LOCALIZATIONS

W. G. DWYER

1. Introduction

The aim of this paper is to describe the concept of localization, as it usually comes up in topology, and give some examples of it. Many of the results we will describe are due to Bousfield.

1.1. **Localization.** We start with the following very simple situation:

- $\mathcal{C}$ is a category, and
- $\mathcal{E} \subset \mathcal{C}$ is a subcategory.

The morphisms in $\mathcal{E}$ are called *equivalences*, and they are maps which for one reason or another we want to treat as honorary isomorphisms. The pair $(\mathcal{C}, \mathcal{E})$ is called a *localization context*.

1.2. **Definition.** An object X of $\mathcal{C}$ is said to be $\mathcal{E}$*-local* (or just *local* if $\mathcal{E}$ is understood) if any morphism $f : A \to B$ in $\mathcal{E}$ induces a bijection

$$f^* : \mathrm{Hom}_{\mathcal{C}}(B, X) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(A, X)\,.$$

Roughly speaking, X is local if equivalent objects of $\mathcal{C}$ cannot be distinguished by mapping them into X.

1.3. **Definition.** A map $\epsilon : A \to X$ in $\mathcal{C}$ is said to be an $\mathcal{E}$*-localization* of A (or just a *localization* of A) if X is local and ϵ is an equivalence. The pair $(\mathcal{C}, \mathcal{E})$ is said to *have good localizations* if every object of $\mathcal{C}$ has a localization.

Usually we refer to X as the localization of A, and leave the map $A \to X$ understood. Suppose that X is a localization of A, Y is a localization of B, and $f : A \to B$ is a map. It is easy to see that there is a unique map $f' : X \to Y$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\epsilon_A} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{\epsilon_B} & Y \end{array}$$

Date: April 28, 2003.

commutes. With a little argument, this shows that any two localizations of A are canonically isomorphic; it also shows that if $(\mathcal{C}, \mathcal{E})$ has good localizations, there is a *localization functor* $L_{\mathcal{E}}$ which assigns to every object A a functorial localization $L_{\mathcal{E}}(A)$. If f is a morphism in $\mathcal{E}$, then $L_{\mathcal{E}}(f)$ is an isomorphism, and it is easy to see that the reverse implication holds if $\mathcal{E}$ satisfies the *two out of three property*, i.e., if whenever f and g are composable morphisms of $\mathcal{C}$ and two of f, g, and gf belongs to $\mathcal{E}$, then the third belongs to $\mathcal{E}$ as well. The intuition is that $L_{\mathcal{E}}(A)$ captures exactly the information in A which is invariant under the notion of equivalence provided by $\mathcal{E}$.

That's the general machinery. There is one main theorem. Let $\mathcal{E}^{-1}\mathcal{C}$ be the category obtained from $\mathcal{C}$ by formally inverting the arrows of $\mathcal{E}$; the functor $\mathcal{C} \to \mathcal{E}^{-1}\mathcal{C}$ is universal among functors $\mathcal{C} \to \mathcal{D}$ which send all of the arrows of $\mathcal{E}$ to isomorphisms. (There may be set theoretic difficulties in forming $\mathcal{E}^{-1}\mathcal{C}$, but the following theorem can be interpreted as saying that these difficulties don't come up in the cases we consider.)

1.4. **Theorem.** *Suppose that $(\mathcal{C}, \mathcal{E})$ has good localizations. Let* $\operatorname{Loc}_{\mathcal{E}}(\mathcal{C})$ *denote the full subcategory of $\mathcal{C}$ given by the local objects. Then the composite functor*

$$\operatorname{Loc}_{\mathcal{E}}(\mathcal{C}) \to \mathcal{C} \to \mathcal{E}^{-1}\mathcal{C}$$

is an equivalence of categories.

Proof. Since the localization functor $L_{\mathcal{E}} : \mathcal{C} \to \operatorname{Loc}_{\mathcal{E}}(\mathcal{C})$ sends all of the morphisms in $\mathcal{E}$ to isomorphisms in $\operatorname{Loc}_{\mathcal{E}}(\mathcal{C})$, it extends to a functor $L'_{\mathcal{E}} : \mathcal{E}^{-1}\mathcal{C} \to \operatorname{Loc}_{\mathcal{E}}(\mathcal{C})$. We leave it to the reader to verify that, up to natural equivalence, the functor $L'_{\mathcal{E}}$ is inverse to the functor appearing in the statement of the theorem. □

1.5. *A slight twist.* Sometimes the localization context $(\mathcal{C}, \mathcal{E})$ is described by specifying the collection of local objects. In this case the equivalences $\mathcal{E}$ are determined by working backwards from 1.2: a map $f : A \to B$ is an equivalence if for each local object X, the map $f^* : \operatorname{Hom}(B, X) \to \operatorname{Hom}(A, X)$ is a bijection.

Next we briefly discuss a variant of the localization idea.

1.6. **Colocalization.** Colocalization is localization in the opposite category, but it's worthwhile to be a little more explicit than this, if only to establish some notation. Suppose that $(\mathcal{C}, \mathcal{E})$ is a pair of categories as above; again, the morphisms in $\mathcal{E}$ are called equivalences.. An object X of $\mathcal{C}$ is said to be *colocal* if any equivalence $f : A \to B$ induces a bijection

$$f_* : \operatorname{Hom}_{\mathcal{C}}(X, A) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{C}}(X, B)\,.$$

Roughly speaking, X is colocal if equivalent objects of $\mathcal{C}$ cannot be distinguished by mapping X into them. If A is an object of $\mathcal{C}$, a map $\eta : X \to A$ is said to be a *colocalization* of A if X is colocal and η is an equivalence. The pair $(\mathcal{C}, \mathcal{E})$ is said to *have good colocalizations* if every object of $\mathcal{C}$ has a colocalization. As before, any two colocalizations of A are canonically isomorphic, and, if $(\mathcal{C}, \mathcal{E})$ has good colocalizations, there is a *colocalization functor* $C_{\mathcal{E}}$ which assigns to every A a functorial colocalization $C_{\mathcal{E}}(A) \to A$. If f in $\mathcal{C}$ is an equivalence then $C_{\mathcal{E}}(f)$ is an isomorphism, and the converse holds if $\mathcal{E}$ satisfies the two-out-of-three condition. Let $\mathrm{Coloc}_{\mathcal{E}}(\mathcal{C})$ denote the full subcategory of $\mathcal{C}$ given by the colocal objects; if $(\mathcal{C}, \mathcal{E})$ has good colocalizations, then the composite functor

$$\mathrm{Coloc}_{\mathcal{E}}(\mathcal{C}) \to \mathcal{C} \to \mathcal{E}^{-1}\mathcal{C}$$

is an equivalence of categories.

1.7. *Remark.* Note that if $(\mathcal{C}, \mathcal{E})$ has both good localizations and good colocalizations, then the category $\mathrm{Coloc}_{\mathcal{E}}(\mathcal{C})$ of $\mathcal{E}$-colocal objects is equivalent to the category $\mathrm{Loc}_{\mathcal{E}}(\mathcal{C})$ of $\mathcal{E}$-local objects, since both are equivalent to $\mathcal{E}^{-1}\mathcal{C}$ [12, 2.6]. The equivalences in question are given by the restricting the localization functor to $\mathrm{Coloc}_{\mathcal{E}}(\mathcal{C})$ and restricting the colocalization functor to $\mathrm{Loc}_{\mathcal{E}}(\mathcal{C})$.

1.8**. Organization of the paper.** Section 2 describes some algebraic (co-)localization constructions, including local homology and local cohomology for commutative rings. Section 3 discusses homological localization for spaces and spectra, while §4 treats the more general idea of localization with respect to a map. In a diagram setting, this kind of localization leads to Goodwillie calculus (4.10) and to motivic homotopy theory (4.13). The next section describes colocalization with respect to an object, and mentions a class of pairs $(\mathcal{C}, \mathcal{E})$ which have both good localizations and good colocalizations (5.6). Section 6 investigates the higher order structure of $\mathcal{E}^{-1}\mathcal{C}$ when $(\mathcal{C}, \mathcal{E})$ has good localizations, and §7 concludes with some remarks on how localizations and colocalizations are constructed.

1.9**. On (not) working up to homotopy.** In most of the examples below, the ambient category $\mathcal{C}$ is the homotopy category associated to some model category $\mathcal{M}$. For instance, $\mathcal{M}$ might be the category of spaces, the category of spectra, or the category of chain complexes over a ring R, in which cases (given the right weak equivalences in $\mathcal{M}$) $\mathcal{C} = \mathrm{Ho}(\mathcal{M})$ is the homotopy category of CW-complexes, the homotopy category of spectra, or the derived category of R (2.2). Suppose in general that $\mathcal{M}$ is a model category and that $\mathcal{E}$ is some subcategory of $\mathrm{Ho}(\mathcal{M})$, so that the localization functor $L_{\mathcal{E}}$, if it exists, assigns to each object X of $\mathcal{M}$ a local object $L_{\mathcal{E}}(X)$ in $\mathrm{Ho}(\mathcal{M})$ together with an $\mathcal{E}$-equivalence $X \to L_{\mathcal{E}}(X)$ in $\mathrm{Ho}(\mathcal{M})$. In most of the interesting cases, the localization functor $L_{\mathcal{E}} : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$ can be covered by a more rigid functor $\hat{L}_{\mathcal{E}}$ defined on $\mathcal{M}$ or at least on the subcategory $\mathcal{M}_{\mathrm{c}}$ of cofibrant objects in $\mathcal{M}$. The functor $\hat{L}_{\mathcal{E}}$ takes values in $\mathcal{M}$ itself, and lies in a commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_{\mathrm{c}} & \xrightarrow{\hat{L}_{\mathcal{E}}} & \mathcal{M} \\
\downarrow & & \downarrow \\
\mathrm{Ho}(\mathcal{M}) & \xrightarrow{L_{\mathcal{E}}} & \mathrm{Ho}(\mathcal{M})
\end{array}
$$

in which the vertical arrows are given by the natural map $\mathcal{M} \to \mathrm{Ho}(\mathcal{M})$. In addition, for each cofibrant object X of $\mathcal{M}$ there is a natural map $X \to \hat{L}_{\mathcal{E}}(X)$ which projects in $\mathrm{Ho}(\mathcal{M})$ to the localization map $X \to L_{\mathcal{E}}(X)$. There are many reasons to seek such a functor $\hat{L}_{\mathcal{E}}$. For instance, if $\mathcal{M}$ is the category of spaces there are ways to apply $\hat{L}_{\mathcal{E}}$ fibrewise in a fibration [18, §1], while there is no good way to do this with a functor like $L_{\mathcal{E}}$ taking values in the homotopy category. Strictly speaking, having a rigid localization functor is even necessary to discuss something like the arithmetic square (3.6), since the homotopy pullback of a diagram in the homotopy category isn't functorially determined.

1.10. *Localized model categories.* Even more is usually true [31] [12, 9.9] [9, 4.6]. Write $\mathcal{M} = (\mathcal{M}, \mathcal{W})$, where $\mathcal{W}$ is the category of weak equivalences in $\mathcal{M}$, so that $\mathrm{Ho}(\mathcal{M}) = \mathcal{W}^{-1}\mathcal{M}$. (See §6 for an explanation of why the pair $(\mathcal{M}, \mathcal{W})$ itself almost never has good localizations or colocalizations.) Let $p : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ be the projection, $\mathcal{E}$ the subcategory of $\mathrm{Ho}(\mathcal{M})$ that is providing the localization context, and $\mathcal{W} + \mathcal{E}$ the subcategory of $\mathcal{M}$ consisting of all maps f in $\mathcal{M}$ with $p(f) \in \mathcal{E}$. If $\mathcal{E}$ contains all of the isomorphisms of $\mathrm{Ho}(\mathcal{M})$, which in practice is always the case, then $\mathcal{W} \subset \mathcal{W} + \mathcal{E}$. Usually it is possible to produce a model category structure on $(\mathcal{M}, \mathcal{W} + \mathcal{E})$ with the same cofibrations as in the original structure on $(\mathcal{M}, \mathcal{W})$, but with fewer fibrations. The homotopy category $\mathrm{Ho}(\mathcal{M}, \mathcal{W} + \mathcal{E}) = (\mathcal{W} + \mathcal{E})^{-1}\mathcal{M}$ is then isomorphic to $\mathcal{E}^{-1}\,\mathrm{Ho}(\mathcal{M}, \mathcal{W})$, and the rigid localization $\hat{L}_{\mathcal{E}}(X)$ is obtained by working in this new model structure and factoring the map $X \to *$ into an acyclic cofibration $X \to \hat{L}_{\mathcal{E}}(X)$ followed by a fibration $\hat{L}_{\mathcal{E}}(X) \to *$.

2. Algebra

In this section we discuss some localization and colocalization constructions in algebra and in homological algebra.

2.1. **Classical localization for modules.** Suppose that R is a commutative ring and that $S \subset R$ is a multiplicative subset, i.e., a subset of R which is closed under taking products. Let $\mathcal{C}$ be the category of R-modules, and say that an R-module M is *S-torsion* if for each $x \in M$ there is an element $s \in S$ such that $sx = 0$. Let $\mathcal{E}$ be the subcategory of $\mathcal{C}$ containing all the objects of $\mathcal{C}$ as well as those maps f such that $\ker(f)$ and $\operatorname{coker}(f)$ are S-torsion; it is easy to argue that the class of maps of this sort is closed under composition. The local objects are the modules M which are uniquely S-divisible, in the sense that for each $x \in M$ and $s \in S$ there is a unique $y \in M$ with $sy = x$. The pair $(\mathcal{C}, \mathcal{E})$ has good localizations, and the localization of a module M is its classical algebraic localization $S^{-1}M$.

In general, the pair $(\mathcal{C}, \mathcal{E})$ does not have good colocalizations. Consider for example the case in which $R = \mathbb{Z}$ and S is the multiplicative set of nonzero integers. If T is a torsion abelian group, then the unique map $T \to 0$ is an $\mathcal{E}$-equivalence, and it follows that if M is a colocal abelian group then $\operatorname{Hom}(M, T) = 0$ for every torsion abelian group T. It is easy to check that this holds only if $M = 0$. But there are special cases; if S contains only the unit in R, for instance, then every module is colocal and the colocalization functor is the identity.

2.2. **An aside on derived categories.** Suppose that R is a ring. Recall that the *bounded derived category* $\mathcal{D}^{\mathrm{b}}(R)$ is the category whose objects are chain complexes of R-modules which vanish in sufficiently low degrees and whose morphisms are *derived chain homotopy classes of maps*. If X and Y are two chain complexes, $\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(R)}(X, Y)$ is computed by finding a projective resolution X' of X (a bounded below chain complex of projective modules which maps to X by an isomorphism on homology) and then computing ordinary chain homotopy classes of maps $X' \to Y$. The *(unbounded) derived category* $\mathcal{D}(R)$ is defined similarly, except that the chain complexes involved are not bounded below, and it is necessary to take some extra care with the definition of projective resolution [44]. If $f : X \to Y$ is a map in the derived category, we let $\operatorname{Cone}(f)$ denote the chain complex mapping cone (cofibre) of f [46, 1.2.8].

2.3. **Classical localization for $\mathcal{D}(R)$.** Suppose that R is a commutative ring and that $S \subset R$ is a multiplicative subset. Let $\mathcal{C}$ be the derived category $\mathcal{D}(R)$, and $\mathcal{E}$ the subcategory containing all maps f such that

$H_* \operatorname{Cone}(f)$ is S-torsion. Then a chain complex X is $\mathcal{E}$-local if and only if the homology groups $H_i(X)$ are uniquely S-divisible. The pair $(\mathcal{C}, \mathcal{E})$ has good localizations; the localization of X is the chain complex $S^{-1}X$ obtained by algebraically localizing X dimension by dimension. Since the algebraic localization process is exact, $H_i L_{\mathcal{E}}(X) = S^{-1}H_i(X)$. In general, $(\mathcal{C}, \mathcal{E})$ does not have good colocalizations.

In contrast to the case above of classical localization, mod p localization for modules is very different from mod p localization for objects of the derived category.

2.4. **Mod p localization for modules.** Let $\mathcal{C}$ be the category of abelian groups, p a prime number, and $\mathcal{E}$ the subcategory of $\mathcal{C}$ containing all maps f such that $\mathbb{Z}/p \otimes f$ is an isomorphism. Then an abelian group is local if and only if it is a module over $\mathbb{Z}/p$. The pair $(\mathcal{C}, \mathcal{E})$ has good localizations, and the localization of an abelian group X is $\mathbb{Z}/p \otimes X$. The only colocal object is the trivial abelian group, and so $(\mathcal{C}, \mathcal{E})$ does not have good colocalizations..

2.5. **Mod p localization for $\mathcal{D}(\mathbb{Z})$.** Let $\mathcal{C} = \mathcal{D}(\mathbb{Z})$ be the derived category of abelian groups, p a prime number, and $\mathcal{E}$ the subcategory of maps $X \to Y$ with the property that $\mathbb{Z}/p \otimes^{\mathrm{h}} X \to \mathbb{Z}/p \otimes^{\mathrm{h}} Y$ is an isomorphism. Here $\mathbb{Z}/p \otimes^{\mathrm{h}} X$ is the *derived tensor product*, i.e., the graded tensor product of X with a projective resolution P of $\mathbb{Z}/p$. To be very explicit, a map $X \to Y$ is in $\mathcal{E}$ if there is a projective resolution X' of X and a chain complex map $X' \to Y$ representing f such that the induced map $P \otimes X' \to P \otimes Y$ gives an isomorphism on homology. It turns out that the pair $(\mathcal{C}, \mathcal{E})$ has both good localizations and good colocalizations. We will describe these in turn.

For any abelian group M, the connecting homomorphism from the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Z}/p^\infty \to 0 \tag{2.6}$$

induces a map $M \to \operatorname{Ext}(\mathbb{Z}/p^\infty, M)$, and M is said to be *Ext-p-complete* if this map is an isomorphism [13, VI]. It turns out that a chain complex is $\mathcal{E}$-local if and only if each of its homology groups is Ext-p-complete. Let $\operatorname{Hom}^{\mathrm{h}}(X, Y)$ denote the derived chain complex of maps from X to Y, constructed by taking a projective resolution of X or an injective resolution of Y and forming the usual mapping chain complex [46, 2.7.4]. The exact sequence 2.6 gives a map $\Sigma^{-1}\mathbb{Z}/p^\infty \to \mathbb{Z}$ in the derived category, and the localization of an object X is the induced map

$$X \cong \operatorname{Hom}^{\mathrm{h}}(\mathbb{Z}, X) \to \operatorname{Hom}^{\mathrm{h}}(\Sigma^{-1}\mathbb{Z}/p^\infty, X).$$

In particular, for each i there is an exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, H_iX) \to H_iL_\mathcal{E}(X) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, H_{i-1}X) \to 0\,.$$

If the homology groups of X are finitely generated, then $H_iL_\mathcal{E}(X)$ is the ordinary p-completion of H_iX, i.e.

$$H_iL_\mathcal{E}(X) \cong \mathbb{Z}_p \otimes H_iX\,,$$

where $\mathbb{Z}_p$ is the ring of p-adic integers.

A chain complex is colocal if and only if each of its homology groups is a p-primary torsion group. The colocalization of X is given by the map

$$\Sigma^{-1}\mathbb{Z}/p^\infty \otimes^{\mathrm{h}} X \to \mathbb{Z} \otimes^{\mathrm{h}} X \cong X$$

induced as above by 2.6. In particular, there are exact sequences

$$0 \to \mathbb{Z}/p^\infty \otimes H_{i+1}X \to H_i\mathcal{C}_\mathcal{E}(X) \to \mathrm{Tor}(\mathbb{Z}/p^\infty, H_iX) \to 0\,.$$

These results can be proved with relatively simple formal arguments [20]. The same arguments cover the following more general situation.

2.7. **Local homology and cohomology.** Suppose that R is a commutative ring and that $I \subset R$ is an ideal generated by a finite number $r_1, \ldots, r_n$ of elements. Let $\mathcal{C} = \mathcal{D}(R)$ be the derived category of R-modules, and $\mathcal{E}$ the subcategory of maps $X \to Y$ with the property that $R/I \otimes^{\mathrm{h}} X \to R/I \otimes^{\mathrm{h}} Y$ is an isomorphism (2.5). Then the pair $(\mathcal{C}, \mathcal{E})$ has both good localizations and good colocalizations.

We will now describe these explicitly. Given a map $M \to N$ of R-modules, let $\langle M \to N\rangle$ denote the chain complex which is trivial except for M in degree 0 and N in degree -1, with the indicated map as differential, and let K denote the tensor product chain complex

$$K = \otimes_i \langle\, R \to R[1/r_i]\,\rangle\,.$$

Note that if $R = \mathbb{Z}$ and $I = (p)$, then K is isomorphic in $\mathcal{D}(\mathbb{Z})$ to $\Sigma^{-1}\mathbb{Z}/p^\infty$. There is an obvious map $K \to R$ which sends all the negative degree components of K to zero (here we are thinking of R as the chain complex consisting of the module R concentrated in degree 0). Then the localization of a chain complex X is the induced map

$$X \to \mathrm{Hom}^{\mathrm{h}}(K, X)$$

and the colocalization is the induced map

$$K \otimes^{\mathrm{h}} X \to X\,.$$

The localization functor amounts to local homology at the ideal I [28] [29] [1], and the colocalization functor to local cohomology at I [30]. For instance, if X is the chain complex consisting of the module M concentrated in degree 0, then $H_iL_\mathcal{E}(X)$ is the local homology group

$H_i^I(M)$, and $H_{-i}C_{\mathcal{E}}(X)$ is the local cohomology group $H_I^i(M)$. If R is noetherian and the homology groups of X are finitely generated over R, then $H_iL_{\mathcal{E}}(X)$ is the I-adic completion of H_iX [28]. A chain complex X is local if and only if for each integer i, the natural map $H_iX \to H_0^I(H_iX)$ is an isomorphism; X is colocal if and only if for each i and each element $x \in H_i(X)$, there exists an integer k with $I^kx = 0$ [20, §6].

3. Homological localization in topology

In this section we describe homological localization constructions for spaces and spectra, some of them parallel to the algebraic constructions from §2. The basic existence theorem is due to Bousfield [4], [6]. In the following statement, the *homotopy category of spaces* means $\mathcal{W}^{-1}\mathcal{M}$, where $\mathcal{M}$ is the category of topological spaces and $\mathcal{W}$ is the subcategory of weak homotopy equivalences. This is equivalent to the ordinary (geometric) homotopy category of CW-complexes, or to the usual homotopy category obatained from simplicial sets.

3.1**. Theorem.** *Suppose that $\mathcal{C}$ is the homotopy category of spaces or the homotopy category of spectra. Let A be a spectrum, and $\mathcal{E}_\otimes(A)$ the collection of maps f in $\mathcal{C}$ such that $A_*(f) = \pi_*(A \wedge f)$ is an isomorphism. Then $(\mathcal{C}, \mathcal{E}_\otimes)$ has good localizations.*

3.2. *Remark.* The notation $\mathcal{E}_\otimes(A)$ is explained by the fact that the smash product $\wedge$ is a kind of tensor product of spectra.

In the above situation, we write L_A for the localization functor; if A is the Eilenberg-MacLane spectrum HR, so that $A_* = H_*(-;R)$ is ordinary homology with coefficients in R, we write L_R. Two spectra A, B are said to be *Bousfield equivalent* if they give rise to the same localization functor on the category of spectra; this happens if and only if $\mathcal{E}_\otimes(A) = \mathcal{E}_\otimes(B)$.

3.3**. Classical localization for spaces.** Here $\mathcal{C}$ is the homotopy category of spaces, R is a subring of $\mathbb{Q}$ (i.e., a ring obtained from $\mathbb{Z}$ by inverting some set of primes), A is the Eilenberg-MacLane spectrum HR, and $A_* = H_*(-;R)$. If X is one-connected there are isomorphisms $\pi_iL_R(X) \cong R \otimes \pi_i(X)$, so that X is local if and only the homotopy groups of X are modules over R. The same principle works if X is nilpotent, as long as $R \otimes \pi_1(X)$ is interpreted correctly [4] [13].

For non-simply connected spaces, $L_R(X)$ can be mysterious even if $R = \mathbb{Z}$; for instance, $L_{\mathbb{Z}}(S^1 \vee S^1)$ is unknown. If the commutator subgroup of π_1X is perfect and X satisfies some other mild conditions, then $L_{\mathbb{Z}}X$ is equivalent to the Quillen plus construction on X.

3.4. **Mod p localization for spaces.** Here $\mathcal{C}$ is the homotopy category of spaces, p is a prime, $A = H\mathbb{Z}/p$, and $A_* = H_*(–;\mathbb{Z}/p)$. If X is one-connected, then X is local if and only if the homotopy groups of X are Ext-p-complete (2.5). More generally, if X is one-connected there are short exact sequences [4] [13]

(3.5)

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_i X) \to \pi_i L_{\mathbb{Z}/p}(X) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{i-1}X) \to 0\,.$$

There are similar formulas for nilpotent spaces. In particular, if X is one-connected and the homotopy groups of X are finitely generated, there are isomorphisms $\pi_i L_{\mathbb{Z}/p}(X) \cong \mathbb{Z}_p \otimes \pi_i(X)$. For this reason, the localization $L_{\mathbb{Z}/p}$ is sometimes referred to as *p-completion*.

3.6. **The arithmetic square.** A good feature of the arithmetic localization functors described above is that there is a way to rebuild a space X, at least if X is nilpotent, from its various localizations. This is provided by the *arithmetic square* (cf. 1.9).

3.7. **Theorem.** [45] [16] *Suppose that X is a nilpotent space. Then there is a homotopy fibre square*

$$\begin{array}{ccc} X & \longrightarrow & \prod_p L_{\mathbb{Z}/p}(X) \\ \downarrow & & \downarrow \\ L_{\mathbb{Q}}(X) & \longrightarrow & L_{\mathbb{Q}}(\prod_p L_{\mathbb{Z}/p}X) \end{array}$$

The bottom row is obtained by applying $L_{\mathbb{Q}}$ to the top row. This square can be viewed as describing how X is determined by p-adic data for each prime p (upper right), rational data (lower left), and coherence information over the rationals (arrows terminating at lower right).

3.8. **Arithmetic localizations of spectra.** Let $\mathcal{C}$ be the homotopy category of spectra, R a subring of $\mathbb{Q}$, and p a prime. If X is *bounded below* in the sense that $\pi_i X = 0$ for $i << 0$, then $\pi_i L_R(X)$ is isomorphic to $R \otimes \pi_i X$, and $\pi_i L_{\mathbb{Z}/p}(X)$ lies in an exact sequence of the form 3.5, so the above results apply as stated to the subcategory of $\mathcal{C}$ consisting of objects which are bounded below. For spectra which are not bounded below, these ordinary homology localizations are more complicated. For instance, let KU be the periodic complex K-theory spectrum. Then the map $H_*(KU;\mathbb{Z}) \to H_*(KU,\mathbb{Q})$ is an isomorphism, and it follows that the map $L_{\mathbb{Z}}(KU) \to L_{\mathbb{Q}}(KU) = \prod_i \Sigma^{2i} H\mathbb{Q}$ is an equivalence. In particular, $\pi_* L_{\mathbb{Z}}(KU) \neq \mathbb{Z} \otimes \pi_*(KU)$. The situation can be repaired by replacing the coefficient Eilenberg-MacLane spectrum HR by the Moore spectrum $M(R) = L_R(S^0)$, and similarly replacing $H\mathbb{Z}/p$ by the Moore spectrum $M(p) = S^0 \cup_p e^1$. Then for any spectrum X,

$\pi_i L_{M(R)}(X)$ is isomorphic to $R \otimes \pi_i X$, and $\pi_i L_{M(p)}(X)$ lies in an exact sequence of the form 3.5.

3.9**. Chromatic localizations of spectra.** We will refer to ideas from [40]. Again, $\mathcal{C}$ is the homotopy category of spectra. Pick a prime p, and take all objects of $\mathcal{C}$ to be localized at p in the sense (3.8) that their homotopy groups are modules over $\mathbb{Z}_{(p)}$. Let $K(n)$ be the n'th Morava K-theory. The *n'th chromatic localization functor* L_n is defined to be localization with respect to the wedge $K(0) \vee \cdots \vee K(n)$. It is easy to see that for any spectrum X there is a tower

$$\cdots \to L_n(X) \to L_{n-1}(X) \to \cdots \to L_0(X)\,,$$

and the chromatic convergence theorem [40, 7.5.7] guarantees that if X is a finite suspension spectrum the homotopy limit of this tower is X. The chromatic approach to $\pi_* S^0$ involves studying how the stable homotopy ring is built up in the tower $\{\pi_* L_n(S^0)\}$. Closely related to L_n is the localization functor $L_{K(n)}$; in fact, $L_{K(n)}(X)$ determines the difference between $L_n(X)$ and $L_{n-1}(X)$, in the sense that there is a homotopy fibre square

$$\begin{array}{ccc} L_n(X) & \longrightarrow & L_{K(n)}(X) \\ \downarrow & & \downarrow \\ L_{n-1}(X) & \longrightarrow & L_{n-1}L_{K(n)}(X) \end{array}$$

rather parallel to 3.6. The spectrum $K(0)$ is Bousfield equivalent to $H\mathbb{Q}$, and so a spectrum X is $K(0)$-local if and only if the homotopy groups of X are uniquely p-divisible, or, in other words, if and only if the degree p map $d_p : S^0 \to S^0$ induces bijections $[S^i, X] \to [S^i, X]$, $i \in \mathbb{Z}$. The *telescope conjecture* can be interpreted as asserting that there is a similar characterization of spectra which are local with respect to $K(0) \vee \cdots \vee K(n)$.

3.10**. Conjecture.** *(Telescope Conjecture) Suppose that $F = F(n)$ is a finite complex of type n, and that $v : \Sigma^a F \to F$ is a v_n-self-map. Then a spectrum X is local with respect to L_n (i.e., $L_n(X) \sim X$) if and only if composition with v induces bijections*

$$[\Sigma^i F, X] \to [\Sigma^{a+i} F, X], \quad i \in \mathbb{Z}.$$

This is true for $n = 0$ (above) and $n = 1$, but the best current evidence suggests that it is false in general [35].

There are a few calculations. For any X, $L_n(X)$ is equivalent to $L_n(S^0) \wedge X$. The spectrum $L_0(S^0)$ is the Eilenberg-MacLane spectrum $H\mathbb{Q}$; $L_1(S^0)$ is a periodic version of the $\operatorname{im} J$ spectrum [6] [39]. The spectrum $L_2(S^0)$ is in a sense known for $p > 3$ [43], and partially known for $p = 2$ or $p = 3$ [42] [25].

3.11. **Chromatic localizations of spaces.** All of the above chromatic localization functors can be applied to spaces as well as to spectra. Again, there are some calculations. Mahowald and Thompson have computed $L_1(X)$ when X is an odd sphere [36], and Bousfield has done the same when X is an infinite loop space [7] or a finite H-space [10]. If X is an infinite loop space with associated spectrum $B^\infty X$, Bousfield has shown that in many cases $L_n X$ agrees except in low dimensions with $\Omega^\infty L_n(B^\infty X)$ [12]. Kuhn has found a way to construct $L_{K(n)}(B^\infty X)$ in terms of X [34], but there's no close relationship between $L_{K(n)}(X)$ and $\Omega^\infty L_{K(n)}(B^\infty X)$. In fact, surprisingly enough, for most spaces the functor $L_{K(n)}$ agrees up to ordinary p-completion with the functor L_n [11].

4. Localization with respect to a map

Suppose that $\mathcal{C}$ is a category with some notion of mapping object $\mathrm{Map}(X, Y)$ for any two objects X, Y in $\mathcal{C}$. Here are some examples:

- $\mathcal{C}$ is the homotopy category of spaces, and $\mathrm{Map}(X, Y) = \mathrm{Map}^{\mathrm{h}}(X, Y)$, the derived space of maps from X to Y, i.e. the space obtained by replacing X by a weakly equivalence CW-complex and taking the usual mapping space.
- $\mathcal{C}$ is the homotopy category of spectra, and $\mathrm{Map}(X, Y)$ is the derived mapping spectrum.
- $\mathcal{C}$ is the derived category of a ring R, and $\mathrm{Map}(X, Y) = \mathrm{Hom}^{\mathrm{h}}(X, Y)$ is the derived chain complex of maps $X \to Y$ (2.5).

Given $f : A \to B$ in $\mathcal{C}$, say that an object X of $\mathcal{C}$ is *f-local* if composition with f induces an equivalence

$$f^* : \mathrm{Map}(B, X) \sim \mathrm{Map}(A, X)\,.$$

The meaning of "equivalence" depends on the category in which the mapping objects lie; for spaces or spectra, equivalence would usually mean weak homotopy equivalence, and for chain complexes it would mean homology isomorphism. Let $\mathcal{E}(f)$ be the category of all maps $g : U \to V$ in $\mathcal{C}$ such that g induces an equivalence $\mathrm{Map}(V, X) \to \mathrm{Map}(U, X)$ for *every* f-local object X.

4.1. **Definition.** In the above situation, if $(\mathcal{C}, \mathcal{E}(f))$ has good localizations, the localization functor is denoted L_f, and called *localization with respect to f*.

4.2. *Remark.* This is a case of working backwards (1.5): we first define the local objects, and then use them to get the equivalences $\mathcal{E}(f)$. The class $\mathcal{E}(f)$ is the smallest class of maps in $\mathcal{C}$ which contains the map f and has a closure property which we leave it to the reader to formulate.

The following theorem of Bousfield and Dror Farjoun is parallel to 3.1

4.3. **Theorem.** [5] [18] *If $\mathcal{C}$ is the homotopy category of spaces, the homotopy category of spectra, or the derived category of a ring R, then for* any *map f the pair $(\mathcal{C}, \mathcal{E}(f))$ has good localizations.*

Theorem 4.3 applies in much greater generality [5] [31]; for instance, it applies if $\mathcal{C}$ is a homotopy category of diagrams of spaces (4.8). Virtually all of the localization functors we have looked at so far can be expressed as localization L_f with respect to some appropriately chosen map f. In fact, in the category of spaces *any* homotopically idempotent functor E with a natural transformation $\mathrm{Id} \to E$ can be expressed as L_f for some map, if you are willing to adopt some exotic axioms for set theory [14].

4.4. *Remark.* If $\mathcal{C}$ is the homotopy category of spectra, there is another possible interpretation for the mapping object $\mathrm{Map}(X, Y)$; rather than the derived mapping spectrum, one could take the zero space in the corresponding infinite loop spectrum, or, more or less equivalently, the (-1)-connective cover of the derived mapping spectrum. Theorem 4.3 remains true with this interpretation of mapping object, but the localization functors change significantly and have more in common with their unstable counterparts. Bousfield uses this notion of localization in [11, 2.2]. Colocalization functors also change under the connective interpretation of "Map": the phrase "and its desuspensions" would have to be deleted from 5.3, and 5.4 would cease in general to be a fibration sequence. For us, though, "Map" in the context of spectra will always mean the entire derived mapping spectrum.

4.5. **Postnikov sections.** Suppose that $\mathcal{C}$ is the homotopy category of spaces and that f is the unique map $S^n \to *$. Then a space X is f-local if and only if for each basepoint $x \in X$ and each $i \geq n$, $\pi_i(X, x)$ is trivial. The class $\mathcal{E}(f)$ is the class of all maps $g : X \to Y$ which for each basepoint $x \in X$ and each $i < n$ induce a bijection $\pi_i(X, x) \to \pi_i(Y, g(x))$. The localization $L_f(X)$ is the Postnikov section $P_{n-1}X$.

4.6. **Homology localization.** If $\mathcal{C}$ is the homotopy category of spaces or spectra and A is a chosen spectrum, then the homology localization functor L_A (§3) is equivalent to L_f for some huge non-canonical A_*-equivalence $f : U \to V$. For instance, if $A = H\mathbb{Z}/p$, then f can be taken to be the disjoint union $\coprod f_\alpha$, where f_α runs over all mod p homology isomorphisms $U_\alpha \to V_\alpha$ between countable CW-complexes (take them to be subspaces of $\mathbb{R}^\infty$, so that the collection $\{f_\alpha\}$ forms a set). In fact, Bousfield essentially constructs L_A by implicitly identifying it as L_f [6]. It was one of Bousfield's great insights that is possible to work with this idea without having a particularly explicit hold on the map f associated to A.

4.7**. Nullification.** Let $\mathcal{C}$ be the homotopy category of spectra, W an object of $\mathcal{C}$, and $f : W \to *$ the unique map from W to the contractible spectrum. The localization functor L_f is called *nullification with respect to* W and assigns to X a universal map $X \to P_W(X)$ with $\mathrm{Map}(W, P_W(X)) \sim *$. Localization with respect to $f : U \to V$ is the same as nullification with respect to the cofibre of f. In particular, the homology localization functor L_A (4.6) can be expressed as nullification P_W with respect to some huge spectrum W with $A_*(W) = 0$. The n'th *finite localization functor* or *telescopic localization functor* functor L_n^f is nullification with respect a finite spectrum $F(n+1)$ of type $n+1$ [37]. The telescope conjecture (3.10) is equivalent to the conjecture that $L_n = L_n^f$, or to the conjecture that a spectrum X is L_n-local if and only if $\mathrm{Map}(F(n+1), X) \sim *$. This is clear from 3.10, since the cofibre of a v_n-self-map of $F(n)$ is a finite complex of type $n+1$.

Unstably, the situation is more complicated. Let $\mathcal{C}$ be the homotopy category of spaces, W an object of $\mathcal{C}$, and $f : W \to *$ the unique map from W to a point. Again the localization L_f is called nullification with respect to W and denoted P_W. The notation P_W comes both from the fact that nullification was originally called periodization [8], and from the fact that for $W = S^n$, P_W is the Postnikov functor P_n (4.5). It is no longer true that localization with respect to a map f is nullification with respect to the cofibre W of f, although there is a natural transformation $P_W \to L_f$ [9, 4.3]. In particular, unstable homology localization functors can not in general be identified as nullification functors. Bousfield and Dror-Farjoun have studied P_W extensively [8], [18]. One of the main results is that the nullification functors preserve fibrations up to an error term which is frequently quite small. Bousfield has used this to prove the remarkable theorem [8, 11.5] that nullification with respect to an finite complex of type $n+1$ preserves the (unstable) v_i-periodic homotopy for $i \le n$.

4.8**. An aside on diagrams.** Suppose that $\mathcal{D}$ is a small category. A $\mathcal{D}$*-space*, or *diagram of spaces with the shape of* $\mathcal{D}$, is a functor from $\mathcal{D}$

to spaces. These functors form a category, in which the morphisms are the natural transformations. Say that a morphism $X \to Y$ is a weak equivalence if for each object $d \in \mathcal{D}$ the induced map $X(d) \to Y(d)$ is a weak homotopy equivalence of spaces. There is a model category structure on $\mathcal{D}$-spaces with these weak equivalences [23] [17] [19, §2]; in the simplest structure like this, the fibrations are the maps $X \to Y$ which for each $d \in \mathcal{D}$ give a Serre fibration $X(d) \to Y(d)$. We will refer to the associated homotopy category as the homotopy category of $\mathcal{D}$-spaces.

For each $d \in \mathcal{D}$ there is a "free $\mathcal{D}$-space based at d" denoted $F\langle d\rangle$ and given by $F\langle d\rangle(x) = \mathrm{Hom}(d, x)$; here we are treating the set $\mathrm{Hom}(d, x)$ as a discrete space. Essentially by Yoneda's lemma, for any $\mathcal{D}$-space X the derived mapping space $\mathrm{Map}^{\mathrm{h}}(F\langle d\rangle, X)$ is canonically weakly equivalent to $X(d)$. Note that $f : d \to d'$ gives a morphism $f^* : F\langle d'\rangle \to F\langle d\rangle$, such that $\mathrm{Map}^{\mathrm{h}}(f^*, X)$ can be identified up to weak equivalence with $X(f) : X(d) \to X(d')$.

4.9**. Inverting morphisms.** Let $\mathcal{C}$ be the homotopy category of $\mathcal{D}$-spaces, where $\mathcal{D}$ is a small category, and suppose that $u : d \to d'$ is a map in $\mathcal{D}$. Let f denote $u^* : F\langle d'\rangle \to F\langle d\rangle$. Then a $\mathcal{D}$-space X is f-local if and only if $X(u) : X(d) \to X(d')$ is a weak homotopy equivalence. It is hard to describe the localization functor L_f in simple terms.

Here's a related example, which we offer without proof, in which it *is* possible to describe L_f. Let f be the disjoint union $\coprod_u u^*$ indexed by *all* of the morphisms u of $\mathcal{D}$. Then a $\mathcal{D}$-space X is f-local if and only if for each morphism u of $\mathcal{D}$, $X(u)$ is a weak homotopy equivalence. The category of f-local objects is in a homotopical sense equivalent to the category of fibrations over the classifying space $B\mathcal{D}$. For an arbitrary $\mathcal{D}$-space X, $L_f(X)$ is a $\mathcal{D}$-space whose value at each object d is equivalent to the homotopy fibre, over the vertex corresponding to d, of the natural map

$$\mathrm{hocolim}_{\mathcal{D}}\, X \to \mathrm{hocolim}_{\mathcal{D}} * = B\mathcal{D}\,.$$

In this case L_f is a kind of averaging or integration functor.

4.10**. Goodwillie calculus.** We begin with something along the lines of 4.9 but a little more general. Suppose that $\mathcal{D}$ is a small category, $\mathcal{P}$ another (usually much smaller) category, and $j : \mathcal{P} \to \mathcal{D}$ a functor. Assume that $\mathcal{P}$ has an initial object ϕ, and let $\mathcal{P} \setminus \phi$ be the subcategory obtained by deleting the initial object. Given a $\mathcal{D}$-space X, we can compose with j to get a $\mathcal{P}$-space X' and ask whether the natural map

$$X'(\phi) \to \mathrm{holim}_{\mathcal{P}\setminus\phi}\, X' \tag{4.11}$$

is a weak equivalence. It is easy to construct a map $L(j)$ of $\mathcal{D}$-spaces such that the answer to this question is "yes" if and only if X is $L(j)$-local. To be explicit, there is a functor F' from $\mathcal{P}^{\mathrm{op}}$ to $\mathcal{D}$-spaces which sends $x \in \mathcal{P}$ to the free diagram $F\langle j(x)\rangle$. Then $L(j)$ is the natural map

$$L(j) : \mathrm{hocolim}_{\mathcal{P}\setminus\phi} F' \to F'(\phi)\,.$$

The homotopy colimit here is calculated in the category of $\mathcal{D}$-spaces, in other words, objectwise in the category of functors from $\mathcal{D}$ to spaces. The point to keep in mind is that $\mathrm{Map}^{\mathrm{h}}$ converts homotopy colimits in the first variable into homotopy limits of mapping spaces.

Now let $\mathcal{D}$ be the category of finite complexes; we choose some small model for this which contains all finite complexes up to homotopy, and is big enough to contain products, homotopy pushouts, etc. Let $\mathcal{C}$ be the homotopy category of $\mathcal{D}$-spaces. A $\mathcal{D}$-space X is said to be a *homotopy functor* if $X(f)$ is a weak equivalence of spaces whenever $f : d \to d'$ is a weak equivalence of CW-complexes. Let f_1 be the obvious map of $\mathcal{D}$-spaces with the property that X is a homotopy functor if and only if X is f_1-local (4.9). Let $\mathcal{P}$ be the poset of subsets of $\{1, \dots, n\}$, so that $\mathcal{P}$ is a category with the empty set as initial object. An *n-cube of finite complexes* is a functor $j : \mathcal{P} \to \mathcal{D}$, and such an n-cube is said to be *strongly homotopy cocartesian* if each of its two-dimensional faces is a homotopy pushout diagram (see [27]). Let f_2 be the map $\coprod_j L(j)$, where j ranges over all strongly homotopy cocartesian n-cubes. A $\mathcal{D}$-space X is f_2-local if and only if it carries each strongly homotopy cartesian n-cube into a homotopy cartesian diagram, or, in the terminology of [27], if and only if X is n-excisive. Let $f = f_1 \coprod f_2$. Then X is f-local if and only if X is an n-excisive homotopy functor. The construction which assigns to a homotopy functor X the n-excisive functor $L_f(X)$ gives the n'th stage P_nX in the Goodwillie tower for X [26].

4.12**. Loop spaces and theories.** Let Δ denote the category of the finite ordered sets $\{0, \dots, m\}$, $m \geq 0$, and weakly monotone maps, so that a Δ^{op}-space is a simplicial object in the category of spaces. Given such a simplicial space X, write $X_m = X(\{0, \dots, m\})$, and say that X is *very special* if for each m the natural map

$$\rho_1 \times \cdots \times \rho_m : X_m \to X_1 \times \cdots \times X_1$$

is a weak homotopy equivalence, where $\rho_i : X_m \to X_1$ is the map corresponding to the function $\{0, 1\} \to \{0, \dots, m\}$ with $\rho_i(0) = 0$ and $\rho_i(1) = i$. (When $m = 0$ the right hand side is the empty product, which is the one-point space, and the condition reads that X_0 is weakly contractible.) This definition is due to Bousfield [8, 3.3]. As in 4.10, there is an evident map f of Δ^{op}-spaces such that X is very special if and only if X is f-local. It follows from ideas of Segal [41, App. B] that the homotopy theory of very special Δ^{op}-spaces is equivalent to the homotopy theory of loop spaces. The equivalence takes a very special Δ^{op}-space X to a loop space canonically weakly equivalent to X_1. In particular, the homotopy category of loop spaces can be obtained as the localization of a diagram category. There is one immediate consequence. If Φ is a functor from spaces to spaces which respects homotopy and preserves products up to weak equivalence, then dimensionwise application of Φ preserves the category of very special Δ^{op}-spaces. It follows that if U is a loop space, the $F(U)$ is canonically weakly equivalent to a loop space [18, §3]. It is possible to treat the theory of n-fold loop spaces, $2 \le n \le \infty$, in a similar way [3].

It is also possible to go a bit further. For the rest of this paragraph, take "space" to mean "simplicial set". Let $\mathcal{A}$ be the category of algebraic objects of some fixed equational type, e.g., the category of groups, abelian groups, groups of nilpotency class $\le n$, monoids, rings, commutative rings, Lie algebras, ... Let F_m be the free object of $\mathcal{A}$ on m generators, and let $\mathcal{T}_{\mathcal{A}}$ be the *opposite* of the full subcategory of $\mathcal{A}$ generated by the F_m. The category $\mathcal{T}_{\mathcal{A}}$ is the "theory" of the objects in $\mathcal{A}$, in the sense that the morphisms of $\mathcal{T}_{\mathcal{A}}$ determine all of the ways of combining elements of such an object by algebraic operations. If X is a $\mathcal{T}_{\mathcal{A}}$-space, write $X_m = X(F_m)$, and say that X is *special* if for each m the natural map

$$\rho_1 \times \cdots \times \rho_m : X_m \to X_1 \times \cdots \times X_1$$

is a weak homotopy equivalence, where $\rho_i : X_m \to X_1$ is the map corresponding to the $\mathcal{A}$-morphism $F_1 \to F_m$ which sends the generator of F_1 to the i'th generator of F_m. Again, there is an evident map f of $\mathcal{T}_{\mathcal{A}}$-spaces such that X is special if and only if X is f-local. Starting with this observation, Badzioch [2] has shown that the homotopy theory of special $\mathcal{T}_{\mathcal{A}}$-spaces is equivalent to the homotopy theory of simplicial objects in $\mathcal{A}$; the equivalence takes a special $\mathcal{T}_{\mathcal{A}}$-space X to a simplicial object which is canonically weakly equivalent, as a simplicial set, to $X(F_1)$. In particular, the homotopy category of simplicial objects in $\mathcal{A}$ can be obtained as the localization of a diagram category. Again, it follows that if Φ is a functor from spaces to spaces which preserves weak equivalences and preserves products up to weak equivalence, and U is a simplicial object in $\mathcal{A}$, then $F(U)$ is canonically weakly equivalent to a simplicial object in $\mathcal{A}$.

If $\mathcal{A}$ is the category of groups, then the category of special $\mathcal{T}_{\mathcal{A}}$-spaces is closely related to the category of very special Δ^{op}-spaces [2, 1.6 ff].

Rather than explain this in detail, we give the following example from [2]. Consider the $\mathcal{T}_{\mathcal{A}}^{\mathrm{op}}$-space B which assigns to a free group F the pointed classifying space BF, where BF is constructed in some functorial way from F. Note that BF_m is equivalent to a wedge of m circles. Now for any pointed space Y there is a special $\mathcal{T}_{\mathcal{A}}$-space Ω_Y given by letting $\Omega_Y(F)$ be the space of basepoint-preserving maps $\mathrm{Map}_*(BF, Y)$; the "special" property amounts to the observation that $\mathrm{Map}_*(BF_m, Y)$ is equivalent to $\mathrm{Map}_*(BF_1, Y)^m$. Since $\Omega_Y(F_1)$ is the loop space ΩY, we conclude from the main theorem of [2] that ΩY is naturally weakly equivalent to a simplicial group.

4.13**. Motivic homotopy theory.** In this paragraph, "space" means simplicial set. We follow [24]. Suppose that S is a noetherian scheme of finite dimension, and $\mathcal{D}$ the category of smooth schemes over S. A *simplicial presheaf* on $\mathcal{D}$ is just a $\mathcal{D}^{\mathrm{op}}$-space; the category of these is denoted $\mathrm{Pre}(\mathcal{D})$. Any scheme M over S gives an object of $\mathrm{Pre}(\mathcal{D})$, which we will denote ρM, according to the formula $\rho M(U) = \mathrm{Hom}(U, M)$. The category $\mathrm{Pre}(\mathcal{D})$ has a homotopy theory as in 4.8 in which the weak equivalences are the maps $X \to Y$ which induce weak equivalences $X(U) \to Y(U)$ for each object $U \in \mathcal{D}$. Call this the *sectionwise homotopy theory*; it turns out to be too rigid for most algebraic purposes. If $\mathcal{D}$ is furnished with a Grothendieck topology T, then there is a more flexible T*-local homotopy theory* [33] on $\mathrm{Pre}(\mathcal{D})$ in which a weak equivalence is a map $X \to Y$ which induces an isomorphism on associated sheaves of homotopy groups for all appropriate choices of basepoints. See [19] for a nice account of how, in a fashion reminiscent of 4.10 above, this can be obtained as the localization of the sectionwise theory with respect to a map f'. The map f' is easy to describe. In the language of [19, §1], for each T-hypercover U of an object X in $\mathcal{D}$ there is a map

$$f_U : \mathrm{hocolim}_n\, \rho U_n \to \rho X\ ,$$

and f' is just the disjoint union $\coprod_U f_U$, where U runs through a large enough set of hypercovers [19, §6]. Now specialize to the case in which T is the Nisnevich topology (a choice explained by the desire to accomodate algebraic K-theory, which has certain patching properties with respect to the Nisnevich topology), and further localize the T-local homotopy theory on $\mathrm{Pre}(\mathcal{D})$ with respect to a map f'' derived as in [24, 4.1] from $\rho\mathbb{A}^1 \to *$, where $\mathbb{A}^1$ is the affine line and $*$ is the constant one-point presheaf. What results is the $\mathbb{A}^1$*-homotopy theory* or *motivic homotopy theory* of Voevodsky and Morel [24, 4.11] [38]. Thus motivic homotopy theory over S is the localization of the ordinary homotopy theory of $\mathcal{D}^{\mathrm{op}}$-spaces with respect to the map $f = f' \coprod f''$.

5. Colocalization with respect to an object

Suppose again that $\mathcal{C}$ is a category with some notion of mapping object (§4), and that W is an object of $\mathcal{C}$. Say that a map $X \to Y$ is a *W-cellular equivalence* if it induces an equivalence $\mathrm{Map}(W, X) \to \mathrm{Map}(W, Y)$, and let $\mathcal{E}_{\mathrm{Map}}(W)$ denote the category of W-cellular equivalences.

5.1**. Definition.** In the above situation, if the pair $(\mathcal{C}, \mathcal{E}_{\mathrm{Map}}(W))$ has good colocalizations, the colocalization functor is written Cell_W and called *cellularization with respect to W*.

The basic existence theorem is again due to Bousfield [5] and Dror Farjoun [18].

5.2**. Theorem.** *If $\mathcal{C}$ is the homotopy category of spaces, the homotopy category of spectra, or the derived category of a ring R, then for* any *object W in $\mathcal{C}$ the pair $(\mathcal{C}, \mathcal{E}_{\mathrm{Map}}(W))$ has good colocalizations.*

Again, the theorem applies in much greater generality [31], and in particular applies to any homotopy category of diagrams of spaces.

5.3**. Stable Cellularization.** Suppose that $\mathcal{C}$ is the homotopy category of spectra or the derived category $\mathcal{D}(R)$, and that $W \in \mathcal{C}$. Then for any X, $\mathrm{Cell}_W(X)$ lies in the smallest subcategory of spectra containing W and its desuspensions and closed under weak equivalence and homotopy colimit; this subcategory is called the *localizing subcategory* generated by W, and the objects in it are said to be *built from* W. In fact, $\mathrm{Cell}_W(X)$ is characterized by the fact that it is built from W and admits a map $\mathrm{Cell}_W(X) \to X$ which lies in $\mathcal{E}_{\mathrm{Map}}(W)$. There is a cofibration sequence (4.7)

$$\mathrm{Cell}_W(X) \to X \to P_W(X) \tag{5.4}$$

which indicates that cellularization and nullification with respect to W capture complementary parts of X. The notation $\mathrm{Cell}_W(X)$ is meant to suggest that this spectrum is built from W and its cones in the same way a cell complex is built from disks and spheres.

5.5**. Unstable cellularization.** If $\mathcal{C}$ is the homotopy category of *unpointed* spaces, then for any nonempty W and any X, $\mathrm{Cell}_W(X) \to X$ is an isomorphism [18, 2.A.4]. Suppose that $\mathcal{C}$ is the homotopy category of *pointed* spaces and that $W \in \mathcal{C}$. Then for any X, $\mathrm{Cell}_W(X)$ belongs to the smallest class of spaces which contains W and is closed under weak equivalence and pointed homotopy colimits; this is the *closed class* of spaces determined by W [18, 2.D], and again the objects in it are said to be built from W. The space $\mathrm{Cell}_W(X)$ is characterized by the fact that it is built from W and admits a map $\mathrm{Cell}_W(X) \to X$ which lies in $\mathcal{E}_{\mathrm{Map}}(W)$. (Here $\mathcal{E}_{\mathrm{Map}}(W)$ is defined using *pointed* mapping spaces $\mathrm{Map}(X, Y)$.) In the sequence

$$\mathrm{Cell}_W(X) \to X \to P_W(X)$$

the composite of the two maps is null, but this is not in general a fibration sequence. Chachólski [15] has studied the relationship between Cell_W and P_W in detail.

5.6**. Localization/colocalization for the same $\mathcal{E}$.** In practice, it is unusual for a pair $(\mathcal{C}, \mathcal{E})$ to have both good localizations and good colocalizations. Suppose for instance that $\mathcal{C}$ is the homotopy category of spectra. Then in localizing with respect to a map f, the equivalences $\mathcal{E}(f)$ are detected by $\mathrm{Map}(–, L)$ for f-local objects L, while in colocalizing with respect to an object W, the equivalences $\mathcal{E}_{\mathrm{Map}}(W)$ are detected by $\mathrm{Map}(W, –)$. It would appear to be hard to find f and W such that $\mathcal{E}(f) = \mathcal{E}_{\mathrm{Map}}(W)$. But there is one case in which this does happen. If W is a finite spectrum, then for any X

$$\mathrm{Map}(W, X) \sim DW \wedge X\ ,$$

where $DW = \mathrm{Map}(W, S^0)$ is the Spanier-Whitehead dual of W. Thus $\mathcal{E}_{\mathrm{Map}}(W) = \mathcal{E}_{\otimes}(DW) = \mathcal{E}(f)$ for some huge f (4.6), and, if $\mathcal{E}$ denotes this class of equivalences, $(\mathcal{C}, \mathcal{E})$ has both good localizations and colocalizations. This is particularly useful in chromatic situations [32]. Suppose that $\mathcal{C}$ is the L_n-local homotopy category of spectra, let W be a finite complex of type n, and let $\mathcal{E} = \mathcal{E}_{\mathrm{Map}}(W) = \mathcal{E}_{\otimes}(DW)$. Then $(\mathcal{C}, \mathcal{E})$ has both good localizations and good colocalizations; the colocalization functor is the monochromatic functor M_n, and the localization functor is localization $L_{K(n)}$ with respect to the Morava K-theory $K(n)$. As always in this situation (1.7) the categories of local and colocal objects are equivalent [32, 6.19]. There is a parallel phenomenon in the telescopic case, where L_n is replaced L_n^f [12, 3.3].

Suppose that R is a commutative ring and that $\mathcal{C} = \mathcal{D}(R)$ is its derived category. Given W in $\mathcal{C}$, let $\mathcal{E}_{\otimes}(W)$ denote the category of all maps $X \to Y$ with the property that $W \otimes_R^{\mathrm{h}} X \to W \otimes_R^{\mathrm{h}} Y$ is an isomorphism in $\mathcal{C}$. Bousfield's arguments [6] show that $(\mathcal{C}, \mathcal{E}_{\otimes}(W))$ has good localizations. Similarly, $(\mathcal{C}, \mathcal{E}_{\mathrm{Map}}(W))$ has good colocalizations. Suppose that $I = \langle r_1, \ldots, r_n \rangle$ is a finitely generated ideal in R. The fact that $(\mathcal{C}, \mathcal{E}_{\otimes}(R/I))$ has good colocalizations as well as good localizations (see 2.7) is explained by the surprising fact that $\mathcal{E}_{\otimes}(R/I) = \mathcal{E}_{\mathrm{Map}}(R/I)$ [20, 6.5]. This is related to the example involving finite spectra described above. In the notation of 2.7, let W be the chain complex given by

$$W = \otimes_i \langle R \xrightarrow{r_i} R \rangle .$$

Then R/I and W can be built from one another, so that $\mathcal{E}_\otimes(R/I) = \mathcal{E}_\otimes(W)$ and $\mathcal{E}_{\mathrm{Map}}(R/I) = \mathcal{E}_{\mathrm{Map}}(W)$. Since W is built from a finite number of copies of R, for any X there is an equivalence

$$\mathrm{Map}(W, X) \sim DW \otimes^{\mathrm{h}} X ,$$

where $DW = \mathrm{Map}(W, R)$ is the Spanier-Whitehead dual of W. This gives $\mathcal{E}_{\mathrm{Map}}(W) = \mathcal{E}_\otimes(DW)$. But by inspection, DW is just a suspension of W itself, so that $\mathcal{E}_\otimes(DW) = \mathcal{E}_\otimes(W)$.

6. Higher invariants of localization

Suppose that $(\mathcal{C}, \mathcal{E})$ is a localization context. Theorem 1.4 indicates that the process of localizing individual objects X of $\mathcal{C}$, say by constructing $L_\mathcal{E}(X)$, is closely connected to the process of “localizing” $\mathcal{C}$ itself by forming $\mathcal{E}^{-1}\mathcal{C}$. But forming $\mathcal{E}^{-1}\mathcal{C}$ creates in general a lot of higher order structure, and the question we want to ask here is how this higher order structure is related to the localization functors.

The nature of this higher order structure is easy to understand. Suppose that $\mathcal{C}$ has a single object, and that $\mathcal{E}$ consists of all of the morphisms in $\mathcal{C}$. Then $\mathcal{C}$ amounts to a monoid M, and forming $\mathcal{E}^{-1}\mathcal{C}$ involves forming the group completion $M^{-1}M$. In this case the higher structure is the topological space ΩBM; the component group $\pi_0\Omega BM$ is $M^{-1}M$, but the other homotopy groups $\pi_i\Omega BM$, $i > 0$, and indeed the whole homotopy type of the space ΩBM, are higher invariants of the group completion process. Another way to obtain these invariants is to form a resolution R of M by a simplicial monoid which is free in each degree, and apply the group completion process degreewise to obtain a simplicial group $R^{-1}R$. The geometric realization of $R^{-1}R$ is then weakly equivalent to ΩBM. This exhibits ΩBM or $R^{-1}R$ as the result of applying to M a kind of total derived functor of the group completion process. Given a general pair $(\mathcal{C}, \mathcal{E})$ of categories it is not hard to construct a simplicial resolution $(\mathcal{R}_\mathcal{C}, \mathcal{R}_\mathcal{E})$ of $(\mathcal{C}, \mathcal{E})$ by free categories (in this context, each category is free on a directed graph with vertex set $\mathrm{Obj}(\mathcal{C})$) and form the *simplicial localization* $\mathcal{L}(\mathcal{C}, \mathcal{E}) = \mathcal{R}_\mathcal{E}^{-1}\mathcal{R}_\mathcal{C}$ [22]. For objects X, Y of $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{L}(\mathcal{C},\mathcal{E})}(X, Y)$ is then a simplicial set with

$$\pi_0 \,\mathrm{Hom}_{\mathcal{L}(\mathcal{C},\mathcal{E})}(X, Y) \cong \mathrm{Hom}_{\mathcal{E}^{-1}\mathcal{C}}(X, Y)$$

The homotopy types of these simplicial function complexes, taken together with the composition maps between them, embody the higher order structure created in forming $\mathcal{E}^{-1}\mathcal{C}$.

6.1. **Proposition.** *Let $(\mathcal{C}, \mathcal{E})$ be a localization context. If $(\mathcal{C}, \mathcal{E})$ has either good localizations or good colocalizations, then the function complexes in the simplicially enriched category $\mathcal{L}(\mathcal{C}, \mathcal{E})$ are homotopically discrete, in the sense that each component is weakly contractible.*

This can be interpreted as saying that if the localization functor $L_{\mathcal{E}}$ exists, then the higher order structure associated to $\mathcal{E}^{-1}\mathcal{C}$ is trivial. As will become clear below (see the proof of 6.1) this follows from the fact that if $L_{\mathcal{E}}$ exists then each map $u : X \to Y$ in $\mathcal{E}^{-1}\mathcal{C}$, represented (1.4) by $u' : L_{\mathcal{E}}(X) \to L_{\mathcal{E}}(Y)$, has a *canonical* zigzag

$$X \xrightarrow{\epsilon_X} L_{\mathcal{E}}(X) \xrightarrow{u'} L_{\mathcal{E}}(Y) \xleftarrow{\epsilon_Y} Y$$

of morphisms in $\mathcal{C}$ which represent it. Any other representing zigzag can be naturally deformed to this canonical one, and so the space of representing zigzags is contractible.

What can be done with a localization context $(\mathcal{C}, \mathcal{E})$ which does not have good localizations or good colocalizations? This is really the purview of homotopy theory, which in practice relies on the theory of model categories to obtain access to the homotopy types of the function complexes in $\mathcal{L}(\mathcal{C}, \mathcal{E})$ in cases in which these function complexes are *not* homotopically discrete.

6.2. *Example.* Suppose that $\mathcal{C}$ is the category of topological spaces, and $\mathcal{E}$ is the subcategory of weak homotopy equivalences. Then $(\mathcal{C}, \mathcal{E})$ does *not* have good localizations or colocalizations; in fact, the only colocal object is the empty space, and the only local objects are the empty space and the one-point space. This is consistent with 6.1, since if X is a CW-complex and Y is an arbitrary space the simplicial set of maps $X \to Y$ in $\mathcal{L}(\mathcal{C}, \mathcal{W})$ is not usually homotopically discrete: its geometric realization has the weak homotopy type of the ordinary topological function space of maps $X \to Y$.

6.3. *Remark.* In most of the localization contexts $(\mathcal{C}, \mathcal{E})$ discussed in this paper, $\mathcal{C} = \mathrm{Ho}(\mathcal{M}) = \mathcal{W}^{-1}\mathcal{M}$, where $(\mathcal{M}, \mathcal{W})$ is the localization context provided by a model category $\mathcal{M}$ and its subcategory $\mathcal{W}$ of weak equivalences. If $L_{\mathcal{E}}$ exists in this situation, then by 6.1 no interesting higher order structure is created in passing from $\mathcal{W}^{-1}\mathcal{M}$ to $(\mathcal{W}+\mathcal{E})^{-1}\mathcal{M}$ (see 1.10 for the notation). This suggests that the higher structure involved in $(\mathcal{W} + \mathcal{E})^{-1}\mathcal{M}$ should be the same as the higher structure involved in $\mathcal{W}^{-1}\mathcal{M}$. In fact, in the situation sketched in 1.10 this is the case: the function complexes in $\mathcal{L}(\mathcal{M}, \mathcal{W}+\mathcal{E})$ are equivalent to the function complexes in $\mathcal{L}(\mathcal{M}, \mathcal{W})$ which involve objects of $\mathcal{M}$ that are $\mathcal{E}$-local in $\mathcal{W}^{-1}\mathcal{M}$.

Proof of 6.1. We sketch a proof based on [21]. Assume that $(\mathcal{C},\mathcal{E})$ has good localizations, let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ given by the $\mathcal{E}$-local objects, and $\mathcal{E}'=\mathcal{E}\cap\mathcal{C}'$. Let $\mathcal{H}=\mathcal{L}^H(\mathcal{C},\mathcal{E})$ be the hammock localization of $(\mathcal{C},\mathcal{E})$ and $\mathcal{H}'=\mathcal{L}^H(\mathcal{C}',\mathcal{E}')$. By [21, 2.2], it is enough to show that $\mathrm{Hom}_{\mathcal{H}}(X,Y)$ is homotopically discrete, and in fact [21, 3.3] it is enough to do this when X and Y are $\mathcal{E}$-local objects. Note that the complex $\mathrm{Hom}_{\mathcal{H}'}(X,Y)$ *is* homotopically discrete, since all the morphisms in $\mathcal{E}'$ are invertible and localizing a category by inverting morphisms which already have inverses does not introduce any higher structure [22, 5.3]. By [21, 5.5], the complex $\mathrm{Hom}_{\mathcal{H}}(X,Y)$ is the colimit of simplicial sets $\mathbf{m}(X,Y)$, where $\mathbf{m}$ is a word $\{\mathcal{C},\mathcal{E}^{-1}\}$ and $\mathbf{m}(X,Y)$ is the nerve of a category [21, 5.1] whose objects are zigzags of pattern $\mathbf{m}$ connecting X to Y in $\mathcal{C}$. Similarly, $\mathrm{Hom}_{\mathcal{H}'}(X,Y)=\operatorname{colim}\mathbf{m}'(X,Y)$. There are inclusions $\mathbf{m}'(X,Y)\to\mathbf{m}(X,Y)$. Applying the functor $L_{\mathcal{E}}$ to the zigzags and using the natural transformation $\mathrm{Id}\to L_{\mathcal{E}}$ gives a deformation retraction $\mathbf{m}(X,Y)\to\mathbf{m}'(X,Y)$, and so by [21, 4.5, 5.4], the colimit map

$$\mathrm{Hom}_{\mathcal{H}'}(X,Y)=\operatorname{colim}\mathbf{m}'(X,Y)\xrightarrow{\sim}\operatorname{colim}\mathbf{m}(X,Y)=\mathrm{Hom}_{\mathcal{H}}(X,Y)$$

is a weak equivalence. □

7. Constructing localizations and colocalizations

The localization map $\epsilon_A : A\to L_{\mathcal{E}}A$ has two universal properties: it is initial among maps $A\to X$ with X local, and terminal among equivalences $A\to A'$. To check the first statement, suppose that $f : A\to X$ is a map with X local. Since ϵ_A is an equivalence, it induces a bijection

$$\mathrm{Hom}_{\mathcal{C}}(L_{\mathcal{E}}(A),X)\xrightarrow{\cong}\mathrm{Hom}_{\mathcal{C}}(A,X)$$

and so there is a unique map $f' : L_{\mathcal{E}}(A)\to X$ such that $f'\cdot\epsilon_A=f$. This can be expressed by saying that ϵ_A is an initial element of the category in which an object is a map $A\to X$ with X local, and a morphism from $A\to X$ to $A\to X'$ is a map $X\to X'$ which makes the evident diagram commute. Similarly, ϵ_A is a terminal element of the analogous category in which an object is an equivalence $A\to A'$.

An initial element in a category is the limit of all of the objects in the category (more precisely, the limit of the identity functor from the category to itself), and a terminal element is the colimit of all the objects. In the present situation this gives two parallel, more or less tautological, formulas for the localization $L_{\mathcal{E}}(A)$.

$$L_{\mathcal{E}}(A) = \begin{cases} \lim\limits_{A\to X} X & (X \text{ local}) \\ \operatorname*{colim}\limits_{A\to A'} A' & (A \to A' \text{ an equivalence}) \end{cases}$$

where in each case the limit or colimit is taken over the indicated category of maps. In general these formulas are not useful for constructing $L_{\mathcal{E}}(A)$; they just translate the problem of building $L_{\mathcal{E}}(A)$ into the more or less equivalent problem of whether the corresponding limit or colimit exists. However, this analysis does suggest two approaches to obtaining $L_{\mathcal{E}}(A)$, which we will label the *Bousfield-Kan approach* and the *Bousfield approach*.

7.1. *The Bousfield-Kan approach.* Find some natural collection of maps $f_\alpha : A \to X_\alpha$ with each X_α is local, and hope that $\lim_{f_\alpha} X_\alpha$ computes $L_{\mathcal{E}}(A)$.

7.2. *The Bousfield approach.* Find some natural collection of equivalences $g_\beta : A \to A_\beta$, and hope that $\operatorname{colim}_{g_\beta} A_\beta$ computes $L_{\mathcal{E}}(A)$.

In situations with a topological flavor, it's natural to adjust the above constructions to use homotopy limits and homotopy colimits instead of limits and colimits. The Bousfield-Kan approach is used in [13] to construct homology localizations of spaces with respect to ordinary homology theories; the approach succeeds in many cases, but not in all. The advantage of this approach is that the homotopy limit diagram involved is very small (just a cosimplicial space) and easy to calculate with. Mahowald and Thompson [36] and Bousfield [10] implicitly use something like this approach to compute unstable L_1-localizations of odd spheres and H-spaces. Bousfield uses the second approach in [4], [5], and [6] to construct arbitrary homology localizations as well as localizations with respect to arbitrary maps. The technique for building the nullification (4.7) $P_W(X)$ of a spectrum X with respect to a finite spectrum W is especially simple. Let $N(X)$ be the result of attaching a cone $C(\Sigma^i W)$ to X for each map $\Sigma^i W \to X$, $i \in \mathbb{Z}$; $P_W(X)$ is then the homotopy colimit of the sequence

$$X \to N(X) \to N^2(X) \to N^3(X) \to \cdots .$$

For larger W it would be necessary to continue the process transfinitely.

Dually, the colocalization map $C_{\mathcal{E}}(A) \to A$ has two universal properties; it is terminal among maps $X \to A$ with X colocal, and initial among equivalences $A' \to A$. This leads to two tautological formulas for $C_{\mathcal{E}}(A)$:

$$C_{\mathcal{E}}(A) = \begin{cases} \operatorname{colim}_{X\to A} X & (X \text{ colocal}) \\ \lim_{A'\to A} A' & (A' \to A \text{ an equivalence}) \end{cases}$$

As above, this suggests two approaches to constructing colocalizations. The (homotopy) colimit approach is the one that is usually used [18] [8]. We do not know of any effective way to build colocalizations with limits, short of constructing a localization in terms of colimits, and then expressing this in the opposite category!

References

1. L. Alonso Tarrío, A. Jeremías López, and J. Lipman, *Studies in duality on Noetherian formal schemes and non-Noetherian ordinary schemes*, Contemporary Mathematics, vol. 244, American Mathematical Society, Providence, RI, 1999.
2. B. Badzioch, *Algebraic theories in homotopy theory*, Ann. of Math. (2) **155** (2002), no. 3, 895–913.
3. A. K. Bousfield, *The simplicial homotopy theory of iterated loop spaces*, unpublished.
4. ______, *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150.
5. ______, *Constructions of factorization systems in categories*, J. Pure Appl. Algebra **9** (1976/77), no. 2, 207–220.
6. ______, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281.
7. ______, *K-localizations and K-equivalences of infinite loop spaces*, Proc. London Math. Soc. (3) **44** (1982), no. 2, 291–311.
8. ______, *Localization and periodicity in unstable homotopy theory*, J. Amer. Math. Soc. **7** (1994), no. 4, 831–873.
9. ______, *Homotopical localizations of spaces*, Amer. J. Math. **119** (1997), no. 6, 1321–1354.
10. ______, *The K-theory localizations and v_1-periodic homotopy groups of H-spaces*, Topology **38** (1999), no. 6, 1239–1264.
11. ______, *On $K(n)$-equivalences of spaces*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 85–89.
12. ______, *On the telescopic homotopy theory of spaces*, Trans. Amer. Math. Soc. **353** (2001), no. 6, 2391–2426 (electronic).
13. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Springer-Verlag, Berlin, 1972, Lecture Notes in Mathematics, Vol. 304.
14. C. Casacuberta, D. Scevenels, and J. H. Smith, *Implications of large-cardinal principles in homotopical localization*, preprint.
15. W. Chachólski, *On the functors CW_A and P_A*, Duke Math. J. **84** (1996), no. 3, 599–631.
16. E. Dror-Farjoun, W. G. Dwyer, and D. M. Kan, *An arithmetic square for virtually nilpotent spaces*, Illinois J. Math. **21** (1977), no. 2, 242–254.

17. E. Dror Farjoun, *Homotopy and homology of diagrams of spaces*, Algebraic topology (Seattle, Wash., 1985), Lecture Notes in Math., vol. 1286, Springer, Berlin, 1987, pp. 93–134.
18. E. Dror-Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996.
19. D. Dugger, S. Hollander, and D. C. Isaksen, *Hypercovers and simplicial presheaves*, preprint (2002) `http://www.math.uiuc.edu/K-theory/0563/spre.dvi`.
20. W. G. Dwyer and J. P. C. Greenlees, *Complete modules and torsion modules*, Amer. J. Math. **124** (2002), no. 1, 199–220.
21. W. G. Dwyer and D. M. Kan, *Calculating simplicial localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35.
22. ______, *Simplicial localizations of categories*, J. Pure Appl. Algebra **17** (1980), no. 3, 267–284.
23. ______, *Function complexes for diagrams of simplicial sets*, Nederl. Akad. Wetensch. Indag. Math. **45** (1983), no. 2, 139–147.
24. P. G. Goerss and J. F. Jardine, *Localization theories for simplicial presheaves*, Canad. J. Math. **50** (1998), no. 5, 1048–1089.
25. P. Goerss, H.-W. Henn, and M. Mahowald, *The homotopy of $L_2V(1)$ for the prime* 3, preprint (2002).
26. T. G. Goodwillie, *Calculus III*, preprint (2002).
27. ______, *Calculus. II. Analytic functors*, K-Theory **5** (1991/92), no. 4, 295–332.
28. J. P. C. Greenlees and J. P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), no. 2, 438–453.
29. ______, *Completions in algebra and topology*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 255–276.
30. A. Grothendieck, *Local cohomology*, Springer-Verlag, Berlin, 1967, Course notes taken by R. Hartshorne, Harvard University, Fall, 1961. Lecture Notes in Mathematics, No. 41.
31. P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.
32. M. Hovey and N. P. Strickland, *Morava K-theories and localisation*, Mem. Amer. Math. Soc. **139** (1999), no. 666, viii+100.
33. J. F. Jardine, *Simplicial presheaves*, J. Pure Appl. Algebra **47** (1987), no. 1, 35–87.
34. N. J. Kuhn, *Morava K-theories and infinite loop spaces*, Algebraic topology (Arcata, CA, 1986), Lecture Notes in Math., vol. 1370, Springer, Berlin, 1989, pp. 243–257.
35. M. Mahowald, D. Ravenel, and P. Shick, *The triple loop space approach to the telescope conjecture*, Homotopy methods in algebraic topology (Boulder, CO, 1999), Contemp. Math., vol. 271, Amer. Math. Soc., Providence, RI, 2001, pp. 217–284.
36. M. Mahowald and R. D. Thompson, *The K-theory localization of an unstable sphere*, Topology **31** (1992), no. 1, 133–141.
37. H. Miller, *Finite localizations*, Bol. Soc. Mat. Mexicana (2) **37** (1992), no. 1-2, 383–389, Papers in honor of José Adem (Spanish).

38. F. Morel and V. Voevodsky, $\mathbf{A}^1$*-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. (1999), no. 90, 45–143 (2001).
39. D. C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), no. 2, 351–414.
40. ______, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies, vol. 128, Princeton University Press, Princeton, NJ, 1992, Appendix C by Jeff Smith.
41. G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.
42. K. Shimomura and X. Wang, *The homotopy groups* $\pi_* L_2(S^0)$ *at the prime* 3, preprint.
43. K. Shimomura and A. Yabe, *The homotopy groups* $\pi_*(L_2 S^0)$, Topology **34** (1995), no. 2, 261–289.
44. N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154.
45. D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. (2) **100** (1974), 1–79.
46. C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

Department of Mathematics, 255 Hurley, University of Notre Dame, Notre Dame IN 46556 USA

E-mail address: dwyer.1@nd.edu