

RATIONAL HOMOTOPY OF THE POLYHEDRAL PRODUCT FUNCTOR

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ABSTRACT. Let $(X, *)$ be a pointed CW-complex, K be a simplicial complex on n vertices and X^K be the associated polyhedral power. In this paper, we construct a Sullivan model of X^K from K and from a model of X .

Let $\mathcal{F}(K, X)$ be the homotopy fiber of the inclusion $X^K \rightarrow X^n$. Recent results of Grbić and Theriault, on one side, and of Denham and Suciu, on the other side, show the diversity of the possible homotopy types for $\mathcal{F}(K, X)$. Here, we prove that the corresponding map between Sullivan models is Golod attached, generalizing a result of J. Backelin. This property is deduced from the existence of a succession of fibrations whose fibers are suspensions.

We consider also the Lusternik-Schnirelmann category of X^K . In the case that $\text{cat } X^n = n \text{ cat } X$, we prove that $\text{cat } X^K = (\text{cat } X)(1 + \dim K)$.

Finally, we mention that this work is written in the case of a sequence of pairs, $\underline{X} = (X_i, A_i)_{1 \leq i \leq n}$, as in a recent work of Bahri, Bendersky, Cohen and Gitler.

The Davis-Januszkiewicz space (see [9]) is defined by the Borel construction $\mathcal{Z}_K \times_{T^n} ET^n$, where \mathcal{Z}_K is the moment angle complex associated to a simplicial complex K on n vertices. This space is an example of the K th power of a space, introduced by Buchstaber and Panov in [7] and by Strickland in [17]. Many properties of this polyhedral power have been developed by Notbohm and Ray in [16], and by Grbić and Theriault in [14]. We are using here a general version defined by Bahri, Bendersky, Cohen and Gitler in [5] that we recall now.

Let K be a simplicial complex on n vertices and $\underline{X} = (X_i, A_i)_{1 \leq i \leq n}$ be a finite sequence of pairs of CW-complexes. For each simplex $\sigma \in K$, we denote by \underline{X}^σ the subspace of the product space $\prod_{i=1}^n X_i$ defined by:

$$\underline{X}^\sigma = \prod_{i=1}^n Y_i \quad \text{where } Y_i = \begin{cases} X_i, & \text{if } i \in \sigma, \\ A_i, & \text{if } i \notin \sigma. \end{cases}$$

The *polyhedral product determined by \underline{X} and K* is the union

$$\underline{X}^K = \bigcup_{\sigma \in K} \underline{X}^\sigma \subset \prod_{i=1}^n X_i.$$

If we need a pointed situation, we choose a base-point x_i in each A_i , and we point \underline{X}^K by (x_1, \dots, x_n) . When $X_i = X$ and $A_i = *$ for all i , we denote the polyhedral product by X^K .

When $X = BS^1$, the homotopy fiber of $X^K \rightarrow X^n$ is the usual moment-angle complex \mathcal{Z}_K and is homotopy equivalent to the complement $U(K)$ of the coordinate subspaces arrangement associated to K , (cf. [7]),

$$U(K) = \mathbb{C}^n - \bigcup_{\sigma \notin K} \{(z_1, \dots, z_n) \mid z_{i_1} = \dots = z_{i_k} = 0, \text{ if } \sigma = \{i_1, \dots, i_k\}\}.$$

In this paper, when \underline{X} is a sequence of nilpotent spaces of finite type, we describe the rational homotopy type of \underline{X}^K and the rational homotopy type of the homotopy

fiber of the injection $\underline{X}^K \rightarrow \prod_{i=1}^n X_i$. Recall that a (rational) model of a connected space X is a commutative differential graded \mathbb{Q} -algebra, \mathcal{M} , which is connected, of finite type and quasi-isomorphic to the minimal model of X . More generally, we refer to [12] for basic informations on rational homotopy theory and Sullivan theory of models. Also, in different parts of this paper, we use the fact that colimits on simplicial sets give homotopy colimits on the associated polyhedral product and refer to [16] for these properties.

For any i , $1 \leq i \leq n$, we consider a surjective model $\varphi_i: \mathcal{M}_i \rightarrow \mathcal{M}'_i$ of the canonical inclusion $A_i \hookrightarrow X_i$. For each $\sigma \notin K$, denote by I_σ the ideal of $\otimes_{i=1}^n \mathcal{M}_i$ defined by

$$I_\sigma = E_1 \otimes \cdots \otimes E_n \quad \text{with} \quad E_i = \begin{cases} \ker(\varphi_i), & \text{if } i \in \sigma, \\ \mathcal{M}_i, & \text{if } i \notin \sigma. \end{cases}$$

The ideal $I(K) = \sum_{\sigma \notin K} I_\sigma$ brings us a model of the polyhedral product as follows.

Theorem 1. *Let $\underline{X} = (X_i, A_i)_{1 \leq i \leq n}$ be a finite sequence of pairs of nilpotent CW-complexes of finite type. If we choose a surjective model, $\varphi_i: \mathcal{M}_i \rightarrow \mathcal{M}'_i$, of the canonical inclusion $A_i \hookrightarrow X_i$, then*

- (1) *the quotient $\otimes_{i=1}^n \mathcal{M}_i / I(K)$ is a model of \underline{X}^K .*
- (2) *Moreover, if $J \subset K$ is a subcomplex, then the projection*

$$p_{K,J}: \otimes_{i=1}^n \mathcal{M}_i / I(K) \rightarrow \otimes_{i=1}^n \mathcal{M}_i / I(J)$$

is a model of the inclusion $\underline{X}^J \hookrightarrow \underline{X}^K$.

As a direct consequence, we see that \underline{X}^K is formal if the maps φ_i are formalisable, see [13] or [18] for a definition and some properties of this notion, as the fact that the cofiber of a formalisable map is formal. In particular, we recover a result of Notbohm and Ray, see [16, Theorem 5.5]: If the space X is formal then X^K is also formal. Note that the result of Notbohm and Ray is more general and works over an arbitrary commutative ring.

Theorem 1 is exactly the rational homotopy version of the result of Bahri, Bendersky, Cohen and Gitler concerning the cohomology of \underline{X}^K . For any field \mathbb{F} , the authors prove, in [5, Theorem 1.22], that there is an isomorphism of algebras $H^*(\underline{X}^K; \mathbb{F}) \cong \otimes_{i=1}^n H^*(X_i; \mathbb{F}) / I(SR)$, where $I(SR)$ is the Stanley-Reisner ideal generated by monomials $x_{i_1} \cdots x_{i_k}$ for which the x_i are elements in $H^*(X_i; \mathbb{F})$ and the sequence $I = (i_1, \dots, i_k)$ for $1 \leq i_1 \leq \dots \leq i_k \leq n$ does not correspond to a simplex of K .

We consider now the homotopy fiber $\mathcal{F}(K, BS^1)$ of $(BS^1)^K \rightarrow (BS^1)^n$. In [14, Theorems 1.2 and 1.3], Grbić and Theriault provide us with sufficient conditions on K which give a fiber $\mathcal{F}(K, BS^1)$ with the homotopy type of a wedge of spheres. Recall that a local homomorphism $f: R \rightarrow S$ is said to be *Golod* if $\text{Tor}^R(S, k)$ has trivial Massey products of all orders ≥ 2 , see [2] for more details. On the topological side of the looking glass described in [1], this notion corresponds to the fact that the homotopy fiber of f has the rational homotopy type of a wedge of spheres. Therefore, in terms of rational models, the result of Grbić and Theriault means: with some hypotheses on K , the canonical map $\mathbb{Q}[x_1, \dots, x_n] \rightarrow \mathbb{Q}[x_1, \dots, x_n] / I(SR)$ is Golod, with x_i of degree 2. Jürgen Backelin has proved (see [3] and [4]) that, in this case, the map $\mathbb{Q}[x_1, \dots, x_n] \rightarrow \mathbb{Q}[x_1, \dots, x_n] / I(SR)$ can always be decomposed in a finite sequence of surjective maps that are Golod. Such property is called *Golod attached*. Here, we prove that this result is true in general: for any sequence, $\underline{X} = (X_i, A_i)_{1 \leq i \leq n}$, of nilpotent spaces of finite type, the map $\otimes_{i=1}^n \mathcal{M}_i \rightarrow \otimes_{i=1}^n \mathcal{M}_i / I(K)$ is Golod attached. In fact, this result comes from the next statement written at the level of spaces.

Theorem 2. *If $\underline{X} = (X_i, A_i)_{1 \leq i \leq n}$ is a finite sequence of pairs of CW-complexes, then there is a sequence of fibrations*

$$\begin{array}{ccccccc} F_2 & & F_3 & & \cdots & & F_n \\ \downarrow & & \downarrow & & & & \downarrow \\ \underline{X}^{K_2} & & \underline{X}^{K_3} & & \cdots & & \underline{X}^K \\ \downarrow & & \downarrow & & & & \downarrow \\ \prod_{i=1}^n X_i & & \underline{X}^{K_2} & & \cdots & & \underline{X}^{K_{n-1}} \end{array}$$

where the F_i are suspensions and the K_i are sub-complexes of K .

A precise description of the K_i is given in the proof. From the ‘‘Golod attached’’ property, we deduce a nice decomposition of the rational Lie algebra of homotopy of $\mathcal{F}(K, \underline{X})$ and the fact that, generically, $\pi_*(\Omega(\mathcal{F}(K, X)))$ contains a free Lie algebra on two generators.

Corollary 1. *Suppose that the CW-pairs $\underline{X} = (X_i, A_i)_{1 \leq i \leq n}$ are nilpotent spaces of finite type and let $\mathcal{F}(K, \underline{X})$ be the homotopy fiber of $\underline{X}^K \rightarrow \prod_{i=1}^n X_i$. Then there exists an integer N and a sequence of short exact sequences of graded Lie algebras where the L_i are free graded Lie algebras defined over the rationals,*

$$\begin{array}{ccccccc} 0 & \rightarrow & L_n & \rightarrow & \pi_{\geq N}(\Omega\mathcal{F}(K, \underline{X})) \otimes \mathbb{Q} & \rightarrow & E_{n-1} \rightarrow 0, \\ 0 & \rightarrow & L_{n-1} & \rightarrow & E_{n-1} & \rightarrow & E_{n-2} \rightarrow 0, \\ & & \cdots & & \cdots & & \cdots \\ 0 & \rightarrow & L_3 & \rightarrow & E_3 & \rightarrow & L_2 \rightarrow 0. \end{array}$$

Corollary 2. *We take over the notation and hypotheses of Corollary 1 and suppose also that $X_i = X$ and $A_i = *$ for $i = 1, \dots, n$. If $K \neq \Delta^{n-1}$ and if the algebra of cohomology $H^*(X; \mathbb{Q})$ is not a polynomial algebra $\mathbb{Q}[\alpha]$ on a generator of even degree, then the rational Lie algebra of homotopy, $\pi_*(\Omega(\mathcal{F}(K, X)))$, contains a free Lie algebra on two generators.*

1. A MODEL FOR THE POLYHEDRAL PRODUCT \underline{X}^K

Proof of Theorem 1. We work by induction on the dimension of K and on the number of simplices of maximal dimension. Let σ be a simplex of maximal dimension and suppose that $\sigma = \{1, 2, \dots, r\}$. Then we decompose K in $K = K' \cup \{1, 2, \dots, r\}$, where K' is obtained from K by deleting the simplex σ .

If we have a pair of simplicial sets, $L \subset M$, we denote by $j_{M,L}: I(M) \rightarrow I(L)$ the canonical inclusion and by $p_{M,L}: \otimes_{i=1}^n \mathcal{M}_i/I(M) \rightarrow \otimes_{i=1}^n \mathcal{M}_i/I(L)$ the associated projection.

Observe that the simplices that are not in K are exactly the simplices that are not in K' and not in $\{1, \dots, r\}$. We thus have a short exact sequence of complexes

$$0 \longrightarrow I(K) \xrightarrow{i_1} I(K') \oplus I(\Delta^{r-1}) \xrightarrow{i_2} I(\partial\Delta^{r-1}) \longrightarrow 0$$

with $i_1 = (j_{K,K'}, j_{K, \Delta^{r-1}})$ and $i_2 = j_{K', \partial\Delta^{r-1}} - j_{\Delta^{r-1}, \partial\Delta^{r-1}}$.

We first prove that the projections $p_{K,K'}: \otimes_{i=1}^n \mathcal{M}_i/I(K) \rightarrow \otimes_{i=1}^n \mathcal{M}_i/I(K')$ and $p_{K, \Delta^{r-1}}: \otimes_{i=1}^n \mathcal{M}_i/I(K) \rightarrow \otimes_{i=1}^n \mathcal{M}_i/I(\Delta^{r-1})$ are models of the injections $\underline{X}^{K'} \hookrightarrow \underline{X}^K$ and $\underline{X}^{\Delta^{r-1}} \hookrightarrow \underline{X}^K$. Since

$$I(\Delta^{r-1}) = \sum_{j=0}^{n-r-1} \left(\otimes_{i=1}^{r+j} \mathcal{M}_i \right) \otimes \ker \varphi_j \otimes \left(\otimes_{i=r+j+2}^n \mathcal{M}_i \right),$$

the quotient $\otimes_{i=1}^n \mathcal{M}_i/I(\Delta^{r-1})$ is isomorphic to $(\otimes_{i=1}^r \mathcal{M}_i) \otimes (\otimes_{i=r+1}^n \mathcal{M}'_i)$. The quotient of $\otimes_{i=1}^n \mathcal{M}_i$ by the short exact sequence of ideals gives a new short exact

sequence of complexes

$$0 \rightarrow \otimes_{i=1}^n \mathcal{M}_i / I(K) \xrightarrow{p_1} (\otimes_{i=1}^n \mathcal{M}_i / I(K')) \oplus ((\otimes_{i=1}^r \mathcal{M}_i) \otimes (\otimes_{i=r+1}^n \mathcal{M}'_i)) \\ \downarrow p_2 \\ (\otimes_{i=1}^r \mathcal{M}_i / \otimes_{i=1}^r \ker \varphi_i) \otimes (\otimes_{i=r+1}^n \mathcal{M}'_i) \longrightarrow 0$$

where $p_1 = (p_{K,K'}, p_{K,\Delta^{r-1}})$ and $p_2 = p_{K',\partial\Delta^{r-1}} - p_{\Delta^{r-1},\partial\Delta^{r-1}}$.

From the induction and from [12, Proposition 13.5], we deduce that $\otimes_{i=1}^n \mathcal{M}_i / I(K)$ is a model for \underline{X}^K , $p_{K,K'}$ a model of the inclusion $\underline{X}^{K'} \hookrightarrow \underline{X}^K$ and $p_{K,\Delta^{r-1}}$ a model of the inclusion $\underline{X}^{\Delta^{r-1}} \hookrightarrow \underline{X}^K$.

When $J \subset K$ is a sub-complex then there exists a simplex of maximal dimension, Δ^{r-1} , such that the injection $J \hookrightarrow K$ factorizes via K' or Δ^{r-1} and this gives the second point of the statement by induction. \square

Remark. In the case $K = \partial\Delta^{n-1}$, $X_i = X$ and $A_i = *$ for $i = 1, \dots, n$, the space X^K is the fat wedge of n copies of X . By Theorem 1, if $(\wedge Z, D)$ is a minimal model for X , then a model for X^K is given by $(\wedge Z, D)^{\otimes n} / (\wedge^+ Z)^{\otimes n}$. This is the model for the fat wedge described in [11].

2. THE HOMOTOPY FIBER OF THE INCLUSION $\underline{X}^K \rightarrow \prod_{i=1}^n X_i$

There exists two kinds of results which show the diversity of the possible homotopy types of this fiber.

- In [14] Grbić and Theriault give sufficient conditions on K for which the fiber of $(BS^1)^K \rightarrow (BS^1)^n$ is a wedge of spheres. This cannot be true in general; it suffices to consider the case where K is the join of two simplicial complexes. More explicitly, if $K = S^0 * S^0$, i.e. K is the simplicial sub-complex on $\{1, 2, 3, 4\}$ whose simplices in maximal degrees are $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$ and $\{2, 4\}$, then the fiber $(BS^1)^K \rightarrow (BS^1)^n$ has the homotopy type of $S^3 \times S^3$.
- On the other hand this fiber is not always a formal space as shown by Denham and Suciu in [10].

In this section, we give the proof of Theorem 2 which provides us with a decomposition of the fiber of $\underline{X}^K \rightarrow \prod_{i=1}^n X_i$ as a succession of fibrations and implies that the algebraic model of this map is Golod attached. We describe also a Sullivan model of the different spaces appearing in this decomposition.

Let $L \subseteq \{1, \dots, n\}$ be any simplicial complex. We consider the following sub-simplicial complexes:

- for $\sigma \in L$, $\text{link}(\sigma) = \{\tau \in L \mid \tau \cup \sigma \in L \text{ and } \tau \cap \sigma = \emptyset\}$,
- for $\sigma \in L$, $\text{star}(\sigma) = \{\tau \in L \mid \tau \cup \sigma \in L\}$,
- for any vertex p , $\text{res}(1, \dots, p) = L \cap \{1, \dots, p\}$.

Proof of Theorem 2. We observe first that K can be written as a pushout:

$$\begin{array}{ccc} \text{link}(\{n\}) & \longrightarrow & \text{star}(\{n\}) \\ \downarrow & & \downarrow \\ \text{res}(1, \dots, n-1) & \longrightarrow & K. \end{array}$$

Recall from [7], that $\underline{X}^{K*L} \simeq \underline{X}^K \times \underline{X}^L$. As $\text{star}(\{n\})$ is the join $\text{link}(\{n\}) * \{n\}$, we have $\underline{X}^{\text{star}(\{n\})} = \underline{X}^{\text{link}(\{n\})} \times X_n$, where $\text{link}(\{n\})$ is considered as a simplicial

complex on $\{1, \dots, n-1\}$. Therefore, we have a homotopy pushout

$$\begin{array}{ccc} Y_n \times A_n & \longrightarrow & Y_n \times X_n \\ \downarrow & & \downarrow \\ Z_n \times A_n & \longrightarrow & \underline{X}^K \end{array}$$

where $Y_n = \underline{X}^{\text{link}(\{n\})}$, $Z_n = \underline{X}^{\text{res}(1, \dots, n-1)}$ and $A_n \subset X_n$ comes from the definition of \underline{X} .

First step: We set $K_{n-1} = \text{res}(1, \dots, n-1) * \{n\}$ which gives $\underline{X}^{K_{n-1}} = Z_n \times X_n$. We are looking at the homotopy fiber F_n of $\underline{X}^K \rightarrow \underline{X}^{K_{n-1}}$. For that, we send each corner of the previous square on $Z_n \times X_n$. Denote by F'_n the homotopy fiber of $Y_n \rightarrow Z_n$ and by F''_n the homotopy fiber of $A_n \rightarrow X_n$. From the classical Cube Lemma (see [15]), we know that the homotopy fibers of the four previous maps onto $Z_n \times X_n$ constitute a homotopy pushout. An easy computation of three of them gives the next homotopy pushout:

$$\begin{array}{ccc} F'_n \times F''_n & \longrightarrow & F'_n \\ \downarrow & & \downarrow \\ F''_n & \longrightarrow & F_n \end{array}$$

Therefore, F_n is the join $F'_n * F''_n$. That is a suspension.

Inductive step: Suppose that the simplicial complex K_{p+1} has been constructed such that $\{p+1, \dots, n\} \in K_{p+1}$ and $K \cap \{1, \dots, p\} = K_{p+1} \cap \{1, \dots, p\}$. We have a pushout

$$\begin{array}{ccc} \text{link}(\{p+1, \dots, n\}) & \longrightarrow & \text{star}(\{p+1, \dots, n\}) \\ \downarrow & & \downarrow \\ \text{res}(1, \dots, p) & \longrightarrow & K_{p+1}. \end{array}$$

We set $K_p = \text{res}(1, \dots, p) * \{p+1, \dots, n\}$ and the same argument than above gives that the fiber of $\underline{X}^{K_{p+1}} \rightarrow \underline{X}^{K_p}$ is a suspension. Observe that at the end of the process, $K_1 = \Delta^{n-1}$ and $X^{K_1} = \prod_{i=1}^n X_i$. \square

Proof of Corollary 1. The sequence of fibrations of Theorem 2 gives information on the rational homotopy of \underline{X}^K and also on the rational homotopy of the fiber of the inclusion $\underline{X}^K \rightarrow \prod_{i=1}^n X_i$. Recall that, in a fibration $F \rightarrow E \rightarrow B$, the image of the connecting map $\pi_*(\Omega B) \otimes \mathbb{Q} \rightarrow \pi_{*-1}(\Omega F) \otimes \mathbb{Q}$ belongs to the center, and that the center of a free Lie algebra on two (or more) generators is zero.

In our case, as the fibers F_i are suspensions, we have three possibilities for their rationalisation: they are either contractible, or homotopy equivalent to a sphere, or else homotopy equivalent to a wedge of at least two spheres. In the two first cases, for some N we have an isomorphism $\pi_{\geq N}(\Omega \underline{X}^{K_i}) \otimes \mathbb{Q} \cong \pi_{\geq N}(\Omega \underline{X}^{K_{i+1}}) \otimes \mathbb{Q}$. In the third case we have a short exact sequence

$$0 \rightarrow L_i \rightarrow \pi_*(\Omega \underline{X}_i^{K_i}) \otimes \mathbb{Q} \rightarrow \pi_*(\Omega \underline{X}^{K_{i+1}}) \otimes \mathbb{Q} \rightarrow 0,$$

where $L_i = \pi_*(\Omega F_i) \otimes \mathbb{Q}$ is a free graded Lie algebra. This gives the statement. \square

Proof of Corollary 2. Since $K \neq \Delta^{n-1}$, there exists an integer i such that $K \neq \{i\} * (\{1, \dots, n\} - \{i\})$. Let $i = n$ for sake of simplicity.

Suppose that one of the L_j of the statement of Corollary 1 is a free Lie algebra on at least two generators. We choose j maximal with this property. Thus, for

some N , the Lie subalgebra $(L_j)_{\geq N}$ of the elements of degree greater than or equal to N injects in $\pi_{\geq N}(\Omega(\mathcal{F}(K, X))) \otimes \mathbb{Q}$.

Finally, using the notation of the proof of Theorem 2, we observe that F'_n is not contractible and that the dimension of the reduced homology of F''_n is greater than 2, from the hypothesis on the cohomology of X . This implies that F_n has the rational homotopy type of a wedge of at least two spheres. \square

Remark. We end this section with a description of a model of the fibrations appearing in Theorem 2. For sake of simplicity, we do it in the case $X_i = X$, $A_i = *$ for all i with X a nilpotent space of finite type. The extension to the case of a sequence of spaces is left to the reader.

Let $(\wedge Z, D)$ be the minimal model of X . It follows from Theorem 1 that a model for the inclusion $X^K \hookrightarrow X^n$ is given by the projection $(\wedge Z, D)^{\otimes n} \rightarrow (\wedge Z, D)^{\otimes n}/I(K)$. Consider a relative minimal model, $(\wedge Z, D) \rightarrow (\wedge Z \otimes \wedge \bar{Z}, D)$, for the path space fibration $\Omega X \rightarrow PX \rightarrow X$. Then, a model of the homotopy fiber of the inclusion $X^K \rightarrow X^n$ is given by the tensor product

$$(\wedge Z, D)^{\otimes n}/I(K) \otimes_{(\wedge Z, D)^{\otimes n}} (\wedge Z \otimes \wedge \bar{Z}, D)^{\otimes n}.$$

Observe that fits with [5, Theorem 1.26] where the fiber of $X^K \hookrightarrow X^n$ is determined as the moment-angle complex associated to $(PX, \Omega X)$ and K .

Denote by T the simplices in $\{1, \dots, n\} \setminus K$ and decompose T as $T = T_2 \cup \dots \cup T_n$, where T_k is the subset consisting of simplices on $\{1, \dots, k\}$ and containing k . Denote $I_k = \sum_{j \leq k} \sum_{\sigma \in T_j} I_\sigma$ and $A_k = (\wedge Z, D)^{\otimes k}/I_k$. The spaces A_k and A_{k+1} can be related as follows.

For each $\sigma \in I_{k+1}/I_k$, write $\sigma = \{i_1, \dots, i_s\} \cup \{k+1\}$, with $1 \leq i_1 < \dots < i_s \leq k$, and denote by J_σ the image of $B_1 \otimes \dots \otimes B_k$ in A_k where

$$B_i = \begin{cases} \wedge^+ Z & \text{if } i \in \{i_1, \dots, i_s\}, \\ \wedge Z & \text{if } i \notin \{i_1, \dots, i_s\}. \end{cases}$$

Set $J_k = \sum_{\sigma \in T_{k+1}} J_\sigma$. Then we have clearly $A_{k+1} = A_k \otimes \wedge Z / J_k \otimes \wedge^+ Z$. The projection $p_k : A_k \otimes (\wedge Z)^{n-k} \rightarrow A_{k+1} \otimes (\wedge Z)^{\otimes n-k-1}$ is a Sullivan model of the fibration $X^{K_k} \rightarrow X^{K_{k-1}}$.

Example 3. Let X be a wedge of spheres with cohomology H and K be a simplicial complex on n vertices, of dimension ≤ 2 and containing all the 1-simplices $\{i, j\}$ for $i \neq j$. Then the homotopy fiber of the injection $X^K \hookrightarrow X^n$ has the rational homotopy type of a wedge of spheres.

For the proof, we use the minimal model of X^K . Denote first by $(H \otimes \wedge \bar{Z}, d)$ a relative minimal model of the augmentation $(H, 0) \rightarrow (\mathbb{Q}, 0)$. Then a model of the fiber is given by $A = (H^{\otimes n}/I(K) \otimes (\wedge \bar{Z})^{\otimes n}, D)$. We introduce a new gradation on $V = H \otimes \wedge \bar{Z}$, by $V = V_0 \oplus V_1$, $V_0 = \mathbb{Q} \otimes \wedge \bar{Z}$, $V_1 = H^+ \otimes \wedge \bar{Z}$. Note that the sequence $0 \longrightarrow \mathbb{Q} \longrightarrow V_0 \xrightarrow{d} V_1 \longrightarrow 0$ is exact. This gradation induces a gradation on A for which $A_4 = 0$ and for which the sequence $0 \longrightarrow \mathbb{Q} \longrightarrow A_0 \xrightarrow{D} A_1 \xrightarrow{D} A_2$ is exact. Therefore we can choose representative for the cohomology in $A_2 \oplus A_3$. Since the product of those cocycles is zero, we get a quasi-isomorphism of algebras $H(A, d) \rightarrow (A, d)$. This implies that the homotopy fiber is a wedge of spheres.

Finally, observe that, as the 2-simplices of K are arbitrary, this complex is not necessarily shifted, see [14, Definition 1.1].

3. THE LUSTERNIK-SCHNIRELMANN CATEGORY OF X^K

The (Lusternik-Schnirelmann) category of a space X , $\text{cat } X$, is the least integer m (or ∞) such that X can be covered by $m+1$ open sets each contractible in X . The category of a power X^n satisfies $\text{cat } X^n \leq n \text{ cat } X$. The equality happens to

be true in many situations, in particular when X is a simply connected rational space [12, Theorem 30.2], see also [6] or [8] for other examples.

Proposition 4. *If X is a simply connected finite type CW-complex such that, for any n , $\text{cat } X^n = n \cdot \text{cat } X$, then the category of the polyhedral power, X^K , is equal to $(\text{cat } X) \cdot (1 + \dim K)$.*

Proof. Suppose σ is a simplex of K , then $X^\sigma \cong X^{|\sigma|} \times \{*\}^{n-|\sigma|}$ and we have a sequence of injections $X^\sigma \hookrightarrow X^K \hookrightarrow X^n$. By using the canonical projection of X^n on $X^{|\sigma|} \times \{*\}^{n-|\sigma|}$, we see that X^σ is a homotopy retract of X^K . Thus we have: $\text{cat } X^K \geq \text{cat } X^\sigma = (\text{cat } X) \cdot (1 + \dim K)$.

On the other hand, if $\text{cat } X = m$, then X is a retract of the m^{th} Ganea space, $G_m(X)$, which is an iterated m -cone. Therefore X^K is a retract of $(G_m(X))^K$ which is a p -cone with $p = n \cdot (1 + \dim K)$. This gives the converse inequality. \square

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