THE SITE R_G^+ FOR A PROFINITE GROUP G

DANIEL G. DAVIS

Abstract. Let G be a non-finite profinite group and let $G - \mathbf{Sets}_{df}$ be the canonical site of finite discrete G-sets. Then the category R_G^+, defined by Devinatz and Hopkins, is the category obtained by considering $G - \mathbf{Sets}_{df}$ together with the profinite G-space G itself, with morphisms being continuous G-equivariant maps. We show that R_G^+ is a site when equipped with the pretopology of epimorphic covers. Also, we explain why the associated topology on R_G^+ is not subcanonical, and hence, not canonical. We note that, since R_G^+ is a site, there is automatically a model category structure on the category of presheaves of spectra on the site. Finally, we point out that such presheaves of spectra are a nice way of organizing the data that is obtained by taking the homotopy fixed points of a continuous G-spectrum with respect to the open subgroups of G.

1. Introduction

Let G be a profinite group that is not a finite group. Let R_G^+ be the category with objects all finite discrete left G-sets together with the left G-space G. The morphisms of R_G^+ are the continuous G-equivariant maps. Since G is not finite, the object G in R_G^+ is very different in character from all the other objects of R_G^+. In this paper, we show that R_G^+ is a site when equipped with the pretopology of epimorphic covers.

As far as the author knows, the category R_G^+ is first defined and used in the paper [Devinatz and Hopkins, 2004], by Ethan Devinatz and Mike Hopkins. Let G_n be the profinite group $S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, where S_n is the nth Morava stabilizer group. In [Devinatz and Hopkins, 2004, Theorem 1], Devinatz and Hopkins construct a contravariant functor - that is, a presheaf -

$$\mathbf{F}\colon (R_{G_n}^+)^{\mathrm{op}} \to (\mathcal{E}_\infty)_{K(n)},$$

to the category $(\mathcal{E}_\infty)_{K(n)}$ of $K(n)$-local commutative S-algebras (see [Elmendorf et. al., 1997]), where $K(n)$ is the nth Morava K-theory (see [Rudyak, 1998, Chapter 9] for an exposition of $K(n)$). The functor $\mathbf{F}$ has the properties that, if U is an open subgroup of G_n, then $\mathbf{F}(G_n/U) = E_n^{dhU}$, and $\mathbf{F}(G_n) = E_n$, where E_n is the nth Lubin-Tate spectrum (for salient facts about E_n and its importance in homotopy theory, see [Devinatz and Hopkins, 1995, Introduction]), and E_n^{dhU} is a spectrum that behaves like the U-homotopy fixed point spectrum of E_n with respect to the continuous U-action. Since $\mathrm{Hom}_{R_{G_n}^+}(G_n, G_n) \cong G_n$, functoriality implies that G_n acts on E_n by maps of commutative S-algebras. In Section

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5, we will give several related examples of presheaves of spectra that illustrate the utility of the category R_G^+.

The pretopology of epimorphic covers on a small category $\mathcal{C}$ is the pretopology $\mathcal{K}$ given by all covering families $\{f_i \colon C_i \to C \mid i \in I\}$ such that $\phi \colon \coprod_{i \in I} C_i \to C$ is onto, where $C_i, C \in \mathcal{C}$, $f_i \in \mathrm{Mor}_{\mathcal{C}}(C_i, C)$, and I is some indexing set. (Of course, one must prove that these covering families actually give a pretopology on $\mathcal{C}$.) We note that we do not require that ϕ be a morphism in $\mathcal{C}$; for our purposes, $\mathcal{C} = R_G^+$ and we only require that ϕ be an epimorphism in the category of all G-sets (so that ϕ does not have to be continuous). This assumption is important for our work, since, for example, $G \coprod G$ is not in R_G^+.

The pretopology $\mathcal{K}$ is a familiar one. For example, for a profinite group G, $\mathcal{K}$ is the standard basis used for the site $G - \mathbf{Sets}_{df}$ of finite discrete G-sets ([Jardine, 1997, pg. 206]). However, there is an important difference between R_G^+ and $G - \mathbf{Sets}_{df}$: the latter category is closed under pullbacks, but it is easy to see that R_G^+ does not have all pullbacks (this point will be discussed later). But in a category with pullbacks, the canonical topology, the finest topology in which every representable presheaf is a sheaf, is given by all covering families of universal effective epimorphisms (see Expose IV, 4.3 of [Demazure, 1970]). This implies that $G - \mathbf{Sets}_{df}$ is a site with the canonical topology when equipped with pretopology $\mathcal{K}$. However, due to the lack of sufficient pullbacks, we cannot conclude that $\mathcal{K}$ gives R_G^+ the canonical topology. In fact, we will show that $\mathcal{K}$ does not generate the canonical topology, since $\mathcal{K}$ does not yield a subcanonical topology.

Note that R_G^+ is built out of the two subcategories $G - \mathbf{Sets}_{df}$ and the groupoid G. Since each of these categories is a site via $\mathcal{K}$ (for G, this is verified in Lemma 2 below), it is natural to think that R_G^+ is also a site via $\mathcal{K}$. Our main result (Theorem 3.1), verifies that this is indeed the case.

As discussed earlier, $\mathbf{F}$ is a presheaf of spectra on the site R_G^+. More generally, there is the category $\mathrm{PreSpt}(R_G^+)$ of presheaves of spectra on R_G^+. Furthermore, since R_G^+ is a site, the work of Jardine (e.g., [Jardine, 1987], [Jardine, 1997]) implies that $\mathrm{PreSpt}(R_G^+)$ is a model category. We recall the definition of this model category in Section 5.

In [Davis, 2006], the author showed that, given a continuous G-spectrum Z, then, for any open subgroup U of G, there is a homotopy fixed point spectrum Z^{hU}, defined with respect to the continuous action of U on Z. In Examples 5.7 and 5.8, we see that there is a presheaf that organizes in a functorial way the following data: Z, Z^{hU} for all U open in G, and the maps between these spectra that are induced by continuous G-equivariant maps between the G-spaces G and G/U. Thus, $\mathrm{PreSpt}(R_G^+)$ is a natural category within which to work with continuous G-spectra. It is our hope that the model category structure on $\mathrm{PreSpt}(R_G^+)$ can be useful for the theory of homotopy fixed points for profinite groups, though we have not yet found any such applications.

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2. Preliminaries

Before we prove our main results, we first collect some easy facts which will be helpful later. As stated in the Introduction, G always refers to an infinite profinite group. (If the profinite group G is finite, then $R_G^+ = G-\mathbf{Sets}_{df}$ and there is nothing to prove.)

2.1. Lemma. *Let* $f\colon C \to G$ *be any morphism in* R_G^+ *with* $C \neq \varnothing$*. Then* $C = G$*.*

Proof. Choose any $c \in C$ and let $f(c) = \gamma$. Choose any $\delta \in G$. Then

$$\delta = (\delta\gamma^{-1})\gamma = (\delta\gamma^{-1}) \cdot f(c) = f((\delta\gamma^{-1}) \cdot c),$$

by the G-equivariance of f. Thus, f is onto and $|\mathrm{im}(f)| = \infty$, so that C cannot be a finite set. ■

2.2. Lemma. *For a topological group* G*, let* G *be the groupoid with the single object* G *and morphisms the* G*-equivariant maps* $G \to G$ *given by right multiplication by some element of* G*. Then* G *is a site with the pretopology* $\mathcal{K}$ *of epimorphic covers.*

Proof. Any diagram $G \xrightarrow{f} G \xleftarrow{g} G$, where f and g are given by multiplication by γ and δ, respectively, can be completed to a commutative square

$$\begin{array}{ccc} G & \xrightarrow{f'} & G \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ G & \xrightarrow{f} & G, \end{array}$$

where f' and g' are given by multiplication by δ^{-1} and γ^{-1}, respectively. This property suffices to show that G is a site with the atomic topology, in which every sieve is a covering sieve if and only if it is nonempty. It is easy to see that the only nonempty sieve of G is $\mathrm{Mor}_G(G, G)$ itself. Thus, the only covering sieve of G is the maximal sieve. Since every morphism of G is a homeomorphism, in the pretopology $\mathcal{K}$, the collection of covers is exactly the collection of all nonempty subsets of $\mathrm{Mor}_G(G, G)$. Then it is easy to check that $\mathcal{K}$ is the maximal basis that generates the atomic topology. ■

Observe that if $f\colon G \to G$ is a morphism in R_G^+, then by G-equivariance, f is the map given by multiplication by $f(1)$ on the right. As mentioned earlier, we have

2.3. Lemma. *The category* G–$\mathbf{Sets}_{df}$*, a full subcategory of* R_G^+*, is closed under pullbacks.*

Proof. The pullback of a diagram in G–$\mathbf{Sets}_{df}$ is formed simply by regarding the diagram as being in the category T_G of discrete G-sets. The category T_G is closed under pullbacks, as explained in [Mac Lane and Moerdijk, 1994, pg. 31]. ■

We recall the following useful result and its proof.

2.4. Lemma. *Let X be any finite set in R_G^+. We write $X = \coprod_{i=1}^n \overline{x_i}$, the disjoint union of all the distinct orbits $\overline{x_i}$, with each x_i a representative. Then X is homeomorphic to $\coprod_{i=1}^n G/U_i$, where $U_i = G_{x_i}$ is the stabilizer in G of x_i.*

Proof. Let $f\colon G/U_i \to \overline{x_i}$ be given by $f(\gamma U_i) = \gamma \cdot x_i$. Since X is a discrete G-set, the stabilizer U_i is an open subgroup of G with finite index, so that G/U_i is a finite set. Then f is open and continuous since it is a map between discrete spaces. Also, it is clear that f is onto. Now suppose $\gamma U_i = \delta U_i$. Then $\gamma^{-1}\delta \in U_i$, so that $(\gamma^{-1}\delta)\cdot x_i = (\gamma^{-1})\cdot(\delta\cdot x_i) = x_i$. Thus, $\gamma \cdot x_i = \delta \cdot x_i$ and f is well-defined. Assume that $\gamma \cdot x_i = \delta \cdot x_i$. Then $\gamma^{-1}\delta \in G_{x_i}$ so that f is a monomorphism. ■

2.5. Lemma. *Let X be a finite discrete G-set in R_G^+ and let $\psi\colon G \to X$ be any G-equivariant function. Then ψ is a morphism in R_G^+.*

Proof. As in Lemma 2.4, we identify X with $\coprod_{i=1}^n G/U_i$. Since ψ is G-equivariant and $\psi(\gamma) = \gamma \cdot \psi(1)$, ψ is determined by $\psi(1)$. Let $\psi(1) = \delta U_j$ for some $\delta \in G$ and some j. Then for any γ in G,

$$\gamma U_j = (\gamma\delta^{-1}\delta)U_j = (\gamma\delta^{-1})\cdot\psi(1) = \psi(\gamma\delta^{-1}),$$

so that $\operatorname{im}\psi = G/U_j$. Since X is discrete, ψ is continuous, if, for any $x \in X$, $\psi^{-1}(x)$ is open in G. It suffices, by the identification, to let $x = \gamma U_j$, for any $\gamma \in G$. Then

$$\begin{aligned}\psi^{-1}(\gamma U_j) &= \{\zeta \in G \mid \psi(\zeta) = \gamma U_j\} = \{\zeta \in G \mid \zeta\cdot(\delta U_j) = \gamma U_j\} \\ &= \{\zeta \in G \mid \delta^{-1}\zeta^{-1}\gamma \in U_j\} = \gamma U_j \delta^{-1}.\end{aligned}$$

Since U_j is open and multiplication on the left or the right is always a homeomorphism in a topological group, we see that $\psi^{-1}(x)$ is an open set in G. ■

3. The proof of the main theorem

With these lemmas in hand, we are ready for

3.1. Theorem. *For any profinite group G, the category R_G^+ equipped with the pretopology $\mathcal{K}$ of epimorphic covers is a small site.*

Before proving the theorem, we first make some remarks about pullbacks in R_G^+ and how this affects our proof. In a category $\mathcal{C}$ with sufficient pullbacks, to prove that a pretopology is given by a function K, which assigns to each object C a collection $K(C)$ of families of morphisms with codomain C, one must prove the stability axiom, which says the following: if $\{f_i\colon C_i \to C \mid i \in I\} \in K(C)$, then for any morphism $g\colon D \to C$, the family of pullbacks

$$\{\pi_L\colon D \times_C C_i \to D \mid i \in I\} \in K(D).$$

Let us examine what this axiom would require of R_G^+.

3.2. Example. The map $G \to *$ forms a covering family and so the stability axiom requires that $G \times_{\{*\}} G = G \times G$ be in R_G^+.

3.3. Example. Let C be any finite discrete G-set with more than one element and with trivial G-action, $g\colon G \to C$ any morphism, and consider the cover

$$\{f_i\colon C_i \to C \mid i \in I\} \in K(C),$$

where $C_j = C$ and $f_j\colon C \to C$ is the morphism mapping C to $g(1)$, for some $j \in I$. Because the action is trivial, f_j is G-equivariant. There certainly exist covers of C of this form, since one could let $f_k = \mathrm{id}_C$, for some $k \neq j$ in I, and then let the other f_i be any morphisms with codomain C. Then the stability axiom requires that $G \times_C C$ exists in R_G^+, but this is impossible, since

$$G \times_C C = \{(\gamma, c) \mid g(\gamma) = f_j(c)\} = \{(\gamma, c) \mid \gamma \cdot g(1) = g(1)\} = G_{g(1)} \times C = G \times C.$$

Thus, the stability axiom for a pretopology must be altered so that one still obtains a topology. We list the correct axioms for our situation below. They are taken from [Mac Lane and Moerdijk, 1994, Exercise 3, pg. 156].

1. If $f : C' \to C$ is an isomorphism, then $\{f : C' \to C\} \in K(C)$.

2. (stability axiom) If $\{f_i\colon C_i \to C \mid i \in I\} \in K(C)$, then for any morphism $g\colon D \to C$, there exists a cover $\{h_j\colon D_j \to D \mid j \in J\} \in K(D)$ such that for each j, $g \circ h_j$ factors through some f_i.

3. (transitivity axiom) If $\{f_i\colon C_i \to C \mid i \in I\} \in K(C)$, and if for each $i \in I$ there is a family $\{g_{ij}\colon D_{ij} \to C_i \mid j \in I_i\} \in K(C_i)$, then the family of composites

$$\{f_i \circ g_{ij}\colon D_{ij} \to C \mid i \in I, j \in I_i\}$$

is in $K(C)$.

Proof of Theorem 3.1. It is clear that the pretopology of epimorphic covers satisfies axiom (1) above. Also, it is easy to see that axiom (3) holds. Indeed, using the above notation, choose any $c \in C$. Then there is some $c_i \in C_i$ for some i, such that $f_i(c_i) = c$. Similarly, there must be some $d_{ij} \in D_{ij}$ for some j, such that $g_{ij}(d_{ij}) = c_i$. Hence, $(f_i \circ g_{ij})(d_{ij}) = f_i(c_i) = c$, so that $\coprod_{i,j} D_{ij} \to C$ is onto. This verifies (3). We verify (2) by considering five cases.

Case (1): Suppose that D and each of the C_i are finite sets in R_G^+. By Lemma 2.1, C must be a finite set. Consider the cover

$$\{\pi_L(i)\colon D \times_C C_i \to D \mid i \in I\},$$

where $\pi_L(i)$ is the obvious map and $g \circ \pi_L(i)$ factors through f_i via the canonical map $\pi_R(i)$. Now choose any $d \in D$ and let $g(d) = c \in C$. Then there exists some i such that

$f_i(c_i) = c$ for $c_i \in C_i$. Thus, $(d, c_i) \in D \times_C C_i$, so that $\coprod_I D \times_C C_i \to D$ maps (d, c_i) to d and is therefore an epimorphism. This shows that $\{\pi_L(i)\}$ is in $K(D)$.

Case (2): Suppose that $D = G$ and that each C_i is a finite set in R_G^+. By Lemma 2.1, C is a finite set and we identify it with $\coprod_{i=1}^n G/U_i$, where $U_i = G_{x_i}$, the stabilizer of x_i in G. The map g is determined by $g(1) = \delta U_k$ for some $\delta \in G$ and some stabilizer U_k. Since $\coprod_I C_i \to C$ is onto and $\mathrm{im}(g) = G/U_k$, there exists some $c_l \in C_l$ such that $f_l(c_l) = U_k$. Since C_l is a finite set, we can identify c_l with some μG_z, where $\mu \in G$ and G_z is the stabilizer of some element $z \in C_l$.

Then define the cover to be $\{\lambda\colon G \to G\}$, where $\lambda(\gamma) = \gamma\delta^{-1}$. Define $\alpha_l\colon G \to C_l$ to be the G-equivariant map given by $1 \mapsto \mu G_z$. By Lemma 2.5, α_l is continuous and is a morphism in R_G^+. Since λ is a homeomorphism, the cover $\{\lambda\}$ is in $K(D)$. Now,

$$(g \circ \lambda)(1) = g(\delta^{-1}) = \delta^{-1} \cdot g(1) = U_k = \mu \cdot f_l(G_z) = \mu \cdot f_l(\mu^{-1} \cdot \alpha_l(1)) = (f_l \circ \alpha_l)(1).$$

This shows that $g \circ \lambda$ factors through f_l via α_l.

Case (3): Suppose not all the C_i are finite sets and that $D = G$. Also, assume that $C = G$. This implies that $C_i = G$ for all $i \in I$. Choose any $k \in I$, let $\alpha_k = \mathrm{id}_G$, and define $\lambda\colon G \to G$ to be multiplication on the right by $f_k(1)g(1)^{-1}$. Then the diagram

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{id}_G} & G \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle f_k} \\ G & \xrightarrow{g} & G \end{array}$$

is commutative, since

$$(g \circ \lambda)(1) = g(f_k(1)g(1)^{-1}) = f_k(1)g(1)^{-1} \cdot g(1) = f_k(1) = (f_k \circ \alpha_k)(1).$$

Thus, $g \circ \lambda$ factors through f_k via α_k, so that the stability axiom is verified by letting the covering family be $\{\lambda\}$.

Case (4): Suppose that not all the C_i are finite sets, $D = G$, and C is a finite set. With C as in Lemma 2.4, let $g(1) = \delta U_k \in C$, as in Case (2). Then there exists some l such that $f_l(c_l) = U_k$, for some $c_l \in C_l$. Now we consider two subcases.

Case (4a): Suppose that C_l is a finite set in R_G^+. Just as in Case (2), we construct maps λ and α_l, so that $g \circ \lambda$ factors through f_l via α_l and $\{\lambda\} \in K(D)$.

Case (4b): Suppose that $C_l = G$. By G-equivariance, $f_l(1) = c_l^{-1} U_k$. Then define $\lambda\colon G \to G$ by $1 \mapsto \delta^{-1}$ and $\alpha_l\colon G \to G$ by $1 \mapsto c_l$. Then $g \circ \lambda$ factors through f_l via α_l, since

$$(g \circ \lambda)(1) = g(\delta^{-1}) = \delta^{-1} \cdot g(1) = U_k = f_l(c_l) = (f_l \circ \alpha_l)(1).$$

Thus, the cover $\{\lambda\}$, as a homeomorphism, is in $K(D)$. This completes Case (4).

Now we consider the final possibility, *Case (5)*: suppose that not all of the C_i are finite sets and suppose that D is a finite set. This implies that C is a finite set. This case is more difficult than the others because the cover consists of more than one morphism and it combines the previous constructions. For each $d \in D$, we make a choice of some $c_l \in C_l$ for some l, such that c_l is in the preimage of $g(d)$ under $\coprod C_i \to C$. Then write $D = D_{df} \coprod D_G$, where D_{df} is the set of all d such that the corresponding C_l is in a finite set, and D_G is the set of all d such that the corresponding $C_l = G$. Now consider the cover $\{h_d \colon D_d \to D | \, d \in D = D_{df} \coprod D_G\}$, where

$$D_d = \begin{cases} D \times_C C_d & \text{if } d \in D_{df}, \\ G & \text{if } d \in D_G. \end{cases}$$

If $d \in D_{df}$, then $h_d = \pi_L$ and $\alpha_d \colon D \times_C C_d \to C_d$ is the canonical map π_R; it is clear that the required square commutes. Now suppose $d \in D_G$. Then there exists $c_l \in C_l = G$ for some l, such that $g(d) = f_l(c_l)$. We write $f_l(1) = \theta U_k \in C$ for some $\theta \in G$ and for some stabilizer U_k. Then we define $\alpha_d \colon G \to C_l = G$ by $1 \mapsto \theta^{-1}$. Also, we define $h_d \colon G \to D$ by $1 \mapsto (\theta^{-1} c_l^{-1}) \cdot d$. Lemma 2.5 shows that h_d is a morphism in R_G^+. Then we have the required commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha_d} & G \\ {\scriptstyle h_d}\downarrow & & \downarrow{\scriptstyle f_l} \\ D & \xrightarrow{g} & C, \end{array}$$

since

$$\begin{aligned} (g \circ h_d)(1) &= g((\theta^{-1} c_l^{-1}) \cdot d) = (\theta^{-1} c_l^{-1}) \cdot g(d) \\ &= (\theta^{-1} c_l^{-1}) \cdot f_l(c_l) = f_l(\theta^{-1}) = (f_l \circ \alpha_d)(1). \end{aligned}$$

The only remaining detail is to show that $\{h_d\} \in K(D)$; that is, we must show that $\phi \colon \coprod_D D_d \to D$ is an epimorphism. Let d be any element in D. Suppose $d \in D_{df}$. Then, using our choice above, there exists some $c_l \in C_l$, a finite set for some l, such that $f_l(c_l) = g(d)$. Then $(d, c_l) \in D \times_C C_l$ and $\phi(d, c_l) = \pi_L(d, c_l) = d$. Now suppose $d \in D_G$. With c_l and θ as above, $c_l\theta \in D_d = G$ and $\phi(c_l\theta) = h_d(c_l\theta) = (c_l\theta) \cdot h_d(1) = d$. Therefore, ϕ is an epimorphism. ∎

4. The site R_G^+ does not have the canonical topology

Now that we have established that R_G^+ is a site with pretopology $\mathcal{K}$, we begin working to show that, contrary to what typically happens with this pretopology, it does not give the canonical topology. We start with a definition.

4.1. Definition. *If T is some collection of morphisms with codomain C, where C is an object in the category $\mathcal{C}$, then (T) denotes the sieve generated by T. Thus,*

$$(T) = \{f \circ g | \, f \in T, \ \mathrm{dom}(f) = \mathrm{cod}(g)\}.$$

4.2. Lemma. *Let K be a pretopology on a category $\mathcal{C}$. Let J be the Grothendieck topology generated by K. Then for any $C \in \mathcal{C}$, $J(C)$ consists exactly of all $(R) \cup (T)$ such that $R \in K(C)$ and T is some collection of morphisms with codomain C.*

Proof. Let S be a covering sieve of C. Then there exists some $R \in K(C)$ such that $R \subset S$. We will prove that $S = (R) \cup (S)$, verifying the forward inclusion. To prove equality it suffices to show that $(R) \cup (S) \subset S$. If $f \in (R)$, $f = g \circ h$ for some $g \in R$ and some h with $\mathrm{dom}(g) = \mathrm{cod}(h)$. Since $g \in S$, $f \in S$. Similarly, if $f \in (S)$, then $f \in S$. Now consider any family of morphisms $(R) \cup (T)$ as described in the statement of the lemma. Since $R \subset (R) \cup (T)$, $(R) \cup (T) \in J(C)$ if it is a sieve. Since (R) and (T) are sieves, it is clear that $(R) \cup (T)$ is also a sieve. ∎

This result is useful for understanding the topology of a site, when the site is defined in terms of a pretopology. For example, $G-\mathbf{Sets}_{df}$ is a site by the pretopology $\mathcal{K}$ and its category of sheaves of sets is equivalent to the category of sheaves on the site $S(G)$ consisting of quotients of G by open subgroups (the morphisms are the G-equivariant maps), where $S(G)$ is given the atomic topology (see [Mac Lane and Moerdijk, 1994, Chapter 3, Section 9]). Thus, one might ask if $G-\mathbf{Sets}_{df}$ also has the atomic topology. However, the lemma allows us to see that $\mathcal{K}$ generates a topology that is coarser than the atomic topology. To see this, let $X = G/U$ and $Y = G/U \coprod G/U$, where U is a proper open subgroup of G. (Since G is an infinite profinite group, the canonical way of writing G as an inverse limit guarantees the existence of such a U.) We define $f \colon X \to Y$ by $f(U) = U$, where U lives in the factor on the left; f is the left inclusion. Now consider the sieve $S = (\{f\})$. Clearly, S does not contain an epimorphic cover, since $\mathrm{im}(\coprod_{g \in S}(\mathrm{dom}(g)) \to Y) = G/U$. The lemma indicates that every sieve of G–$\mathbf{Sets}_{df}$ must contain an epimorphic cover, so that S is not a sieve for Y in the topology generated by $\mathcal{K}$.

Now we consider the site R_G^+ with the pretopology $\mathcal{K}$ of epimorphic covers. We use $\mathrm{Hom}_G(X, Y)$ to denote continuous G-equivariant maps between continuous G-sets X and Y. Recall that a presheaf of sets P on a site $(\mathcal{C}, J)$ is a sheaf, if for each object $C \in \mathcal{C}$ and each covering sieve $S \in J(C)$, the diagram

$$P(C) \xrightarrow{e} \prod_{f \in S} P(\mathrm{dom}(f)) \underset{a}{\overset{p}{\rightrightarrows}} \prod P(\mathrm{dom}(g))$$

is an equalizer of sets, where the second product is over all f, g, with $f \in S$, $\mathrm{dom}(f) = \mathrm{cod}(g)$. Here, e is the map $e(x) = \{P(f)(x)\}_f$, p is given by

$$\{x^f\}_f \mapsto \{x^{fg}\}_{f,g},$$

and a is given by

$$\{x^f\}_f \mapsto \{P(g)(x^f)\}_{f,g} = \{x^f \circ g\}_{f,g}.$$

Recall that a representable presheaf of R_G^+ is any presheaf which, up to isomorphism, has the form of $\mathrm{Hom}_G(-, C)$ for some $C \in R_G^+$. Also, the Yoneda embedding

$$R_G^+ \to \mathbf{Sets}^{(R_G^+)^{\mathrm{op}}}, \quad C \mapsto \mathrm{Hom}_G(-, C)$$

is a full and faithful functor, so that one can identify C with an object of $\mathbf{Sets}^{(R_G^+)^{\rm op}}$. We now consider which objects of R_G^+ yield sheaves of sets on R_G^+.

Noting that the empty set is a discrete G-set, we have

4.3. LEMMA. *The presheaf* $\mathrm{Hom}_G(-,\varnothing)$ *is a sheaf of sets on the site* R_G^+.

PROOF. Let $\bullet : \varnothing \to X$ denote the vacuous map, for any $X \in R_G^+$. Since $\bullet : \varnothing \to \varnothing$ is vacuously an epimorphism, $\{\bullet\}$ is the unique covering sieve for $\varnothing$. Let $C = \varnothing$. Then the desired equalizer diagram has the form

$$\mathrm{Hom}_G(\varnothing,\varnothing) = \{\bullet\} \xrightarrow{e} \{\bullet\} \overset{p}{\underset{a}{\rightrightarrows}} \{\bullet\}.$$

It is clear that this is an equalizer diagram.

Now let C be a nonempty finite set in $G-\mathbf{Sets}_{df}$. Let S be any covering sieve of C. There must exist a morphism in S with domain equal to a nonempty object in R_G^+. Therefore, since $\varnothing \times Z = \varnothing$ for any space Z, we have

$$\mathrm{Hom}_G(C,\varnothing) = \varnothing \xrightarrow{e} \varnothing \overset{p}{\underset{a}{\rightrightarrows}} \varnothing.$$

Since the equalizer must exist and the vacuous map $\bullet : \varnothing \to \varnothing$ is the unique map with codomain $\varnothing$, this must be an equalizer diagram.

Finally, letting $C = G$, we get

$$\mathrm{Hom}_G(G,\varnothing) = \varnothing \xrightarrow{e} \prod_{f\in \mathrm{Hom}_G(G,G)} \varnothing = \varnothing \overset{p}{\underset{a}{\rightrightarrows}} \varnothing.$$

Again, this is an equalizer diagram. ∎

To prove the next theorem, we need the following lemma.

4.4. LEMMA. *If G is a compact topological group, U an open subgroup of G, and $X \neq \varnothing$ a finite discrete G-set, then*

$$\mathrm{Hom}_G(G/U, X) \cong \{x \in X \,|\, U < G_x\},$$

where G_x is the stabilizer of x in G.

PROOF. Let $f\colon G/U \to X$. It is clear that f is G-equivariant if and only if it is completely determined by $f(U)$ in the obvious way. Since U is an open subgroup, it has finite index in G, so that G/U is a discrete space. Thus, any G-equivariant map $G/U \to X$ is continuous. The key is that f is well-defined if and only if $U < G_{f(U)}$. To see this, first assume that f is well-defined; let $\gamma \in U$. Then $\gamma U = U$, so that $\gamma \cdot f(U) = f(\gamma U) = f(U)$. Hence, $\gamma \in G_{f(U)}$ and $U < G_{f(U)}$. Now suppose that $U < G_{f(U)}$ and take any $\gamma U = \delta U$. This implies that $\gamma^{-1}\delta \in U$ and hence, in $G_{f(U)}$. Thus, $(\gamma^{-1}\delta)\cdot f(U) = f(U)$, so that $\gamma \cdot f(U) = \delta \cdot f(U)$. Equivariance gives $f(\gamma U) = f(\delta U)$ and f is well-defined. Thus,

$$\mathrm{Hom}_G(G/U, X) \cong \{f(U) \in X \,|\, U < G_{f(U)}\}.$$

∎

Henceforth, let $\mathcal{J}$ denote the topology of R_G^+ generated by $\mathcal{K}$.

4.5. Theorem. *Let X be any object in R_G^+ that is not a finite discrete trivial G-set, where G is an infinite profinite group. Then the presheaf* $\mathrm{Hom}_G(-, X)$ *is not a sheaf of sets on the site R_G^+.*

Proof. Suppose $\mathrm{Hom}_G(-, X)$ is a sheaf of sets on the site R_G^+. The equalizer condition says that for every object $C \in R_G^+$ and for every covering sieve $S \in \mathcal{J}(C)$,

$$\mathrm{Hom}_G(C, X) \cong \{\{h^f\}_f \,|\, h^{fg} = h^f \circ g, f, g, f \in S, \mathrm{dom}(f) = \mathrm{cod}(g)\},$$

where for $f \in S$, $h^f \in \mathrm{Hom}_G(\mathrm{dom}(f), X)$. We will construct an example of some C and S such that this sheaf condition fails to be true with X as above.

Let $C \in G\mathrm{-}\mathbf{Sets}_{df}$; we identify C with $\coprod_{i=1}^n G/U_i$, where each U_i is an open subgroup of G. For each i, define $f_i \colon G \to C$ by $1 \mapsto U_i$. Thus, $\mathrm{im}(f_i) = G/U_i$ and $\{f_i\}$ is an epimorphic cover of C. The preceding lemma tells us that $S = (\{f_i\})$ is a covering sieve of C. For this S, we will examine the sheaf condition. Let $S = S' \cup S''$, where $S' = \{f_i\}$ and S'' is the complement of S' in S. Thus, every $k \in S''$ has the form $k = f_i \circ g$ for some g with $\mathrm{dom}(f_i) = \mathrm{cod}(g)$. Then

$$\begin{aligned}
&\{\{h^f\}_f \,|\, h^{fg} = h^f \circ g, f, g, f \in S, \mathrm{dom}(f) = \mathrm{cod}(g)\} \\
&\quad = \{\{h^{f_i}\}_{f_i} \times \{h^k\}_{k \in S''} \,|\, h^{fg} = h^f \circ g, f, g, f \in S, \mathrm{dom}(f) = \mathrm{cod}(g)\} \\
&\quad = \{\{h^{f_i}\}_{f_i} \times \{h^{f_i} \circ g\}_{f_i \circ g \in S''} \,|\, h^{fg} = h^f \circ g, f, g, f \in S, \mathrm{dom}(f) = \mathrm{cod}(g)\} \\
&\quad = \{\{h^{f_i}\}_{f_i} \times \{h^{f_i} \circ g\}_{f_i \circ g \in S''} \,|\, h^{f_i} \in \mathrm{Hom}_G(G, X), f_i \in S', g, \mathrm{dom}(f_i) = \mathrm{cod}(g)\}.
\end{aligned}$$

We verify the last equality. Suppose h^{f_i} is any morphism in $\mathrm{Hom}_G(G, X)$. Now take any f and g with $f \in S$ and $\mathrm{dom}(f) = \mathrm{cod}(g)$. If $f = f_i \in S'$, then $h^f \circ g = h^{f_i} \circ g = h^{f_i \circ g} = h^{f \circ g}$, by construction. Now suppose $f \in S''$. Then $f = f_i \circ k$ for some $k \colon G \to G$. Thus,

$$h^{fg} = h^{f_i \circ (k \circ g)} = (h^{f_i} \circ k) \circ g = h^{f_i \circ k} \circ g = h^f \circ g.$$

Since $h^{f_i} \circ g$ is determined by h^{f_i} and $f_i \circ g$, we see that the set

$$\{\{h^{f_i}\}_{f_i} \times \{h^{f_i} \circ g\}_{f_i \circ g \in S''} \,|\, h^{f_i} \in \mathrm{Hom}_G(G, X), f_i \in S', g, \mathrm{dom}(f_i) = \mathrm{cod}(g)\}$$

is isomorphic to the set

$$\{\{h^{f_i}\}_{f_i} \,|\, h^{f_i} \in \mathrm{Hom}_G(G, X), f_i \in S'\} = \mathrm{Hom}_G(G, X)^n,$$

where $\mathrm{Hom}_G(G, X)^n$ is the n-fold Cartesian product of $\mathrm{Hom}_G(G, X)$. Now, there is an isomorphism $\mathrm{Hom}_G(G, X)^n \cong X^n$. Therefore, for $\mathrm{Hom}_G(-, X)$ to be a sheaf, it must be that $\mathrm{Hom}_G(C, X) \cong X^n$ for every $C \in G - \mathbf{Sets}_{df}$. If $X = G$ and $C \neq \varnothing$ is in $G - \mathbf{Sets}_{df}$, then $\mathrm{Hom}_G(C, G) = \varnothing$, whereas, since $|C| \geq 1$, $n \geq 1$ and $X^n = G^n$. Thus, $\mathrm{Hom}_G(-, G)$ is not a sheaf.

Now we consider $X \neq G$ and assume that $\mathrm{Hom}_G(C, X) \cong X^n$ for every $C \in G$–$\mathbf{Sets}_{df}$. This implies that

$$\begin{aligned} X^n &\cong \mathrm{Hom}_G(C, X) \cong \mathrm{Hom}_G(\textstyle\coprod_{i=1}^n G/U_i, X) \\ &\cong \textstyle\prod_{i=1}^n \mathrm{Hom}_G(G/U_i, X) \cong \prod_{i=1}^n \{x \in X \,|\, U_i < G_x\} \subset X^n. \end{aligned}$$

Therefore, it must be that $\{x \in X \,|\, U_i < G_x\} = X$, for all $i = 1, ..., n$. Thus, $U_i < G_x$ for all $x \in X$ and each i. Now let us write $X \cong \coprod_{j=1}^m G/G_{x_j}$, where each x_j is a representative from a distinct orbit of X. Let C be a trivial G-set so that every stabilizer of $c \in C$ in G is equal to G. This implies that $G < G_{x_j}$ for all j. Thus, each $G_{x_j} = G$. This indicates that X must be a trivial G-set. This contradiction shows that every X violates the sheaf condition for some C and S. ∎

This result immediately yields

4.6. Corollary. *For an infinite profinite group G, the site R_G^+ with the pretopology $\mathcal{K}$ of epimorphic covers is not subcanonical.*

Proof. There exists a proper open subgroup U of G satisfying $[G : U] > 1$. Thus, the representable presheaves $\mathrm{Hom}_G(-, G)$ and $\mathrm{Hom}_G(-, \coprod_{i=1}^n G/U)$, for any $n \geq 1$, are not sheaves. ∎

Since a canonical topology is, by definition, subcanonical, we obtain

4.7. Corollary. *For an infinite profinite group G, the site R_G^+, with the pretopology $\mathcal{K}$, is not canonical.*

The next result is an elementary fact about profinite groups that helps us understand "how often" representable presheaves fail to be sheaves in R_G^+ and what such "failing" presheaves can look like, based on what we know from Theorem 4.5.

4.8. Lemma. *If G is an infinite profinite group, then G contains an infinite number of distinct proper open subgroups.*

Proof. We have already seen that G has at least one proper open subgroup. Suppose that G has only a finite number of distinct proper open subgroups. Then G has a finite number of distinct proper open normal subgroups $N_1, ..., N_k$. Since G is profinite, $N = \bigcap_{i=1}^k N_i = \{1\}$. Because N is an open subgroup with finite index, it has uncountable order. This contradiction gives the conclusion. ∎

4.9. Remark. Since any topology finer than $\mathcal{J}$ would contain the covering sieve $(\{f_i\})$ that was the key to Theorem 4.5, no topology finer than $\mathcal{J}$ can be subcanonical.

5. Presheaves of spectra on the site R_G^+

Let **Ab** be the category of abelian groups, and let Spt denote the model category of Bousfield-Friedlander spectra of pointed simplicial sets. We refer to the objects of Spt as simply "spectra." Now that R_G^+ is a site, we can consider the category $\mathrm{PreSpt}(R_G^+)$ of presheaves of spectra on the site R_G^+. By applying the work of Jardine ([Jardine, 1987], [Jardine, 1997, Section 2.3]), $\mathrm{PreSpt}(R_G^+)$ is a model category. We recall the critical definitions that give the model category structure and then we state Jardine's result, when it is applied to R_G^+.

5.1. Definition. *Let* $P\colon (R_G^+)^{\mathrm{op}} \to \mathrm{Spt}$ *be a presheaf of spectra. Then, for each* $n \in \mathbb{Z}$,

$$\pi_n(P)\colon (R_G^+)^{\mathrm{op}} \to \mathbf{Ab}, \quad C \mapsto \pi_n(P(C)),$$

is a presheaf of abelian groups. Then the associated sheaf $\tilde{\pi}_n(P)$ *of abelian groups is the sheafification of* $\pi_n(P)$.

Let $f\colon P \to Q$ be a morphism of presheaves of spectra on R_G^+. Then f is a *weak equivalence* if the induced map $\tilde{\pi}_n(P) \to \tilde{\pi}_n(Q)$ of sheaves is an isomorphism, for all $n \in \mathbb{Z}$. The map f is a *cofibration* if $f(C)$ is a cofibration of spectra, for all $C \in R_G^+$. Also, f is a *global fibration* if f has the right lifting property with respect to all morphisms which are weak equivalences and cofibrations.

5.2. Theorem. [Jardine, 1997, Theorem 2.34] *The category* $\mathrm{PreSpt}(R_G^+)$*, together with the classes of weak equivalences, cofibrations, and global fibrations, is a model category.*

Now we give some interesting examples of presheaves of spectra on the site R_G^+.

5.3. Example. In the Introduction, we saw that the Devinatz-Hopkins functor $\mathbf{F}$ is an example of an object in $\mathrm{PreSpt}(R_{G_n}^+)$.

For the next example, if X is a spectrum, then, for each $k \geq 0$, we let X_k be the kth pointed simplicial set constituting X, and, for each $l \geq 0$, $X_{k,l}$ is the pointed set of l-simplices of X_k.

5.4. Example. Let X be a discrete G-spectrum (see [Davis, 2006] for a definition of this term), so that each $X_{k,l}$ is a pointed discrete G-set. If $C \in R_G^+$, then let $\mathrm{Hom}_G(C, X)$ be the spectrum, such that

$$\mathrm{Hom}_G(C, X)_k = \mathrm{Hom}_G(C, X_k),$$

where

$$\mathrm{Hom}_G(C, X)_{k,l} = \mathrm{Hom}_G(C, X_k)_l = \mathrm{Hom}_G(C, X_{k,l}).$$

Above, the set $X_{k,l}$ is given the discrete topology, since it is naturally a discrete G-set. Then $\mathrm{Hom}_G(-, X)$ is an object in $\mathrm{PreSpt}(R_G^+)$. It is easy to see that if U is an open subgroup of G, then $\mathrm{Hom}_G(G/U, X) \cong X^U$, the U-fixed point spectrum of X. Also, $\mathrm{Hom}_G(G, X) \cong X$.

Now we recall part of [Behrens and Davis, 2005, Proposition 3.3.1], since this result (and its corollary) will be helpful in our next example. We note that this result is only a slight extension of [Jardine, 1997, Remark 6.26]: if U is normal in G, then the lemma below is an immediate consequence of Jardine's remark.

5.5. Lemma. *Let X be a discrete G-spectrum. Also, let $f: X \to X_{f,G}$ be a trivial cofibration, such that $X_{f,G}$ is fibrant, where all this takes place in the model category of discrete G-spectra (see* [Davis, 2006]*). If U is an open subgroup of G, then $X_{f,G}$ is fibrant in the model category of discrete U-spectra.*

5.6. Corollary. *Let X and U be as in the preceding lemma. Then $X^{hU} = (X_{f,G})^U$.*

Proof. Let f be as in the above lemma. Since f is G-equivariant, it is U-equivariant. Also, since f is a trivial cofibration in the model category of discrete G-spectra, it is a trivial cofibration in the model category of spectra. The preceding two facts imply that f is a trivial cofibration in the model category of discrete U-spectra. By the lemma, $X_{f,G}$ is fibrant in this model category. Thus, $X^{hU} = (X_{f,G})^U$. ∎

5.7. Example. Let X be a discrete G-spectrum. Then $\mathrm{Hom}_G(-, X_{f,G})$ is a presheaf in $\mathrm{PreSpt}(R_G^+)$. In particular, notice that

$$\mathrm{Hom}_G(G/U, X_{f,G}) \cong (X_{f,G})^U = X^{hU}$$

and

$$\mathrm{Hom}_G(G, X_{f,G}) \cong X_{f,G} \simeq X.$$

5.8. Example. For any unfamiliar concepts in this example, we refer the reader to [Davis, 2006]. Let $Z = \mathrm{holim}_i Z_i$ be a continuous G-spectrum, so that $\{Z_i\}_{i \geq 0}$ is a tower of discrete G-spectra, such that each Z_i is a fibrant spectrum. Then

$$P(-) = \operatorname*{holim}_i \mathrm{Hom}_G(-, (Z_i)_{f,G}) \in \mathrm{PreSpt}(R_G^+),$$

where

$$P(G/U) \cong \operatorname*{holim}_i ((Z_i)_{f,G})^U = \operatorname*{holim}_i (Z_i)^{hU} = Z^{hU}$$

and

$$P(G) \cong \operatorname*{holim}_i (Z_i)_{f,G} \simeq Z.$$

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Department of Mathematics
Wesleyan University
265 Church St.
Middletown, CT 06459-0128

Email: dgdavis@wesleyan.edu