THE E_2-TERM OF THE DESCENT SPECTRAL SEQUENCE FOR CONTINUOUS G-SPECTRA

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Abstract. Let $\{X_i\}$ be a tower of discrete G-spectra, each of which is fibrant as a spectrum, so that $X = \operatorname{holim}_i X_i$ is a continuous G-spectrum, with homotopy fixed point spectrum X^{hG}. The E_2-term of the descent spectral sequence for $\pi_*(X^{hG})$ cannot always be expressed as continuous cohomology. However, we show that the E_2-term is always built out of a certain complex of spectra, that, in the context of abelian groups, is used to compute the continuous cochain cohomology of G with coefficients in $\lim_i M_i$, where $\{M_i\}$ is a tower of discrete G-modules.

1. Introduction

In this note, G always denotes a profinite group. Let $H^*_c(G; M)$ denote the continuous cohomology of G with coefficients in the discrete G-module M. This cohomology is defined as the right derived functors of G-fixed points. Then we always assume that G has finite virtual cohomological dimension; that is, there exists an open subgroup H and a non-negative integer m, such that $H^s_c(H; M) = 0$, for all discrete H-modules M and all $s \geq m$.

All of our spectra are Bousfield-Friedlander spectra of simplicial sets. In particular, a *discrete G-spectrum* is a G-spectrum such that each simplicial set X_k is a simplicial object in the category of discrete G-sets (thus, for any $l \geq 0$, the action map on the l-simplices, $G \times (X_k)_l \to (X_k)_l$, is continuous when $(X_k)_l$ is regarded as a discrete space). The category of discrete G-spectra, with morphisms being G-equivariant maps of spectra, is denoted by Spt_G.

Discrete G-spectra are considered in more detail in [3], which shows (see [3, Theorem 3.6]) that Spt_G is a model category, where a morphism f in Spt_G is a weak equivalence (cofibration) if and only if f is a weak equivalence (cofibration) in Spt, the category of spectra. Given a discrete G-spectrum X, the *homotopy fixed point spectrum* X^{hG} is obtained as the total right derived functor of fixed points: $X^{hG} = (X_{f,G})^G$, where $X \to X_{f,G}$ is a trivial cofibration and $X_{f,G}$ is fibrant, all in Spt_G.

Let $X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ be a tower of discrete G-spectra, such that each X_i is a fibrant spectrum. As explained in [3, Lemma 4.4], there exists a tower $\{X'_i\}$ of discrete G-spectra, such that there are weak equivalences

$$\operatorname*{holim}_i X_i \xrightarrow{\simeq} \operatorname*{holim}_i X'_i \xleftarrow{\simeq} \lim_i X'_i.$$

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(In this paper, holim always denotes the version of the homotopy limit of spectra that is constructed levelwise in the category of simplicial sets, as defined in [1] and [7, 5.6].) Since the inverse limit of a tower of discrete G-sets is a topological G-space, and because $\mathrm{holim}_i X_i$ can be identified with $\lim_i X_i'$, in the eyes of homotopy, $\mathrm{holim}_i X_i$ is a *continuous G-spectrum.* Notice that, under this identification, the continuous G-action respects the topology of both G and all the X_i together. Continuous G-spectra and examples of such in chromatic stable homotopy theory are considered in [3, 2].

Given the continuous G-spectrum $\mathrm{holim}_i X_i$, there is the *homotopy fixed point spectrum*

$$(\operatorname*{holim}_i X_i)^{hG} = \operatorname*{holim}_i (X_i)^{hG}.$$

This construction is called homotopy fixed points because it is equivalent to the usual definition when G is a finite group and it is the total right derived functor of fixed points in the appropriate sense (see [3, Remark 8.4]).

By [3, Theorem 8.8], thanks to the assumption of finite virtual cohomological dimension, there is a descent spectral sequence

$$E_2^{s,t} \Rightarrow \pi_{t-s}((\operatorname*{holim}_i X_i)^{hG}), \tag{1.1}$$

where

$$E_2^{s,t} = \pi^s \pi_t(\operatorname*{holim}_i (\Gamma^\bullet_G(X_i)_{f,G})^G), \tag{1.2}$$

and, if the tower of abelian groups $\{\pi_t(X_i)\}$ satisfies the Mittag-Leffler condition for every integer t, then $E_2^{s,t} \cong H^s_{\mathrm{cont}}(G; \{\pi_t(X_i)\})$, which is continuous cohomology in the sense of Jannsen. (This cohomology is obtained by taking the right derived functors of $\lim_i(-)^G$, a functor from towers of discrete G-modules to abelian groups; see [5].)

In expression (1.2), since $\pi_t(\mathrm{holim}_i(-))$ is not necessarily $\lim_i \pi_t(-)$, the E_2-term of descent spectral sequence (1.1), in general, can not be expressed as continuous cohomology, and, in general, it has no compact algebraic description. However, in this note, we show that the E_2-term (1.2) can always be described in an interesting way.

In more detail, Theorem 2.5 gives a particular cochain complex $\mathcal{C}^*$ for computing the continuous cochain cohomology of G for a topological G-module $\lim_i M_i$, where $\{M_i\}$ is a tower of discrete G-modules. In Corollary 3.4, we show that the E_2-term of (1.2) can always be given by taking the cohomology of the homotopy groups of the complex $\mathcal{C}^*$, where $\lim_i M_i$ is replaced by the continuous G-spectrum $\mathrm{holim}_i X_i$, in an appropriate sense. This presentation of the E_2-term shows that $E_2^{*,*}$ always takes into account the topology of the continuous G-spectrum, even when it cannot be expressed as continuous cohomology.

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2. THE PRO-DISCRETE COCHAIN COMPLEX AND CONTINUOUS COHOMOLOGY

We begin this section with some terminology. If $\mathcal{C}$ is a category, then $\mathbf{tow}(\mathcal{C})$ is the category of towers

$$C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$$

in $\mathcal{C}$. The morphisms $\{f_i\}$ are natural transformations such that each f_i is a morphism in $\mathcal{C}$. In this note, we will be working with $\mathbf{tow}(\mathrm{DMod}(G))$, where $\mathrm{DMod}(G)$ is the category of discrete G-modules, and $\mathbf{tow}(\mathrm{Spt}_G)$.

If A is an abelian group with the discrete topology, let $\mathrm{Map}_c(G, A)$ be the abelian group of continuous maps from G to A. If X is a spectrum, one can also define $\mathrm{Map}_c(G, X)$, where the l-simplices of the kth simplicial set $(\mathrm{Map}_c(G, X)_k)_l$ are given by $\mathrm{Map}_c(G, (X_k)_l)$, where $(X_k)_l$ is given the discrete topology.

Consider the functor

$$\Gamma_G \colon \mathrm{Spt}_G \to \mathrm{Spt}_G, \quad X \mapsto \Gamma_G(X) = \mathrm{Map}_c(G, X),$$

where the action of G on $\mathrm{Map}_c(G, X)$ is induced on the level of sets by $(g \cdot f)(g') = f(g'g)$, for $g, g' \in G$ and $f \in \mathrm{Map}_c(G, (X_k)_l)$, for each $k, l \geq 0$. As explained in [3, Definition 7.1], the functor Γ_G forms a triple and there is a cosimplicial discrete G-spectrum $\Gamma_G^\bullet X$. Also, it is clear that $\Gamma_G \colon \mathrm{DMod}(G) \to \mathrm{DMod}(G)$ can be defined as above, so that, given a discrete G-module M, $\Gamma_G^\bullet M$ is a cosimplicial discrete G-module.

We do not claim any originality in the following definition, where here and elsewhere colim_N is a colimit indexed over all open normal subgroups N of G.

Definition 2.1. Let $\{X_i\}$ be an object in $\mathbf{tow}(\mathrm{DMod}(G))$ or in $\mathbf{tow}(\mathrm{Spt}_G)$. Then the *pro-discrete cochain complex* is defined to be the complex

$$\mathcal{C}^*(G; \{X_i\}) = \lim_i \operatorname*{colim}_N ((\Gamma^*_{G/N}((X_i)^N))^{G/N}),$$

where $\Gamma^*_{G/N}((X_i)^N)$ is the canonical complex associated to $\Gamma^\bullet_{G/N}((X_i)^N)$. The pro-discrete cochain complex is a complex of abelian groups or spectra, respectively, and the limit and colimit are both formed in abelian groups or spectra, respectively.

Lemma 2.2. *Let $\{X_i\}$ be in $\mathbf{tow}(\mathrm{DMod}(G))$ or in $\mathbf{tow}(\mathrm{Spt}_G)$. Then there is a natural isomorphism*

$$\mathcal{C}^*(G; \{X_i\}) \to \lim_i (\Gamma^*_G X_i)^G.$$

Remark 2.3. In the isomorphism above, if the $\lim_i$ is removed from both the source and target, then Lemma 2.2 is basically a version of the isomorphism in the first sentence of [6, proof of Proposition (1.2.6)], which is in the context of discrete G-modules only. We give a proof of Lemma 2.2 below because this is easier than translating from [op. cit.].

Proof of Lemma 2.2. Let X be a discrete G-module or a discrete G-spectrum. Note that

$$\mathrm{Map}_c(G, X)^G \cong X \cong \operatorname*{colim}_N X^N \cong \operatorname*{colim}_N \mathrm{Map}_c(G/N, X^N)^{G/N}.$$

Similarly, notice that

$$\begin{aligned}\mathrm{Map}_c(G, \mathrm{Map}_c(G, X))^G &\cong \mathrm{Map}_c(G, X) \cong \operatorname*{colim}_N \mathrm{Map}_c(G/N, X^N)\\ &\cong \operatorname*{colim}_N \mathrm{Map}_c(G/N, \mathrm{Map}_c(G/N, X^N))^{G/N}.\end{aligned}$$

Also, we have

$$\mathrm{Map}_c(G, \mathrm{Map}_c(G, \mathrm{Map}_c(G, X)))^G \cong \mathrm{Map}_c(G^2, X) \cong \operatorname*{colim}_N \mathrm{Map}_c((G/N)^2, X^N),$$

and the last expression is isomorphic to

$$\operatorname*{colim}_N \mathrm{Map}_c(G/N, \mathrm{Map}_c(G/N, \mathrm{Map}_c(G/N, X^N)))^{G/N}.$$

This verifies the isomorphism for 0-, 1-, and 2-cochains.

By tracing through and iterating the above ingredients, we see that there is a natural isomorphism

$$\operatorname*{colim}_N((\Gamma^\bullet_{G/N}(X^N))^{G/N}) \to (\Gamma^\bullet_G X)^G$$

of cosimplicial objects. Thus, there is an isomorphism of associated cochain complexes

$$\operatorname*{colim}_N((\Gamma^*_{G/N}(X^N))^{G/N}) \to (\Gamma^*_G X)^G.$$

Now let $\{X_i\}$ be as described in the statement of the theorem. Then there is an isomorphism

$$\lim_i \operatorname*{colim}_N((\Gamma^*_{G/N}((X_i)^N))^{G/N}) \to \lim_i (\Gamma^*_G X_i)^G.$$

□

Let M be any topological G-module. Then the *continuous cochain cohomology of G with coefficients in M*, $H^*_{\mathrm{cts}}(G;M)$, is the cohomology of a cochain complex that has the form

$$M \to \mathrm{Map}_c(G,M) \to \mathrm{Map}_c(G^2,M) \to \cdots \tag{2.4}$$

(see [6, pg. 106] for details). We note that, by [5, Theorem (2.2)], if $\{M_i\}$ is a tower of discrete G-modules that satisfies the Mittag-Leffler condition, then

$$H^s_{\mathrm{cts}}(G; \lim_i M_i) \cong H^s_{\mathrm{cont}}(G; \{M_i\}),$$

for all $s \geq 0$, but, in general, these two versions of continuous cohomology need not be isomorphic. Also, if M is a discrete G-module, then $H^s_{\mathrm{cts}}(G;M) = H^s_c(G;M)$.

Now we show that the pro-discrete cochain complex can be used to compute continuous cochain cohomology.

Theorem 2.5. *If $\{M_i\}$ is a tower of discrete G-modules, then*

$$H^s_{\mathrm{cts}}(G; \lim_i M_i) \cong H^s[\mathcal{C}^*(G; \{M_i\})].$$

Proof. As explained in [6, pg. 106], for a topological G-module M, the chain complex in (2.4) is defined by taking the G-fixed points of the complex

$$X^*(G;M) = [\mathrm{Map}_c(G,M) \to \mathrm{Map}_c(G^2,M) \to \cdots], \tag{2.6}$$

where $X^n(G;M) = \mathrm{Map}_c(G^{n+1},M)$ has a G-action that is defined by

$$(g \cdot f)(g_1, ..., g_{n+1}) = g \cdot f(g^{-1}g_1, ..., g^{-1}g_{n+1}).$$

Now let M be a discrete G-module. Then it is a standard fact that the cochain complex $(X^*(G,M))^G$ is naturally isomorphic as a complex to the cochain complex $(\Gamma^*_G M)^G$. This isomorphism uses the fact that the abelian group of n-cochains of $(\Gamma^*_G M)^G$ is isomorphic to $\mathrm{Map}_c(G^{n+1},M)^G$, where $\mathrm{Map}_c(G^{n+1},M)$ has a G-action that is given by

$$(g \cdot f)(g_1, g_2, g_3, ..., g_{n+1}) = f(g_1 g, g_2, g_3, ..., g_{n+1}).$$

Since

$$(X^n(G; \lim_i M_i))^G \cong \lim_i((X^n(G; M_i))^G) \cong \lim_i(\Gamma_G^{n+1} M_i)^G,$$

we have:

$$H^s_{\mathrm{cts}}(G; \lim_i M_i) = H^s[(X^*(G; \lim_i M_i))^G] = H^s[\lim_i(\Gamma^*_G M_i)^G],$$

where we used the aforementioned fact that $(X^*(G, M_i))^G$ and $(\Gamma^*_G M_i)^G$ are naturally isomorphic cochain complexes. The proof is completed by applying Lemma 2.2. □

3. The E_2-term and the pro-discrete cochain complex

In this section, we show that the E_2-term of (1.2) can be built out of the same complex that computes continuous cochain cohomology. More precisely, given a continuous G-spectrum $\mathrm{holim}_i X_i$, there exists a tower $\{X'_i\}$ of discrete G-spectra, such that

$$E_2^{s,t} \cong H^s[\pi_t(\mathcal{C}^*(G; \{X'_i\}))]. \tag{3.1}$$

We find the expression on the right-hand side in (3.1) interesting for the following reason. The homotopy fixed point spectrum is defined with respect to a continuous action of G on the spectrum. Thus, homotopy fixed points take into account the topology of the spectrum. Similarly, since the E_2-term is built out of the pro-discrete cochain complex of spectra, the E_2-term is always taking into account the topology of the spectrum.

By [4, VI, Proposition 1.3], $\mathbf{tow}(\mathrm{Spt}_G)$ is a model category, where $\{f_i\}$ is a weak equivalence (cofibration) if and only if each f_i is a weak equivalence (cofibration) in Spt_G.

Theorem 3.2. *The E_2-term (1.2) of descent spectral sequence (1.1) has the form*

$$E_2^{s,t} \cong \pi^s\pi_t(\lim_i(\Gamma^\bullet_G X'_i)^G), \tag{3.3}$$

where $\{X_i\} \to \{X'_i\}$ is a trivial cofibration with $\{X'_i\}$ fibrant, all in $\mathbf{tow}(\mathrm{Spt}_G)$.

Proof. Let $\{X'_i\}$ be as stated in the theorem. By [4, VI, Remark 1.5], each X'_i is fibrant and each map $X'_i \to X'_{i-1}$ is a fibration, all in Spt_G.

For any $k \geq 0$, we consider the expression

$$\mathrm{holim}_i((\Gamma^\bullet_G X'_i)^G)^k = \mathrm{holim}_i(\mathrm{Map}_c(G, \mathrm{Map}_c(G, \cdots, \mathrm{Map}_c(G, X'_i)\cdots)))^G,$$

where $\mathrm{Map}_c(G, -)$ appears $k+1$ times. By [3, Section 3], the forgetful functor $U\colon \mathrm{Spt}_G \to \mathrm{Spt}$, $\mathrm{Map}_c(G, -)\colon \mathrm{Spt} \to \mathrm{Spt}_G$, where $\mathrm{Map}_c(G, X) = \Gamma_G(X)$, and the functor $(-)^G\colon \mathrm{Spt}_G \to \mathrm{Spt}$ all preserve fibrations. Thus, $\{X'_i\}$ is a tower of fibrations of fibrant spectra, all in Spt. This implies that $\{\mathrm{Map}_c(G, X'_i)\}$ is a tower of fibrations of fibrant spectra, in Spt_G, and hence, in Spt. By iteration,

$$\{\mathrm{Map}_c(G, \mathrm{Map}_c(G, \cdots, \mathrm{Map}_c(G, X'_i)\cdots))\}$$

is a tower of fibrations of fibrant spectra, in Spt_G, so that

$$\{(\mathrm{Map}_c(G, \mathrm{Map}_c(G, \cdots, \mathrm{Map}_c(G, X'_i)\cdots)))^G\}$$

is a tower of fibrations of fibrant spectra in Spt. Therefore, the canonical map

$$\lim_i((\Gamma^\bullet_G X'_i)^G)^k \to \mathrm{holim}_i((\Gamma^\bullet_G X'_i)^G)^k$$

is a weak equivalence.

Since $\{((\Gamma^\bullet_G X_i)^G)^k\}$ and $\{((\Gamma^\bullet_G (X_i)_{f,G})^G)^k\}$ are towers of fibrant spectra, there is a zigzag of weak equivalences

$$\lim_i((\Gamma^\bullet X_i')^G)^k \to \operatorname{holim}_i((\Gamma^\bullet X_i')^G)^k \leftarrow \operatorname{holim}_i((\Gamma^\bullet X_i)^G)^k \to \operatorname{holim}_i((\Gamma^\bullet (X_i)_{f,G})^G)^k,$$

where $\Gamma = \Gamma_G$. This zigzag of weak equivalences implies that

$$\pi^s\pi_t(\lim_i((\Gamma^\bullet_G X_i')^G)) \cong \pi^s\pi_t(\operatorname{holim}_i((\Gamma^\bullet_G (X_i)_{f,G})^G)).$$

□

Corollary 3.4. *Let $\{X_i'\}$ be as in Theorem 3.2. Then there is an isomorphism*

$$E_2^{s,t} \cong H^s[\pi_t(\mathcal{C}^*(G;\{X_i'\}))],$$

where $E_2^{s,t}$ is the E_2-term of (1.2).

Proof. This follows immediately from applying Lemma 2.2 to Theorem 3.2. □

Remark 3.5. By Theorem 2.5, $H^s[\mathcal{C}^*(G;\{\pi_t(X_i')\})] \cong H^s_{\mathrm{cts}}(G;\lim_i \pi_t(X_i))$.

4. THE FAILURE OF OTHER POSSIBLE DESCRIPTIONS OF THE E_2-TERM

After studying the expression in (3.3) further, one recalls that $\lim_i(-)^G$ is the functor used to define $H^s_{\mathrm{cont}}(G;-)$, and, if M is any discrete G-module, then

$$0 \to M \to \Gamma^*_G M$$

is a $(-)^G$-acyclic resolution of M, so that $H^s[(\Gamma^*_G M)^G] = H^s_c(G;M)$.

Let $\{M_i\}$ be a tower of discrete G-modules. If

$$\{0\} \to \{M_i\} \to \{\Gamma^*_G M_i\}$$

is a $\lim_i(-)^G$-acyclic resolution of $\{M_i\}$ in $\mathbf{tow}(\mathrm{DMod}(G))$, then

$$H^s[(\lim_i(-)^G)(\{\Gamma^*_G M_i\})] = H^s_{\mathrm{cont}}(G;\{M_i\}).$$

This would imply that $E_2^{s,t} \cong H^s[\pi_t(\lim_i(\Gamma^*_G X_i')^G)]$ is computed by taking the cohomology of the homotopy groups of a complex of spectra that, in the context of abelian groups, computes continuous cohomology. This would be an interesting presentation of the E_2-term.

However, it is not hard to show that

$$\{0\} \to \{M_i\} \to \{\Gamma^*_G M_i\}$$

need not be a $\lim_i(-)^G$-acyclic resolution of $\{M_i\}$ in $\mathbf{tow}(\mathrm{DMod}(G))$, so that the above interpretation of the E_2-term does not work out. For example, by [5, (2.1)], there is a short exact sequence

$$0 \to \lim_i{}^1 H^{s-1}_c(G;\Gamma_G M_i) \to H^s_{\mathrm{cont}}(G;\{\Gamma_G M_i\}) \to \lim_i H^s_c(G;\Gamma_G M_i) \to 0,$$

for each $s \geq 0$, where $H^{-1}_c(G;-) = 0$. Therefore, when $s \geq 1$, $H^s_c(G;\Gamma_G M_i) = 0$, so that, for all $s \geq 2$, $H^s_{\mathrm{cont}}(G;\{\Gamma_G M_i\}) = 0$. But, the short exact sequence also implies that

$$H^1_{\mathrm{cont}}(G;\{\Gamma_G M_i\}) \cong \lim_i{}^1 M_i,$$

which need not vanish. Thus, $\{\Gamma_G M_i\}$, the first object in the complex $\{\Gamma^*_G M_i\}$, need not be $\lim_i(-)^G$-acyclic in $\mathbf{tow}(\mathrm{DMod}(G))$.

Upon further consideration of the expression in (3.3), one notices that, for any $k, l, m \geq 0$,

$$((\lim_i (\Gamma_G^{m+1} X_i')^G)_k)_l = \lim_i (\Gamma_G^{m+1}((X_i')_k)_l)^G \cong \mathrm{Map}_c(G^m, \lim_i ((X_i')_k)_l)$$

is an isomorphism of sets. If one could promote this isomorphism to

$$\lim_i (\Gamma_G^{m+1} X_i')^G \cong \mathrm{Map}_c(G^m, \lim_i X_i'), \tag{4.1}$$

then one could use this to interpret the expression in (3.3) as being the cohomology of homotopy groups applied to the complex of continuous cochains with target ("coefficients") the continuous G-spectrum $\lim_i X_i'$.

But notice that, in this interpretation, the expression $\mathrm{Map}_c(G^m, \lim_i X_i')$ does not have the desired meaning. For isomorphism (4.1) to hold, $\lim_i X_i'$ must be a spectrum whose simplicial sets have simplices with the pro-discrete topology. But, as a Bousfield-Friedlander spectrum, in the construction $\mathrm{Map}_c(G^m, \lim_i X_i')$, $\lim_i X_i'$ consists of simplicial sets whose simplices all have the discrete topology, by default. This conflict means that this interpretation also fails to work.

References

[1] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.

[2] Daniel G. Davis. Iterated homotopy fixed points for the Lubin-Tate spectrum. *In preparation, Purdue University*, 2006.

[3] Daniel G. Davis. Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action. *To appear in the Journal of Pure and Applied Algebra*, accepted July 13, 2005.

[4] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory.* Birkhäuser Verlag, Basel, 1999.

[5] Uwe Jannsen. Continuous étale cohomology. *Math. Ann.*, 280(2):207–245, 1988.

[6] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 2000.

[7] R. W. Thomason. Algebraic K-theory and étale cohomology. *Ann. Sci. École Norm. Sup. (4)*, 18(3):437–552, 1985.