $x \in (K_*)_0 = K_0$, we define $T_*(x) : \phi(x) \to \psi(x)$ as the arrow in $\mathcal{G}_*$ given by $T_*(x) = [T(x)]$. This arrow verifies the equality $\psi([g_0, \alpha_1, g_1, \dots, \alpha_n, g_n])T_*(x) = T_*(y)\phi([g_0, \alpha_1, g_1, \dots, \alpha_n, g_n]$. Then two morphisms ϕ and ψ are homotopic if their images by π are equivalent. All the (strong) equivalences in $\mathfrak{G}$ are homotopy equivalences in $\mathfrak{H}$, but there are homotopy equivalences that do not come from (strong) equivalences.

Now we introduce the *essential homotopy equivalences* as the morphisms that induce an essential equivalence between the fundamental groupoids.

Definition 5.4. A morphism $\phi : \mathcal{K} \to \mathcal{G}$ is an essential homotopy equivalence if $\phi_* : \mathcal{K}_* \to \mathcal{G}_*$ is an essential equivalence.

In this case, ϕ_* defines an isomorphism between fundamental groups.

Remark 5.5. If $\epsilon : \mathcal{K} \to \mathcal{G}$ is an essential equivalence, then ϵ is an essential homotopy equivalence since it induces an essential equivalence between fundamental groupoids by Proposition 4.4(1).

Let E be the set of essential equivalences and W the set of essential homotopy equivalences.

Proposition 5.6. *Every (strong) equivalence is a essential homotopy equivalence.*

Proof. Every strong equivalence is an essential equivalence [18] and $E \subset W$ by Remark 5.5. □

Proposition 5.7. *Every homotopy equivalence is a essential homotopy equivalence*

Proof. By Proposition 5.3, we have that $\phi_* : \mathcal{K}_* \to \mathcal{G}_*$ is an equivalence. Then ϕ_* is an essential equivalence. □

Strong equivalences ⇒ Essential equivalences ⇒ Essential homotopy equivalences
Strong equivalences ⇒ Homotopy equivalences ⇒ Essential homotopy equivalences.

Definition 5.8. Two Lie groupoids $\mathcal{K}$ and $\mathcal{G}$ are *weakly homotopic* if there exist essential homotopy equivalences:

$$\mathcal{K} \overset{\omega}{\leftarrow} \mathcal{L} \overset{\theta}{\rightarrow} \mathcal{G}$$

for a third Lie groupoid $\mathcal{L}$.

This defines an equivalence relation that we denote by $\simeq_W$.

5.2. **Bicategories of fractions.** Given a bicategory $\mathfrak{B}$ and a subset $S \subset B_1$ of morphisms verifying certain conditions, there exists a bicategory $\mathfrak{B}(S^{-1})$ having the same objects as $\mathfrak{B}$ but inverse morphisms of morphisms in S have been added as well as more 2-morphisms. This new bicategory is called a bicategory of fractions of $\mathfrak{B}$ with respect to S and was constructed by Pronk in [22]. The bicategory of fractions $\mathfrak{B}(S^{-1})$ is characterized by the universal property that any homomorphism $F : \mathfrak{B} \to \mathfrak{D}$ sending elements of S into equivalences factors in a unique way as $F = \tilde{F} \circ U$

$$\begin{tikzcd}
\mathfrak{B} \arrow[r, "U"] \arrow[d, "F"] & \mathfrak{B}(S^{-1}) \arrow[dl, "\tilde{F}"] \\
\mathfrak{D} &
\end{tikzcd}$$

The following conditions are needed on S to admit a bicalculus of right fractions [22]:

BF1 All equivalences are in S.
BF2 If ϕ and ψ are in S, then $\phi\psi \in S$
BF3 For all $\epsilon : \mathcal{J} \to \mathcal{G}$ and $\phi : \mathcal{K} \to \mathcal{G}$ with $\epsilon \in S$ there exists an object $\mathcal{P}$ and morphisms $\delta : \mathcal{P} \to \mathcal{K}$ and $\psi : \mathcal{P} \to \mathcal{J}$ with $\delta \in S$ such that the following square commutes up to a 2-equivalence:

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\psi"] \arrow[d, "\delta"] & \mathcal{J} \arrow[d, "\epsilon"] \\
\mathcal{K} \arrow[r, "\phi"] & \mathcal{G}
\end{tikzcd}$$

BF4 If $H : \eta\phi \Rightarrow \eta\varphi$ is a 2-morphism with $\eta \in S$ then there exists a morphism $\epsilon \in S$ and a 2-morphism $G : \phi\epsilon \Rightarrow \varphi\epsilon$ such that $H{\cdot}\epsilon = \eta{\cdot}G$.
BF5 If there is a 2-equivalence $\epsilon \Rightarrow \delta$ with $\delta \in S$ then $\epsilon \in S$.

We will prove now that the set of essential homotopy equivalences W allows a bicalculus of right fractions.

Conditions BF1 and BF2 follow from Proposition 5.7 and from the fact that the set of essential equivalences is closed under composition. To prove BF3 we start by defining the *weak homotopy pullback* $\mathcal{P}$ of the morphisms

$$\begin{tikzcd}
 & \mathcal{J} \arrow[d, "\epsilon"] \\
\mathcal{K} \arrow[r, "\phi"] & \mathcal{G}
\end{tikzcd}$$

as follows. Objects are triples $(x, [g_0, \alpha_1, \dots, \alpha_n, g_n], y)$ where $x \in J_0$, $y \in K_0$ and $[g_0, \alpha_1, \dots, \alpha_n, g_n]$ is a $\mathcal{G}$-path from $\epsilon(x)$ to $\phi(y)$. Arrows in $\mathcal{P}$ from $(x, [g_0, \alpha_1, \dots, \alpha_n, g_n], y)$ to $(x', [g'_0, \alpha'_1, \dots, \alpha'_n, g'_n], y')$ are pairs (j, k) of arrows $j \in J_1$ and $k \in K_1$ such that

$$[g'_0, \alpha'_1, \dots, \alpha'_n, g'_n]\epsilon(j) = \phi(k)[g_0, \alpha_1, \dots, \alpha_n, g_n]$$

In other words, $\mathcal{P}$ is the weak pullback $\mathcal{J} \times_{\mathcal{G}_*} \mathcal{K}$ of the morphisms $\phi_* i_{\mathcal{K}}$ and $\epsilon_* i_{\mathcal{J}}$. Since ϵ_* is an essential equivalence and $i_{\mathcal{J}}$ is the identity on objects we can assume that $\epsilon_* i_{\mathcal{J}}$ is a submersion on objects and $\mathcal{P}$ is a Lie groupoid. The square

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "p_1"] \arrow[d, "p_3"] & \mathcal{J} \arrow[d, "\epsilon"] \\
\mathcal{K} \arrow[r, "\phi"] & \mathcal{G}
\end{tikzcd}$$

does not necessarily commute but it does when taken the induced morphisms in the fundamental groupoids. Consider the weak pullback $\mathcal{J}_* \times_{\mathcal{G}_*} \mathcal{K}_*$ of groupoids

$$\begin{tikzcd}
\mathcal{J}_* \times_{\mathcal{G}_*} \mathcal{K}_* \arrow[r, "\pi_1"] \arrow[d, "\pi_3"] & \mathcal{J}_* \arrow[d, "\epsilon_*"] \\
\mathcal{K}_* \arrow[r, "\phi_*"] & \mathcal{G}_*
\end{tikzcd}$$

where $\epsilon_*\pi_1 \sim \phi_*\pi_3$ and π_3 is an essential equivalence. We have that $\mathcal{J}_* \times_{\mathcal{G}_*} \mathcal{K}_* = \mathcal{P}_*$, $\pi_1 = p_{1*}$ and $\pi_3 = p_{3*}$, then $\mathcal{P}_*$ is the weak pullback of ϵ_* and ϕ_*. Since the weak pullback square commutes up to natural transformation, we have that there is a 2-morphism $H : \epsilon p_1 \Rightarrow \phi p_3$ with $p_3 \in W$.

If $\phi, \psi : \mathcal{K} \to \mathcal{G}$ are morphisms, $\eta : \mathcal{J} \to \mathcal{K}$ is an essential homotopy equivalence and $H : \eta\phi \Rightarrow \eta\varphi$ is a 2-morphism, we have that η_* is an essential equivalence. Then there exist a natural transformation b [16] with $\phi_* \sim_b \psi_*$ and $\eta_* b = a$, where a is the natural transformation between $\eta_*\phi_*$ and $\eta_*\varphi_*$. The natural transformation b defines a 2-morphisms $G : \phi \Rightarrow \psi$ such that $H = \eta G$ and condition BF4 follows.

Finally, if $\epsilon \Rightarrow \delta$ is a 2-equivalence, then there is a natural transformation between ϵ_* and δ_*. Since δ is an essential homotopy equivalence, δ_* is an essential equivalence and this implies that ϵ_* is an essential equivalence as well. Then, ϵ is an essential homotopy equivalence.

Therefore, there exists a bicategory of fractions $\mathfrak{H}(W^{-1})$ inverting the essential homotopy equivalences.

5.3. **The bicategory $\mathfrak{H}(W^{-1})$.** The objects of $\mathfrak{H}(W^{-1})$ are Lie groupoids. The morphisms from $\mathcal{K}$ to $\mathcal{G}$ are formed by pairs (ω, ϕ)

$$\mathcal{K} \overset{\omega}{\leftarrow} \mathcal{J} \overset{\phi}{\rightarrow} \mathcal{G}$$

such that ω is an essential homotopy equivalence. We call *weak generalized maps* to the morphisms (ω, ϕ) in this bicategory.

The composition of weak generalized maps $(\mathcal{G} \overset{\omega'}{\leftarrow} \mathcal{J}' \overset{\phi'}{\rightarrow} \mathcal{L}) \circ (\mathcal{K} \overset{\omega}{\leftarrow} \mathcal{J} \overset{\phi}{\rightarrow} \mathcal{G})$ is given by a morphism

$$\mathcal{K} \xleftarrow{\omega p_3} \mathcal{P} \xrightarrow{\phi' p_1} \mathcal{L}$$

where $\mathcal{P}$ is the weak homotopic pullback of ω' and ϕ.

A 2-morphisms from (ω, ϕ) to (ω', ϕ') is given by the following diagram:

$$\begin{tikzcd}
& \mathcal{J} \arrow[dl, "\omega"'] \arrow[dr, "\phi"] & \\
\mathcal{K} \arrow[r, phantom, "\Downarrow H"] & \mathcal{L} \arrow[u, "u"'] \arrow[d, "v"] \arrow[r, phantom, "H' \Downarrow"] & \mathcal{G} \\
& \mathcal{J}' \arrow[ul, "\omega'"] \arrow[ur, "\phi'"'] &
\end{tikzcd}$$

where $\mathcal{L}$ is a Lie groupoid, u and v are essential homotopy equivalences and $H : \omega u \Rightarrow \omega' v$ and $H' : \phi u \Rightarrow \phi' v$ are 2-equivalences in $\mathfrak{H}$. The horizontal and vertical composition of diagrams are defined as in [22].

The notion of homotopy we propose corresponds to 2-morphisms in the bicategory $\mathfrak{H}(W^{-1})$.

Definition 5.9. Two weak generalized maps (ω, ϕ) and (ω', ϕ') are *homotopic* if there is a 2-morphism between them.

In this case, we write $(\omega, \phi) \simeq (\omega', \phi')$ and we say that there is a *groupoid homotopy* between (ω, ϕ) and (ω', ϕ').

In particular, when ω and ω' are essential equivalences, we have a notion of homotopy for generalized maps and when they are identities, we have a notion of homotopy for strict maps.

Two objects $\mathcal{K}$ and $\mathcal{G}$ are equivalent in $\mathfrak{H}(W^{-1})$ if there are morphisms (ω, ϕ) from $\mathcal{K}$ to $\mathcal{G}$ and (θ, ψ) from $\mathcal{G}$ to $\mathcal{K}$ such that $(\omega, \phi) \circ (\theta, \psi)$ is homotopic to the identity $(id_{\mathcal{G}}, id_{\mathcal{G}})$ and $(\theta, \psi) \circ (\omega, \phi) \simeq (id_{\mathcal{K}}, id_{\mathcal{K}})$.

Proposition 5.10. *A morphism* (ω, ϕ) *in* $\mathfrak{H}(W^{-1})$ *is invertible if and only if* ϕ *is an essential homotopy equivalence. In this case, the inverse of* (ω, ϕ) *is the morphism* (ϕ, ω)*.*

In other words, the definition of weak homotopy equivalence in subsection 5.1 amounts to isomorphism of objects in the bicategory $\mathfrak{H}(W^{-1})$. So, we write $\mathcal{K} \simeq_W \mathcal{G}$ for equivalence of objects in $\mathfrak{H}(W^{-1})$. The *groupoid homotopy type* (or weak homotopy type) of $\mathcal{G}$ is the class of $\mathcal{G}$ under the equivalence relation $\simeq_W$.

We show now that the groupoid homotopy type is invariant of Morita equivalence.

Proposition 5.11. *If* $\mathcal{K} \sim_M \mathcal{G}$ *then* $\mathcal{K} \simeq_W \mathcal{G}$*.*

Proof. If $\mathcal{K}$ and $\mathcal{G}$ are Morita equivalent, then there is Lie groupoid $\mathcal{L}$ and essential equivalences:

$$\mathcal{K} \xleftarrow{\epsilon} \mathcal{L} \xrightarrow{\delta} \mathcal{G}.$$

The morphisms ϵ and δ are also weak homotopy equivalences by Remark 5.5. Then the morphisms (ϵ, δ) and (δ, ϵ) are inverse and $\mathcal{K}$ is equivalent to $\mathcal{G}$ in the bicategory $\mathfrak{H}(W^{-1})$. □

The bicategory $\mathfrak{H}(W^{-1})$ comes equipped with a universal homomorphism $U : \mathfrak{H} \to \mathfrak{H}(W^{-1})$ [22] sending essential homotopy equivalences to equivalences and with the following universal property: any homomorphism of bicategories $F : \mathfrak{H} \to \mathfrak{B}$ sending essential homotopy equivalences to equivalences factors in a unique way as $F = G \circ U$ where $G : \mathfrak{H}(W^{-1}) \to \mathfrak{B}$ is a homomorphism of bicategories.

We have the following square of bicategories which relates in particular the notions of strong equivalence, Morita equivalence, homotopy equivalence and groupoid homotopy equivalence (weak homotopy equivalence):

$$\begin{tikzcd}
\mathfrak{G} \arrow[r, "U|_{\mathfrak{G}}"] \arrow[d, "i_{\mathfrak{G}}"'] & \mathfrak{G}[E^{-1}] \arrow[d, "i_{\mathfrak{G}[E^{-1}]}"] \\
\mathfrak{H} \arrow[r, "U"] & \mathfrak{H}[W^{-1}]
\end{tikzcd}$$

5.4. **The classifying space.** We will show that the homotopy groups of a groupoid defined in terms of the classifying space are invariant of the groupoid homotopy type as defined above. First we recall the construction of the homotopy groups of a groupoid as the homotopy groups of its classifying space [24, 8].

The classifying space $B\mathcal{G}$ of a groupoid $\mathcal{G}$ is the geometric realization of the simplicial manifold whose n-simplices are the contravariant functors $[n] \to \mathcal{G}$ where $[n]$ is the linearly ordered set $\{0, 1, \ldots, n\}$. We can describe the simplicial manifold G_* as a sequence of manifolds of composable strings of arrows:

$$G_n = \{(g_1, \ldots, g_n) \mid g_i \in G_1, s(g_i) = t(g_{i+1}), i = 1, \ldots, n-1\}$$

connected by face operators

$$d_i : G_n \to G_{n-1}$$

given by $d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & 0 < i < n \\ (g_1, \ldots, g_{n-1}) & i = n \end{cases}$

when $n > 1$ and $d_0(g) = s(g)$, $d_1(g) = t(g)$ when $n = 1$.

The classifying space of $\mathcal{G}$ is the geometric realization of the simplicial manifold G_*:

$$B\mathcal{G} = |G_*| = \bigsqcup_n (G_n \times \Delta^n)/(d_i(g), x) \sim (g, \delta_i(x))$$

where Δ^n is the standard topological n-simplex and $\delta_i : \Delta^{n-1} \to \Delta^n$ is the linear embedding of the i-th face.

Definition 5.12. The fundamental group of a groupoid $\mathcal{G}$ is defined as the fundamental group of $B\mathcal{G}$:

$$\pi_1(\mathcal{G}, x) = \pi_1(B\mathcal{G}, x)$$

where $x \in G_0$.

Proposition 5.13. [24, 16] *A morphism $\phi : \mathcal{K} \to \mathcal{G}$ induces a map $B\phi : B\mathcal{K} \to B\mathcal{G}$. If ϕ is an essential equivalence, then $B\phi$ is a weak homotopy equivalence.*

If two groupoids are Morita equivalent, then they have isomorphic homotopy groups. We will show that this remains true for the much weaker invariant of groupoid homotopy type.

Proposition 5.14. *If $\phi : \mathcal{K} \to \mathcal{G}$ is an essential homotopy equivalence, then $\pi_1(\mathcal{K}, x) = \pi_1(\mathcal{G}, \phi(x))$*

Proof. The morphism $\phi : \mathcal{K} \to \mathcal{G}$ induces an essential equivalence $\phi_* : \mathcal{K}_* \to \mathcal{G}_*$ on the fundamental groupoids. Then $B\phi_* : B\mathcal{K}_* \to B\mathcal{G}_*$ is a weak homotopy equivalence and we have that $\pi_1(B\mathcal{K}_*, x) = \pi_1(B\mathcal{G}_*, \phi(x))$. Since $\pi_1(\mathcal{K}_*, x) = \pi_1(B\mathcal{K}_*, x)$ by definition and $\pi_1(\mathcal{K}_*) = \mathcal{K}_*$ by Proposition 4.4 (3), we have that $\pi_1(B\mathcal{K}_*, x) = \pi_1(\mathcal{K}, x)$. Analogously, $\pi_1(B\mathcal{G}_*, \phi(x)) = \pi_1(\mathcal{G}, \phi(x))$ and the claim follows. ☐

6. Orbifolds as groupoids

We recall now the description of orbifolds as groupoids due to Moerdijk and Pronk [20, 17]. Orbifolds were first introduced by Satake [23] as a generalization of a manifold defined in terms of local quotients. The groupoid approach provides a global language to reformulate the notion of orbifold.

We follow the exposition in [1]. A groupoid $\mathcal{G}$ is *proper* if $(s, t) : G_1 \to G_0 \times G_0$ is a proper map and it is a *foliation* groupoid is each isotropy group is discrete.

Definition 6.1. An *orbifold* groupoid is a proper foliation groupoid.

For instance the holonomy group of a foliation $\mathcal{F}$ is always a foliation groupoid but it is an orbifold groupoid if and only if $\mathcal{F}$ is a compact-Hausdorff foliation.

Given an orbifold groupoid $\mathcal{G}$, its orbit space $|\mathcal{G}|$ is a locally compact Hausdorff space. Given an arbitrary locally compact Hausdorff space X we can equip it with an orbifold structure as follows:

Definition 6.2. An *orbifold structure* on a locally compact Hausdorff space X is given by an orbifold groupoid $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to X$.

If $\epsilon : \mathcal{K} \to \mathcal{G}$ is an essential equivalence and $|\epsilon| : |\mathcal{K}| \to |\mathcal{G}|$ is the induced homeomorphism between orbit spaces, we say that the composition $f \circ |\epsilon| : |\mathcal{K}| \to X$ defines an *equivalent* orbifold structure.

Definition 6.3. An *orbifold* $\mathcal{X}$ is a space X equipped with an equivalence class of orbifold structures. A specific such structure, given by $\mathcal{G}$ and $f : |\mathcal{G}| \to X$ is a *presentation* of the orbifold $\mathcal{X}$.

If two groupoids are Morita equivalent, then they define the same orbifold. Therefore any structure or invariant for orbifolds, if defined through groupoids, should be invariant of Morita equivalence.

We define an orbifold map $\mathcal{Y} \to \mathcal{X}$ as a generalized map (ϵ, ϕ) from $\mathcal{K}$ to $\mathcal{G}$ between presentations of the orbifolds such that the diagram commutes:

$$\begin{tikzcd}
|\mathcal{K}| \arrow[r] \arrow[d] & |\mathcal{G}| \arrow[d, "f"] \\
Y \arrow[r] & X
\end{tikzcd}$$

A specific such generalized map $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ is called *presentation* of the orbifold map.

And then, the notion of groupoid homotopy gives a notion of homotopy for orbifolds:

Proposition 6.4. *Two orbifold maps* $\mathcal{Y} \rightrightarrows \mathcal{X}$ *are homotopic if their presentations* (ϵ, ϕ) *and* (ν, ψ) *are groupoid homotopic.*

In other words, if $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K} \xleftarrow{\nu} \mathcal{J}' \xrightarrow{\psi} \mathcal{G}$ are the presentations of the orbifold maps, they are homotopic if there exists a Lie groupoid $\mathcal{L}$ and essential homotopy equivalences

$$\mathcal{J} \xleftarrow{\omega} \mathcal{L} \xrightarrow{\omega'} \mathcal{J}'$$

such that $\epsilon_*\omega_* \sim \nu_*\omega'_*$ and $\phi_*\omega_* \sim \psi_*\omega'_*$.

Proposition 6.5. *Orbifold homotopy equivalence is an equivalence relation in the set of orbifolds maps.*

A weak generalized map (ω, ϕ) defines a homotopy class of orbifold maps. The bicategory of orbifolds and homotopy classes of orbifold maps is biequivalent to a full subbicategory of $\mathfrak{H}[W^{-1}]$.

With the obvious notion of *orbifold homotopy type*, we have that the orbifold fundamental group of $\mathcal{X}$ defined as the fundamental group of a presentation groupoid $\mathcal{G}$ [8, 7, 9, 5],

$$\pi_1^{orb}(\mathcal{X}, \bar{x}) = \pi_1(\mathcal{G}, x)$$

is an invariant of orbifold homotopy type.

Example 6.6. Consider the orbifold $\mathcal{X}$ having as a presentation groupoid the holonomy groupoid associated to the Seifert fibration on the Möbius band, $\mathcal{G} = \mathrm{Hol}(M, \mathcal{F}_S)$. The orbifold $\mathcal{X}$ has the same homotopy type as the orbifold $\mathcal{Y}$ represented by the point groupoid $\bullet^{\mathbb{Z}_2}$. Consider the weak generalized map $\bullet^{\mathbb{Z}_2} \xleftarrow{c} \mathcal{J} \xrightarrow{i} \mathcal{G}$ where $\mathcal{J} = \mathrm{Hol}_I(M, \mathcal{F}_S)$ is the reduced holonomy groupoid to a transversal interval I, and c is the constant map on objects $J_0 = I$ and the constant map on each connected component of the manifold of arrows $J_1 = I \sqcup I$. This weak generalized map is invertible since both the constant map c and the inclusion map i are weak homotopy equivalences. The inverse weak generalized map is

$$\mathcal{G} \xleftarrow{i} \mathcal{J} \xrightarrow{c} \bullet^{\mathbb{Z}_2}$$

The orbifold $\mathcal{Y}$ has the same underlying space $Y = \bullet$ than the trivial point orbifold **1**, but they are not homotopically equivalent.

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