

# EVALUATION MAPS IN RATIONAL HOMOTOPY

YVES FÉLIX AND GREGORY LUPTON

ABSTRACT. Let  $E$  be an  $H$ -space acting on a based space  $X$ . Then we refer to  $w: E \rightarrow X$ , the map obtained by acting on the base point of  $X$ , as a “generalized evaluation map” (see Definition 1.1 for a precise definition). We establish several fundamental results about the rational homotopy behaviour of a generalized evaluation map, all of which apply to the usual evaluation map  $\text{Map}(X, X; 1) \rightarrow X$ . With mild hypotheses on  $X$ , we show that a generalized evaluation map  $w$  factors, up to rational homotopy, through a map  $\Gamma_w: S_w \rightarrow X$  where  $S_w$  is a (relatively small) finite product of odd-dimensional spheres and  $\pi_{\#}(\Gamma_w) \otimes \mathbb{Q}$  is injective. This result has strong consequences: if the image in rational homotopy groups of  $w$  is trivial, then *the generalized evaluation map is null-homotopic* after rationalization; unless  $X$  satisfies a very strong splitting condition, any generalized evaluation map *induces the trivial homomorphism in rational cohomology*; the map  $\Gamma_w$  is *rationally a homotopy monomorphism* and a generalized evaluation map may be written as a composition of a homotopy epimorphism and this homotopy monomorphism. We include illustrative examples and prove numerous subsidiary results of interest.

## 1. INTRODUCTION

Let  $X$  be a based space and let  $\text{Map}(X, X)$  be the space of unbased, or free, maps from  $X$  to itself. In general  $\text{Map}(X, X)$  is disconnected; we denote by  $\text{Map}(X, X; 1)$  its identity component, that is, the path component that consists of self maps that are (freely) homotopic to the identity. Then we have *the evaluation map*  $\omega: \text{Map}(X, X; 1) \rightarrow X$  defined by evaluation at the basepoint of  $X$ . This map occupies a central place in the homotopy theory of fibrations (cf. [5, 6, 7, 8]).

The evaluation map  $\omega$  and its rationalization will play a distinguished role in this paper. However, we find that our methods and results apply equally well to other contexts in which one has an “evaluation map.” For example, it is often of interest to consider the space  $\text{Top}(X, X)$  of self-homeomorphisms of  $X$  and the corresponding evaluation map  $w: \text{Top}(X, X; 1) \rightarrow X$ . Here,  $\text{Top}(X, X; 1)$  denotes the component of  $\text{Top}(X, X)$  that consists of self-homeomorphisms homotopic (*via* self-homeomorphisms) to the identity. Likewise, if  $X$  is a smooth manifold, then one may replace  $\text{Top}(X, X)$  with  $\text{Diff}(X, X)$ , and so-forth. A further example of an “evaluation map” to which our methods apply concerns *configuration spaces*. Let  $F(X, k)$  denote the configuration space that consists of ordered  $k$ -tuples of distinct points in a space  $X$ , and let  $(p_1, \dots, p_k)$  be a choice of basepoint in  $F(X, k)$ . Then we have a map

$$\theta: \text{Top}(X, X; 1) \rightarrow F(X, k)$$

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given by  $\theta(\alpha) = (\alpha(p_1), \dots, \alpha(p_k))$ . Actually, here we have  $\theta = w \circ \Theta$ , where

$$\Theta: \text{Top}(X, X; 1) \rightarrow \text{Top}(F(X, k), F(X, k); 1)$$

is the natural injection defined by  $\Theta(\alpha)(q_1, \dots, q_k) = (\alpha(q_1), \dots, \alpha(q_k))$ , and

$$w: \text{Top}(F(X, k), F(X, k); 1) \rightarrow F(X, k)$$

is an evaluation map for  $F(X, k)$  in the preceding sense, with “Top” replacing “Map.”

Motivated by the preceding examples, we now make a formal definition of the evaluation maps that we consider. Recall that an  $H$ -space is a pair  $(E, \mu)$  with  $E$  a based space and multiplication  $\mu: E \times E \rightarrow E$  a based map that satisfies  $\mu \circ J \sim \nabla: E \vee E \rightarrow E$ . Here,  $\nabla: E \vee E \rightarrow E$  denotes the folding map and  $J: E \vee E \rightarrow E \times E$  the obvious inclusion. We say that the multiplication has *strict identity* if  $\mu \circ J = \nabla$  (equals, not just homotopic). Note that  $\text{Map}(X, X; 1)$  is an  $H$ -space with strict identity. Now let  $i_1: E \rightarrow E \times X$  and  $i_2: X \rightarrow E \times X$  denote the inclusions. By an *action of  $E$  on  $X$*  we mean a map  $A: E \times X \rightarrow X$  that satisfies  $A \circ i_2 = 1: X \rightarrow X$ . We say that *the action is associative* if in addition we have  $A \circ (\mu \times 1) = A \circ (1 \times A)$ .

**Definition 1.1.** A *generalized evaluation map* is any (based) map  $w: E \rightarrow X$ , from a connected  $H$ -space with strict identity  $E$  to a space  $X$ , for which there exists an associative action  $A: E \times X \rightarrow X$  that restricts to  $w$ , that is, that satisfies  $A \circ i_1 = w: E \rightarrow X$ .

**Examples 1.2.** (1) The action  $A: \text{Map}(X, X; 1) \times X \rightarrow X$  given by  $A(f, x) = f(x)$  makes  $\omega: \text{Map}(X, X; 1) \rightarrow X$  a generalized evaluation map according to Definition 1.1. Similarly for all the other examples mentioned above.

(2) Suppose  $G$  is a connected topological group and  $A: G \times X \rightarrow X$  is a group action in the usual sense. Then the *orbit map* of the action is a generalized evaluation map  $G \rightarrow X$ .

(3) More generally, suppose given a fibration  $X \rightarrow Y \rightarrow B$ . Then the connecting map  $\partial: \Omega B \rightarrow X$  is a generalized evaluation map. This follows from the usual action of  $\Omega B$  on the fibre  $X$ . Note, however, that we must take Moore loops in  $\Omega X$  to obtain an  $H$ -space with strict identity.

Revert now to the ordinary evaluation map  $\omega: \text{Map}(X, X; 1) \rightarrow X$ . *For the remainder of the paper, we assume that  $X$  is a nilpotent, finite complex.* Since  $X$  is finite, a result of Milnor [17] implies that  $\text{Map}(X, X; 1)$  is a CW complex. Since  $X$  is nilpotent, we may choose and fix a rationalization  $e: X \rightarrow X_{\mathbb{Q}}$ . Now results of [12] imply that  $e_*: \text{Map}(X, X; 1) \rightarrow \text{Map}(X, X_{\mathbb{Q}}; e)$  is a rationalization. Thus, the map  $\omega_{\mathbb{Q}}: \text{Map}(X, X_{\mathbb{Q}}; e) \rightarrow X_{\mathbb{Q}}$ , also defined by evaluation at the basepoint of  $X$ , may be taken to be the rationalization of  $\omega$ . We refer to  $\omega_{\mathbb{Q}}$  as *the rationalized evaluation map*. Recall that the  *$n$ th Gottlieb group of  $X$* , denoted  $G_n(X)$ , is the subgroup of  $\pi_n(X)$  defined as the image of  $\omega_{\#}: \pi_n(\text{Map}(X, X; 1)) \rightarrow \pi_n(X)$  [7]. The subgroup of  $\pi_n(X_{\mathbb{Q}})$  defined as the image of  $(\omega_{\mathbb{Q}})_{\#}: \pi_n(\text{Map}(X, X_{\mathbb{Q}}; e)) \rightarrow \pi_n(X_{\mathbb{Q}})$  is called the  *$n$ th rationalized Gottlieb group of  $X$*  and denoted by  $G_n(X_{\mathbb{Q}})$ . By a theorem of Lang [13], we have  $G_n(X_{\mathbb{Q}}) \cong G_n(X) \otimes \mathbb{Q}$  under our assumption that  $X$  is finite. The rationalized Gottlieb groups have played an important role in some of the major developments of rational homotopy theory (cf. [1, 11]). Our results in this paper show that the rationalized Gottlieb groups exercise a very strong determining effect on the rationalized evaluation map.

A result of Félix-Halperin ([1, Th.III]) implies that  $G_{2i}(X_{\mathbb{Q}}) = 0$  for all  $i$  and  $G_{2i+1}(X_{\mathbb{Q}})$  is non-zero for only finitely many  $i$ . Suppose  $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$  is a basis of  $G_*(X_{\mathbb{Q}}) = G_{\text{odd}}(X_{\mathbb{Q}})$  with  $\alpha_i \in G_{n_i}(X_{\mathbb{Q}})$  (here we regard an element of  $\pi_n(X_{\mathbb{Q}})$  as represented by a map  $\alpha: S_{\mathbb{Q}}^n \rightarrow X_{\mathbb{Q}}$ ). For each  $\alpha_i$ , we may choose a  $\beta_i \in \pi_{n_i}(\text{Map}(X, X_{\mathbb{Q}}; e))$  such that  $\omega_{\mathbb{Q}} \circ \beta_i = \alpha_i$ . The adjoint of  $\beta_i$  gives a map  $F_i: S_{\mathbb{Q}}^{n_i} \times X \rightarrow X_{\mathbb{Q}}$  that extends the map  $(\alpha_i | e): S_{\mathbb{Q}}^{n_i} \vee X \rightarrow X_{\mathbb{Q}}$ . Denote by  $S_X$  the product of odd-dimensional rational spheres  $S_{\mathbb{Q}}^{n_1} \times \dots \times S_{\mathbb{Q}}^{n_r}$  whose factors correspond to the domains of the basis of  $G_*(X_{\mathbb{Q}})$ . Then we form a map  $F: S_X \times X \rightarrow X_{\mathbb{Q}}$  as the composition

$$F = F_1 \circ (1 \times F_2) \circ \dots \circ (1 \times \dots \times 1 \times F_r).$$

Now set  $\Gamma_X = F \circ i: S_X \rightarrow X_{\mathbb{Q}}$ , where  $i$  denotes the inclusion of the product of spheres as the first  $r$  factors. We refer to  $\Gamma_X$  as a *total Gottlieb element of  $X_{\mathbb{Q}}$* . By taking the adjoint of  $F$ , we obtain a lift  $\tilde{\Gamma}_X: S_X \rightarrow \text{Map}(X, X_{\mathbb{Q}}; e)$  of  $\Gamma_X$  through the rationalized evaluation map  $\omega_{\mathbb{Q}}$ .

We prove the following result:

**Theorem 1.3.** *Let  $X$  be any nilpotent, finite complex. The rationalized evaluation map  $\omega_{\mathbb{Q}}: \text{Map}(X, X_{\mathbb{Q}}; e) \rightarrow X_{\mathbb{Q}}$  factors up to homotopy through the total Gottlieb element  $\Gamma_X: S_X \rightarrow X_{\mathbb{Q}}$ . In fact, there is a retraction  $r$  of  $\tilde{\Gamma}_X$ , that is, a map  $r: \text{Map}(X, X_{\mathbb{Q}}; e) \rightarrow S_X$  with  $r \circ \tilde{\Gamma}_X = 1$ , such that  $\omega_{\mathbb{Q}} = \Gamma_X \circ r$ .*

This basic result has several strong consequences. An immediate one is the following striking illustration of the effect that the homomorphism induced on rational homotopy groups has on the rationalized evaluation map.

**Corollary 1.4.** *The evaluation map  $\omega: \text{Map}(X, X; 1) \rightarrow X$  is rationally null-homotopic if and only if  $G_*(X_{\mathbb{Q}}) = 0$ .*

Now the evaluation map  $\omega$  may be viewed as a “universal connecting map” for fibrations with fibre  $X$ , in that any connecting map  $\Omega B \rightarrow X$  of a fibration  $X \rightarrow E \rightarrow B$  factors through  $\omega$  [6]. A further immediate consequence of Theorem 1.3, therefore, is the following result.

**Corollary 1.5.** *Let  $X \rightarrow E \rightarrow B$  be any fibration with fibre  $X$  a nilpotent, finite space. If  $G_*(X_{\mathbb{Q}}) = 0$ , then the connecting map  $\partial: \Omega B \rightarrow X$  is rationally null-homotopic.*

There are many spaces to which these corollaries may be applied. For instance, any suspension that is not rationally equivalent to a sphere has trivial rationalized Gottlieb groups. Roughly speaking, a typical wedge or connected sum of spaces has trivial Gottlieb groups, as do many non-elliptic, coformal spaces. More precisely, a space whose rational homotopy Lie algebra has trivial centre has trivial rationalized Gottlieb group. Therefore, by Corollary 1.4, the rationalized evaluation map is null-homotopic in all such cases.

The preceding discussion of  $\omega$  and the Gottlieb groups extends naturally to generalized evaluation maps. Suppose given  $w: E \rightarrow X$  any generalized evaluation map. In Section 2 we construct a map  $\Gamma_w: S_w \rightarrow X_{\mathbb{Q}}$  such that  $\text{im}(\Gamma_w)_{\#} \otimes \mathbb{Q} = \text{im}(w)_{\#} \otimes \mathbb{Q}$ . As with  $S_X$  above,  $S_w$  is a product of a relatively small number of odd-dimensional rational spheres. We refer to  $\Gamma_w$  as a *total Gottlieb element of  $X_{\mathbb{Q}}$  with respect to  $w$* . Furthermore,  $\Gamma_w$  admits a lift through  $\omega_{\mathbb{Q}}$ , the rationalization

of  $w$ . That is, there exists a map  $\tilde{\Gamma}_w: S_w \rightarrow E_{\mathbb{Q}}$  that satisfies  $w_{\mathbb{Q}} \circ \tilde{\Gamma}_w = \Gamma_w$ . Then we have generalizations of Theorem 1.3 and Corollary 1.4 as follows.

**Theorem 1.6.** *Let  $w: E \rightarrow X$  be any generalized evaluation map with  $X$  a nilpotent, finite complex. Suppose that  $\Gamma_w: S_w \rightarrow X_{\mathbb{Q}}$  is a total Gottlieb element of  $X_{\mathbb{Q}}$  with respect to  $w$ . Then  $w_{\mathbb{Q}}$  factors up to homotopy through  $\Gamma_w$ . More precisely, suppose that  $\tilde{\Gamma}_w: S_w \rightarrow E_{\mathbb{Q}}$  is a lift of  $\Gamma_w$  through  $w_{\mathbb{Q}}$ . Then there is a retraction  $r: E_{\mathbb{Q}} \rightarrow S_w$  of  $\tilde{\Gamma}_w$  such that  $w_{\mathbb{Q}} = \Gamma_w \circ r$ .*

**Corollary 1.7.** *Let  $w: E \rightarrow X$  be any generalized evaluation map. Then  $w_{\#} \otimes \mathbb{Q} = 0: \pi_*(E_{\mathbb{Q}}) \rightarrow \pi_*(X_{\mathbb{Q}})$  if and only if  $w: E \rightarrow X$  is rationally null-homotopic.*

We continue with a theorem related to the homotopy behaviour of the maps  $\Gamma_w: S_w \rightarrow X_{\mathbb{Q}}$ . Recall that a map  $f: X \rightarrow Y$  is a *homotopy monomorphism* if, for any  $A$ , the induced map of homotopy sets  $f_*: [A, X] \rightarrow [A, Y]$  is injective [3]. In general it is a difficult problem to identify when a map is a homotopy monomorphism. We say that a map of nilpotent spaces  $f: X \rightarrow Y$  is a *homotopy monomorphism in the nilpotent category* if  $f_*: [A, X] \rightarrow [A, Y]$  is injective whenever  $A$  is a nilpotent space.

**Theorem 1.8.** *Let  $X$  be a nilpotent, finite complex and  $w: E \rightarrow X$  be any generalized evaluation map. Then  $\Gamma_w: S_w \rightarrow X_{\mathbb{Q}}$  is a homotopy monomorphism in the nilpotent category.*

Theorem 1.8 is proved towards the end of Section 3. As a consequence, together with Theorem 1.6 we find that, after rationalization, a generalized evaluation map may be written as a composition  $w_{\mathbb{Q}} = \Gamma_w \circ r$  of a homotopy epimorphism and a homotopy monomorphism in the nilpotent category (Corollary 3.6). We also note the following immediate consequence of Theorem 1.8.

**Corollary 1.9.** *Let  $X$  be a nilpotent, finite complex and let  $\alpha: S^n \rightarrow X_{\mathbb{Q}}$  be any rationalized Gottlieb element. Then  $\alpha$  is a homotopy monomorphism in the nilpotent category.*

In particular, this implies that the rationalized Hopf maps are homotopy monomorphisms in the nilpotent category. By contrast, the Hopf map  $\eta: S^7 \rightarrow S^4$  is not a homotopy monomorphism [3].

A further consequence of Theorem 1.8 is the classification up to rational homotopy of cyclic maps. A map  $f: A \rightarrow X$  is called *cyclic* if  $(f | 1): A \vee X \rightarrow X$  extends to a map  $A \times X \rightarrow X$  [20]. Denote by  $G(A, X)$  the set of homotopy classes of cyclic maps from  $A$  into  $X$ . This is a generalization of the  $n$ th Gottlieb group of  $X$ , which we obtain by taking  $A = S^n$ . Upon rationalizing a cyclic map, we obtain a map  $f_{\mathbb{Q}}: A \rightarrow X_{\mathbb{Q}}$  in  $G(A, X_{\mathbb{Q}})$ .

**Theorem 1.10.** *Let  $X$  be a nilpotent, finite complex and let  $A$  be any nilpotent space. Then there is a bijection of sets*

$$G(A, X_{\mathbb{Q}}) \cong [A, S_X] \cong \bigoplus_r \text{Hom}(H_r(A; \mathbb{Q}), G_r(X_{\mathbb{Q}})).$$

This classification allows us, for instance, to easily identify situations in which  $G(A, X_{\mathbb{Q}})$  is trivial and, hence,  $G(A, X)$  is finite. In Theorem 3.8, we extend this result to apply to any generalized evaluation map.

Our last topic is the (co)homological behaviour of generalized evaluation maps. For the ordinary evaluation map  $\omega: \text{Map}(X, X; 1) \rightarrow X$ , this behaviour has been studied by Gottlieb [9] and Oprea [18, 19]. From [18] we have the following result:

**Theorem 1.11** (Oprea). *Let  $F \rightarrow E \rightarrow B$  be a fibration with connecting map  $\partial: \Omega B \rightarrow F$ . Suppose that  $B$  is 1-connected and that  $B$  and  $F$  have finite type rational homology. Then there is a splitting, up to rational homotopy,  $F \simeq_{\mathbb{Q}} S \times Y$  with  $S$  a product of Eilenberg-Mac Lane spaces and*

$$\dim \pi_*(S) \otimes \mathbb{Q} = \dim \text{Image}(h_F \circ \partial_{\#}: \pi_*(\Omega B) \otimes \mathbb{Q} \rightarrow H_*(F; \mathbb{Q})).$$

Here,  $h_F: \pi_*(F) \otimes \mathbb{Q} \rightarrow H_*(F; \mathbb{Q})$  denotes the rational Hurewicz homomorphism.

Oprea's result may be applied to the evaluation map  $\omega$  by considering it as the connecting map in the universal fibration for fibrations with fibre  $X$ . Our main result about the homological behaviour of a generalized evaluation map is the following composite theorem, which gives a complete description for rational coefficients.

**Theorem 1.12.** *Let  $w: E \rightarrow X$  be any generalized evaluation map with  $X$  a nilpotent, finite complex. Then we have:*

- (1)  $\tilde{H}_*(w; \mathbb{Q}) \neq 0: \tilde{H}_*(E; \mathbb{Q}) \rightarrow \tilde{H}_*(X; \mathbb{Q})$  if and only if  $h_X \circ (w_{\#} \otimes \mathbb{Q}) \neq 0: \pi_*(E) \otimes \mathbb{Q} \rightarrow \pi_*(X) \otimes \mathbb{Q} \rightarrow H_*(X; \mathbb{Q})$ ;
- (2) if  $h_X \circ (w_{\#} \otimes \mathbb{Q})$  has image in  $H_*(X; \mathbb{Q})$  of dimension  $r > 0$ , then  $H_*(w; \mathbb{Q})$  has image in  $H_*(X; \mathbb{Q})$  of dimension  $2^r$  and there is a rational homotopy equivalence  $X \simeq_{\mathbb{Q}} S \times Y$ , with  $S$  a product of odd-dimensional spheres such that  $H_*(S; \mathbb{Q}) \cong \text{Image } H_*(w; \mathbb{Q})$  and  $\pi_*(S) \otimes \mathbb{Q} \cong \text{Image } h_X \circ (w_{\#} \otimes \mathbb{Q})$ ;
- (3) if  $X \simeq_{\mathbb{Q}} S^{2n+1} \times Y$ , then  $\tilde{H}_*(w; \mathbb{Q}) \neq 0$ , where  $\omega: \text{Map}(X, X; 1) \rightarrow X$  is the ordinary evaluation map.

Our treatment here extends Oprea's theorem to a generalized evaluation map. Theorem 1.12 shows that, in most cases, the rank of  $H_*(w; \mathbb{Q})$  is relatively small. We also deduce that  $H_*(w; \mathbb{Q})$  is surjective only when  $X$  is an  $H_0$ -space. Theorem 1.12 has various interesting corollaries, such as the following sharpening of a result of Gottlieb [9, Th.3] for rational coefficients.

**Corollary 1.13.** *Suppose that  $\chi(X) \neq 0$ . Then for every generalized evaluation map  $w: E \rightarrow X$ , we have  $\tilde{H}_*(w; \mathbb{Q}) = 0: \tilde{H}_*(E; \mathbb{Q}) \rightarrow \tilde{H}_*(X; \mathbb{Q})$ .*

A further consequence is the following result:

**Corollary 1.14.** *Let  $M$  be a simply connected, symplectic manifold. Then every generalized evaluation map  $w: E \rightarrow M$  is trivial on rational homology, that is,  $\tilde{H}_*(w; \mathbb{Q}) = 0: \tilde{H}_*(E; \mathbb{Q}) \rightarrow \tilde{H}_*(M; \mathbb{Q})$ . Consequently, if  $G$  is a connected Lie group and  $a: G \rightarrow M$  is the orbit map of any  $G$ -action on  $M$ , we have  $\tilde{H}_*(a; \mathbb{Q}) = 0$ .*

These corollaries appear as Corollary 4.1 and Corollary 4.2, respectively.

The text is divided into five parts. In Section 2 we present the factorization results. Section 3 contains some technical lemmas on Gottlieb groups, and the monomorphism theorem. The homological behaviour of generalized evaluation maps is discussed in Section 4. Section 5 is a brief, concluding section in which we mention several problems that suggest directions for future work.

We finish this introduction with some terminology and notation. We work in the homotopy category, and so we often do not distinguish between a map and the homotopy class it represents. We use  $\simeq$  to denote that two spaces are homotopy equivalent, or that a map is a homotopy equivalence. If  $f: A \rightarrow B$  is a map, then  $f^*$  denotes pre-composition by  $f$  and  $f_*$  denotes post-composition by  $f$ . We use  $H_*(f)$

and  $H^*(f)$  to denote the map induced on homology, respectively cohomology, by the map of spaces  $f$ , and  $f_{\#}$  to denote the map induced on homotopy groups. Likewise,  $\tilde{H}_*(f)$  and  $\tilde{H}^*(f)$  denote reduced (co)homology. We denote the rationalization of a space  $X$  by  $X_{\mathbb{Q}}$  and of a map  $f$  by  $f_{\mathbb{Q}}$  (cf. [12]). By an  $H_0$ -space, we mean a space whose rationalization is an  $H$ -space. We say that maps  $f, g: X \rightarrow Y$  are rationally homotopic if their rationalizations are homotopic. We denote this relation either by  $f =_{\mathbb{Q}} g: X \rightarrow Y$  or by  $f_{\mathbb{Q}} = g_{\mathbb{Q}}: X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$ . We reserve  $\omega$  to denote the evaluation map  $\omega: \text{Map}(X, X; 1) \rightarrow X$ . Generalized evaluation maps will be denoted with a generic  $w$ . For the remainder of the paper, we will usually drop the “generalized” and refer simply to *an evaluation map*.

We assume familiarity with rational homotopy theory and use the standard notation and terminology for minimal models as presented in [2]. The basic facts that we use are as follows: Each nilpotent space  $X$  has a unique Sullivan minimal model  $(\mathcal{M}_X, d_X)$  in the category of commutative DG (differential graded) algebras over  $\mathbb{Q}$ . This DG algebra  $(\mathcal{M}_X, d_X)$  is of the form  $\mathcal{M}_X = \wedge V$ , a free graded commutative algebra generated by a positively graded vector space  $V$  of finite type. The differential  $d_X$  is decomposable, in that  $d_X(V) \subseteq \wedge^{\geq 2} V$ , and  $V$  admits a basis  $\{v_{\alpha}\}$  indexed by a well ordered set such that  $d_X(v_{\alpha}) \in \wedge(\{v_{\beta}\}_{\beta < \alpha})$ . A fact that we use very frequently here is that an  $H_0$ -space has a minimal model with zero differential. Each map  $f: X \rightarrow Y$  also has a Sullivan minimal model which is a DG algebra map  $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$ . The Sullivan minimal model is a complete rational homotopy invariant for a space or a map. If  $f, g: X \rightarrow Y$  are maps of *rational spaces*, then  $f$  and  $g$  are homotopic if and only if their Sullivan minimal models  $\mathcal{M}_f$  and  $\mathcal{M}_g$  are homotopic in an algebraic sense. Rational cohomology is readily retrieved from Sullivan minimal models: We have a natural isomorphism  $H(\mathcal{M}_X, d_X) \cong H^*(X; \mathbb{Q})$  and this isomorphism identifies  $H(\mathcal{M}_f): H(\mathcal{M}_Y) \rightarrow H(\mathcal{M}_X)$  with  $H^*(f): H^*(Y; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$ . Rational homotopy groups are retrieved as follows: Let  $Q(\mathcal{M}_X) \cong V$  be the (quotient) module of indecomposables of  $\mathcal{M}_X$ . There is a natural isomorphism  $Q(\mathcal{M}_X) \cong \text{Hom}(\pi_*(X), \mathbb{Q})$ , that identifies  $Q(\mathcal{M}_f): Q(\mathcal{M}_Y) \rightarrow Q(\mathcal{M}_X)$  with  $(f_{\#} \otimes \mathbb{Q})^*: \text{Hom}(\pi_*(Y), \mathbb{Q}) \rightarrow \text{Hom}(\pi_*(X), \mathbb{Q})$ .

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## 2. FACTORIZATION OF AN EVALUATION FIBRATION

The main purpose of this section is the proof of Theorem 1.3, Theorem 1.6 and their corollaries. The results will flow from some general considerations about fibrations of nilpotent spaces  $F \rightarrow E \rightarrow X$  in which both  $F$  and  $E$  are  $H_0$ -spaces. The evaluation fibration  $\omega: \text{Map}(X, X; 1) \rightarrow X$  is of this form with fibre the subspace of  $\text{Map}(X, X; 1)$  consisting of based maps, which we denote by  $\text{Map}_*(X, X; 1)$ .

First we focus on the fibre inclusion of such a fibration.

**Proposition 2.1.** *Suppose  $j: F \rightarrow E$  is any map between  $H_0$ -spaces. Then  $E_{\mathbb{Q}}$  decomposes up to homotopy equivalence as  $E_{\mathbb{Q}} \simeq Y \times Z$ , with  $Y$  and  $Z$  products of rational Eilenberg-Mac Lane spaces, so that there is a corresponding map  $\phi: Y \rightarrow$*

$F_{\mathbb{Q}}$  with  $(j_{\mathbb{Q}})_{\#}(\pi_*(F_{\mathbb{Q}})) = i_{\#}(\pi_*(Y))$  and  $j_{\mathbb{Q}} \circ \phi = i$ , where  $i: Y \rightarrow E_{\mathbb{Q}}$  denotes the inclusion of the first factor.

*Proof.* Decompose  $\pi_*(E_{\mathbb{Q}})$  as  $V \oplus W$  with  $V = \text{im}(j_{\mathbb{Q}})_{\#}$  and  $W$  a complement. Set  $Y = \prod_i K(V_i, i)$  and  $Z = \prod_i K(W_i, i)$ . Choose a basis  $\{\alpha_t\}_{t \in T}$  for  $V$  so that  $Y = \prod_t K(\mathbb{Q}, |\alpha_t|)$ . If  $|\alpha_t|$  is odd, then we identify  $K(\mathbb{Q}, |\alpha_t|) \simeq S_{\mathbb{Q}}^{|\alpha_t|}$  and we have a map  $\alpha_t: K(\mathbb{Q}, |\alpha_t|) \rightarrow E_{\mathbb{Q}}$ . If  $|\alpha_t|$  is even, we construct a corresponding map as follows. First, we identify  $K(\mathbb{Q}, |\alpha_t|) \simeq \Omega \Sigma S_{\mathbb{Q}}^{|\alpha_t|}$ . Let  $\epsilon: X \rightarrow \Omega \Sigma X$  denote the adjoint of the (suspension of the) identity. Since  $E_{\mathbb{Q}}$  is an  $H$ -space, we may choose a retraction  $r: \Omega \Sigma E_{\mathbb{Q}} \rightarrow E_{\mathbb{Q}}$  of  $\epsilon: E_{\mathbb{Q}} \rightarrow \Omega \Sigma E_{\mathbb{Q}}$  so that  $r \circ \epsilon = 1$  and the following diagram commutes:

$$\begin{array}{ccc} \Omega \Sigma S_{\mathbb{Q}}^{|\alpha_t|} & \xrightarrow{\Omega \Sigma \alpha_t} & \Omega \Sigma E_{\mathbb{Q}} \\ \epsilon \uparrow & & \uparrow \epsilon \\ S_{\mathbb{Q}}^{|\alpha_t|} & \xrightarrow{\alpha_t} & E_{\mathbb{Q}} \end{array} \quad \left. \vphantom{\begin{array}{ccc} \Omega \Sigma S_{\mathbb{Q}}^{|\alpha_t|} & \xrightarrow{\Omega \Sigma \alpha_t} & \Omega \Sigma E_{\mathbb{Q}} \\ \epsilon \uparrow & & \uparrow \epsilon \\ S_{\mathbb{Q}}^{|\alpha_t|} & \xrightarrow{\alpha_t} & E_{\mathbb{Q}} \end{array}} \right) r$$

That is, each  $\alpha_t$  extends to map  $\tilde{\alpha}_t = r \circ \Omega \Sigma \alpha_t: K(\mathbb{Q}, |\alpha_t|) \rightarrow E_{\mathbb{Q}}$ . So far, we have a map  $a: \bigvee_t K(\mathbb{Q}, |\alpha_t|) \rightarrow E_{\mathbb{Q}}$  defined as  $\alpha_t$  on odd-degree summands and  $\tilde{\alpha}_t$  on even-degree summands. Now we may use the multiplication of  $E_{\mathbb{Q}}$  to extend this map to the product, yielding a map  $A: Y \rightarrow E_{\mathbb{Q}}$ . From the construction, we have that  $\text{im } A_{\#} = V = \text{im}(j_{\mathbb{Q}})_{\#}$ .

An identical construction yields a map  $B: Z \rightarrow E_{\mathbb{Q}}$  that satisfies  $\text{im } B_{\#} = W$ . Finally, one more use of the multiplication  $m$  of  $E_{\mathbb{Q}}$  gives a map  $m \circ (A \times B): Y \times Z \rightarrow E_{\mathbb{Q}}$  that is a homotopy equivalence.

For each  $\alpha_t$ , choose a  $\beta_t \in \pi_*(F_{\mathbb{Q}})$  such that  $(j_{\mathbb{Q}})_{\#}(\beta_t) = \alpha_t$ . Repeating the above argument with the  $\beta_t$  replacing the  $\alpha_t$  yields a map  $\phi: Y \rightarrow F_{\mathbb{Q}}$  with the desired properties, namely that  $j_{\mathbb{Q}} \circ \phi = A = m \circ (A \times B) \circ i$ .  $\square$

Now consider any map  $p: E \rightarrow X$  with  $E$  an  $H_0$ -space. We will construct a counterpart to the total Gottlieb element that depends on the map  $p$ . We first show that, under the hypothesis that  $X$  is finite—or more generally of finite rational category, the image of  $p_{\#}$  in rational homotopy groups is restricted exactly as in the Félix-Halperin result about Gottlieb groups mentioned in the introduction. Indeed, the following result generalizes that result. Here, we denote the rational category of  $X$  by  $\text{cat}_0(X)$  (see [1] or [2] for details of this invariant).

**Proposition 2.2.** *Let  $X$  be a nilpotent space and  $p: E \rightarrow X$  be any map with  $E$  an  $H_0$ -space. If  $X$  has finite rational category, that is, if  $\text{cat}_0(X) = r < \infty$ , then  $p_{\#}(\pi_{\text{even}}(E_{\mathbb{Q}})) = 0$  and  $p_{\#}(\pi_{\text{odd}}(E_{\mathbb{Q}}))$  is of (finite) dimension no more than  $r$ .*

*Proof.* For the first assertion, suppose that  $\beta \in \pi_{2i}(E_{\mathbb{Q}})$ . Because  $E_{\mathbb{Q}}$  is an  $H$ -space the map  $\beta$  extends, exactly as in the proof of the previous result, to a map  $\tilde{\beta}: \Omega S^{2i+1} \rightarrow E_{\mathbb{Q}}$  that is injective in (rational) homotopy in degree  $2i$ . If  $p_{\#}(\beta) \neq 0$ , then  $p \circ \tilde{\beta}: \Omega S^{2i+1} \rightarrow X$  is a map that is injective in rational homotopy—recall that  $\Omega S^{2i+1}$  rationalizes to an Eilenberg-Mac Lane space  $K(\mathbb{Q}, 2i)$ . But then the mapping theorem of [1] implies that  $\infty = \text{cat}_0(K(\mathbb{Q}, 2i)) \leq \text{cat}_0(X) = r$ , which is a contradiction. Therefore, we have  $p_{\#}(\pi_{\text{even}}(E_{\mathbb{Q}})) = 0$ . For the second assertion, consider any finite, linearly independent subset  $\{\alpha_1, \dots, \alpha_k\}$  of  $\pi_{\text{odd}}(X_{\mathbb{Q}})$  such that each  $\alpha_i \in \pi_{n_i}(X_{\mathbb{Q}})$  is in the image of  $p_{\#}$ . Choose a  $\beta_i \in \pi_{n_i}(E_{\mathbb{Q}})$  with  $p_{\#}(\beta_i) = \alpha_i$

for each  $i$ . Write the corresponding product of odd-dimensional rational spheres  $\prod_{i=1}^k S_{\mathbb{Q}}^{n_i}$  as  $S_p$ . Using the multiplication of  $E_{\mathbb{Q}}$ , we may extend the map  $\bigvee_i S_{\mathbb{Q}}^{n_i} \rightarrow E_{\mathbb{Q}}$  defined as  $\beta_i$  on each summand into a map  $\tilde{\Gamma}_p: S_p \rightarrow E_{\mathbb{Q}}$ . Specifically, let  $M: E_{\mathbb{Q}} \times \cdots \times E_{\mathbb{Q}} \rightarrow E_{\mathbb{Q}}$  denote the association  $m \circ (1 \times m) \circ \cdots \circ (1 \times \cdots \times 1 \times m)$ , where  $m$  denotes the multiplication on  $E_{\mathbb{Q}}$ . (Recall that we are not assuming  $E_{\mathbb{Q}}$  to be associative.) Then we set

$$\tilde{\Gamma}_p = M \circ (\beta_1 \times \cdots \times \beta_k): S_p \rightarrow E_{\mathbb{Q}}.$$

Now an odd-dimensional rational sphere  $S_{\mathbb{Q}}^{n_i}$  is a rational Eilenberg-Mac Lane space  $K(\mathbb{Q}, n_i)$ . From the construction, therefore, we have that  $p \circ \tilde{\Gamma}_p: S_p \rightarrow X_{\mathbb{Q}}$  is injective in rational homotopy groups. Once again, the mapping theorem implies that  $k = \text{cat}_0(S_p) \leq r$ . The second assertion follows.  $\square$

So now suppose that  $p: E \rightarrow X$  is any map from an  $H_0$ -space  $E$  to a nilpotent, finite space  $X$ . The image of  $p$  in rational homotopy groups is of finite dimension and we may pick a finite basis  $\{\alpha_1, \dots, \alpha_k\}$  in  $\pi_{\text{odd}}(X_{\mathbb{Q}})$  for this image. Exactly as in the above proof, we construct a map  $\tilde{\Gamma}_p: S_p \rightarrow E_{\mathbb{Q}}$  and then set  $\Gamma_p = p_{\mathbb{Q}} \circ \tilde{\Gamma}_p: S_p \rightarrow X_{\mathbb{Q}}$ . (In the case in which  $p$  has trivial image in rational homotopy groups, we may take  $\tilde{\Gamma}_p$  and  $\Gamma_p$  to be the trivial map.) In all cases, our construction gives a commutative diagram

$$\begin{array}{ccc} & & E_{\mathbb{Q}} \\ & \nearrow \tilde{\Gamma}_p & \downarrow p_{\mathbb{Q}} \\ S_p & \xrightarrow{\Gamma_p} & X_{\mathbb{Q}} \end{array}$$

in which  $\Gamma_p$  is both injective and onto the image of  $p$  in rational homotopy groups.

**Definition 2.3.** Suppose given any map  $p: E \rightarrow X$  from an  $H_0$ -space  $E$  to a nilpotent, finite space  $X$ . A *total Gottlieb element for  $X_{\mathbb{Q}}$  with respect to  $p$*  is a map  $\Gamma_p: S_p \rightarrow X_{\mathbb{Q}}$  that admits a lift  $\tilde{\Gamma}_p: S_p \rightarrow E_{\mathbb{Q}}$  through  $p_{\mathbb{Q}}$ , where

- (1)  $S_p$  is a product of rational Eilenberg-Mac Lane spaces with homotopy isomorphic to  $\text{im}(p_{\mathbb{Q}})_{\#}: \pi_*(E_{\mathbb{Q}}) \rightarrow \pi_*(X_{\mathbb{Q}})$ ; and
- (2)  $\Gamma_p$  is injective in (rational) homotopy groups.

In general, there may be many choices of total Gottlieb elements with respect to  $p$  and different lifts of each. By the above discussion, we see that such always exist. We keep the notation  $\Gamma_X: S_X \rightarrow X_{\mathbb{Q}}$  for a total Gottlieb element with respect to the ordinary evaluation fibration  $\omega: \text{Map}(X, X; 1) \rightarrow X$ .

**Theorem 2.4.** *Let*

$$F \xrightarrow{j} E \xrightarrow{p} X$$

*be a fibration sequence of nilpotent spaces in which  $F$  and  $E$  are  $H_0$ -spaces and  $X$  is a nilpotent, finite space. Let  $\Gamma_p: S_p \rightarrow X_{\mathbb{Q}}$  be any total Gottlieb element for  $X_{\mathbb{Q}}$  with respect to  $p$  and  $\tilde{\Gamma}_p$  any lift of  $\Gamma_p$  through  $p_{\mathbb{Q}}$ . Assume there is an action  $A: F_{\mathbb{Q}} \times E_{\mathbb{Q}} \rightarrow E_{\mathbb{Q}}$  of  $F_{\mathbb{Q}}$  on  $E_{\mathbb{Q}}$  that satisfies  $A \circ i_1 = j_{\mathbb{Q}}$  and  $p_{\mathbb{Q}} \circ A = p_{\mathbb{Q}} \circ p_2: F_{\mathbb{Q}} \times E_{\mathbb{Q}} \rightarrow X_{\mathbb{Q}}$ . Then there is a retraction  $r: E_{\mathbb{Q}} \rightarrow S_p$  of  $\tilde{\Gamma}_p$  such that  $p_{\mathbb{Q}} = \Gamma_p \circ r: E_{\mathbb{Q}} \rightarrow X_{\mathbb{Q}}$ .*

*Proof.* From Proposition 2.1, we assume an identification  $E_{\mathbb{Q}} \simeq Y \times Z$ , with  $Y$  and  $Z$  rational  $H$ -spaces, together with maps  $i: Y \rightarrow E_{\mathbb{Q}}$  and  $\phi: Y \rightarrow F_{\mathbb{Q}}$  with  $i_{\#}$  an injection onto  $\text{im}(j_{\mathbb{Q}})_{\#}$  and  $j_{\mathbb{Q}} \circ \phi = i$ . Now consider the following commutative diagram:

$$\begin{array}{ccc} Y \times S_p & \xrightarrow{p_2} & S_p \\ \downarrow A \circ (\phi \times \tilde{\Gamma}_p) & \simeq H & \downarrow \Gamma_p \\ E_{\mathbb{Q}} & \xrightarrow{p_{\mathbb{Q}}} & X_{\mathbb{Q}} \end{array}$$

Observe that  $A \circ (\phi \times \tilde{\Gamma}_p) \circ i_1 = i: Y \rightarrow E_{\mathbb{Q}}$ . Furthermore, from the long exact sequence in homotopy of the fibration, we find that  $A \circ (\phi \times \tilde{\Gamma}_p) \circ i_2: S_p \rightarrow E_{\mathbb{Q}}$  has image in homotopy that is complementary to  $\text{im}(j_{\mathbb{Q}})_{\#}$ . Hence  $A \circ (\phi \times \tilde{\Gamma}_p)$  induces an isomorphism in rational homotopy and thus is a homotopy equivalence. Consequently, there is an inverse (rational) homotopy equivalence  $H: E_{\mathbb{Q}} \rightarrow Y \times S_p$  as indicated in the diagram. Now set  $r = p_2 \circ H: E_{\mathbb{Q}} \rightarrow S_p$ . Then we have  $r \circ \tilde{\Gamma}_p = p_2 \circ H \circ A \circ (\phi \times \tilde{\Gamma}_p) \circ i_2 = 1: S_p \rightarrow S_p$ , so that  $r$  is a retraction of  $\tilde{\Gamma}_p$ . Furthermore, since  $p \circ A(\phi \times \tilde{\Gamma}_p) = \Gamma_p \circ p_2$ , we have  $\Gamma_p \circ r = \Gamma_p \circ p_2 \circ H = p_{\mathbb{Q}}: E_{\mathbb{Q}} \rightarrow X_{\mathbb{Q}}$ , which gives the desired factorization.  $\square$

We obtain Theorem 1.3 by specializing as follows:

*Proof of Theorem 1.3.* The action

$$A: \text{Map}_*(X, X; 1) \times \text{Map}(X, X; 1) \rightarrow \text{Map}(X, X; 1),$$

defined by  $A(f, g) = g \circ f$ , restricts to the inclusion  $\text{Map}_*(X, X; 1) \rightarrow \text{Map}(X, X; 1)$  and satisfies the hypothesis of Theorem 2.4. Therefore, we may apply the result to the evaluation fibration sequence

$$\text{Map}_*(X, X; 1) \rightarrow \text{Map}(X, X; 1) \xrightarrow{\omega} X$$

and the total Gottlieb element for this evaluation map constructed from the Gottlieb groups as in the introduction.  $\square$

By the same argument, we obtain Theorem 1.3 for each of the evaluation fibrations in which  $\text{Top}$ ,  $\text{Diff}$ , and so-forth, replaces  $\text{Map}$ , as in the introduction.

The following observation allows us to strengthen Theorem 1.3 in certain circumstances. We will also use it in Section 4. Roughly speaking, we may say that if  $X$  decomposes up to homotopy equivalence as a product, then *the evaluation map decomposes as a corresponding product of evaluation maps*.

More precisely, suppose we have a homotopy equivalence  $h: X \rightarrow A \times B$ . Then we have homotopy equivalences

$$h_*: \text{Map}(X, X; 1) \rightarrow \text{Map}(X, A \times B; h)$$

and

$$h^*: \text{Map}(A \times B, A \times B; 1) \rightarrow \text{Map}(X, A \times B; h).$$

Let  $p_1: A \times B \rightarrow A$  and  $p_2: A \times B \rightarrow B$  denote the projections, and  $i_1: A \rightarrow A \times B$  and  $i_2: B \rightarrow A \times B$  the inclusions. We write  $h = (h_1, h_2)$ , with  $h_1 = p_1 \circ h$  and  $h_2 = p_2 \circ h$ , and define evaluation maps  $\omega_1: \text{Map}(X, A; h_1) \rightarrow A$  and  $\omega_2: \text{Map}(X, B; h_2) \rightarrow B$  by evaluation at the basepoint of  $X$ . Let  $I: \text{Map}(A \times$

$B, A \times B; 1) \rightarrow \text{Map}(A \times B, A; p_1) \times \text{Map}(A \times B, B; p_2)$  be the standard homeomorphism. One checks easily that

$$h \circ \omega_X = (\omega_1 \times \omega_2) \circ I \circ (h^*)^{-1} \circ h_* : \text{Map}(X, X; 1) \rightarrow A \times B,$$

and thus we may identify  $\omega_X$  with  $h^{-1} \circ (\omega_1 \times \omega_2) \circ I \circ h_*$ . Furthermore, we observe that  $\omega_1$  factors through  $\omega_A$  as  $\omega_1 = \omega_A \circ (i_1)^*$  and likewise  $\omega_2 = \omega_B \circ (i_2)^*$ . Therefore, we have the following commutative diagram:

$$(1) \quad \begin{array}{ccc} \text{Map}(X, X; 1) & \xrightarrow[\simeq]{I \circ (h^*)^{-1} \circ h_*} & \text{Map}(A \times B, A; p_1) \times \text{Map}(A \times B, B; p_2) \\ \omega_X \downarrow & & \downarrow (i_1)^* \times (i_2)^* \\ & & \text{Map}(A, A; 1_A) \times \text{Map}(B, B; 1_B) \\ & & \downarrow \omega_A \times \omega_B \\ X & \xrightarrow[\underset{h}{\simeq}]{} & A \times B. \end{array}$$

This discussion leads to the following result, which should be compared with the well-known fact that  $G_*(A \times B) \cong G_*(A) \oplus G_*(B)$  [7].

**Theorem 2.5.** *Suppose that we have a homotopy equivalence  $X \simeq A \times B$ . Then the evaluation map  $\omega_X$  factors through the product of evaluation maps  $\omega_A \times \omega_B$ .*

We now continue with the main results. In order to study generalized evaluation maps  $w: E \rightarrow X$ , we first present a global structure result concerning maps between  $H_0$ -spaces.

**Proposition 2.6.** *Let  $f: X \rightarrow Y$  be a map between  $H_0$ -spaces.*

- (a) *The map  $f$  admits a Sullivan minimal model of the form  $\varphi: (\wedge(V \oplus R), 0) \rightarrow (\wedge(V \oplus S), 0)$  with  $\varphi(v) = v$  for  $v \in V$  and such that  $\varphi(R) \in \wedge^{\geq 2}(V \oplus S) \cap \wedge V \otimes \wedge^+(S)$ .*
- (b) *If  $f_{\mathbb{Q}}$  is an  $H$ -map then  $f$  admits a model of the form  $\varphi: (\wedge(V \oplus K), 0) \rightarrow (\wedge(V \oplus S), 0)$  with  $\varphi(v) = v$  for  $v \in V$  and  $\varphi(K) = 0$ .*

*Proof.* Let  $\varphi: (\wedge T, 0) \rightarrow (\wedge W, 0)$  be any model of  $f$ . We will use standard tricks from rational homotopy to change generators in  $\wedge T$  and  $\wedge W$  so that, with respect to the new generators, the minimal model of  $f$  has the desired form.

(a) We denote by  $V$  a maximal subspace of  $T$  such that  $Q(\varphi): V \rightarrow W$  is injective. Denote by  $R \subseteq T$  a complement of  $V$  and by  $S \subseteq W$  a complement of  $\text{im } Q(\varphi)$  in  $W$ . Let  $\{v_i\}_{i \in I}$  be a graded basis for  $V$ . Then the elements  $\varphi(v_i)$  are linearly independent indecomposable elements in  $\wedge W$ . Denote by  $\{r_j\}_{j \in J}$  a graded basis for  $R$  and  $\{s_k\}_{k \in K}$  a graded basis for  $S$ . With respect to the generators  $\{v_i, r_j\}$  for  $\wedge T$  and  $\{v'_i = \varphi(v_i), s_k\}$  for  $\wedge W$ , the map  $\varphi$  satisfies  $\varphi(v_i) = v'_i$  and  $\varphi(R) \subset \wedge^{\geq 2}(W)$ . We can thus suppose  $\varphi(v) = v$  and that  $\varphi(R)$  is decomposable. We now change generators in  $R$  so that  $\varphi(R)$  also belongs to the ideal generated by  $S$ . Suppose that this is true for  $R^{<n}$ , and let  $r$  be a generator in  $R^n$ . If  $\varphi(r) = a + b$  with  $a \in \wedge V$  and  $b$  in the ideal generated by  $S$ , we change the generator to  $r' = r - a$ . The result follows by induction.

(b) Here, we apply the previous step to write  $\varphi: \wedge(V \oplus K) \rightarrow \wedge(V \oplus S)$  with  $\varphi(v) = v$  for  $v \in V$  and  $\varphi(k)$  both decomposable and in  $\wedge V \otimes \wedge^+(S)$  for  $k \in K$ . We now prove by induction that  $\varphi$  is zero on  $K$ .

The existence of multiplications on  $X_{\mathbb{Q}}$  and  $Y_{\mathbb{Q}}$  is reflected in their Sullivan models by morphisms of algebras  $\Delta_1: \wedge T \rightarrow \wedge T \otimes \wedge T$  and  $\Delta_2: \wedge W \rightarrow \wedge W \otimes \wedge W$  that satisfy  $\Delta_1(v) - (v \otimes 1 + 1 \otimes v) \in \wedge^+ T \otimes \wedge^+ T$  and likewise for  $\Delta_2$ . Furthermore, since  $f_{\mathbb{Q}}$  is an  $H$ -map, we have the following commutative diagram after the previous step:

$$\begin{array}{ccc} \wedge(V \oplus K) & \xrightarrow{\Delta_1} & \wedge(V \oplus K) \otimes \wedge(V \oplus K) \\ \varphi \downarrow & & \downarrow \varphi \otimes \varphi \\ \wedge(V \oplus S) & \xrightarrow{\Delta_2} & \wedge(V \oplus S) \otimes \wedge(V \oplus S) \end{array}$$

Assume inductively that we have  $\varphi(K^{\leq n}) = 0$  and suppose that  $u \in K^{n+1}$ . We write

$$\varphi(u) = \varphi_r(u) + \varphi_{r+1}(u) + \cdots + \varphi_m(u)$$

with  $\varphi_r(u) \in \wedge^r(V \oplus S)$ . By the definition of  $K$ , we have  $r \geq 2$ . Consider a term in  $\varphi_r(u)$  that is of minimal length  $q$  in  $\wedge S$ , for some  $1 \leq q \leq r$ . Let  $\{s_i\}$  be a basis of  $S$  and write such a minimal term as  $s_{i_1} s_{i_2} \cdots s_{i_q} \nu$  for some  $\nu \in \wedge^{r-q} V$ . Then  $\Delta_2 \varphi(u)$  contains a contribution  $s_{i_1} \otimes s_{i_2} \cdots s_{i_q} \nu$  and this term will appear uniquely as such in  $\Delta_2 \varphi(u) - (1 \otimes \varphi(u) + \varphi(u) \otimes 1)$ . On the other hand,  $\Delta_1(u) - (1 \otimes u + u \otimes 1) \in \wedge^+(V \oplus K^{\leq n}) \otimes \wedge^+(V \oplus K^{\leq n})$  and so  $(\varphi \otimes \varphi) \Delta_1(u) - (1 \otimes \varphi(u) + \varphi(u) \otimes 1)$  cannot contain any occurrence of a term such as  $s_{i_1} \otimes s_{i_2} \cdots s_{i_q} \nu$ , by our induction hypothesis. In summary, if  $\varphi_r(u)$  contains some non-zero term, then we cannot have  $(\varphi \otimes \varphi) \Delta_1(u) = \Delta_2 \varphi(u)$ , which is a contradiction. It follows by induction that  $\varphi(K) = 0$ .  $\square$

We remark in passing that Proposition 2.6 implies the following result:

**Corollary 2.7.** *Let  $f: X \rightarrow Y$  be a map between  $H_0$ -spaces that is an  $H$ -map after rationalization. If  $(f_{\mathbb{Q}})_{\#}$  is zero, then  $f$  is rationally null-homotopic.*

We also observe that the conclusion of Proposition 2.6 (b) holds for certain compositions. We will use this observation in the following form in the sequel,:

**Corollary 2.8.** *Let  $g: X \rightarrow Y$  and  $r: Y \rightarrow Z$  be maps between  $H_0$ -spaces. If  $g_{\mathbb{Q}}$  is an  $H$ -map and  $(r_{\mathbb{Q}})_{\#}$  is surjective, then their composition  $r \circ g$  admits a Sullivan minimal model of the form  $\varphi: (\wedge(V \oplus K), 0) \rightarrow (\wedge(V \oplus W), 0)$  with  $\varphi(v) = v$  for  $v \in V$  and  $\varphi(K) = 0$ .*

*Proof.* Denote by  $\varphi: (\wedge W_1, 0) \rightarrow (\wedge W_2, 0)$  a minimal model of  $r \circ g$ , and by  $V \subset W_1$  a maximal subspace such that  $Q(\varphi): V \rightarrow W_2$  is injective. Then by part (a) of Proposition 2.6, we have models for  $r$  and  $g$

$$(\wedge(V \oplus R), 0) \xrightarrow{\theta_1} (\wedge(V \oplus R \oplus S), 0) \xrightarrow{\theta_2} (\wedge(V \oplus W), 0),$$

with  $\theta_1(v) = \theta_2(v) = v$ . Using part (b) of Proposition 2.6, we can suppose that  $\theta_2(R \oplus S) = 0$ . By a change of generators in  $(\wedge(V \oplus R), 0)$  we can suppose that  $\theta_1(R)$  is contained in the ideal generated by  $R \oplus S$ , so that  $\theta_2 \circ \theta_1(R) = 0$ .  $\square$

We now proceed to the proof of our second main result, namely Theorem 1.6,

*Proof of Theorem 1.6.* Suppose  $w: E \rightarrow X$  is an evaluation map. Then there is an action  $A: E \times X \rightarrow X$  that restricts to  $w$ . The adjoint  $g: E \rightarrow \text{Map}(X, X; 1)$  of this action, defined by  $g(y)(x) = A(y, x)$ , is a lift of  $w$  through  $\omega: \text{Map}(X, X; 1) \rightarrow$

$X$ . Since we assume the action is associative, the adjoint  $g$  is an  $H$ -map. Upon rationalizing, we obtain the commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{g} & \text{Map}(X, X_{\mathbb{Q}}; e) & \xrightarrow{r} & S_X \\ & \searrow w_{\mathbb{Q}} & \downarrow \omega_{\mathbb{Q}} & \swarrow \Gamma_X & \\ & & X_{\mathbb{Q}} & & \end{array}$$

in which  $r: \text{Map}(X, X_{\mathbb{Q}}; e) \rightarrow S_X$  is a retraction of  $\tilde{\Gamma}_X: S_X \rightarrow \text{Map}(X, X_{\mathbb{Q}}; e)$  as in Theorem 1.3. Since  $r$  is a retraction,  $r_{\#}$  is surjective. So we may apply Corollary 2.8 and assume a model of  $r \circ g: E \rightarrow S_X$  has the form  $\varphi: (\wedge(V \oplus K), 0) \rightarrow (\wedge(V \oplus W), 0)$  with  $\varphi(v) = v$  for  $v \in V$  and  $\varphi(K) = 0$ . Thus  $\varphi$  factors in the form

$$\wedge(V \oplus K) \xrightarrow{\text{proj}} \wedge V \xrightarrow{\text{incl}} \wedge(V \oplus W)$$

$\xleftarrow{\text{proj}}$

together with the evident retraction of the inclusion  $\wedge V \rightarrow \wedge(V \oplus W)$  as indicated. When translated into spaces, this implies that  $r \circ g$  factors rationally through a rational  $H$ -space  $Y$

$$E \xrightarrow{q} Y \xrightarrow{j} S_X .$$

$\xleftarrow{i}$

Notice that  $j: Y \rightarrow S_X$  has minimal model the projection  $\wedge(V \oplus K) \rightarrow \wedge V$  and hence is injective in (rational) homotopy. Furthermore, we have the right inverse  $i$  for  $q$  as indicated. That is, we have maps that satisfy  $j \circ q = r \circ g$  and  $q \circ i = 1$ . Now consider the diagram

$$\begin{array}{ccc} & E & \\ q \nearrow & \downarrow w_{\mathbb{Q}} & \\ Y & \xrightarrow{\Gamma_X \circ j} & X_{\mathbb{Q}} \end{array}$$

$\searrow i$

in which we have  $\Gamma_X \circ j \circ q = \Gamma_X \circ r \circ g = \omega_{\mathbb{Q}} \circ g = w_{\mathbb{Q}}$  and hence  $w_{\mathbb{Q}} \circ i = \Gamma_X \circ j \circ q \circ i = \Gamma_X \circ j$ . We see that  $\Gamma_X \circ j: Y \rightarrow X_{\mathbb{Q}}$  satisfies the requirements of a total Gottlieb element for  $X_{\mathbb{Q}}$  with respect to  $w$ . Since we have a retraction  $q$  of  $i$ , which here serves as our lift of  $\Gamma_X \circ j$  through  $w_{\mathbb{Q}}$ , this total Gottlieb element satisfies the conclusion of the theorem.

The conclusion now follows for every total Gottlieb element. For suppose given another total Gottlieb element  $\Gamma'_p: S'_p \rightarrow X_{\mathbb{Q}}$  for  $X_{\mathbb{Q}}$  with lift  $\tilde{\Gamma}'_p: S'_p \rightarrow E$ . Then the map  $h = q \circ \tilde{\Gamma}'_p: S'_p \rightarrow Y$  is a homotopy equivalence. This follows since  $S'_p$  and  $Y$  have isomorphic (rational) homotopy groups, and  $h$  is injective in (rational) homotopy groups. Therefore, we may define  $r' = h^{-1} \circ q: E \rightarrow S'_p$ , which is easily checked to be a retraction of  $\tilde{\Gamma}'_p$  that satisfies  $\Gamma'_p \circ r' = w_{\mathbb{Q}}$ .  $\square$

We may supplement the vocabulary of Definition 2.3 with the following: Suppose given any map  $p: E \rightarrow X$  from an  $H_0$ -space  $E$  to a nilpotent, finite space  $X$ . Then we define the  $n$ th Gottlieb group of  $X$  with respect to  $p$  as the subgroup of  $\pi_n(X)$  that is the image of  $p_{\#}: \pi_n(E) \rightarrow \pi_n(X)$ . We denote this subgroup by  $G_n^p(X)$ . Then we have Corollary 1.7, phrased using this notation, as an immediate consequence of Theorem 1.6.

**Corollary 2.9** (Corollary 1.7). *Let  $X$  be any nilpotent, finite complex and  $w: E \rightarrow X$  an evaluation map. Then  $G_*^w(X_{\mathbb{Q}}) = 0$  if and only if  $w_{\mathbb{Q}}$  is null-homotopic.*

Before we present some examples, we notice the following generalization of Corollary 1.7 that does not require the fibration to be “principal” in the sense required by Theorem 2.4:

**Theorem 2.10.** *Suppose given any fibration sequence of nilpotent spaces*

$$F \xrightarrow{j} E \xrightarrow{p} B$$

*in which  $F$  and  $E$  are  $H_0$ -spaces. If  $(p_{\mathbb{Q}})_{\#} = 0: \pi_*(E_{\mathbb{Q}}) \rightarrow \pi_*(B_{\mathbb{Q}})$ , then  $p_{\mathbb{Q}} = *$ .*

*Proof.* From the long exact sequence in rational homotopy groups induced by the fibration sequence, we have that  $(j_{\mathbb{Q}})_{\#}: \pi_*(F_{\mathbb{Q}}) \rightarrow \pi_*(E_{\mathbb{Q}})$  is surjective. This gives a section  $\sigma: E_{\mathbb{Q}} \rightarrow F_{\mathbb{Q}}$  of the rationalized fibre inclusion  $j_{\mathbb{Q}}: F_{\mathbb{Q}} \rightarrow E_{\mathbb{Q}}$ . Thus we have  $p_{\mathbb{Q}} = p_{\mathbb{Q}} \circ j_{\mathbb{Q}} \circ \sigma = *$ , since  $p_{\mathbb{Q}} \circ j_{\mathbb{Q}} = *$ .  $\square$

**Example 2.11.** We give first an example of a fibration that satisfies the hypotheses of Theorem 2.4, and yet is not a cyclic map and therefore, in particular, does not satisfy the hypotheses of Theorem 1.6. For this, let  $B$  denote a space whose minimal model is  $\Lambda(a, b, c)$ , with  $|a| = |b| = 3$  and  $|c| = 5$ , and differential given by  $d(a) = d(b) = 0$  and  $d(c) = ab$ . Then consider the map  $p: S^3 \rightarrow B$  that corresponds to one of the homotopy elements of  $\pi_3(B)$ . We find that, up to rational equivalence, the homotopy fibre of  $p$  is the  $H$ -space  $F = \Omega(S^3 \times S^5)$ . Furthermore, again up to rational equivalence, the fibre inclusion  $j: F \rightarrow S^3$  is null-homotopic. The fibre sequence  $F \rightarrow S^3 \rightarrow B$ , therefore, admits an action of  $F$  on  $S^3$  that is principal in the sense required by the hypotheses of Theorem 2.4. Namely, the projection  $p_2: F \times S^3 \rightarrow S^3$  is such an action. Observe, however, that the fibre map  $p: S^3 \rightarrow B$  cannot be a cyclic map, since  $G_3(B) = 0$ . In particular, this example does not satisfy the hypotheses of Theorem 1.6.

**Example 2.12.** Observe, however, that there are maps from an  $H_0$ -space that induce zero on (rational) homotopy groups, and yet are not (rationally) null-homotopic. For instance, the quotient map  $q: S^3 \times S^3 \rightarrow S^6$  is a map from an  $H$ -space that induces zero on rational homotopy groups, yet is non-zero on rational cohomology groups and so is not rationally trivial. Of course, here the homotopy fibre of  $q$  is not an  $H_0$ -space. It is interesting to note that Corollary 2.9 implies  $q_{\mathbb{Q}}$  cannot occur as the connecting map of any fibration (cf. Corollary 1.5).

Allowing a non-trivial image in homotopy for  $p$  appears to make a fundamental change in the situation. In particular, if we simply assume  $F$  and  $E$  are  $H_0$ -spaces, as in Theorem 2.10, but allow the image of  $p$  in rational homotopy groups to have dimension 1, then it may be impossible to factor  $p$  through an odd-dimensional sphere, or any finite product of odd-dimensional spheres. We give examples to illustrate this point:

**Example 2.13.** Let  $q: S^3 \times S^3 \times S^3 \rightarrow S^9$  be the map obtained by pinching out all but the top cell of the product. As may be checked by a direct computation, the fibre sequence

$$F \xrightarrow{j} S^3 \times S^3 \times S^3 \xrightarrow{p} S^3 \times S^9$$

with  $p = (p_1, q)$  has fibre that is rationally equivalent to the  $H$ -space  $S^3 \times S^3 \times K(\mathbb{Q}, 8)$ . Hence, the fibre inclusion  $j$  is a map of  $H_0$ -spaces. Now  $p$  has image of dimension 1 on rational homotopy groups. Evidently, however,  $p$  does not factor through  $S^3$  (or any single odd-dimensional sphere).

**Example 2.14.** We describe a rational fibre sequence of the form

$$F \xrightarrow{j} E \xrightarrow{p} S^3 \vee S^9,$$

in which  $E$  and  $F$  are  $H_0$ -spaces and where  $p$  does not factor through any finite product of odd-dimensional spheres. First we specify a map of minimal models  $\mathcal{M}_p: \mathcal{M}_{S^3 \vee S^9} \rightarrow \mathcal{M}_E$  by writing  $\mathcal{M}_{S^3 \vee S^9} = \Lambda(b, \{u_i\}_{i \geq 1}; d)$ , with  $|b| = 3$  and  $|u_i| \geq 9$ , and then setting  $\mathcal{M}_E$  to be the minimal model  $\Lambda(b, y, \{v_i\}_{i \geq 1}; d_E = 0)$ , with  $|b| = |y| = 3$  and  $|v_i| = |u_i| - 6$ . The map  $\mathcal{M}_p$  is given by  $\mathcal{M}_p(b) = b$ , and  $\mathcal{M}_p(u_i) = byv_i$  for  $i \geq 1$ . Since  $d$  is decomposable, we have  $\mathcal{M}_p \circ d = 0$  and thus  $\mathcal{M}_p$  is a map of DG algebras. Hence it defines a map of rational spaces  $p: E \rightarrow (S^3 \vee S^9)_{\mathbb{Q}}$ . By standard rational homotopy techniques, one checks that the homotopy fibre of  $p$  is an  $H_0$ -space. However, one may see from the minimal models that the map  $p$  does not factor through any finite product of odd-dimensional spheres.

### 3. GOTTLIEB GROUPS AND HOMOTOPY MONOMORPHISMS

Let  $w: E \rightarrow X$  be an evaluation map. By Theorem 1.6,  $w$  factors as  $w = \Gamma_w \circ r$  where  $r: E_{\mathbb{Q}} \rightarrow S_w$  is a left inverse of  $\widetilde{\Gamma}_w$ . As a retraction,  $r$  has  $\widetilde{\Gamma}_w$  as a right inverse and so *is a homotopy epimorphism*. That is, the map of homotopy sets

$$r_*: [A, E_{\mathbb{Q}}] \rightarrow [A, S_w]$$

is surjective for any space  $A$ . On the other hand, a total Gottlieb element  $\Gamma_w: S_w \rightarrow X_{\mathbb{Q}}$  generally does not admit a left inverse. For instance, take  $X = S^2$  so that  $G_*(X_{\mathbb{Q}}) = G_3(S_{\mathbb{Q}}^2) \cong \mathbb{Q}$ . Then  $S_X = S_{\mathbb{Q}}^3$  and we may take  $\Gamma_X: S_X \rightarrow X_{\mathbb{Q}}$  to be the rationalized Hopf map, which does not admit a left inverse. Nonetheless, we will show that

$$(\Gamma_w)_*: [A, S_w] \rightarrow [A, X_{\mathbb{Q}}]$$

is injective for any nilpotent space  $A$ . In order to show this, and in addition to obtain our results about cohomology, we need to establish some technical points concerning Gottlieb groups and rational homotopy monomorphisms.

The following discussion will fix our notation for the remainder of the paper. Suppose  $X$  has minimal model  $(\wedge W, d_X)$ . The Gottlieb group  $G_*(X_{\mathbb{Q}})$  may be identified with the subspace of  $\text{Hom}(W, \mathbb{Q})$  formed by those linear maps that extend to derivations of  $\wedge W$  that commute with  $d_X$  (see [2] for a discussion of this). Denote by  $\bar{\theta}_i$  a linear basis of  $G_*(X_{\mathbb{Q}})$ , and by  $v_i$  elements of  $W$  with  $\bar{\theta}_i(v_j) = \delta_{ij}$ . We denote by  $\theta_i$  an extension of  $\bar{\theta}_i$  to a derivation of  $\wedge W$  that satisfies  $d_X \theta_i = (-1)^{|v_i|} \theta_i d_X$ . We suppose, without loss of generality, that  $|v_i| \leq |v_j|$  for  $i < j$ . Then we may—and do—suppose that  $\theta_i(v_j) = 0$  for  $i > j$ . Other than this, however, we have very little control over how the  $\bar{\theta}_i$  extend. This point is the main source of the technicalities. We denote by  $V$  the vector space generated by the  $v_i$ , and  $Z$  a choice of complement in  $W$ . Thus the minimal model of  $X$  is  $(\wedge(V \oplus Z), d_X)$  with  $V = \langle v_1, \dots, v_r \rangle$  corresponding to the Gottlieb group, accompanying derivations  $\theta_1, \dots, \theta_r$ , and  $Z$  a complement to  $V$  in  $W$ .

**Lemma 3.1.** *With notation as above, we may choose  $Z$ ,  $V$ , and the  $\theta_i$  such that  $\theta_i(Z \oplus \langle v_{i+1}, \dots, v_r \rangle) \subset \wedge V \otimes \wedge^+ Z$ . In particular the ideal  $I(Z)$  is  $\theta_i$ -stable for each  $i$ .*

*Proof.* Let  $\mathcal{L}$  denote the Lie algebra of derivations of  $\mathcal{M}_X$  generated by the derivations  $\theta_1, \dots, \theta_r$ . We prove by induction on  $k$  that we may choose  $Z$  and  $V$  for which we have  $\theta(W) \subseteq \mathbb{Q} \oplus (\wedge^{\geq k} V + \wedge V \otimes \wedge^+ Z)$  for any  $\theta \in \mathcal{L}$ , for all  $k$ . Since  $\wedge V$  is finite dimensional, taking  $k > r$  establishes the result.

For  $k = 1$ , we choose  $Z = \cap_{i=1}^r \ker(\bar{\theta}_i: W \rightarrow \mathbb{Q})$ . We have directly that  $\theta_i(Z) \subseteq \wedge^+(V \oplus Z)$ , and hence  $\theta(Z) \subseteq \wedge^+(V \oplus Z)$  for any  $\theta \in \mathcal{L}$ .

Now suppose that, for some  $k \geq 1$ , we have  $\theta(W) \subseteq \mathbb{Q} \oplus (\wedge^k V + \wedge V \otimes \wedge^+ Z)$  for any  $\theta \in \mathcal{L}$ . For each generating derivation  $\theta_j$ , and for  $z \in W$  a basis element, with  $\theta(z) \notin \mathbb{Q}$ , we write

$$\theta_j(z) \equiv \sum_{i_1 < i_2 < \dots < i_k} \lambda_j^{(i_1, i_2, \dots, i_k)} v_{i_1} v_{i_2} \dots v_{i_k}$$

modulo terms in  $\wedge^{\geq k+1} V + \wedge V \otimes \wedge^+ Z$ . Then we make a change of basis for  $W$ —in effect, a different choice of complement—by replacing each basis element  $z$  with  $z'$ , where

$$z' = z - \sum_{j=k+1}^r \sum_{i_1 < i_2 < \dots < i_k < j} \lambda_j^{(i_1, i_2, \dots, i_k)} v_j v_{i_1} v_{i_2} \dots v_{i_k}.$$

The effect of this basis change in  $W$  is that we may now suppose

$$(2) \quad \theta_j(z) \equiv \sum_{i_1 < i_2 < \dots < i_k | i_k \geq j} \lambda_j^{(i_1, i_2, \dots, i_k)} v_{i_1} v_{i_2} \dots v_{i_k}$$

modulo terms in  $\wedge^{\geq k+1} V + \wedge V \otimes \wedge^+ Z$ , for each generating derivation  $\theta_j$  and each element  $z \in W$  such that  $\theta(z) \notin \mathbb{Q}$ . We now claim that all the coefficients  $\lambda_j^{(i_1, i_2, \dots, i_k)}$  that appear in (2) are in fact zero. For suppose that this is not the case, and let  $j$  be the least index for which some  $\lambda_j^{(i_1, i_2, \dots, i_k)}$  in (2) is non-zero. Denote by  $n \geq j$  the maximum of the  $i_k$  with  $\lambda_j^{(i_1, i_2, \dots, i_k)} \neq 0$ . Then  $\theta_n \circ \theta_j(z) = \alpha + \beta$ , with  $\alpha \neq 0 \in \wedge^{r-1}(v_{i_1}, v_{i_2}, \dots, v_{n-1})$  and  $\beta \in \wedge^{\geq r} V + \wedge V \otimes \wedge^+ Z$ . If  $n = j$ , then  $\theta_n \circ \theta_j = \frac{1}{2}[\theta_n, \theta_j] \in \mathcal{L}$ , and this contradicts the induction hypothesis on  $\mathcal{L}$ . However, if  $n > j$ , then  $\theta_j \circ \theta_n(z) = \gamma + \delta$ , with  $\gamma$  of length  $k-1$  but in  $\wedge(v_{i_1}, v_{i_2}, \dots, \widehat{v}_j, \dots, v_{n-1}) \otimes \wedge^+(v_n, v_{n+1}, \dots, v_r)$  and  $\delta \in \wedge^{\geq r} V + \wedge V \otimes \wedge^+ Z$ . This shows again that  $[\theta_n, \theta_j] \in \mathcal{L}$  contradicts the induction hypothesis on  $\mathcal{L}$ . It follows that all the coefficients  $\lambda_j^{(i_1, i_2, \dots, i_k)}$  that appear in (2) are zero. Therefore, we have  $\theta_j(W) \subseteq \mathbb{Q} \oplus (\wedge^{k+1} V + \wedge V \otimes \wedge^+ Z)$  for any  $z \in W$ , for each generating derivation  $\theta_j$ .

To complete the inductive step, we must also consider a general  $\theta \in \mathcal{L}$ . Suppose that, for some  $z \in W$ , we have

$$\theta(z) \equiv \sum_{i_1 < i_2 < \dots < i_k} \mu^{(i_1, i_2, \dots, i_r)} v_{i_1} v_{i_2} \dots v_{i_k}$$

modulo terms in  $\wedge^{\geq k+1} V + \wedge V \otimes \wedge^+ Z$ . We claim that all the coefficients  $\mu^{(i_1, i_2, \dots, i_k)}$  that appear in this expression are zero. For suppose not, and once again, denote by  $n$  the maximum of the  $i_k$  for which some  $\mu^{(i_1, i_2, \dots, i_k)} \neq 0$ . The composition  $\theta_n \circ \theta(z)$  then contains a non-zero term in  $\wedge^{r-1} V$ . On the other hand, since  $\theta$  is a derivation, and we have just shown that  $\theta_n(z) \in \wedge^{\geq k+1} V + \wedge V \otimes \wedge^+ Z$ , we have

$\theta \circ \theta_n(z) \in \Lambda^{\geq k+1}V + \Lambda V \otimes \Lambda^+Z$ . Therefore,  $[\theta_n, \theta] \in \mathcal{L}$  contradicts the induction hypothesis on  $\mathcal{L}$ . This shows that all the  $\mu^{(i_1, i_2, \dots, i_k)} = 0$ , and hence the induction is complete.  $\square$

**Proposition 3.2.** *Suppose  $V$ ,  $Z$ , and the  $\theta_i$  satisfy  $\theta_i(W) \subseteq \mathbb{Q} \oplus (\Lambda V \otimes \Lambda^+Z)$  for each  $i$ . Then,*

- (1)  $d_X(W) \subseteq \Lambda V \otimes \Lambda^{\geq 2}Z$ . In particular, the ideal  $I(Z)$  itself is  $d_X$ -stable.
- (2) There exists a choice of total Gottlieb element  $\Gamma_X: S_X \rightarrow X_{\mathbb{Q}}$  with minimal model  $\mathcal{M}_{\Gamma}: (\Lambda(V \oplus Z), d_X) \rightarrow (\Lambda V, d = 0)$  that satisfies  $\mathcal{M}_{\Gamma}(Z) = 0$  and  $\mathcal{M}_{\Gamma}(v) = v$  for  $v \in V$ .

*Proof.* (1) We first argue by contradiction to prove that  $d_X(W) \subset \Lambda V \otimes \Lambda^+Z$ . Suppose this is not true, and that  $m \geq 1$  is the minimal length for which any  $d(\chi)$  contains a non-zero term in  $\Lambda^m V$ . For such a  $\chi \in \Lambda V \otimes \Lambda Z$ , write  $d(\chi) = \alpha + \beta$  with  $\alpha \neq 0 \in \Lambda^m V$  and  $\beta \in \Lambda^{\geq m+1}V + \Lambda V \otimes \Lambda^+Z$ . Further, suppose that  $\alpha \in \Lambda^m(v_1, \dots, v_s)$  for some  $s \leq r$  such that

$$d(\chi) = \alpha' + \alpha''v_s + \beta,$$

with  $\alpha' \in \Lambda^m(v_1, \dots, v_{s-1})$  and  $\alpha'' \neq 0 \in \Lambda^{m-1}(v_1, \dots, v_{s-1})$  ( $\alpha'' \in \mathbb{Q}$  if  $m = 1$ ). Then  $\theta_s d(\chi) = \pm \alpha'' + \theta_s(\beta)$  (recall that  $\theta_i(v_j) = 0$  for  $i > j$ ). However, we have  $\theta_s d(\chi) = -\theta_s d(\chi)$ . Using Lemma 3.1 and the fact that  $\theta_s$  is a derivation, we also have  $\theta_s(\beta) \in \Lambda^{\geq m}V + \Lambda V \otimes \Lambda^+Z$ . This contradicts our minimal length assumption.

We claim that  $d_X(W) \subset \Lambda V \otimes \Lambda^{\geq 2}Z$ . Suppose this is not the case and let  $w$  be an element of lowest degree such that

$$d_X(w) = \sum_{i=1}^q z_i \omega_i + \alpha,$$

with  $z_i \in Z$ ,  $|z_1| \leq |z_2| \leq \dots \leq |z_q|$ ,  $\omega_i \in \Lambda V$  and  $\alpha \in \Lambda^{\geq 2}Z \otimes \Lambda V$ . We choose then an element  $v_s$  of highest degree such that  $\omega_q = v_s \gamma + \delta$ ,  $\gamma \neq 0$ ,  $\gamma, \delta \in \Lambda(v_1, \dots, v_{s-1})$ . Then

$$\theta_q d_X(w) = z_q v_s \text{ mod } \Lambda V \otimes (\Lambda^{\geq 2}Z + Z^{<|z_q|} + (z_1, \dots, z_{q-1})).$$

Since  $\theta_q d_X(w) = d_X \theta_q(w)$ , there exists an element  $w' \in W$  with  $|w'| < |w|$  such that  $d_X(w') \notin \Lambda V \otimes \Lambda^{\geq 2}Z$ . This is impossible by our assumption.

(2) We will define a map  $\phi: \mathcal{M}_X \rightarrow \mathcal{M}_{S_X} \otimes \mathcal{M}_X$  whose composition with the projection onto the first factor

$$(1 \cdot \epsilon) \circ \phi: \mathcal{M}_X \rightarrow \mathcal{M}_{S_X} \otimes \mathcal{M}_X \rightarrow \mathcal{M}_{S_X}$$

is surjective and satisfies  $(1 \cdot \epsilon) \circ \phi(Z) = 0$ , and whose composition with the projection onto the second factor is the identity,  $(\epsilon \cdot 1) \circ \phi = 1: \mathcal{M}_X \rightarrow \mathcal{M}_X$ .

Translating this into topological terms,  $\phi$  is the minimal model of a map  $F: S_X \times X_{\mathbb{Q}} \rightarrow X_{\mathbb{Q}}$  such that  $F \circ i_1: S_X \rightarrow X_{\mathbb{Q}}$  is injective in rational homotopy and  $F \circ i_2 = 1: X_{\mathbb{Q}} \rightarrow X_{\mathbb{Q}}$ . In other words, we may choose  $F \circ i_1$  as a total Gottlieb element (the corresponding lift through  $\omega_{\mathbb{Q}}$  is given by the adjoint of  $F$ ). Furthermore, the model of  $F \circ i_1$  is  $(1 \cdot \epsilon) \circ \phi$  by construction, which satisfies  $(1 \cdot \epsilon) \circ \phi(Z) = 0$ .

So as to avoid confusion, we write  $\mathcal{M}_{S_X} \otimes \mathcal{M}_X$  as  $\Lambda V' \otimes \Lambda V \otimes \Lambda Z$ , with  $V' = \langle v'_1, \dots, v'_r \rangle$ . First, define a sequence of maps  $\phi_1, \dots, \phi_r: \mathcal{M}_X \rightarrow \mathcal{M}_{S_X} \otimes \mathcal{M}_X$  by  $\phi_1(\chi) = \chi + v'_1 \theta_1(\chi)$ , and

$$\phi_s(\chi) = \phi_{s-1}(\chi) + v'_s \theta_s(\phi_{s-1}(\chi))$$

for  $s = 2, \dots, r$ . Then we set  $\phi = \phi_r$ . An inductive argument shows that  $\phi$  so defined is a DG algebra map. For it is straightforward to check that  $\phi_1$  is a DG algebra map. Supposing inductively that  $\phi_{s-1}$  is a DG algebra map, the computation

$$\begin{aligned} \phi_{s-1}(\chi_1)\phi_{s-1}(\chi_2) &= (\phi_{s-1}(\chi_1) + v'_s\theta_s(\phi_{s-1}(\chi_1)))(\phi_{s-1}(\chi_2) + v'_s\theta_s(\phi_{s-1}(\chi_2))) \\ &= \phi_{s-1}(\chi_1)\phi_{s-1}(\chi_2) + v'_s\theta_s(\phi_{s-1}(\chi_1))\phi_{s-1}(\chi_2) \\ &\quad + (-1)^{|\chi_1|}v'_s\phi_{s-1}(\chi_1)\theta_s(\phi_{s-1}(\chi_2)) \\ &= \phi_{s-1}(\chi_1\chi_2) + v'_s\theta_s(\phi_{s-1}(\chi_1\chi_2)) \end{aligned}$$

shows that  $\phi_s$  is an algebra map. A similar computation, using that  $\phi_{s-1}$  and  $\theta_s$  commute with  $d_X$ , and also that  $d(V') = 0$ , shows that  $\phi_s$  also commutes with  $d_X$ , and hence is a DG algebra map. Thus, each  $\phi_1, \dots, \phi_r$  is a DG algebra map and in particular so is  $\phi = \phi_r$ .

Next, we show the following: That  $\phi(v_1) = v_1 + v'_1$  and, for  $i = 2, \dots, r$ ,

$$\phi(v_i) = v_i + v'_i + I(v'_1, \dots, v'_{i-1}).$$

This we do by induction on  $s$ . Suppose inductively that we have  $\phi_s(v_1) = v_1 + v'_1$  and

$$\phi_s(v_i) = \begin{cases} v_i + v'_i + I(v'_1, \dots, v'_{i-1}) & \text{if } i = 2, \dots, s \\ v_i + I(v'_1, \dots, v'_{i-1}) & \text{if } i = s+1, \dots, r \end{cases}$$

Induction starts with  $s = 1$ , where the formulas

$$\phi_1(v_1) = v_1 + v'_1 \quad \text{and} \quad \phi_1(v_i) = v_i + v'_1\theta_1(v_i)$$

give the result. For the inductive step, we compute as follows:  $\phi_{s+1}(v_1) = \phi_s(v_1) + v'_{s+1}\theta_{s+1}(v_1) = v_1 + v'_1$ , since  $1 < s+1$  and hence  $\theta_{s+1}(v_1) = 0$ . For  $i = 2, \dots, s$ , we have

$$\begin{aligned} \phi_{s+1}(v_i) &= \phi_{s+1}(v_i) + v'_{s+1}\theta_{s+1}(\phi_s(v_i)) \\ &= v_i + v'_i + I(v'_1, \dots, v'_{i-1}) + v'_{s+1}\theta_{s+1}(v_i + v'_i + I(v'_1, \dots, v'_{i-1})) \\ &= v_i + v'_i + I(v'_1, \dots, v'_{i-1}) \end{aligned}$$

since  $i < s+1$  and thus  $\theta_{s+1}(v_i) = 0$ , and also the ideal  $I(v'_1, \dots, v'_{i-1})$  is  $\theta_{s+1}$ -stable, as  $\theta_{s+1}(v'_i) = 0$ . Further,  $\phi_{s+1}(v_{s+1}) = \phi_{s+1}(v_{s+1}) + v'_{s+1}\theta_{s+1}(\phi_s(v_{s+1})) = v_{s+1} + I(v'_1, \dots, v'_s) + v'_{s+1}\theta_{s+1}(v_{s+1} + I(v'_1, \dots, v'_s)) = v_{s+1} + v'_{s+1} + I(v'_1, \dots, v'_s)$ . Finally, for  $i = s+2, \dots, r$ , we have

$$\begin{aligned} \phi_{s+1}(v_i) &= \phi_s(v_i) + v'_{s+1}\theta_{s+1}(\phi_s(v_i)) \\ &= v_i + I(v'_1, \dots, v'_{i-1}) + v'_{s+1}\theta_{s+1}(v_i + I(v'_1, \dots, v'_{i-1})) \\ &= v_i + I(v'_1, \dots, v'_{i-1}) \end{aligned}$$

since  $s+1 \leq i-1$ . This completes the induction.

Finally, we observe that, for any  $z \in Z$ , we have  $\phi(z) \in I(Z)$ . This follows easily from the fact that  $Z$  is  $\theta_i$ -stable for each  $i$ .

From these facts, it is evident that  $(1 \cdot \epsilon) \circ \phi$  satisfies  $(1 \cdot \epsilon) \circ \phi(v_1) = v'_1$ , and  $(1 \cdot \epsilon) \circ \phi(v_i) = v'_i + I(v'_1, \dots, v'_{i-1})$  for  $i = 2, \dots, r$ . It follows that  $(1 \cdot \epsilon) \circ \phi$  is surjective. Furthermore, we have  $(1 \cdot \epsilon) \circ \phi(z) = 0$ . For the other projection, it is evident from the definition of  $\phi$  that we have  $(\epsilon \cdot 1) \circ \phi = 1$ .  $\square$

We deduce the following technical proposition.

**Proposition 3.3.** *Suppose  $V$  decomposes as  $V = V' \oplus V''$ , with  $d_X(V') = 0$  and  $V''$  satisfying the following: For any cycle of the form  $v + z + \chi$ , with  $v \in V$ ,  $z \in Z$ , and  $\chi \in \wedge^{\geq 2}(V \oplus Z)$ , we have  $v \in V'$ . Suppose the complement  $Z$  has been chosen to satisfy  $\theta_i(Z) \subseteq \wedge V \otimes \wedge^+ Z$  for each  $i$ . Then any cycle of  $\wedge^+(V \oplus Z)$  is in the ideal  $I(V', Z)$  generated by  $V' \oplus Z$ .*

*Proof.* The proof is similar to that of part (1) of Proposition 3.2. We argue by contradiction. Suppose this is not true, and that amongst cycles of the form  $\alpha + \beta$ , with  $\alpha \neq 0 \in \wedge V''$ ,  $\beta \in I(V', Z)$ , that the shortest length term in any such  $\alpha$  is  $m \geq 2$ . Since each  $\theta_i$  commutes with the differential,  $\theta_i(v')$  is a cycle for each  $i$ . Therefore, we must have that  $\theta_i(V') \subseteq \wedge^{\geq m} V'' + I(V', Z)$ . Now adjust our notation slightly for this situation. Write  $V'' = \langle v''_1, \dots, v''_s \rangle$  for suitable  $s \leq r$ , with corresponding derivations  $\theta''_1, \dots, \theta''_s$ . Let  $\chi$  be a cycle that displays a shortest length part in  $\wedge V''$ , and suppose that  $t \leq s$  is the highest index for which  $v''_t$  occurs in this shortest length part. Then write

$$\chi = \alpha' + \alpha'' v''_t + \alpha''' + \beta,$$

with  $\alpha' \in \wedge^m(v''_1, \dots, v''_{t-1})$ ,  $\alpha'' \neq 0 \in \wedge^{m-1}(v''_1, \dots, v''_{t-1})$ ,  $\alpha''' \in \wedge^{m+1} V''$ , and  $\beta \in I(V', Z)$ . Since  $\theta''_t$  commutes with the differential,  $\theta''_t(\chi)$  is again a cycle. However, we have  $\theta''_t(\chi) = \alpha'' + \theta''_t(\alpha''' + \beta)$  (recall that  $\theta_i(v_j) = 0$  for  $i > j$ ). Using Lemma 3.1 and the fact that  $\theta_t$  is a derivation, we have  $\theta_t(\alpha''' + \beta) \in I(\wedge^m V'', V', Z)$ . This contradicts our minimal length assumption.  $\square$

The next result is a consequence of Oprea's Theorem 1.11. In order to be self-contained we include here a short proof.

**Proposition 3.4.** *Suppose  $\mathcal{M}_X$  is written as  $\wedge(V' \oplus V'' \oplus Z)$  as in Proposition 3.3. Then we may identify  $V'$  with  $\text{im } h_X \circ (\omega_{\mathbb{Q}})_{\#}$ . Furthermore,  $X_{\mathbb{Q}}$  decomposes as a product  $X_{\mathbb{Q}} \simeq S \times Y$  with  $S$  a product of odd-dimensional rational spheres whose minimal model is  $(\wedge V', 0)$ .*

*Proof.* Suppose  $(\wedge V, d)$  is a minimal model for  $X$ ,  $x \in V$  is a cocycle of odd degree and that there is a derivation  $\theta$  of  $\wedge V$  such that  $[\theta, d] = 0$  and  $\theta(x) = 1$ . Write  $\wedge V = \wedge(x) \otimes \wedge W$ . Then by induction on the degree we can modify the choice of  $W$  in order to have  $d(W) \subset \wedge W$ , as follows. Suppose that  $d(W^{<n}) \subset \wedge W$  and let  $v \in W^n$ . We write  $d(v) = x\alpha + \beta$  with  $\alpha$  and  $\beta \in \wedge W$ . Then  $[\theta, d] = 0$  implies  $\alpha = -d(\theta(v))$ . Replacing  $v$  by  $v' = v + x\theta(v)$  we obtain  $d(v') \in \wedge W$ . In this way, we may assume that  $(\wedge V, d) = (\wedge x, 0) \otimes (\wedge W, d)$ , i.e.,  $X_{\mathbb{Q}} \simeq S_{\mathbb{Q}}^n \times Y$ . Since we then have  $G_*(X_{\mathbb{Q}}) \cong G_*(S_{\mathbb{Q}}^n) \oplus G_*(Y)$ , we can proceed in the same way with  $Y$ . This results in the required decomposition.  $\square$

We may now prove Theorem 1.8 of the introduction. In her thesis [4], Sonia Ghorbal has obtained a criterion for a map to be a homotopy monomorphism in the nilpotent category. In order to be self-contained, we reproduce here the statement and the proof of this criterion.

**Proposition 3.5** (S. Ghorbal). *Let  $f: X \rightarrow Y$  be a map of rational spaces that admits a minimal model of the form  $\gamma: (\wedge(V \oplus W), d) \rightarrow (\wedge V, \bar{d})$  such that  $\gamma(W) = 0$ ,  $\gamma(v) = v$  for  $v \in V$ ,  $d(W) \subseteq \wedge V \otimes \wedge^{\geq 2} W$ , and  $d(V) \subseteq \wedge V + \wedge V \otimes \wedge^{\geq 2} W$ . Then  $f$  is a homotopy monomorphism in the nilpotent category.*

*Proof.* We first recall from [10] that two morphisms  $k, l: (\wedge V, d) \rightarrow (A, d)$  are homotopic if there exists a map  $H: \wedge(V \oplus \bar{V} \oplus V') \rightarrow (A, d)$  with  $k(v) = H(v)$  and

$$l(v) = H(e^{sd+ds}(v)) = H(v) + dH(\bar{v}) + \sum_{r \geq 1} \frac{1}{r!} H((sd)^r(v))$$

for each  $v \in V$ . In this definition  $\bar{Z}$  and  $Z'$  are graded vector spaces,  $\bar{Z}^p = Z^{p+1}$ ,  $(Z')^p = Z^p$ . The differential  $d$  on  $\wedge V$  is extended to  $\wedge(V \oplus \bar{V} \oplus V')$  by  $d(\bar{v}) = v'$  and  $d(v') = 0$  for  $v \in V$ . The map  $s$  is a degree  $-1$  derivation defined by  $s(v) = \bar{v}$  and  $s(\bar{v}) = s(v') = 0$ .

Now suppose that  $g, h: (\wedge V, \bar{d}) \rightarrow (\wedge T, d)$  are DG algebra maps and that

$$\Phi: (\wedge(V \oplus W \oplus \bar{V} \oplus \bar{W} \oplus V' \oplus W'), d) \rightarrow (\wedge T, d)$$

is a homotopy between  $g \circ \gamma$  and  $h \circ \gamma$ . We denote by  $I$  the ideal of  $\wedge(V \oplus W \oplus \bar{V} \oplus \bar{W} \oplus V' \oplus W')$  generated by the vector spaces  $\wedge^2 W$ ,  $s(\wedge^2 W)$ , and  $W'$ .

First we show that  $sd(I) \subseteq I$ . For this, write a typical element of  $I$  as  $aA + bB + cC$  with  $a \in \wedge^2 W$ ,  $b \in s(\wedge^2 W)$ ,  $c \in W'$ , and  $A, B, C$  general elements of  $\wedge(V \oplus W \oplus \bar{V} \oplus \bar{W} \oplus V' \oplus W')$ . Then we have

$$sd(aA) = s((da)A \pm a(dA)) = (sda)A \pm (da)(sA) \pm (sa)(dA) \pm a(sdA).$$

The last two terms are automatically in  $I$ . The second, and hence the first, is in  $I$  due to the hypothesis on  $d$ . A similar analysis of the terms that occur shows that  $sd(bB)$  and  $sd(cC)$  are also in  $I$ .

Next, we show by induction that  $\Phi(I) = 0$ . For this, choose a basis  $y_1, y_2, \dots, y_r, \dots$  of  $W$  with  $|y_i| \leq |y_{i+1}|$ . Also, denote by  $W_{(n)}$  the subspace  $\langle y_1, \dots, y_n \rangle$  of  $W$  and by  $I_{(n)}$  the ideal generated by the vector spaces  $\wedge^2(W_{(n)})$ ,  $s(\wedge^2(W_{(n)}))$ , and  $W'_{(n)}$ .

When  $n = 1$ , we have  $d(y_1) = 0$  from our hypothesis on  $d$ . Thus we have

$$0 = h \circ \gamma(y_1) = \Phi(e^{sd+ds}(y_1)) = \Phi(y_1) + \Phi(y'_1) = g \circ \gamma(y_1) + \Phi(y'_1) = \Phi(y'_1),$$

which starts the induction. Now suppose that the result is true for  $i < n$ . Then we have

$$0 = \Phi(e^{sd+ds}(y_n)) = \Phi(y_n) + \Phi(y'_n) + \sum_{r \geq 1} \frac{1}{r!} \Phi((sd)^r(y_n)).$$

The hypothesis on  $d$  implies that  $sd(y_n) \in I_{(n-1)}$ . A refinement of the argument in the previous part shows that, in fact, each  $I_{(n-1)}$  is stable under  $sd$ . Therefore, we have  $(sd)^r(y_n) \in I_{(n-1)}$  for  $r \geq 1$ . Since  $\Phi(I_{(n-1)}) = 0$  by our induction hypothesis, we have that  $\Phi((sd)^r(y_n)) = 0$  and therefore  $\Phi(y'_n) = 0$ . Of course,  $\Phi$  is already zero on  $W$  and hence vanishes on both  $\wedge^2(W_{(n)})$  and  $s(\wedge^2(W_{(n)}))$ . Therefore, we have  $\Phi(I_{(n)}) = 0$  and the induction is complete. It follows that the ideal  $I$  possesses two key properties, namely  $sd(I) \subset I$  and  $\Phi(I) = 0$ . We now define a homotopy

$$\Psi: (\wedge(V \oplus \bar{V} \oplus V'), \bar{d}) \rightarrow (\wedge T, d)$$

simply by restricting  $\Phi$ . We remark that  $(sd)^r(v) - (s\bar{d})^r(v) \in I$  for  $v \in V$ , for  $r \geq 1$ . Therefore the homotopy ends at  $\Psi(e^{s\bar{d}+d\bar{s}}(v)) = \Phi(e^{sd+ds}(v)) = h(v)$ . Furthermore, we have  $\Psi(v) = \Phi(v) = g(v)$  for  $v \in V$ . Thus  $\Psi$  is a homotopy between  $g$  and  $h$ .

The argument so far shows that  $f$  is a homotopy monomorphism in the rational category. That is, if  $A$  is any rational space, then

$$f_*: [A, X] \rightarrow [A, Y]$$

is one-to-one. From the universal properties of localization, it follows that  $f$  is a homotopy monomorphism in the nilpotent category.  $\square$

*Proof of Theorem 1.8.* For the ordinary evaluation map  $\omega: \text{Map}(X, X; 1) \rightarrow X$ , we have that  $\Gamma_X: S_X \rightarrow X_{\mathbb{Q}}$  is a homotopy monomorphism in the nilpotent category by Proposition 3.5 and Proposition 3.2. Now suppose that  $w: E \rightarrow X$  is any evaluation map. From Theorem 1.6, we have the following commutative diagram of solid arrows

$$\begin{array}{ccc}
 & E & \xrightarrow{g} \text{Map}(X, X_{\mathbb{Q}}; e) \\
 \begin{array}{c} \swarrow r_w \\ \downarrow \tilde{\Gamma}_w \\ \downarrow \Gamma_w \end{array} & & \begin{array}{c} \searrow r_X \\ \downarrow \tilde{\Gamma}_X \\ \downarrow \Gamma_X \end{array} \\
 S_w & & S_X \\
 & \downarrow w_{\mathbb{Q}} & \downarrow \omega_{\mathbb{Q}} \\
 & X_{\mathbb{Q}} & = X_{\mathbb{Q}}
 \end{array}$$

with retractions  $r_X$  and  $r_w$  of  $\tilde{\Gamma}_X$  and  $\tilde{\Gamma}_w$  respectively. We define  $j: S_w \rightarrow S_X$  by  $j = r_X \circ g \circ \tilde{\Gamma}_w$  and claim that this map admits a retraction. Recall that both  $S_w$  and  $S_X$  are (finite) products of odd-dimensional rational spheres. Also, since  $\Gamma_w$  and  $\Gamma_X$  are both injective in rational homotopy and  $\Gamma_X \circ j = \Gamma_w$ , it follows that  $j$  is injective in rational homotopy. In terms of minimal models, then, we have a map  $\mathcal{M}_j: (\wedge V, d=0) \rightarrow (\wedge W, d=0)$  with  $Q(\mathcal{M}_j)$  surjective. But if  $Q(\mathcal{M}_j)$  is surjective, so too is  $\mathcal{M}_j$ . Therefore, we may choose a splitting of  $\mathcal{M}_j$  which corresponds to a retraction of  $j$ . Since  $j$  admits a retraction, it is a homotopy monomorphism. Finally, it follows that  $\Gamma_w$  is a composition of homotopy monomorphisms and hence is a homotopy monomorphism.  $\square$

**Corollary 3.6.** *Let  $w: E \rightarrow X$  be any evaluation map. Then  $w_{\mathbb{Q}}$  factors as a composition  $w_{\mathbb{Q}} = \Gamma_w \circ r_w$  with  $r_w$  a homotopy epimorphism and  $\Gamma_w$  a homotopy monomorphism in the nilpotent category.*

*Proof.* The discussion at the start of this section concluded that  $r_w$  is a homotopy epimorphism and the remainder follows immediately from Theorem 1.6 and Theorem 1.8.  $\square$

We remark that the fact that  $\Gamma_w$  is associated to an evaluation map is key in Theorem 1.8. In particular, we may give the following example of a map  $\gamma: S \rightarrow X$  from an  $H_0$ -space  $S$  into  $X$  that is injective in rational homotopy but is not a homotopy monomorphism in the nilpotent category.

**Example 3.7.** Let  $S = S_a^3 \times S^5$  and  $X = S_a^3 \vee S_b^3 \cup_{\alpha} e^8$ , where  $\alpha$  is the triple Whitehead bracket  $[a, [a, b]]$ . Then  $\gamma: S \rightarrow X$  is an extension of  $(1 \mid [a, b]): S_a^3 \vee S^5 \rightarrow X$  obtained using the fact that  $[a, [a, b]] = 0$  in  $\pi_*(X)$ . Consider two maps  $h, k: S^2 \times S^3 \rightarrow S_a^3 \times S^5$ . The map  $h$  is the composition

$$S^2 \times S^3 \xrightarrow{p_2} S^3 \xrightarrow{i_1} S_a^3 \times S^5$$

and  $k$  is the composition of the inclusion  $S^3 \vee S^5 \rightarrow S^3 \times S^5$  with the map that consists of collapsing the cell  $S^2$  into a point:

$$S^2 \times S^3 \longrightarrow S^2 \times S^3 / S^2 = S^3 \vee S^5 \longrightarrow S^3 \times S^5.$$

Clearly  $h_{\mathbb{Q}}$  and  $k_{\mathbb{Q}}$  are not homotopic because they do not induce the same map in rational homology. However a simple computation using minimal models show that the compositions  $f_{\mathbb{Q}} \circ h_{\mathbb{Q}}$  and  $f_{\mathbb{Q}} \circ k_{\mathbb{Q}}$  are homotopic.

We finish this section with the topic of cyclic maps. A *cyclic map*  $f: A \rightarrow X$  may be defined as a map that lifts through the evaluation map  $\omega: \text{Map}(X, X; 1) \rightarrow X$ . This definition is easily seen to be equivalent to that given above Theorem 1.10 *via* the adjoint correspondence between a map  $A \rightarrow \text{Map}(X, X; 1)$  that lifts  $f$  and a map  $A \times X \rightarrow X$  that extends  $(f | 1)$ . Together with Sam Smith, the second-named author has studied cyclic maps from the rational homotopy point of view in [16]. As we mentioned in the introduction, our interest in the results of this paper arose from that earlier work.

To state Corollary 2.9 we defined the Gottlieb groups of a space relative to an evaluation map. We say that a map  $f: A \rightarrow X$  is *cyclic with respect to an evaluation map*  $w: E \rightarrow X$  if  $f$  lifts through the evaluation map  $w$ . Denote the set of homotopy classes of such maps by  $G^w(A, X)$ . Upon rationalizing such a map, we obtain a map in  $G^{w_{\mathbb{Q}}}(A, X_{\mathbb{Q}})$ .

**Theorem 3.8.** *Let  $w: E \rightarrow X$  be an evaluation map with  $X$  a nilpotent, finite complex and let  $A$  be a nilpotent space. Then there are bijections of sets*

$$G^{w_{\mathbb{Q}}}(A, X_{\mathbb{Q}}) \cong [A, S_w] \cong \bigoplus_r \text{Hom}(H_r(A; \mathbb{Q}), G_r^w(X_{\mathbb{Q}})).$$

*Proof.* The first bijection is given by  $(\Gamma_w)_*: [A, S_w] \rightarrow G^{w_{\mathbb{Q}}}(A, X_{\mathbb{Q}})$ . This is a bijection by Theorem 1.6 and Theorem 1.8. Now remark that  $S_w$  has the homotopy type of a product of rational Eilenberg-Mac Lane spaces,  $S_w = \prod_{i=1}^r K(\mathbb{Q}, n_i)$ . By taking cohomology classes we thus obtain a bijection

$$[A, S_w] \xrightarrow{\cong} \bigoplus_{i=1}^r H^{n_i}(A; \mathbb{Q})$$

and the result follows.  $\square$

Thus, for instance, we retrieve [16, Th.3.2]: If  $A$  is a space with non-zero rational cohomology in even degrees only, then any map  $g: A \rightarrow S_w$  must be null-homotopic, as  $S_w$  is a product of odd-dimensional rational Eilenberg-Mac Lane spaces. Consequently, this hypothesis on  $A$  entails the triviality of the set  $G^{w_{\mathbb{Q}}}(A, X_{\mathbb{Q}})$ . Many of the other results of [16] may be placed in context with the results of this paper.

If  $X$  is a suspension, or more generally a  $\text{co-}H_0$ -space, then its rationalized Gottlieb groups are generally trivial. Indeed, this is the case as long as  $X$  does not have the rational homotopy type of a single sphere. Therefore, it follows from Theorem 2.10 that any cyclic map into a  $\text{co-}H_0$ -space that does not have the rational homotopy type of a sphere is rationally trivial. Basic finiteness results, such as those of [14], follow from this.

Note, however, that a general cyclic map does not factor through the product of odd spheres that corresponds to its image in rational homotopy. That is, we are not able to extend Theorem 1.6 to cyclic maps. In particular, we note that there exist cyclic maps that are trivial in rational homotopy and yet not null-homotopic (e.g. [16, Ex.4.1]).

#### 4. EVALUATION MAPS AND HOMOLOGY

After the preparatory results of Section 3, we prove in this section the results concerning the homomorphism induced in rational homology by an evaluation map.

*Proof of Theorem 1.12.* Consider  $\omega : \text{Map}(X, X; 1) \rightarrow X$  as a special case first. If  $X$  is an  $H_0$ -space, then the multiplication of  $X_{\mathbb{Q}}$  provides a section of  $\omega_{\mathbb{Q}}$ , so that  $H_*(\omega; \mathbb{Q})$  is surjective. If we have  $X_{\mathbb{Q}} \simeq S_{\mathbb{Q}}^{2n+1} \times Y$ , then we may apply Theorem 2.5. As  $S^{2n+1}$  is an  $H_0$ -space, the above observation gives that  $\omega_{S^{2n+1}}$  is surjective on rational homology. Furthermore, the map  $(i_1)^*$  in diagram (1) immediately preceding Theorem 2.5 admits a section, namely  $(p_1)^*$ , and so it too is surjective on rational homology. It follows that  $\text{im } H_*(\omega; \mathbb{Q})$  contains at least the  $H_*(S^{2n+1}; \mathbb{Q})$  factor and thus is non-zero. This establishes item (3) of Theorem 1.12.

Next, suppose that  $h_X \circ (\omega_{\mathbb{Q}})_{\#} = 0$ . We deduce from Lemma 3.1 and Proposition 3.3 that a model of  $\Gamma_X$  is given by

$$\mu : (\wedge(V \oplus Z), d_X) \rightarrow (\wedge V, 0)$$

with all cocycles of  $\wedge(V \oplus Z)$  in the ideal generated by  $Z$  and  $\mu(Z) = 0$ . Now Proposition 3.2 (2) shows that the total Gottlieb element  $\Gamma_X$  induces the trivial homomorphism in rational cohomology.

On the other hand, suppose that  $h_X \circ (\omega_{\mathbb{Q}})_{\#}$  has image of dimension  $r > 0$ . Then Proposition 3.4 implies that we have  $X_{\mathbb{Q}} \simeq S \times Y$  where  $S$  is an  $r$ -fold product of rational spheres of odd dimensions that correspond to the image of  $h_X \circ (\omega_{\mathbb{Q}})_{\#}$ . Now we apply Theorem 2.5 and conclude that  $\text{im } H_*(\omega; \mathbb{Q})$  contains the  $H_*(S; \mathbb{Q})$  factor. Furthermore, we have  $h_Y \circ (\omega_Y)_{\#} = 0$ , otherwise the image of  $h_X \circ (\omega_{\mathbb{Q}})_{\#}$  would be of dimension  $> r$ . Therefore,  $\tilde{H}_*(\omega_Y; \mathbb{Q}) = 0$  and the image of  $H_*(\omega_{\mathbb{Q}}; \mathbb{Q})$  is precisely the  $H_*(S; \mathbb{Q})$  factor. This establishes the remaining items of Theorem 1.12 for  $\omega$ .

Now consider a generalized evaluation map  $w : E \rightarrow X$ . We suppose that  $\text{im } h_X \circ (\omega_{\mathbb{Q}})_{\#}$  is of dimension  $r$  and  $\text{im } h_X \circ (w_{\mathbb{Q}})_{\#}$  is of dimension  $s$ . Since  $w$  factors through  $\omega$ , we have  $s \leq r$ . We write  $X_{\mathbb{Q}} \simeq S \times Y$  as above, and we obtain a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & \text{Map}(X, X_{\mathbb{Q}}; e) \\ \downarrow w_{\mathbb{Q}} & & \downarrow \omega_{\mathbb{Q}} \\ X_{\mathbb{Q}} & \xrightarrow[h]{\simeq} & S \times Y \end{array}$$

where  $g$  is the  $H$ -map obtained from the definition of a generalized evaluation map. By Theorem 2.5 the coordinate maps  $p_1 \circ \omega_{\mathbb{Q}}$  and  $p_2 \circ \omega_{\mathbb{Q}}$  factor through  $(\omega_S)_{\mathbb{Q}}$  and  $(\omega_Y)_{\mathbb{Q}}$  respectively. Because of this factorization, and the fact that  $\tilde{H}_*(\omega_Y; \mathbb{Q}) = 0$ , we may make the following identifications:

$$\text{im } H_*(w_{\mathbb{Q}}; \mathbb{Q}) \cong \text{im } H_*(\omega_{\mathbb{Q}} \circ g; \mathbb{Q}) \cong \text{im } H_*(p_1 \circ \omega_{\mathbb{Q}} \circ g; \mathbb{Q}) \subseteq H_*(S; \mathbb{Q}).$$

Since the composition  $p_1 \circ \omega_{\mathbb{Q}} \circ g : E \rightarrow S$  satisfies the hypotheses of Corollary 2.8, it admits a minimal model of the form  $\varphi : (\wedge V, 0) \rightarrow (\wedge W, 0)$  with  $\varphi(V) \subset W$ . Then the image of  $p_1 \circ \omega_{\mathbb{Q}} \circ g : E \rightarrow S$  in rational homotopy has dimension  $s$  and we may factor its minimal model  $\varphi : (\wedge V, 0) \rightarrow (\wedge W, 0)$  as the composition of a surjection and an injection  $\wedge(V_s \oplus K) \rightarrow \wedge V_s \rightarrow \wedge(V_s \oplus K')$ , with  $V_s$  a vector space of dimension  $s$  isomorphic to the image of  $\text{im } h_X \circ (w_{\mathbb{Q}})_{\#}$ . This corresponds to a

factorization of  $p_1 \circ \omega_{\mathbb{Q}} \circ g: E \rightarrow S$  as

$$\begin{array}{ccc} E & \xrightarrow{p_1 \circ \omega_{\mathbb{Q}} \circ g} & S \simeq S' \times S'' \\ & \searrow q & \nearrow i_1 \\ & & S' \end{array}$$

with  $S'$  a product of odd-dimensional rational spheres with minimal model  $(\wedge V_s, 0)$ . It is now clear that the image in homology of  $w_{\mathbb{Q}}$  is isomorphic to  $H_*(S'; \mathbb{Q})$ .  $\square$

For  $X$  a finite complex, a result of Gottlieb ([9, Th.3]) says that if  $\chi(X) \neq 0$ , then the first degree in which the homomorphism induced by the evaluation map on rational cohomology may be non-zero is even. With Theorem 1.12, we sharpen this result in a very significant way.

**Corollary 4.1** (Corollary 1.13). *Let  $X$  be a nilpotent, finite space. Suppose that  $\chi(X) \neq 0$  or, more generally, that  $X$  does not factor up to rational homotopy as  $X_{\mathbb{Q}} \simeq S_{\mathbb{Q}}^{2n+1} \times Y$ . Then for every evaluation map  $w: E \rightarrow X$ , we have  $\tilde{H}_*(w; \mathbb{Q}) = 0$ .*

Recall that  $X$  is called a *c-symplectic space* if it is an even-dimensional rational Poincaré duality space that possesses some class  $x \in H^2(X; \mathbb{Q})$ , some power of which is a fundamental class [15].

**Corollary 4.2** (Corollary 1.14). *Let  $X$  be a simply connected, c-symplectic space. Then every evaluation map  $w: E \rightarrow X$  satisfies  $\tilde{H}_*(w; \mathbb{Q}) = 0$ .*

*Proof.* It is evident that the cohomology algebra structure does not allow a decomposition of the form  $X \simeq_{\mathbb{Q}} S^{2n+1} \times X'$ , and so Theorem 1.12 implies the evaluation map is trivial in rational homology.  $\square$

At the other extreme from the situation described in these corollaries, we have the following:

**Corollary 4.3.** *Let  $w: E \rightarrow X$  be an evaluation map with  $X$  a nilpotent, finite complex. The following are equivalent:*

- (1) *The homomorphism  $H_*(w): H_*(E; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$  is surjective;*
- (2)  *$\Gamma_w: S_w \rightarrow X$  is a rational homotopy equivalence.*

*When (1) and (2) pertain,  $X$  is an  $H_0$ -space and the evaluation map admits a section.*

*Proof.* All parts follow easily from Theorem 1.6 and Theorem 1.12.  $\square$

## 5. CONCLUSION: SOME OPEN PROBLEMS

At present, we have very little information about the map  $w: \text{Top}(X, X; 1) \rightarrow X$  or the other variations on the evaluation map  $\omega$  mentioned at the start of the introduction. It would be most interesting to identify  $G_*^w(X_{\mathbb{Q}})$ , the image in rational homotopy of  $w$ , or, more generally the rational homotopy groups of  $\text{Top}(X, X; 1)$ . As specific instances of this kind of problem, we offer the following.

**Problem 5.1.** *Let  $M$  be a compact smooth manifold. Is the image in (rational) homotopy of the evaluation map  $w: \text{Diff}(M, M; 1) \rightarrow M$  strictly contained in, or equal to, the (rational) Gottlieb groups of  $M$ ?*

**Problem 5.2.** *Let  $X$  be an  $H$ -space and recall that  $G_*(X) = \pi_*(X)$  in this case. Let  $H(X, X; 1)$  denote the subspace of  $\text{Map}(X, X; 1)$  that consists of  $H$ -equivalences. Is the evaluation map  $w: H(X, X; 1) \rightarrow X$  surjective in (rational) homotopy?*

Assuming that  $G_*^w(X)$  and  $G_*(X)$  are generally different from each other, it would be interesting to know whether there are structural results for  $G_*^w(X_{\mathbb{Q}})$  comparable to those of Félix-Halperin for the ordinary Gottlieb groups.

Corollary 1.5 and Corollary 2.9 may be used to give necessary conditions for certain maps to be the connecting map of a fibration (cf. Example 2.12). This suggests the following particular version of an old problem of Massey:

**Problem 5.3.** *Let  $p: \Omega B \rightarrow X$  be a map from a loop space to a nilpotent, finite complex  $X$ . When is  $p$  the connecting map of some fibration sequence  $X \rightarrow E \rightarrow B$ ?*

It would be nice to find other situations in which the image in rational homotopy groups of a map led to factorizations analogous to those of Section 2. In this direction, we offer the following rather general problem:

**Problem 5.4.** *Suppose given a map  $f: X \rightarrow Y$  with  $Y$  finite-dimensional. If the image of  $f_{\#}$  in rational homotopy groups is finite-dimensional, does  $f$  factor through an elliptic space?*

We have restricted ourselves entirely to the rational homotopy context in this paper. But it could be feasible to investigate similar results working either integrally or localized at different sets of primes. We end with two “moonshots” that indicate how little we know outside the rational situation.

**Problem 5.5.** *Let  $X$  be a space with trivial Gottlieb groups (integrally). Is the evaluation map  $\omega: \text{Map}(X, X; 1) \rightarrow X$  null-homotopic?*

**Problem 5.6.** *Let  $X$  be a nilpotent, finite complex. When is a Gottlieb element  $S^n \rightarrow X$  a homotopy monomorphism, and not just a rational homotopy monomorphism?*

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INSTITUT MATHÉMATIQUE, UNIVERSITÉ CATHOLIQUE DE LOUVAIN, B-1348 LOUVAIN-LA-NEUVE  
BELGIQUE

*E-mail address:* `felix@math.ucl.ac.be`

DEPARTMENT OF MATHEMATICS, CLEVELAND STATE UNIVERSITY, CLEVELAND OH 44115 U.S.A.

*E-mail address:* `G.Lupton@csuohio.edu`