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NULLIFICATION AND CELLULARIZATION OF CLASSIFYING SPACES OF FINITE GROUPS

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ABSTRACT. In this note we discuss the effect of the $\mathrm{B}\mathbb{Z}/p$-nullification $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}$ and the $\mathrm{B}\mathbb{Z}/p$-cellularization $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}$ over classifying spaces of finite groups, and we relate them with the corresponding functors with regard to Moore spaces, that have been intensively studied in the last years. We describe $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ by means of a Postnikov fibration, and we classify all finite groups G for which $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular. We also study carefully the analogous functors in the category of groups, and its relationship with the fundamental groups of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ and $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.

1. INTRODUCTION

Let A be a pointed connected space. P. Bousfield defined in [Bou94] the A-nullification functor $\mathbf{P}_A$ as the localization L_f with regard to the constant map $f : A \longrightarrow *$. Roughly speaking, if X is another pointed space, $\mathbf{P}_A(X)$ is the biggest "quotient" of X that has no essential map from any suspension of A. A little bit later, Dror-Farjoun defined ([DF95]) the somewhat dual notion of A-cellularization of a space X as the largest space $\mathbf{CW}_A X$ endowed of a map $\mathbf{CW}_A X \xrightarrow{cw} X$ such that every map from a n-suspension of A to X lifts to $\mathbf{CW}_A X$ via cw. The A-cellularization of X can also be viewed as the closest approximation of X that can be built from A taking pointed homotopy colimits. The very close relationship that exists between these two functors was clarified by the work of Chachólski ([Cha96]) where he explains how to describe anyone of these functors in terms of the other, and he introduces the crucial concept of closed class. Moreover, these functors have been widely studied and used since they appeared; see for example [Dwy94], [Bou97], and [CDI02].

Dror-Farjoun also defines the A-homotopy theory of a space, where A and its suspensions play the role that S^0 and its suspensions play in classical homotopy theory. For example, he defines the A-homotopy groups $\pi_i(X; A)$ as the homotopy classes of pointed maps $[\Sigma^i A, X]_*$. In this framework, the functors $\mathbf{P}_{\Sigma^i A}(X)$ and $\mathbf{CW}_{\Sigma^i A}(X)$ can be viewed, respectively, as the i-th Postnikov section and the i-connective cover of X. Moreover the spaces for which $X \simeq \mathbf{P}_A(X)$ (A-null spaces) play the role of weakly contractible spaces, and the spaces such that $A \simeq \mathbf{CW}_A X$ (A-cellular spaces) are the analogues of the CW-complexes. Indeed, the A-cellularization is nothing but the A-cellular approximation of X.

Our main interest is $\mathrm{B}\mathbb{Z}/p$-homotopy theory. After the positive resolution by Miller ([Mil84]) of the Sullivan conjecture, there has been a lot of research which has allowed deep knowledge of the space $\mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, X)$ when the target is nilpotent, and hence it has given clues about the value of the $\mathrm{B}\mathbb{Z}/p$-nullification functor over X; a good survey on this topic can be found in the book [Sch94]. For example, we have the result of Lannes-Schwartz ([LS89]) which points out that if moreover the $\mathbb{Z}/p$-cohomology of X has finite type and $\pi_1(X)$ is finite, then X is $\mathrm{B}\mathbb{Z}/p$-null

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if and only if X is locally finite with regard to the action of the Steenrod algebra. Unfortunately, not much is known if we do not impose the nilpotence hypothesis.

We have focused our study in the $\mathrm{B}\mathbb{Z}/p$-nullification and $\mathrm{B}\mathbb{Z}/p$-cellularization of classifying spaces of finite groups. Even if the group G is not nilpotent (and hence $\mathrm{B}G$ is not nilpotent as a space, and we cannot apply the mentioned results), this seems to be an accesible case, because we have a precise description of the space $\mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathrm{B}G)$ as the space of group homomorphisms $\mathrm{Hom}(\mathbb{Z}/p, G)$, which is homotopically discrete. In particular, this gives a feeling that the value of these functors over $\mathrm{B}G$ will depend strongly on the group theory of G.

One of the classical invariants that measure the p-primary part of the homotopy structure of a space X are the homotopy groups with coefficients in $\mathbb{Z}/p$. Recall that the $(n+1)$-th homotopy group of X with coefficients in $\mathbb{Z}/p$ is defined to be the group of homotopy classes of pointed maps $[M(\mathbb{Z}/p, n), X]_*$, where $M(\mathbb{Z}/p, n)$ is the corresponding Moore space and $n \geq 1$. Thus, understand these homotopy groups amounts to describe the $M(\mathbb{Z}/p, 1)$-homotopy theory of X. This problem has been attacked succesfully by Bousfield ([Bou97]) and Rodríguez-Scherer ([RS98]), who describe precisely the $M(\mathbb{Z}/p, 1)$-nullification and $M(\mathbb{Z}/p, 1)$-cellularization of $\mathrm{B}G$. Then it seems natural to ask if there exists any similar description of the $\mathrm{B}\mathbb{Z}/p$-homotopy theory of $\mathrm{B}G$ and if so, what relationship does it have with the mentioned p-primary part of the classical homotopy structure of $\mathrm{B}G$.

Our first result gives a partial answer to this question by characterizing the $\mathrm{B}\mathbb{Z}/p$-nullification of the classifying space of a finite group G by means of a Postnikov fibration. Recall that Bousfield defines the $\mathbb{Z}/p$-radical $\mathrm{T}_{\mathbb{Z}/p}G$ of G (sometimes denoted by O_pG in a framework of group theory) as the smallest normal subgroup of G that contains all the p-torsion.

Theorem 1.1. *Let G be a finite group, p a prime number; then, the $\mathrm{B}\mathbb{Z}/p$-nullification $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ of $\mathrm{B}G$ fits in the fibration sequence:*

$$\prod_{q \neq p} \mathrm{B}(\mathrm{T}_{\mathbb{Z}/p}G)^\wedge_q \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_{\mathbb{Z}/p}G).$$

where $\mathrm{B}(\mathrm{T}_{\mathbb{Z}/p}G)^\wedge_q$ denotes the Bousfield-Kan $\mathbb{Z}/q$-completion of $\mathrm{B}(\mathrm{T}_{\mathbb{Z}/p}G)$ (see [BK72] *for the definition and main properties of this functor).*

It is easy to see from this result that the only groups G for which $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-acyclic are the p-groups, and, according to [Lev95], $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ has nonzero homotopy groups in highly arbitrary dimensions if and only if the $\mathbb{Z}/p$-radical of G is a p-group. Moreover, if G is simple, the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$ is simply-connected, and it is not hard to prove that if G is nilpotent, $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ is an Eilenberg-MacLane space that is nilpotent too.

If $\mathrm{M}(\mathbb{Z}/p, 1)$ is a Moore space of dimension 2, the results ([Bou97], section 7) and the previous theorem guarantee that the $\mathrm{M}(\mathbb{Z}/p, 1)$-nullification of $\mathrm{B}G$ is homotopy equivalent, via the map induced by the inclusion $\mathrm{M}(\mathbb{Z}/p, 1) \hookrightarrow \mathrm{B}\mathbb{Z}/p$, to the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$. Hence, these functors coincide over classifying spaces of finite groups, and this proves that the aforementioned relationship between the p-primary part of $\mathrm{B}G$ and its $\mathrm{B}\mathbb{Z}/p$-homotopy theory is really close. On the other hand, it is easy to see that these two functors do not coincide over every X: take for example $p = 2$ and $X = \mathbb{R}\mathbf{P}^2$, which is a model for $\mathrm{M}(\mathbb{Z}/2, 1)$ and is $\mathrm{B}\mathbb{Z}/2$-null by Miller's theorem ([Mil84]).

An important consequence of the previous observation is that a great part of the results of ([RS98]) relative to Moore spaces remain valid for the case of $\mathrm{B}\mathbb{Z}/p$-nullification, and in particular they allow us to obtain a very precise description of the value of the acyclic functor $\bar{\mathbf{P}}_{\mathrm{B}\mathbb{Z}/p}$ over $\mathrm{B}G$. We recall that the acyclic functor is the colocalization associated to the nullification.

We finish the study of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ using our description of it for proving (3.7) that the nullification functors $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}$ and $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}$ commute over $\mathrm{B}G$ for different primes p and q (see [RS00] for an overview of the problem of commutation in localization theory), and establishing some relations (3.10 and 3.12) between the $\mathrm{B}\mathbb{Z}/p$-nullification functor and the Bousfield-Kan $\mathbb{Z}/p$-completion and $\mathbb{Z}[1/p]$-completion.

The second part of this note is devoted to the analysis of the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}G$. Our main result on this topic has been the characterization of the class of finite groups G such that $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular.

Theorem 1.2. *Let G be a finite $\mathbb{Z}/p$-cellular group. Then $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular if and only if G is a p-group generated by order p elements.*

The proof relies essentially on a result of Chachólski (2.4) about preservation of cellularity under fibrations.

In general the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}G$ is related very closely with the $\mathbb{Z}/p$-cellularization $\mathrm{CW}_{\mathbb{Z}/p}G$ of G in the category of groups (see section 4 for the precise definition and [RS98] for the original reference) by means of the following lemma:

Lemma 1.3. *If G is a finite group, the natural map $\mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow G$ induces a homotopy equivalence $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}\mathrm{CW}_{\mathbb{Z}/p}G \simeq \mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.*

From this result, it is desirable to have an algorithm for computing the $\mathbb{Z}/p$-cellularization $\mathrm{CW}_{\mathbb{Z}/p}G$ of G, which allows to restrict ourselves to the calculation of $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ to the case of G $\mathbb{Z}/p$-cellular. According to ([RS98], 2.7) the group $\mathrm{CW}_{\mathbb{Z}/p}G$ fits in a central extension

$$0 \longrightarrow A \longrightarrow \mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow \mathrm{S}_{\mathbb{Z}/p}G \longrightarrow 0$$

where $\mathrm{S}_{\mathbb{Z}/p}G$ is the the $\mathbb{Z}/p$-*socle* of G, v.g. the (normal) subgroup of G generated by the order p elements, and A is the second homotopy group of the $\mathrm{M}(\mathbb{Z}/p,2)$-cellularization of the cofibre C_f of the evaluation map $\bigvee_{[\mathrm{B}\mathbb{Z}/p,\mathrm{B}G]_*}\mathrm{B}\mathbb{Z}/p \longrightarrow \mathrm{B}G$. In fact, it is easily proved that $A = \pi_2(C_f)/\mathrm{T}_{\mathbb{Z}/p}(\pi_2(C_f))$, so our goal has been to calculate $\pi_2(C_f)$ and we have obtained the following:

Proposition 1.4. *Let G a finite group generated by order p elements, K the kernel of the evaluation map $*\mathbb{Z}/p \longrightarrow G$, where the free product is extended to all the homomorphisms $\mathbb{Z}/p \longrightarrow G$; then $\pi_2(C_f) = K/[K, *\mathbb{Z}/p]$.*

This group is computable, because it is finitely generated and hence we can use the Reidemeister-Schreier method to obtain a presentation of it. It is worth to point out that this result is applicable to every finite group G, because of the fact that $\mathrm{CW}_{\mathbb{Z}/p}G = \mathrm{CW}_{\mathbb{Z}/p}\mathrm{S}_{\mathbb{Z}/p}G$ and the $\mathbb{Z}/p$-socle is always generated by order p elements.

Once we have computed A, the universal properties characterizing the previous extension ([RS98], 2.7) allows a complete description of it in most cases, and sometimes give a precise hold of the extension class. On the other hand, we prove that the $\mathbb{Z}/p$-cellularization of G can give a lot of information about some group-theoretic invariants of the group G, such that its Schur multiplier, or its universal central extension with coefficients in $\mathbb{Z}$ or $\mathbb{Z}[1/p]$ (if G is respectively perfect or $\mathbb{Z}[1/p]$-perfect). In fact, the latter turns out to be the fundamental group of $\bar{\mathbf{P}}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$, and this is used to prove that the $\mathrm{B}\mathbb{Z}/p$ acyclic functor "commutes" with fundamental group, in a similar way that $\mathrm{B}\mathbb{Z}/p$-nullification does.

Our study of the cellularization ends up with a description of the fundamental group of the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}G$ as a central extension of G by a p-group.

In last section we compute the value of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}$ and $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}$ over the classifying spaces of various concrete families of finite groups. In particular, we analyze the effect of these functors over classifying spaces of dihedral, semidihedral, symmetric or quaternionic groups.

Notation. In all this note the word "space" will stand for "CW-complex", and usually we will suppose that these spaces are pointed. The notation $X_p^\wedge$ will denote Bousfield-Kan $\mathbb{Z}/p$-completion of X, whereas $\mathbb{Z}[1/p]_\infty X$ will stand for the $\mathbb{Z}[1/p]$-completion.

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2. Preliminaries

In this paragraph we recall briefly the definition and properties of the nullification functor $\mathbf{P}_A$ and the cellularization functor $\mathbf{CW}_A$. We only comment here the results that are somewhat relevant for our work; the main references on this topic are [Bou94], [DF95] and [Cha96].

In all the section, A will stand for a connected space, and all the spaces will be pointed, unless express mention against.

A space X is called *A-null* if the mapping space $\mathrm{map}_*(A, X)$ is weakly contractible, which roughly speaking means that every pointed map of a n-suspension of A into X is inessential. The *A-nullification* $\mathbf{P}_A X$ (sometimes called *A-periodization*) is the only A-null space, up to homotopy equivalence, endowed with a map $X \longrightarrow \mathbf{P}_A X$ which induces, for every A-null space Y, a weak homotopy equivalence

$$\mathrm{map}_*(\mathbf{P}_A X, Y) \simeq \mathrm{map}_*(X, Y).$$

In this way it is defined a functor $\mathbf{P}_A : \mathbf{Spaces}_* \longrightarrow \mathbf{Spaces}_*$ which is coaugmented and idempotent. We remark that there is no problem for defining the nullification functor in the unpointed category, with no reference to the base point.

There are some constructions of $\mathbf{P}_A$, and the easiest is probably the following: take a space X_λ for every ordinal λ; define $X_0 = X$ and by induction let $X_{\lambda+1}$ be the space obtained from X_λ gluing cones over all posible homotopy classes of maps from n-suspensions of A into X_λ. Then, the A-nullification of X is defined as the colimit of all these spaces. Other constructions can be found in Bousfield ([Bou94],2.8) and Chachólski ([Cha96],17.1).

The A-nullification can also be defined as the localization in the sense of Dror-Farjoun with regard to the trivial map $A \longrightarrow *$, and using that the following properties are easy to prove ([DF95], 1.A.8):

- The natural map $\mathbf{P}_A(X \times Y) \longrightarrow \mathbf{P}_A X \times \mathbf{P}_A Y$ is a homotopy equivalence.
- If X is A-null then it is indeed ΣA-null for every n, and $\Omega^n X$ is A-null.
- If $\mathbf{P}_A X \simeq *$, then $\mathbf{P}_{\Sigma A} \Sigma X \simeq *$.
- If we have a fibration where the base and the fibre are A-null, then the total space is A-null.
- If X is n-connected, $\mathbf{P}_A X$ is n-connected too.

One of the most importants achievements of the work of Dror-Farjoun is the description of the behaviour of the localization functors with regard to fibrations. In particular, he defines the fibrewise nullification, and proves the following results, that will be crucial and intensively used in our work:

Theorem 2.1 ([DF95],1.H.1 and 3.D.3). *Let $F \longrightarrow E \longrightarrow B$ be a fibration. Then*

(1) *If* $\mathbf{P}_A(F)$ *is contractible, then the induced map* $\mathbf{P}_A(E) \longrightarrow \mathbf{P}_A(B)$ *is a homotopy equivalence.*
(2) *If* B *is* A*-null then the fibration is preserved under* A*-nullification.*

The most simple example of nullification functor is the Postnikov n-section, which is exactly the S^{n+1}-nullification. Other widely studied examples have been Quillen plus-construction $X \longrightarrow X^+$, which is the nullification with regard to a large space that is acyclic for a certain homology theory (see [BC99], for example), the nullification with regard to Moore spaces, or the $B\mathbb{Z}/p$-nullification of classifying spaces of compact Lie groups such that the group of components is a p-group. As we will see, our work is closely related with these two examples.

To conclude with nullification, we will comment briefly the concept of A-periodic equivalence, due to Bousfield:

Definition 2.2. A map $f : X \longrightarrow Y$ is called an *A-periodic equivalence* if for any A-null space Z and any choice basepoints in X, Y, Z, the map f induces a weak homotopy equivalence $\mathrm{map}_*(Y,Z) \simeq \mathrm{map}_*(X,Z)$. In particular, the coaugmentation $X \longrightarrow \mathbf{P}_A X$ is always an A-periodic equivalence, because the functor $\mathbf{P}_A$ is idempotent.

In ([Cha96], section 13), one can find the main properties of A-periodic equivalences.

In the same way as the A-nullification isolates "the part" of a space that is not visible by means of maps $\Sigma^n A \longrightarrow X$, the A-cellularization describe to what extent a space can be built using A as building blocks. We will begin our overhaul of cellularization defining the concept of cellular space.

Definition 2.3. A space X is called *A-cellular* if for any choice of basepoint in X and for every pointed map $f : Y \longrightarrow Z$ such that $f_* : \mathrm{map}_*(A,Y) \longrightarrow \mathrm{map}_*(A,Z)$ is a weak equivalence, we have that the induced map $f_* : \mathrm{map}_*(X,Y) \longrightarrow \mathrm{map}_*(X,Z)$ is also a weak equivalence. It can be proved that this is equivalent to say that X can be built as an (iterated) pointed homotopy colimit of copies of A.

Hence, the *A-cellularization* of X is defined as the unique A-cellular space $\mathbf{CW}_A X$ (up to homotopy) such that it exists a canonical augmentation $cw : \mathbf{CW}_A X \longrightarrow X$ which induces a weak homotopy equivalence $\mathrm{map}_*(A, \mathbf{CW}_A X) \simeq \mathrm{map}_*(A,X)$. In particular, the augmentation cw has the following two features ([DF95], 2.E.8):

(1) If $f : Y \longrightarrow X$ induces a weak homotopy equivalence $\mathrm{map}_*(A,Y) \simeq \mathrm{map}_*(A,X)$, there is a map $f' : \mathbf{CW}_A X \longrightarrow Y$ such that $f \circ f'$ is homotopic to cw. Moreover, f' is unique up to homotopy.
(2) If Z is A-cellular and $g : Z \longrightarrow X$ is a map, then it exists $g' : Z \longrightarrow \mathbf{CW}_A X$ such that $cw \circ g'$ is homotopic to g and g' is unique up to homotopy.

Dror Farjoun also gave the first two constructions of the functor $\mathbf{CW}_A$, the standard ([DF95] 2.E.3), and another one that is more intuitive but it has the disadvantage of being not functorial ([DF95] 2.E.5). However, we will recall here the construction of Chachólski ([Cha96], section 7) because it will be more useful for our purposes.

Proposition 2.4. *If* A *is connected, the* A*-cellularization of* X *has the homotopy type of the homotopy fibre of the map* $\eta : X \longrightarrow LX$*, where* η *is the composition of the inclusion* $X \hookrightarrow Cf$ *into the homotopy cofibre of the evaluation map* $\bigvee_{[A,X]_*} A \longrightarrow X$*, with the nullification* $Cf \longrightarrow \mathbf{P}_{\Sigma A} Cf$*.*

This can be interpreted as a definition of the functor $\mathbf{CW}_A$ in terms of $\mathbf{P}_A$, and in fact it is also possible to describe $\mathbf{P}_A$ in terms of the A-cellularization. It is worth to point out that any construction of $\mathbf{CW}_A$ must be worked out in the *pointed* category, because it is not possible to define $\mathbf{CW}_A$ over unpointed spaces ([Cha96], 7.4).

Chachólski also makes the key observation that A-cellular spaces constitute a *closed class*, v.g. a class of spaces that is closed under weak equivalences and pointed homotopy colimits. In particular, the class of A-cellular spaces is the smallest closed class that contains A. Now we list some important properties of A-cellular spaces, which are nothing but the translations of the corresponding properties of closed classes; the last one will be particularly important in this note, because it allows to build new cellular spaces from old ones. The proofs can be found in ([Cha96], section 4):

Proposition 2.5. *Let A be a space. Then*

- *If X is weakly contractible, it is A-cellular.*
- *If B is A-cellular and X is B-cellular, then X is A-cellular.*
- *If X is A-cellular, then the n-suspension $\Sigma^n A$ is A-cellular.*
- *If $F \longrightarrow E \longrightarrow B$ is a fibration with a section and F and B are A-cellular, then E is A-cellular. In particular, the product of two A-cellular spaces is A-cellular, and for every pair of spaces X, Y; we have a weak equivalence $\mathbf{CW}_A(X \times Y) \simeq \mathbf{CW}_A X \times \mathbf{CW}_A Y$.*

In ([DF95], 3.5) one can find a lot of examples of interesting A-cellular spaces. For example, the James construction $\Omega\Sigma A$ is A-cellular, the Dold-Thom functor $SP^\infty A$ is A-cellular, the classifying space of a group $\mathrm{B}G$ is ΣG-cellular for every group G, etc. In particular, he proves that for every $n \geq 1$ the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}\mathbb{Z}/p^n$ is $\mathrm{B}\mathbb{Z}/p$, a fact that can be considered a starting point for our work.

It is also worth to recall the concept of *A-cellular equivalence*:

Definition 2.6. A map $X \longrightarrow Y$ is called an A-cellular equivalence if it induces a weak equivalence $\mathrm{map}_*(A, X) \simeq \mathrm{map}_*(A, Y)$.

The main properties of the A-cellular equivalences can be found in ([Cha96], section 6).

We finish this sample, which certainly doesn't pretend to be exhaustive, by commenting on the relationship between the functors $\mathbf{P}_A$ and $\mathbf{CW}_A$. For this is necessary to define the acyclics, which at any rate have their own interest.

Definition 2.7. An space X is called *A-acyclic* if $\mathbf{P}_A X$ is contractible. The functor $\bar{\mathbf{P}}_A : \mathbf{Spaces}_* \longrightarrow \mathbf{Spaces}_*$ which sends every space to the homotopy fiber of its A-nullification is augmented and idempotent (in fact, it is a colocalization) and its image is the class of A-acyclic spaces.

The class of A-acyclic spaces was the crucial ingredient that Chachólski used for stating the main result of [Cha96], which strongly generalized a theorem of Dror-Farjoun, and gave an amazingly sharp description of to what extent the functors $\mathbf{P}_A$ and $\mathbf{CW}_A$ can be considered "dual":

Theorem 2.8. *Let A be a space. Then*

1. *A space X is A-null if and only if its A-cellularization is a point.*
2. *A space X is A-acyclic if and only if it belongs to the smallest closed class that contains A and is closed by extension by fibrations. In particular, every A-cellular space is A-acyclic.*

We recall that a closed class $\mathcal{C}$ is closed by extensions by fibrations if for every fibration $F \longrightarrow E \longrightarrow B$ such that $F \in \mathcal{C}$ and $B \in \mathcal{C}$, we have $E \in \mathcal{C}$.

This result and ([DF95] 9.A.6), which leave the taste that the A-cellularization is a kind of mixing process between the functors $\bar{\mathbf{P}}_A$ and $\bar{\mathbf{P}}_{\Sigma A}$, have made much more accesible the computation of the value of the A-cellularization of a space, and have greatly stimulated the research on this field. Among the most recent works on it, we can quote [CDI02], where the authors generalize the notion of dimension of a CW-complex to the A-cellular framework, or [DGI02], where they generalize the notion of cellularization to algebraic categories of R-modules.

3. $\mathrm{B}\mathbb{Z}/p$-NULLIFICATION OF CLASSIFYING SPACES OF FINITE GROUPS

3.1. **Computing the nullification of $\mathbf{B}G$.** Let G be a finite group, p a prime number. As we have stated in the introduction, our interest has been focused on studying the p-primary part of the classifying space of G using the functors $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}$ and $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}$; in this section we will be concerned with the former one. So, our first result on this topic is a characterization of the space $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ by means of a Postnikov fibration. As far as we know, the unique description already done of this space is for the case of G nilpotent, and in this case the proof is really easy: if H is the p-torsion subgroup of G, it is enough to $\mathrm{B}\mathbb{Z}/p$-nullify the fibration

$$\mathrm{B}H \longrightarrow \mathrm{B}G \longrightarrow \mathrm{B}(G/H)$$

for obtaining that $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \mathrm{B}(G/H)$. The difficulty of the general case take root in the fact that in general the minimal subgroup that contains the p-torsion have elements that are *not* of p-torsion. We get round the trouble identifying first the case in which the $\mathrm{B}\mathbb{Z}/p$-nullification is simply-connected, and then passing to the general case.

Proposition 3.1. *Let G be a finite group, p prime. Suppose that G has no non-trivial quotients of order prime to p. Then we have $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \prod_{q\neq p}\mathrm{B}G^\wedge_q$ (which is in fact homotopy equivalent to $\mathbb{Z}[1/p]_\infty\mathrm{B}G$).*

Proof. First of all, we have to prove that $\prod_{q\neq p}\mathrm{B}G^\wedge_q$ is a $\mathrm{B}\mathbb{Z}/p$-null space. But this is clear because

$$\mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathrm{B}G^\wedge_q) \simeq \mathrm{map}_*((\mathrm{B}\mathbb{Z}/p)^\wedge_q, \mathrm{B}G^\wedge_q) \simeq *$$

where the first equivalence holds because $\mathrm{B}G^\wedge_q$ is $\mathbb{Z}/q$-complete.

So, now we must see that if X is another $\mathrm{B}\mathbb{Z}/p$-null space, it exists a homotopy equivalence

$$\mathrm{map}_*(\mathrm{B}G, X) \simeq \mathrm{map}_*(\prod_{q\notin S}\mathrm{B}G^\wedge_q, X).$$

We will consider two cases.

First, we will suppose that X is a simply-connected space. In this case, Sullivan's arithmetic square gives us a homotopy equivalence

$$\mathrm{map}_*(\mathrm{B}G, X) \simeq \mathrm{map}_*(\mathrm{B}G, \prod_{q \text{ prime}} X^\wedge_q).$$

We shall check that $\mathrm{map}_*(\mathrm{B}G, X^\wedge_p)$ is contractible. The space $X^\wedge_p$ is $\mathbb{Z}/p$-complete, so we have an equivalence $\mathrm{map}_*(\mathrm{B}G, X^\wedge_p) \simeq \mathrm{map}_*(\mathrm{B}G^\wedge_p, X^\wedge_p)$. Using Jackowski-McClure-Oliver subgroup decomposition ([JMO92], see also [Dwy97]), we obtain that

$$\mathrm{map}_*(\mathrm{B}G^\wedge_p, X^\wedge_p) \simeq \mathrm{map}_*((\mathrm{hocolim}_{\mathbf{O}_\mathcal{C}}\beta_\mathcal{C})^\wedge_p, X^\wedge_p)$$

where $\beta_\mathcal{C}$ is a functor whose values have the homotopy type of classifying spaces of p-subgroups of G, and $\mathbf{O}_\mathcal{C}$ is a $\mathbb{Z}/p$-acyclic category. Again, because of $X^\wedge_p$ is $\mathbb{Z}/p$-complete, we have

$$\mathrm{map}_*((\mathrm{hocolim}_{\mathbf{O}_\mathcal{C}}\beta_\mathcal{C})^\wedge_p, X^\wedge_p) \simeq \mathrm{map}_*(\mathrm{hocolim}_{\mathbf{O}_\mathcal{C}}\beta_\mathcal{C}, X^\wedge_p).$$

By the classical result of Bousfield-Kan ([BK72], XII, 4.1), we have

$$\mathrm{map}_*(\mathrm{hocolim}_{\mathbf{O}_{\mathcal{C}}}\beta_{\mathcal{C}}, X_p^\wedge) \simeq \mathrm{holim}_{\mathbf{O}_{\mathcal{C}}^{op}}\mathrm{map}_*(\beta_{\mathcal{C}}(-), X_p^\wedge).$$

But X is $\mathrm{B}\mathbb{Z}/p$-null, and by a theorem of Miller ([Mil84] 9.9), its $\mathbb{Z}/p$-completion is; so, as the functor $\beta_{\mathcal{C}}$ take its image over p-groups, the space $\mathrm{map}_*(\beta_{\mathcal{C}}(-), X_p^\wedge)$ is contractible for every value of the mentioned functor. This means that

$$\mathrm{map}_*(\mathrm{B}G, X_p^\wedge) \simeq \mathrm{holim}_{\mathbf{O}_{\mathcal{C}}{}^{op}} * \simeq *,$$

as we wanted. So, we have now the following string of weak equivalences:

$$\mathrm{map}_*(\mathrm{B}G, X) \simeq \mathrm{map}_*(\mathrm{B}G, \prod_{q\neq p} X_q^\wedge) \simeq \prod_{q\neq p} \mathrm{map}_*(\mathrm{B}G, X_q^\wedge) \overset{(*)}{\simeq}$$

$$\prod_{q\neq p} \mathrm{map}_*(\mathrm{B}G_q^\wedge, X_q^\wedge) \overset{(**)}{\simeq} \mathrm{map}_*(\prod_{q\neq p} \mathrm{B}G_q^\wedge, X)$$

where the equivalence (*) holds because of X is simply-connected (and therefore $X_q^\wedge$ is $\mathbb{Z}/q$-complete for every q) and (**) holds because of the space $\mathrm{map}_*(\mathrm{B}G_q^\wedge, X_r^\wedge)$ is contractible if q and r are different primes.

Now we can attack the general case. Let X be a $\mathrm{B}\mathbb{Z}/p$-null space. If $\tilde{X}$ is the universal cover of X, we have the Postnikov fibration

$$\tilde{X} \longrightarrow X \xrightarrow{h} \mathrm{B}\pi_1(X).$$

Our first goal will be to see that the map

$$\mathrm{map}_*(\mathrm{B}G, X) \xrightarrow{h_\natural} \mathrm{map}_*(\mathrm{B}G, \mathrm{B}\pi_1(X))$$

takes every map of the space $\mathrm{map}_*(\mathrm{B}G, X)$ to the homotopy class of the constant map. So, let $f\colon \mathrm{B}G \longrightarrow X$ be such a map, and consider the following commutative diagram

$$\begin{array}{ccc} & & X \\ & \overset{f}{\nearrow} & \big\downarrow h \\ \mathrm{B}G & \xrightarrow[h_\natural f]{} & \mathrm{B}\pi_1(X) \end{array}$$

We must prove that $h_\natural f \simeq *$, so let $g : \mathrm{B}\mathbb{Z}/p \longrightarrow \mathrm{B}G$ be a continuous map. As X is a $\mathrm{B}\mathbb{Z}/p$-null space, we know the composition $f \circ g \simeq *$, and in particular $h_\natural f \circ g \simeq *$. On the other hand, there exist two group homomorphisms, $\mu : \mathbb{Z}/p \longrightarrow G$ and $\rho : G \longrightarrow \pi_1(X)$ such that $\mathrm{B}\mu \simeq g$ and $\mathrm{B}\rho \simeq h_\natural f$. Thus, it is clear that the composition

$$\mathbb{Z}/p \xrightarrow{\mu} G \xrightarrow{\rho} \pi_1(X)$$

is the zero homomorphism, and this happen for *every* homomorphism $\mathbb{Z}/p \longrightarrow G$; so, we obtain Im ρ should be a quotient of G whose order is coprime to p. By our hypothesis, G does not have such nontrivial quotients, so ρ is zero, and its induced map at the level of classifying spaces is homotopic to the constant, as we wanted to know. Consider, then, the following diagram, where the left column is a fibration, the horizontal maps are all induced by the product of the $\mathbb{Z}/q$-completions of $\mathrm{B}G$, and $\tilde{X}$ is the universal cover of X:

$$\begin{array}{ccc}
\mathrm{map}_*(\prod_{q\neq p} \mathrm{B}G^\wedge_q, \tilde{X}) & \xrightarrow{\simeq} & \mathrm{map}_*(\mathrm{B}G, \tilde{X}) \\
\downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (3)} \\
\mathrm{map}_*(\prod_{q\neq p} \mathrm{B}G^\wedge_q, X) & \xrightarrow{(1)} & \mathrm{map}_*(\mathrm{B}G, X) \\
\downarrow{\scriptstyle h_\natural} & & \downarrow{\scriptstyle h_\natural} \\
\mathrm{map}_*(\prod_{q\neq p} \mathrm{B}G^\wedge_q, \mathrm{B}\pi_1(X)) & \longrightarrow & \mathrm{map}_*(\mathrm{B}G, \mathrm{B}\pi_1(X))_c
\end{array}$$

The top-horizontal map is an equivalence by the first case done before and ([ABN94] 9.7), and the down-right map has been seen to take values in the component of the constant map, so it is a fibration. It is known that this component is contractible, and the same is true for $\mathrm{map}_*(\prod_{q\neq p} \mathrm{B}G^\wedge_q, \mathrm{B}\pi_1(X))$, because $\prod_{q\neq p} \mathrm{B}G^\wedge_q$ is 1-connected and $\mathrm{B}\pi_1(X)$ is an Eilenberg-MacLane space. Hence, the maps (2) and (3) are weak equivalences; by the commutativity of the diagram, this means that (1) is a weak equivalence, and we have finished.

□

For the general case of the theorem we will need to identify in some way the p-torsion of G, and this will lead us to the concept of $\mathbb{Z}/p$-radical.

Definition 3.2. Let G be a finite group, p a prime. The *$\mathbb{Z}/p$-radical* of G (sometimes called the $\mathbb{Z}/p$-isolator) is the minimal normal subgroup $\mathrm{T}_{\mathbb{Z}/p}G$ that contains all the p-torsion elements of G.

The following features of this subgroup are easy to prove:

- The index of $\mathrm{T}_{\mathbb{Z}/p}G$ in G is coprime with p, and $\mathrm{T}_{\mathbb{Z}/p}G$ is minimal among the normal subgroups of G for which this condition holds.
- $\mathrm{T}_{\mathbb{Z}/p}G$ is a characteristic subgroup of G, v.g., every automorphism of G reduces by restriction to an automorphism of $\mathrm{T}_{\mathbb{Z}/p}G$.
- $\mathrm{T}_{\mathbb{Z}/p}G$ has no normal subgroups whose index in $\mathrm{T}_{\mathbb{Z}/p}G$ is coprime with p.

Now we are prepared to prove the general case of the theorem:

Theorem 3.3. *Let G be a finite group, p a prime; then we have that the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$ fits in the following fibration sequence:*

$$\prod_{q\neq p} \mathrm{B}(\mathrm{T}_{\mathbb{Z}/p}G)^\wedge_q \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_{\mathbb{Z}/p}G).$$

Proof. The $\mathbb{Z}/p$-radical is normal in G, so we can consider the fibration of classifying spaces

$$\mathrm{BT}_{\mathbb{Z}/p}G \longrightarrow \mathrm{B}G \longrightarrow \mathrm{BT}_{\mathbb{Z}/p}G.$$

The quotient group $G/\mathrm{T}_{\mathbb{Z}/p}G$ has order coprime with p, so its classifying space is $\mathbb{Z}/p$-null. Now, by (2.1), the sequence of nullifications

$$\mathbf{P}_{\mathbb{Z}/p}\mathrm{BT}_{\mathbb{Z}/p}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_{\mathbb{Z}/p}G)$$

is a fibration sequence. But $\mathbf{P}_{\mathbb{Z}/p}\mathrm{BT}_{\mathbb{Z}/p}G \simeq \prod_{q\neq p} \mathrm{B}(\mathrm{T}_{\mathbb{Z}/p}G)^\wedge_q$ by 3.1, so the theorem is proved.

□

Recall that if G is a finite group and p is a prime number, it is known ([BK72], II.5, see also [Lev95]) that the fundamental group of $\mathrm{B}G^\wedge_q$ is the quotient of G by its p-perfect maximal normal subgroup $O^p(G)$; in particular, the fundamental group is a p-group. So, in the case of the theorem, this means that $\prod_{q\neq p} \mathrm{B}(\mathrm{T}_{\mathbb{Z}/p}G)^{\wedge}_q$ is a simply-connected space, and thus it is the universal cover of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.

If $S = p_1 \ldots p_r$ is a finite collection of prime numbers, it can be defined the S-radical of G in the same lines of 3.2 as the minimal normal subgroup $\mathrm{T}_S G$ of G that contains all the S-torsion. This group verifies analogous properties for those previously quoted for $\mathrm{T}_{\mathbb{Z}/p}G$, and in fact it is a normal subgroup of the $\mathbb{Z}/p_i$-radical for every $p_i \in S$. Using this object, we can establish the following generalization of the previous theorem, that can be proved using exactly the same line of reason as before:

Proposition 3.4. *Let G be a finite group, $S = p_1 \ldots p_n$ a finite collection of prime numbers, and $W = \mathrm{B}\mathbb{Z}/p_1 \vee \ldots \vee \mathrm{B}\mathbb{Z}/p_n$; then we have that the W-nullification of $\mathrm{B}G$ fits in the following fibration sequence:*

$$\prod_{q\notin S} \mathrm{B}(\mathrm{T}_S G)^{\wedge}_q \longrightarrow \mathbf{P}_W \mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_S G).$$

Following ([RS00] 1.1), it is enough to consider only the case in which the primes are different, because there is a homotopy equivalence $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.

It is also interesting to note that the $\mathrm{B}\mathbb{Z}/p$-nullification of the classifying space of a finite simple group is nothing but a completion:

Corollary 3.5. *If p is a prime number and G is a finite simple group, then we have $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G = \mathbb{Z}[1/p]_\infty \mathrm{B}G$.*

Proof. It is a direct consequence of the fibration that gives the theorem 3.3. □

It is greatly remarkable the fact that, if $M(\mathbb{Z}/p,1)$ is a 2-dimensional Moore space, then the inclusion $M(\mathbb{Z}/p,1) \hookrightarrow \mathrm{B}\mathbb{Z}/p$ induces a map $\mathbf{P}_{M(\mathbb{Z}/p,1)}\mathrm{B}G \longrightarrow \mathbf{P}_{M(\mathrm{B}\mathbb{Z}/p)}\mathrm{B}G$ which according to 3.3 and ([RS98], 1.3) is a homotopy equivalence. In other words, this statement tells us that the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$ depends only on the 2-skeleton of $\mathrm{B}G$. On the other hand, it proves the following beautiful result, that concerns localization of groups:

Corollary 3.6. *If G is a finite group and p is a prime number, $L_{\mathbb{Z}/p}\pi_1(\mathrm{B}G)$ is isomorphic to $\pi_1(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G)$. Here $L_{\mathbb{Z}/p}$ denotes the usual localization of G with regard to the null map $\mathbb{Z}/p \longrightarrow *$.*

Proof. It only must be pointed out that $\pi_1(\mathbf{P}_{M(\mathrm{B}\mathbb{Z}/p)}\mathrm{B}G) \simeq G/\mathrm{T}_{\mathbb{Z}/p}G \simeq L_{\mathbb{Z}/p}G$. See ([Cas94], 3.2) for details. □

In section 4 we will use these last results for giving a precise description of the fundamental group of $\bar{\mathbf{P}}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$, a way of compute it and a characterization of the finite groups whose classifying space is $\mathrm{B}\mathbb{Z}/p$-acyclic.

We would like to finish this section by mentioning the article [Dwy94], where the author proves that the $\mathrm{B}\mathbb{Z}/p$-nullification of the classifying space of a compact Lie group G whose group of components is a p-group is homotopy equivalent to its $\mathbb{Z}[1/p]$-localization. We consider our work complementary to that, and it would be desirable to find a way to arrange all these data to find a description of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ for every compact Lie group G.

3.2. **Commutation of the nullification functors.** It is known that localization functors usually do not commute, not even in the case of nullifications. There are several examples of this in [RS00], where the authors also try to elucidate what happens if we apply in succession two localization functors to a certain space.

We will prove now that for different primes p and q, the functors $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}$ and $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}$ *do* commute if we apply them over $\mathrm{B}G$. In view of what we have said, this is an interesting exception to the general case.

Proposition 3.7. *Let G be a finite group, p and q different primes. Then we have homotopy equivalences*

$$\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G.$$

Proof. The proof of this result will be divided in two cases: in the first one we will suppose that G coincides with its S-radical for $S=\{p,q\}$, and then we will pass to the general case.

So, let $G=\mathrm{T}_S G$. We will prove that $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is homotopy equivalent to $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ and the other equivalence will follow interchanging the roles of p and q. We want to check that $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$, i.e. $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-null, and for every $\mathrm{B}\mathbb{Z}/p$-null space X we have a homotopy equivalence

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \simeq \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X)$$

which should be given by a coaugmentation $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G$. The first statement is trivial, because $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is by definition a $\mathrm{B}\mathbb{Z}/p$-null space. So we only must verify the previously mentioned homotopy equivalence between the mapping spaces. So, let X be a $\mathrm{B}\mathbb{Z}/p$-null space.

First, as $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/q$-null, we have a natural map

$$\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G$$

which induces another one between the mapping spaces

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G, Y) \longrightarrow \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, Y)$$

for every space Y. So, if we consider the $\mathrm{B}\mathbb{Z}/p$-null space X and its universal cover $\tilde{X}$ we have the following commutative diagram:

(3.2.1)
$$\begin{array}{ccc}
\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G, \tilde{X}) & \longrightarrow & \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, \tilde{X}) \\
\downarrow & & \downarrow \\
\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) & \longrightarrow & \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, Y) \\
\downarrow{\scriptstyle \tilde{p}_{p,q}} & & \downarrow{\scriptstyle \tilde{p}_q} \\
\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G, \mathrm{B}\pi_1(X))_{\tilde{p}_{p,q}} & \xrightarrow{r_\natural} & \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, \mathrm{B}\pi_1(X))_{\tilde{p}_q}
\end{array}$$

For proving that the columns are fibrations, we must check that the maps $\tilde{p}_{p,q}$ y $\tilde{p}_q$ take value in both cases in the component of the constant map. The first one is trivial, because by 2.1, the space $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee \mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is simply connected. In the other case, we must verify that if we have a map $f:\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \longrightarrow X$, the composition with the projection $\pi : X \longrightarrow \mathrm{B}\pi_1(X)$ is homotopic to the constant map. We will denote by g the composition

$$\mathrm{B}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \xrightarrow{f} X \xrightarrow{\pi} \mathrm{B}\pi_1(X),$$

where $\mathrm{B}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is the coaugmentation. It is clear that for every map $\mathrm{B}\mathbb{Z}/p \longrightarrow \mathrm{B}G$ the composition

$$\mathrm{B}\mathbb{Z}/p \longrightarrow \mathrm{B}G \xrightarrow{g} \mathrm{B}\pi_1(X)$$

must be inessential, because X is $\mathrm{B}\mathbb{Z}/p$-null, and same happens for every map $\mathrm{B}\mathbb{Z}/q \longrightarrow \mathrm{B}G$, in this case because $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/q$-null. As G has no quotients coprimes with pq, the map $\mathrm{B}g$ is trivial. Moreover, g factors through the projection $G \longrightarrow G/\mathrm{T}_{\mathbb{Z}/q}G$, and the induced map $G/\mathrm{T}_{\mathbb{Z}/q}G \longrightarrow \pi_1(X)$ is trivial too. Hence, the commutative diagram

$$\begin{array}{ccccc} \mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G & \xrightarrow{f} & X & \xrightarrow{\pi} & \mathrm{B}\pi_1(X) \\ \downarrow & & & \nearrow_{*} & \\ \mathrm{B}(G/\mathrm{T}_{\mathbb{Z}/q}G) & & & & \end{array}$$

where $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_{\mathbb{Z}/q}G)$ is the natural projection over $\mathrm{B}\pi_1(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G)$ proves that the composition $\pi \circ f$ is trivial, as we wanted.

To finish the proof of the first case, we must see the isomorphism of the top horizontal level of the diagram 3.2.1. We know ([ABN94], 9.4) that if X is a $\mathrm{B}\mathbb{Z}/p$-null space, its universal cover is too, so we only need to check that the previously described map

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \longrightarrow \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X)$$

is a homotopy equivalence if X is a simply-connected, $\mathrm{B}\mathbb{Z}/p$-null space. As X is simply-connected, it is homotopy equivalent to the pullback of the Sullivan arithmetic square ([BK72] V,6). By 3.3 and 3.4, the rationalizations of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ and $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ are homotopy equivalent to a point, so

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \simeq \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, \prod_{r \text{ prime}} X_r^\wedge),$$

and same is true for maps which come from $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$.

It is clear that

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, \prod_{r \text{ prime}} X_r^\wedge) \simeq \prod_{r \text{ prime}} \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X_r^\wedge)$$

and, as $X_r^\wedge$ is $\mathbb{Z}/r$-complete for every prime r, the latter is homotopy equivalent to $\prod_{r \text{ prime}} \mathrm{map}_*((\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G)_r^\wedge, X_r^\wedge)$. If $r \neq q$, by the result (3.10) we have that $(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G)_r^\wedge \simeq \mathrm{B}G_r^\wedge$, and on the other hand, the $\mathbb{Z}/q$-completion of the Postnikov fibration of (3.3) gives us that $(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G)_q^\wedge$ is contractible (see 3.12). In addition, observe that $\mathrm{map}_*(\mathrm{B}G_p^\wedge, X_p^\wedge)$ is contractible too, because X is $\mathrm{B}\mathbb{Z}/p$-null (see the proof of 3.3). So we obtain

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \simeq \prod_{r\neq p,q} \mathrm{map}_*(\mathrm{B}G_r^\wedge, X_r^\wedge).$$

In the other case, we have again

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \simeq \mathrm{map}_*((\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G)_r^\wedge, \prod_{r \text{ prime}} X_r^\wedge).$$

Just like before, if $r \neq p, q$, we have a homotopy equivalence $(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G)_r^\wedge \simeq \mathrm{B}G_r^\wedge$, and on the other hand, using again the Postnikov fibration of (3.3),

$$(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G)_p^\wedge \simeq * \simeq (\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G)_q^\wedge.$$

So we have

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \simeq \prod_{r\neq p,q} \mathrm{map}_*(\mathrm{B}G_r^\wedge, X_r^\wedge) \simeq \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X)$$

and this chain of equivalences finishes the proof of the first case.

Let G now be a finite group, and let us consider the fibration

$$\mathrm{BT}_S G \longrightarrow \mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_S G).$$

As the base is $\mathrm{B}\mathbb{Z}/p \vee \mathrm{B}\mathbb{Z}/q$-null, the fibration is preserved after $\mathrm{B}\mathbb{Z}/p \vee \mathrm{B}\mathbb{Z}/q$-nullification (2.1), and also after $\mathrm{B}\mathbb{Z}/p$-nullification and $\mathrm{B}\mathbb{Z}/q$-nullification. Moreover, $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-null and $\mathrm{B}\mathbb{Z}/q$-null, and so we have a commutative diagram where the rows are fibrations:

$$\begin{array}{ccccc}
\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{BT}_S G & \longrightarrow & \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G & \longrightarrow & \mathrm{B}(G/\mathrm{T}_S G) \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \mathrm{Id}} \\
\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{BT}_S G & \longrightarrow & \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G & \longrightarrow & \mathrm{BT}_S G.
\end{array}$$

Now, the left vertical map is an equivalence by the case already proved, so the natural map $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is an equivalence too and we have finished.

□

We finish the section by showing a slight generalization of the previous proposition.

Corollary 3.8. *Let $p_1 \dots p_r$ and $q_1 \dots q_s$ be two families of prime numbers, and denote $W = \mathrm{B}\mathbb{Z}/p_1 \vee \dots \vee \mathrm{B}\mathbb{Z}/p_r$ and $W' = \mathrm{B}\mathbb{Z}/q_1 \vee \dots \vee \mathrm{B}\mathbb{Z}/q_s$; then we have $\mathbf{P}_W\mathbf{P}'_W\mathrm{B}G \simeq \mathbf{P}_{W\vee W'}\mathrm{B}G \simeq \mathbf{P}'_W\mathbf{P}_W\mathrm{B}G$.*

Proof. It is proved exactly in the lines of the previous theorem. We leave the details to the reader.

□

3.3. **Relation between the nullification and the completion.** It seems quite natural to ask for the relation between the effect on classifying spaces of finite groups of the $\mathrm{B}\mathbb{Z}/p$-nullification and the $\mathbb{Z}[1/p]_\infty$-completion, because these two functors "kill" the p-primary part of $\mathrm{B}G$. We have already seen, for instance, that they coincide if G is a simple group, but now we will see by means of an easy example that this is not always true.

Example 3.9. Consider the dihedral group D_{15}, which is isomorphic to the semidirect product $\mathbb{Z}/15 \rtimes \mathbb{Z}/2$. It will be seen in 5.1 that the $\mathrm{B}\mathbb{Z}/3$-nullification of $\mathrm{B}D_{15}$ is homotopy equivalent to $\mathrm{B}D_{10}$.

On the other hand, the group $\mathbb{Z}/15$ is normal in the semidirect product, and we can consider the associated fibration

$$\mathrm{B}\mathbb{Z}/15 \longrightarrow \mathrm{B}D_{15} \longrightarrow \mathrm{B}\mathbb{Z}/2.$$

The space $\mathrm{B}\mathbb{Z}/2$ is $\mathbb{Z}/2$-complete, so this fibration is preserved by $\mathbb{Z}/2$-completion, and we obtain the homotopy equivalence $(\mathrm{B}D_{15})^\wedge_2 \simeq \mathrm{B}\mathbb{Z}/2$.

Now, it is obvious that $\pi_1(\mathbf{P}_{\mathrm{B}\mathbb{Z}/3}\mathrm{B}D_{15}) = D_{10}$, while $\pi_1((\mathrm{B}D_{15})^\wedge_5 \times \mathrm{B}\mathbb{Z}/2) = \mathbb{Z}/2$, because $(\mathrm{B}D_{15})^\wedge_5$ is simply-connected (D_{15} has no quotient groups that are 5-groups). So, in this case, we obtain that the $\mathrm{B}\mathbb{Z}/3$-nullification cannot be the product of the $\mathbb{Z}/2$-completion and the $\mathbb{Z}/5$-completion of $\mathrm{B}D_{15}$, and in particular $\mathbf{P}_{\mathrm{B}\mathbb{Z}/3}\mathrm{B}D_{15} \neq \mathbb{Z}[1/3]_\infty\mathrm{B}D_{15}$, as we wanted to see.

Now we will study what happens if we apply in succession the $\mathrm{B}\mathbb{Z}/p$-nullification and $\mathbb{Z}/q$-completion functors to $\mathrm{B}G$, in the two possible orders, and for primes p and q not necessarily different. We come to the conclusion that the functor $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}$ is quite sensitive, in the sense that it kills the p-primary structure of $\mathrm{B}G$ leaving untouched the q-primary part which is detected by the $\mathbb{Z}/q$-completion functor.

We consider first the case $p \neq q$.

Proposition 3.10. *Let G be a finite group, p and q different primes. Then we have homotopy equivalences*

$$\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}(\mathrm{B}G^\wedge_q) \simeq \mathrm{B}G^\wedge_q \simeq (\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G)^\wedge_q$$

Proof. Every map $\mathrm{B}\mathbb{Z}/p \longrightarrow \mathrm{B}G^\wedge_q$ factors through the $\mathbb{Z}/q$-completion of $\mathrm{B}\mathbb{Z}/p$, because $\mathrm{B}G^\wedge_q$ is $\mathbb{Z}/q$-complete. But $(\mathrm{B}\mathbb{Z}/p)^\wedge_q$ is contractible, so $\mathrm{B}G^\wedge_q$ is $\mathrm{B}\mathbb{Z}/p$-null and the first equivalence is proved.

For the second, notice that the map $\mathrm{B}\mathbb{Z}/p \longrightarrow *$ is a $\mathbb{Z}/q$-equivalence, and so, the canonical coaugmentation $\mathrm{B}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ is again a $\mathbb{Z}/q$-equivalence. □

Remark 3.11. The arguments of the last proof remain valid if we replace $\mathrm{B}G$ by any $\mathbb{Z}/q$-good space.

If we now consider the case $p = q$, we obtain the following:

Proposition 3.12. *If G is a finite group and p is a prime number, then $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}(\mathrm{B}G^\wedge_p)$ is contractible, and the same happens to $(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G)^\wedge_p$.*

Proof. We must check that, for every $\mathrm{B}\mathbb{Z}/p$-null space, the space $\mathrm{map}_*(\mathrm{B}G^\wedge_p, X)$ is contractible. Consider the Postnikov fibration

$$\tilde{X} \xrightarrow{h} X \xrightarrow{f} \mathrm{B}\pi_1(X).$$

Recall that if X is $\mathrm{B}\mathbb{Z}/p$-null, its universal cover $\tilde{X}$ is $\mathrm{B}\mathbb{Z}/p$-null too (see [ABN94], 9.7). Now we consider a finite group G such that $\mathrm{B}G^\wedge_p$ is a simply-connected space. In this case we have $\mathrm{map}_*(\mathrm{B}G^\wedge_p, \mathrm{B}\pi_1(X)) \simeq *$, so for proving that $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G^\wedge_p$ is contractible too we only must check $\mathrm{map}_*(\mathrm{B}G^\wedge_p, \tilde{X})$ is. But this is proved exactly in the same way as in the proof of (3.1), taking care of the fact that if $q \neq p$, $\mathrm{map}_*(\mathrm{B}G^\wedge_p, \tilde{X}^\wedge_q)$ is contractible.

Now, let G be any finite group. If we denote by $O^p(G)$ the p-perfect maximal normal subgroup of G, it is known that $\pi_1(\mathrm{B}G^\wedge_p)$ is isomorphic to $G/O^p(G)$. Now we have a fibration sequence:

$$\mathrm{B}O^p(G) \longrightarrow \mathrm{B}G \longrightarrow \mathrm{B}(G/O^p(G)).$$

The quotient $G/O^p(G)$ is a p-group, so $\mathrm{B}(G/O^p(G))$ is $\mathbb{Z}/p$-complete, and we have the correspondent sequence of the $\mathbb{Z}/p$-completions is a fibration:

$$\mathrm{B}O^p(G)^\wedge_p \longrightarrow \mathrm{B}G^\wedge_p \longrightarrow \mathrm{B}(G/O^p(G))$$

As $O^p(G)$ has no quotients of order a power of p, the $\mathbb{Z}/p$-completion $\mathrm{B}O^p(G)^\wedge_p$ is a simply-connected space, and so the last fibration is the Postnikov fibration of $\mathrm{B}G^\wedge_p$. But we have proven that the fiber $\mathrm{B}O^p(G)^\wedge_p$ is $\mathrm{B}\mathbb{Z}/p$-acyclic, so by 2.1, $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}(\mathrm{B}G^\wedge_p)$ is homotopy equivalent to $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}(\mathrm{B}(G/O^p(G)))$, and this last one is contractible because $G/O^p(G)$ is a p-group. So we have finished the proof of the first statement.

For proving the second, notice that the universal cover of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$, that is $\prod_{q\neq p}(\mathrm{B}G)^\wedge_q$ for q prime, is $\mathbb{Z}/p$-homology equivalent to a point. So by the fibre lemma 5.1 of [BK72], the Postnikov fibration

$$\prod_{q\neq p}(\mathrm{B}G)^\wedge_q \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_{\mathbb{Z}/p}G)$$

is preserved by $\mathbb{Z}/p$-completion. But the $\mathbb{Z}/p$-completions of the base space and the fibre are contractible, so $(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G)^\wedge_p$ is contractible too and we have finished. □

In conclusion, we will establish the relationship between the $\mathrm{B}\mathbb{Z}/p$-nullification and the Bousfield-Kan completion with coefficients in the ring $\mathbb{Z}[1/p]$.

Proposition 3.13. *Let G be a finite group, p and q two different primes. Then the following relations hold:*

(1) $\mathbb{Z}[1/p]_\infty \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbb{Z}[1/p]_\infty \mathrm{B}G \simeq \mathbb{Z}[1/p]_\infty \mathrm{B}G$.
(2) $\mathbb{Z}[1/p]_\infty \mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathbb{Z}[1/p]_\infty \mathrm{B}G \simeq \mathbb{Z}[1/p, 1/q]_\infty \mathrm{B}G$

Proof. It is an immediate consequence of the previous results of this section, taking into account the results ([BK72], VII, 4.2 and 4.3) that allow us to express the $\mathbb{Z}[1/p]$-completion of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ as the product of their $\mathbb{Z}/p'$-completions in the rest of primes. □

4. Cellularization

Let G be a finite group. In the proposition 3.6 we have seen that the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$ is intimately related with the $\mathbb{Z}/p$-localization of G as a group. In this way, it turns out to be interesting to study the $\mathbb{Z}/p$-cellularization of the group G for obtaining information about the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}G$. This is with broad strokes our approach to this subject, and it is worth to recall the main definitions concerning the cellularization in the category of groups. Recall that a group G is *$\mathbb{Z}/p$-cellular* if and only if it can be built from G by (maybe iterated) colimits, and the *$\mathbb{Z}/p$-cellularization* of G is the unique $\mathbb{Z}/p$-cellular group $\mathrm{CW}_{\mathbb{Z}/p}G$ endowed with an augmentation $\mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow G$ which induces an isomorphism $\mathrm{Hom}(\mathbb{Z}/p, \mathrm{CW}_{\mathbb{Z}/p}G) \simeq \mathrm{Hom}(\mathbb{Z}/p, G)$. This concept was first defined in [RS98] and mainly used for describing the cellularization with regard to Moore spaces.

We begin our study by showing that the problem of the $\mathrm{B}\mathbb{Z}/p$-cellularization of classifying spaces of finite groups can be reduced to the problem of the $\mathrm{B}\mathbb{Z}/p$-cellularization of classifying spaces of finite *$\mathbb{Z}/p$-cellular* groups.

Proposition 4.1. *If G is a finite group, the natural map $\mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow G$ induces a homotopy equivalence $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}\mathrm{CW}_{\mathbb{Z}/p}G \simeq \mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.*

Proof. By the functoriality of $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}$, the mentioned map $\mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow G$ induces another one $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}\mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow \mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ which composed with the canonical augmentation gives rise to the map $f : \mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}\mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow \mathrm{B}G$. The source is $\mathrm{B}\mathbb{Z}/p$-cellular, so we only need to see that the last map induces a weak equivalence between the pointed mapping spaces $\mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}\mathrm{CW}_{\mathbb{Z}/p}G)$ and $\mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathrm{B}G)$. This is proved by the following string of weak equivalences:

$$\mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}\mathrm{CW}_{\mathbb{Z}/p}G) \simeq \mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathrm{B}\mathrm{CW}_{\mathbb{Z}/p}G) \simeq$$
$$\mathrm{Hom}(\mathbb{Z}/p, \mathrm{CW}_{\mathbb{Z}/p}G) \simeq \mathrm{Hom}(\mathbb{Z}/p, G) \simeq \mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathrm{B}G).$$

what finishes the proof. Note that we have build explicitly the natural augmentation f.

□

Hence, it becomes interesting to find appropriate tools for computing the $\mathbb{Z}/p$-cellularization of a finite group G. Aiming to this, we need to recall the following concept of group theory, that will be crucial in the sequel:

Definition 4.2. If G is a finite group, the *$\mathbb{Z}/p$-socle* $\mathrm{S}_{\mathbb{Z}/p}G$ of G is the subgroup of G generated by the order p elements of G.

It can be seen that this subgroup is always normal and characteristic, and it is contained in the $\mathbb{Z}/p$-radical $\mathrm{T}_{\mathbb{Z}/p}G$. In fact, the following holds:

Proposition 4.3. *If G is a $\mathbb{Z}/p$-cellular group, then $G = \mathrm{S}_{\mathbb{Z}/p}G = \mathrm{T}_{\mathbb{Z}/p}G$.*

Proof. If G is $\mathbb{Z}/p$-cellular, it is a colimit of $\mathbb{Z}/p$'s, and hence it is generated by order p elements. As the $\mathbb{Z}/p$-socle is the group generated by the order p elements of G, $G = \mathrm{S}_{\mathbb{Z}/p}G$, and the other equality is obvious from the inclusions $\mathrm{S}_{\mathbb{Z}/p}G \subseteq \mathrm{T}_{\mathbb{Z}/p}G \subseteq G$.

□

For our work, the most relevant property of the $\mathbb{Z}/p$-socle is the fact that the inclusion $\mathrm{S}_{\mathbb{Z}/p}G \subseteq G$ always induces an isomorphism $\mathrm{Hom}(\mathbb{Z}/p, \mathrm{S}_{\mathbb{Z}/p}G) \simeq \mathrm{Hom}(\mathbb{Z}/p, G)$, and this implies that $\mathrm{CW}_{\mathbb{Z}/p}G \simeq \mathrm{CW}_{\mathbb{Z}/p}\mathrm{S}_{\mathbb{Z}/p}G$. This last assertion shows that the computation of the $\mathbb{Z}/p$-cellularization of groups can be again reduced to the case of groups generated by order p elements. Moreover, it is worth to point out that we are interested only in finite groups, and in this case it is not hard to calculate the $\mathbb{Z}/p$-socle of a group starting from a presentation of G (using GAP, for example).

Now, the main tool we are going to use in our description of the $\mathbb{Z}/p$-cellularization is the following version of a theorem of Rodríguez-Scherer ([RS98] 2.7):

Theorem 4.4. *For each group G, there is a central extension*

$$0 \longrightarrow A \longrightarrow \mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow \mathrm{S}_{\mathbb{Z}/p}G \longrightarrow 0$$

such that A has no order p elements and is universal with regard to this property.

The key case is G finite and $\mathbb{Z}/p$-cellular; thus, $G = \mathrm{S}_{\mathbb{Z}/p}G$ and the essential problem here is computing A. According to (2.4), this group is the second homotopy group of the $\Sigma M(\mathbb{Z}/p,1)$-nullification of the cofibre C_f of the evaluation map $f : \bigvee_{[M(\mathbb{Z}/p,1),BG]} M(\mathbb{Z}/p,1) \longrightarrow BG$, where $M(\mathbb{Z}/p,1)$ stands here for a two-dimensional Moore space. It is not hard to see that $\pi_2(\mathbf{P}_{\Sigma M(\mathbb{Z}/p,1)}C_f) = \pi_2(C_f)/\mathrm{T}_{\mathbb{Z}/p}\pi_2(C_f)$, so our problem is to describe the second homotopy group of this homotopy cofibre. As G is generated by order p elements, C_f is simply-connected, and then $\pi_2(C_f) = H_2(C_f)$, the Schur multilplier of the cofibre. Using this property, we have computed the group in the following way:

Proposition 4.5. *Let G be a finite group generated by order p elements, denote by H the free product $*\mathbb{Z}/p$ extended over all the homomorphisms $\mathbb{Z} \longrightarrow G$, and let K be the kernel of the evaluation map $H \longrightarrow G$. If C_f is the cofiber of the evaluation map at the level of classifying spaces, we have $\pi_2(C_f) = K/[K,H]$.*

Proof. It is an easy consequence of ([BL87], 3.4) taking, in the notation of there, $P = H, M = H$ and $N = K$.

□

As we said in the introduction, using this proposition it is not hard to calculate a presentation of the group $\pi_2(C_f)$ from presentations of K and H using the Reidemeister-Schreier method ([MKS76], 2.3). In this way we obtained the value of $\pi_2(C_f)$ in the case $G = PSL(2,3)$ the tetrahedral group (see 5.2) and the computation of this pathological example served as motivation for further work. This formula is particularly useful for calculating the $\mathbb{Z}/p$-cellularization of G if we don't know the Schur multiplier of the group G, and indeed gives a way of computing directly this last invariant:

Corollary 4.6. *With the notation of the previous proposition, $H_2(G) = K \cap [H,H]/[K,H]$.*

Proof. It is an immediate consequence of the previous proposition and the Mayer-Vietoris sequence associated to the cofibration f.

□

On the other hand, if the second homology group of G is known, the computations are easily simplified using the following lemma, which in fact will lead us to a complete classification of $\mathbb{Z}/p$-cellular groups:

Lemma 4.7. *Let G be a finite group generated by order p elements. Then the group A of 4.4 is isomorphic to $H_2(G)/T_{\mathbb{Z}/p}(H_2(G))$.*

Proof. The Mayer-Vietoris sequence of the cofibration f has the form

$$0 \longrightarrow H_2(G) \longrightarrow \pi_2(C_f) \longrightarrow \bigoplus \mathbb{Z}/p \longrightarrow 0.$$

But as $A = \pi_2(C_f)/\mathrm{T}_{\mathbb{Z}/p}\pi_2(C_f)$, the result follows.

□

Proposition 4.8. *Let G be a finite group. Then G is $\mathbb{Z}/p$-cellular if and only if is generated by order p elements and $H_2(G)$ is a p-group.*

Proof. If G is $\mathbb{Z}/p$-cellular, it a colimit of $\mathbb{Z}/p's$, and hence is generated by order p elements. According to 4.4 and the previous proposition, H_2G must be equal to its $\mathbb{Z}/p$-radical, and hence it must be a p-group. Reciprocally, if G is generated by order p elements, then $G = \mathrm{S}_{\mathbb{Z}/p}G$, and the fact that $H_2(G)$ is a p-group implies that the group A of 4.4 is trivial; so G is $\mathbb{Z}/p$-cellular.

□

Once we have computed A, the only thing that remains is to know what is the extension that identifies the $\mathbb{Z}/p$-cellularization. This is usually not hard, because the key result 4.4 describes that extension with great precision. Furthermore, in some favorable cases, we can walk one step up and find explicitly the cohomology class associated to this extension. For this reason we turn now our attention to *perfect* groups. Recall that a group G is called perfect if it is equal to its commutator subgroup or, equivalently, if the first integer homology group is trivial.

Proposition 4.9. *Let G be a finite perfect group generated by order p elements, let*

$$0 \longrightarrow H_2G \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 0$$

be the universal central extension of G, let B be the quotient of H_2G by the p-torsion (which is in fact a subgroup of H_2G), and let

$$0 \longrightarrow B \longrightarrow H \longrightarrow G \longrightarrow 0$$

be the central extension $\mathcal{E}$ induced by the previous one. Then, the latter is equivalent to the extension of the theorem 4.4, and in particular H is isomorphic to the $\mathbb{Z}/p$-cellularization of G.

Proof. The group B has no p-torsion, so by the universality of the extension $\mathcal{E}'$ of 4.4 that defines the $\mathbb{Z}/p$-cellularization there is an unique morphism of extensions $\mathcal{E}' \longrightarrow \mathcal{E}$ that is the identity over G.

On the other hand, it is clear that for any central extension whose kernel has not p-torsion, the unique morphism that come from the universal central extension to this one factors through $\mathcal{E}$. This proves that there is again an unique morphism over G from $\mathcal{E}$ to $\mathcal{E}'$, and is easy to check, by universality, that the two morphisms that we have defined are one inverse to each other. Hence, the extensions $\mathcal{E}$ and $\mathcal{E}'$ are equivalent and we have finished. □

Hence, if G is a perfect group, the cohomology class in $H^2(G; A)$ that correspond to the extension $\mathcal{E}'$ that defines the $\mathbb{Z}/p$-cellularization of G is the image of the identity map of A under the universal coefficient isomorphism

$$\mathrm{Hom}(A, A) \simeq \mathrm{Hom}(H_2G, A) \simeq H^2(G; A).$$

On the other hand, the methods of computation of $\mathrm{CW}_{\mathbb{Z}/p}G$ developed above can be reinterpreted in a framework of group theory as tools for describing the universal central extension of a finite perfect group G. For example, we can get this easy and interesting consequence:

Corollary 4.10. *If G is a perfect group generated by order p elements and whose Schur multiplier has no p-torsion, then the $\mathbb{Z}/p$-cellularization of G is isomorphic to its universal covering group $\tilde{G}$.*

A somewhat similar line of reasoning can be applied sometimes to non-perfect groups, as we see in the next proposition:

Proposition 4.11. *Let p be an odd prime, G a finite group such that the Schur multiplier of its $\mathbb{Z}/p$-socle is $\mathbb{Z}/2$. Then the central extension of 4.4 that identify the $\mathbb{Z}/p$-cellularization of G is the one that is not trivial.*

Proof. The $\mathbb{Z}/p$-socle $\mathrm{S}_{\mathbb{Z}/p}G$ of G is generated by order p elements, so its abelianization is an elementary abelian p-group. Hence, we have $H^1(\mathrm{S}^{ab}_{\mathbb{Z}/p}G, \mathbb{Z}/2) = 0$. By the universal coefficient theorem, we have the isomorphisms

$$H^2(\mathrm{S}_{\mathbb{Z}/p}G, \mathbb{Z}/2) \simeq \mathrm{Hom}(H_2(\mathrm{S}_{\mathbb{Z}/p}G), H_2(\mathrm{S}_{\mathbb{Z}/p}G)) = \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/2) \simeq \mathbb{Z}/2.$$

This implies that there are only two central extensions of $\mathrm{S}_{\mathbb{Z}/p}G$ by $\mathbb{Z}/2$. Now we observe that the group defined by the trivial central extension $\mathbb{Z}/2 \times \mathrm{S}_{\mathbb{Z}/p}G$ is not $\mathbb{Z}/p$-cellular, because it cannot be generated by order p elements. Thus, the group defined by the nontrivial extension that corresponds to the identity element in $\mathrm{Hom}(H_2(\mathrm{S}_{\mathbb{Z}/p}G), H_2(\mathrm{S}_{\mathbb{Z}/p}G))$ is the $\mathbb{Z}/p$-cellularization of G. □

This result will be very useful in certain cases, as we will see in the next section.

Once we have studied carefully the $\mathbb{Z}/p$-cellularization functor in the category of finite groups, it turns out to be interesting to relate it to the other group colocalization involved in this work, namely the homotopy fiber $\bar{L}_{\mathbb{Z}/p}G$ of the localization map $G \longrightarrow L_{\mathbb{Z}/p}G$ described in 3.1. This is established in the next proposition.

Proposition 4.12. *If G is a finite group such that $\mathrm{T}_{\mathbb{Z}/p}G = \mathrm{S}_{\mathbb{Z}/p}G$, the $\mathbb{Z}/p$-cellularization of G is isomorphic to the fundamental group of $\bar{\mathbf{P}}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$, which is usually called $\mathrm{D}_{\mathbb{Z}/p}G$.*

Proof. Following ([RS98],3.3), the group $\mathrm{D}_{\mathbb{Z}/p}G$ is defined by a central extension $\mathcal{D}$ given by

$$0 \longrightarrow H_2(\mathrm{T}_{\mathbb{Z}/p}G; \mathbb{Z}[1/p]) \longrightarrow \mathrm{D}_{\mathbb{Z}/p}G \longrightarrow \mathrm{T}_{\mathbb{Z}/p}G \longrightarrow 0$$

which is universal among the central extensions

$$0 \longrightarrow B \longrightarrow E \longrightarrow \mathrm{T}_{\mathbb{Z}/p}G \longrightarrow 0$$

such that $\mathrm{Hom}(\mathbb{Z}/p, B) = 0$ and $\mathrm{Ext}(\mathbb{Z}/p, B) = 0$.

Now, these conditions hold for the extension $\mathcal{E}'$ of 4.4 that defines $\mathrm{CW}_{\mathbb{Z}/p}G$, because p doesn't divide the order of A. So, there is just one map $f : \mathcal{D} \longrightarrow \mathcal{E}'$ that is the identity over G.

On the other hand, as $H_2(\mathrm{T}_{\mathbb{Z}/p}G; \mathbb{Z}[1/p])$ has no p-torsion it exists by 4.4 an unique map $g : \mathcal{E}' \longrightarrow \mathcal{D}$ which is again the identity over G. By universality, $\mathcal{E}' \simeq \mathcal{D}$ and the result follows.

□

The last proposition states that the tools we have developed above for the computation of the $\mathbb{Z}/p$-cellularization of a group G remain useful for calculating $\mathrm{D}_{\mathbb{Z}/p}G$, or in the language of group theory, the universal central extension of the $\mathbb{Z}/p$-radical of G with coefficients in $\mathbb{Z}[1/p]$. See ([RS98], 3.5) and ([MP01]) for more details on universal central extensions with coefficients.

We can also prove an analogous to the commutation result 3.1, which appears as an easy consequence of the last statement.

Corollary 4.13. *If G is a finite $\mathbb{Z}/p$-cellular group , then there is an isomorphism*

$$\pi_1 \bar{\mathbf{P}}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \bar{L}_{\mathbb{Z}/p}G.$$

Proof. According to the previous proposition, the fundamental group of $\bar{\mathbf{P}}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ is $\mathbb{Z}/p$-cellular, so by 4.8 we obtain that $H_2(G;\mathbb{Z}/p)=0$. This implies that $\mathrm{D}_{\mathbb{Z}/p}G = G$, and G coincides with its $\mathbb{Z}/p$-radical because it is $\mathbb{Z}/p$-cellular. But the $\mathbb{Z}/p$-radical is precisely the homotopy fiber of the $\mathbb{Z}/p$-localization of G, so we have finished.

□

It is worth to point out that the hypothesis of G $\mathbb{Z}/p$-cellular is essential, as you can see taking for example $G=\mathbb{Z}/p^2$.

Once we have completed the description of the $\mathbb{Z}/p$-cellularization of finite groups, we are prepared to present what is probably the main result of this section, a complete characterization of the finite groups G such that its classifying space $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular.

Proposition 4.14. *Let G be a finite $\mathbb{Z}/p$-cellular group. Then $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular if and only if G is a p-group.*

Proof. If G is not a p-group, $H_n(G)$ is not p-torsion for a certain $n \geq 2$. Using the result ([RS98], 6.3), it is not $\mathrm{M}(\mathbb{Z}/p,1)$-cellular for any two-dimensional Moore space $\mathrm{M}(\mathbb{Z}/p,1)$. Hence by 2.5 it cannot be $\mathrm{B}\mathbb{Z}/p$-cellular, because $\mathrm{B}\mathbb{Z}/p$ is itself $\mathrm{M}(\mathbb{Z}/p,1)$-cellular.

Conversely, suppose G is a p-group. We use induction over the order of the group G. It is clear that $\mathrm{B}\mathbb{Z}/p$ is $\mathrm{B}\mathbb{Z}/p$-cellular, so we admit the hypothesis is true for every group whose order is strictly smaller than p^k. Let G be a group of order p^k, then, and consider a minimal system of order p generators $\{x_1,\ldots,x_r,y\}$, that exists because the group is finite and $\mathbb{Z}/p$-cellular. Denote by H the minimal subgroup of G generated by $\{x_1,\ldots,x_r\}$. H is different from G, because the system of generators is minimal, and in addition it is normal, because it is a maximal subgroup of a nilpotent group. Now, it is easy to see that G/H is isomorphic to $\mathbb{Z}/p$ and it is generated by the image of y. So, we can write G as a *split* extension:

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0.$$

By the induction hypothesis, $\mathrm{B}H$ is $\mathrm{B}\mathbb{Z}/p$-cellular, $\mathrm{B}(G/H)$ is $\mathrm{B}\mathbb{Z}/p$-cellular too because it is isomorphic to $\mathbb{Z}/p$, and as the extension is split, the associated fibration has a section. Then 2.5 implies that $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular.

□

We will conclude this section by describing the fundamental group of $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.

Proposition 4.15. *Let G be a finite $\mathbb{Z}/p$-cellular group. Let us call r the order of $H_2(G)$ and s the number of different homomorphisms $\mathbb{Z}/p \longrightarrow G$. Then the fundamental group π of the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}G$ fits in the following central extension:*

$$0 \longrightarrow H \longrightarrow \pi \longrightarrow G \longrightarrow 0$$

where π is a finite p-group whose order is bounded by prs.

Proof. We consider again the evaluation map $\bigvee_{[\mathrm{B}\mathbb{Z}/p,\mathrm{B}G]_*}\longrightarrow \mathrm{B}G$, where the wedge is extended over all the elements of $[\mathrm{B}\mathbb{Z}/p,\mathrm{B}G]_*$, and let C_f be again the homotopy cofibre of this map. The Mayer-Vietoris sequence of this cofibration is as follows:

$$0\longrightarrow H_2(\mathrm{B}G)\longrightarrow H_2(C_f)\longrightarrow\bigoplus\mathbb{Z}/p\longrightarrow G^{ab}\longrightarrow 0$$

where the rank of the elementary abelian p-group $\bigoplus\mathbb{Z}/p$ is the number of homomorphisms $\mathbb{Z}/p\longrightarrow G$. The cofibre C_f is simply-connected, so $H_2(C_f)=\pi_2(C_f)$, and its order is clearly bounded by prs. Now, the fundamental group π is defined by a central extension

$$0\longrightarrow A\longrightarrow\pi\longrightarrow \mathrm{S}_{\mathbb{Z}/p}G\longrightarrow 0$$

where A is the second homotopy group of the $\Sigma\mathrm{B}\mathbb{Z}/p$-nullification of C_f. Using the fact that the suspension and C_f are simply-connected and the construction of the nullification functor described in the preliminaries we obtain that A is a quotient of $\pi_2(C_f)$. We finish by observing that the $\mathbb{Z}/p$-socle of G is $\mathbb{Z}/p$-cellular and hence equal to G.

□

The problem of determining exactly how many $\mathbb{Z}/p$'s appear in the kernel of this extension seems by no means easy stuff, because it depends essentially on how many maps from $\mathrm{M}(\mathbb{Z}/p,2)$ to the succesive cofibers that appear in the construction of the $\Sigma\mathrm{B}\mathbb{Z}/p$-nullification of C_f can be lifted to $\Sigma\mathrm{B}\mathbb{Z}/p$. However, in next section we will see some examples of groups for which is possible to determine exactly who is the fundamental group of $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.

The previous result allows us to characterize the finite groups such that the $\mathrm{B}\mathbb{Z}/p$-cellularization of its classifying space is a $K(G,1)$:

Proposition 4.16. *Let G be a finite group. Then $\pi_n(\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G)=0$ for $n\geq 2$ if and only if the $\mathbb{Z}/p$-cellularization of G is a p-group.*

Proof. If $\mathrm{CW}_{\mathbb{Z}/p}G$ is a p-group, then by 4.14 and 4.3

$$\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G\simeq\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}\mathrm{CW}_{\mathbb{Z}/p}G\simeq\mathrm{B}\mathrm{CW}_{\mathbb{Z}/p}G.$$

Conversely, if $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ is an aspherical space $\mathrm{B}H$, the group H is $\mathbb{Z}/p$-cellular, because taking fundamental group "commutes" with (homotopy) colimits. The result follows now from 4.3 and the previous proposition.

□

5. Examples

In this section we will apply the theorem 3.3 for calculating explicitly the $\mathrm{B}\mathbb{Z}/p$-nullification and $\mathrm{B}\mathbb{Z}/p$-cellularization of the classifying spaces of some well-known finite groups. The omitted details of the structure of the groups involved can be found in [Wei77], [Gor80] or [Rob96]. We always suppose that the primes that appear divide the order of G; otherwise classifying space of the group is $\mathrm{B}\mathbb{Z}/p$-null and hence its $\mathrm{B}\mathbb{Z}/p$-cellularization is a point.

5.1. **Dihedral groups.** Let $D_n=\{X,Y;X^n=1,Y^2=1,(XY)^2=1\}$ be the dihedral group of order $2n$.

a) *Nullification.*

For computing $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}D_n$, we will distinguish the cases $p\neq 2$ and $p=2$.

Suppose firstly that p is different of 2. Then $|D_n|=p^rq$, p coprime with q. If $r=0$ the classifying space is $\mathrm{B}\mathbb{Z}/p$-null and we have nothing to say. Suppose $r>0$.

Now, the $\mathbb{Z}/p$-radical $\mathrm{T}_{\mathbb{Z}/p}G$ of D_n is the subgroup generated by X^{n/p^r}, which is easily seen to be the unique p-Sylow subgroup of D_n (in particular is normal). Moreover, $< X^{n/p^r} >$ is isomorphic to $\mathbb{Z}/p^r$, and $D_n/< X^{n/p^r} >$ is isomorphic to D_{n/p^r}. So, by the result 3.3 we have the following Postnikov fibration:

$$\prod_{s\neq p}(\mathrm{B}\mathbb{Z}/p^r)^\wedge_s \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}D_n \longrightarrow \mathrm{B}D_{n/p^r}.$$

Now, as $(\mathrm{B}\mathbb{Z}/p^r)^\wedge_s$ is contractible if $s \neq p$ we obtain $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}D_n$ is homotopy equivalent to $\mathrm{B}D_{n/p^r}$.

Now we attack the case $p = 2$. It is clear by the relations that define the group that Y and XY belong to the $\mathbb{Z}/2$-radical of D_n. But this implies that X belongs too, and then $\mathrm{T}_{\mathbb{Z}/2}(D_n) = D_n$. Hence, $\mathbf{P}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}D_n = \prod_{q\neq 2}(\mathrm{B}D_n)^\wedge_q$.

b) *Cellularization.*

If $p \neq 2$, the $\mathbb{Z}/p$-socle of D_n is the cyclic group of order p, whose generator is identified inside D_n with $X^{n/p}$. Hence, the $\mathbb{Z}/p$-cellularization of D_n is $\mathbb{Z}/p$, and then $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}D_n = \mathrm{B}\mathbb{Z}/p$.

If $p = 2$, we can make the change $Z = XY$ to obtain the presentation $D_n = \{X, Z; X^2 = 1, Z^2 = 1, (XZ)^n = 1\}$. This proves that D_n is always generated by order two elements. In particular, if $n = 2^j$ for a certain natural number j, the corresponding dihedral group is a 2-group, and hence by 4.14 its classifying space is $\mathrm{B}\mathbb{Z}/2$-cellular.

In the case $n \neq 2^j$, we can assure $\mathrm{B}D_n$ is not $\mathrm{B}\mathbb{Z}/2$-cellular, because it has torsion in other prime, but we can prove that the group is $\mathbb{Z}/2$-cellular. In fact, we give an explicit construction of D_n, for every n, as a colimit of copies of $\mathbb{Z}/2$.

Consider the second presentation given above, the usual presentations $\mathbb{Z}/2 = \{A; A^2 = 1\}$, $\mathbb{Z}/2*\mathbb{Z}/2 = \{B, C; B^2 = 1, C^2 = 1\}$, and suppose n is odd. Then D_n is the coequalizer of the homomorphisms

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2*\mathbb{Z}/2 \\ A & \longrightarrow & BCB\ldots CB \end{array}$$

where B appears $\frac{n+1}{2}$ times, and

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2*\mathbb{Z}/2 \\ A & \longrightarrow & CB\ldots BC \end{array}$$

where now C appears $\frac{n+1}{2}$ times. The case n even is similar, with B appearing $\frac{n}{2}$ times in the first homomorphism, and C appearing $\frac{n}{2}+1$ times in the second. As every coequalizer is a colimit, we have proved that the dihedral groups are always $\mathbb{Z}/2$-cellular.

5.2. **Finite projective special linear groups and special linear groups.** Next we study the family of groups $SL(2, q)$, q prime, and its quotient groups $PSL(2, q)$. If $q = 2$, then $SL(2, q) = PSL(2, q) = D_6$, and this case has been already studied in the previous section. Rememeber in the sequel that $PSL(2, q)$ is simple if $q \geq 5$.

a) *Nullification.*

Consider the presentation of the tetrahedral group $PSL(2, 3)$ given by $\{X, Y; X^3 = 1, Y^3 = 1, (XY)^2 = 1\}$. The unique normal subgroup of $PSL(2, 3)$ that is not trivial is $H = \{1, XY, YX, XY^2X\}$, which is isomorphic to the Klein group $\mathbb{Z}/2\times\mathbb{Z}/2$, and it is easy to see that this is precisely the $\mathbb{Z}/2$-radical of $PSL(2, 3)$. The associated extension gives rise to a fibration of classifying spaces

$$\mathrm{B}\mathbb{Z}/2 \times \mathrm{B}\mathbb{Z}/2 \longrightarrow \mathrm{B}PSL(2, 3) \longrightarrow \mathrm{B}\mathbb{Z}/3.$$

As the fibre is $\mathrm{B}\mathbb{Z}/2$-acyclic, we have a homotopy equivalence $\mathbf{P}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}PSL(2, 3) \simeq \mathrm{B}\mathbb{Z}/3$. For $p = 3$, $PSL(2, 3)$ is generated by order 3 elements, so it is equal to its $\mathbb{Z}/3$-radical, and hence $\mathbf{P}_{\mathrm{B}\mathbb{Z}/3}\mathrm{B}PSL(2,3) \simeq \mathrm{B}PSL(2,3)^{\wedge}_{2}$. If $q \geq 5$, then $\mathrm{T}_{\mathbb{Z}/p}(PSL(2,q)) = PSL(2,q)$ for every prime p dividing the order of $PSL(2,q)$, because the group is simple. Then, by 3.3, the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}PSL(2,q)$ is homotopy equivalent to the $\mathbb{Z}[1/p]$-completion of $\mathrm{B}PSL(2,q)$. We consider now the non-projective case. We have always the fibration

$$\mathrm{B}\mathbb{Z}/2 \longrightarrow \mathrm{B}SL(2,q) \longrightarrow \mathrm{B}PSL(2,q).$$

Now the base is $\mathrm{B}\mathbb{Z}/2$-acyclic, and we have $\mathbf{P}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}PSL(2,q) \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}SL(2,q)$, which is again simply connected except for the pathological case $q = 3$. If $p \neq 2$ and divides the order of $PSL(2,q)$, this group is always generated by the transvections of order p ([Rob96],3.2.10) and again the $\mathrm{B}\mathbb{Z}/p$-nullification of its classifying space is homotopy equivalent to its $\mathbb{Z}[1/p]$-completion.

b) *Cellularization.*

We will again consider first the case $q = 3$, the tetrahedral group.

If $p = 2$, we have said that the $\mathbb{Z}/2$-radical is the Klein group $\mathbb{Z}/2 \times \mathbb{Z}/2$, which is equal to its $\mathbb{Z}/2$-socle, because it is generated by order 2 elements. As the Klein group is indeed a 2-group, we have $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}PSL(2,3) = \mathrm{B}\mathbb{Z}/2 \times \mathrm{B}\mathbb{Z}/2$.

In the case $p = 3$, recall $PSL(2,3)$ is generated by the order 3 transvections, so it is equal to its $\mathbb{Z}/3$-socle. It is easy to see that the Schur multiplier $H_2(PSL(2,3))$ is $\mathbb{Z}/2$, and then according to 4.4, the $\mathbb{Z}/3$-cellularization of $PSL(2,3)$ is defined by a central extension

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{CW}_{\mathbb{Z}/3}PSL(2,3) \longrightarrow PSL(2,q) \longrightarrow 0.$$

This extension is nontrivial by 4.11, and it is known that the only nontrivial central extension of $PSL(2,3)$ by $\mathbb{Z}/2$ is $SL(2,3)$, so the $\mathbb{Z}/3$-cellularization of $PSL(2,3)$ is $SL(2,3)$ and hence the latter group is $\mathbb{Z}/3$-cellular, because cellularization is an idempotent functor.

If $q \geq 5$, the group $PSL(2,3)$ is perfect, and its universal central extension is precisely

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow SL(2,3) \longrightarrow PSL(2,q) \longrightarrow 0.$$

Hence, the Schur multiplier of $PSL(2,q)$ is $\mathbb{Z}/2$, and by 4.7 this group is $\mathbb{Z}/2$-cellular. On the other hand, $SL(2,q)$ is the universal covering group of itself, and in particular $H_2(SL(2,q)) = 0$, so by corollary 4.10 it is $\mathbb{Z}/2$-cellular. For p odd, the same corollary proves that if p divides the order of $PSL(2,3)$, the $\mathbb{Z}/p$-cellularization of $PSL(2,q)$ is $SL(2,q)$, and this group is $\mathbb{Z}/p$-cellular.

To conclude, observe that neither $SL(2,q)$ nor $PSL(2,q)$ are p-groups for any primes $p \geq 2$ and $q > 2$; hence, its classifying spaces cannot be $\mathrm{B}\mathbb{Z}/p$-cellular. It is interesting to point out that $\mathrm{B}SL(2,q)$ cannot be the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}PSL(2,q)$, not even in the case p odd, in which we know $SL(2,q)$ to be the $\mathbb{Z}/p$-cellularization of $PSL(2,q)$.

5.3. **Symmetric and alternating groups.** In this section we will put our attention over the permutation group S_n and the alternating group A_n, the subgroup of permutations of even signature of S_n.

a) *Nullification.*

The symmetric group of n-letters ($n \geq 2$) admits the following presentation:

$$S_n = \{X_1, \ldots, X_{n-1}; X_iX_{i+1}X_i = X_{i+1}X_iX_{i+1}, X_iX_{i+j}X_i = X_{i+j} \text{ for } j \geq 2\}.$$

This shows that S_n is generated by order 2 elements, and in particular is equal to its $\mathbb{Z}/2$-radical. Hence, its $\mathrm{B}\mathbb{Z}/2$-nullification is homotopy equivalent to its $\mathbb{Z}[1/2]$-completion. If p is odd, we can consider the fibration defined by the inclusion of the alternating group:

$$\mathrm{B}A_n \longrightarrow \mathrm{B}S_n \longrightarrow \mathrm{B}\mathbb{Z}/2.$$

As the base is $\mathrm{B}\mathbb{Z}/p$-null for p odd, the fibration is preserved under $\mathrm{B}\mathbb{Z}/p$-nullification. If $n = 3$, $A_3 = \mathbb{Z}/3$, and then $\mathbf{P}_{\mathrm{B}\mathbb{Z}/3}\mathrm{B}S_3 \simeq \mathrm{B}\mathbb{Z}/2$. In the case $n \geq 4$, A_n is always generated by order p elements: if $n = 4$, it is known that $A_4 = PSL(2,3)$, and hence is generated by the 3-transvections, as we have said before; if $n > 4$, A_n is simple, and then the $\mathbb{Z}/p$-radical, which is normal, is the whole group. So, these considerations prove that for $n \geq 4$ and p odd, $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}A_n$ is homotopy equivalent to $\mathbb{Z}[1/p]_\infty\mathrm{B}A_n$. In particular, the fibration which defines the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}S_n$ takes the form

$$\mathbb{Z}[1/p]_\infty\mathrm{B}A_n \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}S_n \longrightarrow \mathrm{B}\mathbb{Z}/2$$

which turns to be a Postnikov fibration, because $\mathbb{Z}[1/p]_\infty\mathrm{B}A_n$ is simply connected.

The previous arguments also prove that $\mathbf{P}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}A_n \simeq \mathbb{Z}[1/2]_\infty\mathrm{B}A_n$ if $n \geq 5$. If $n = 4$ the alternating group is isomorphic to the tetrahedral group and this case have been studied in the previous section.

b) *Cellularization.*

In ([RS98], 6.5) Rodríguez-Scherer build effectively S_n as a colimit of copies of $\mathbb{Z}/2$, so S_n is $\mathbb{Z}/2$-cellular. If p is odd, let us see that every permutation of order p is even. Let σ be such a permutation. Then

$$1 = \mathrm{sig}(\mathrm{Id}) = \mathrm{sig}(\sigma^p) = (\mathrm{sig}(\sigma))^p$$

and, if p is odd, this is only possible if $\mathrm{sig}(\sigma) = 1$, what amounts to say that σ is even. This means that the $\mathbb{Z}/p$-socle of S_n is a subgroup of A_n, which is normal because the socle is always a characteristic subgroup; this already proves that $\mathrm{S}_{\mathbb{Z}/p}S_n = A_n$, and in particular $\mathrm{CW}_{\mathbb{Z}/3}S_4 = \mathrm{CW}_{\mathbb{Z}/3}A_4 = SL(2,3)$. So, we fix our attention again in alternating groups A_n with $n \geq 5$. It is known ([Asc00], 33.15) that the Schur multiplier of A_n is $\mathbb{Z}/2$ for every $n \neq 6,7$, and $H_2(A_6) = H_2(A_7) = \mathbb{Z}/6$. So, if $p \geq 5$, 4.10 imply that the $\mathbb{Z}/p$-cellularization of A_n is in this case isomorphic to the universal central extension of A_n. If $p = 3$, according to 4.11 the $\mathbb{Z}/3$-cellularization is given by the unique nontrivial extension of A_n by $\mathbb{Z}/2$, and finally, if $p = 2$, A_n is $\mathbb{Z}/2$-cellular for every n different from 6 or 7; in the pathological case, 4.9 tells us that the $\mathbb{Z}/2$-cellularization of A_n for $n = 6,7$ is the extension of A_n by $\mathbb{Z}/3$ that corresponds to element of $H^2(A_n;\mathbb{F}_3)$ that goes to the identity via the isomorphism $H^2(A_n;\mathbb{F}_3) \simeq \mathrm{Hom}(\mathbb{F}_3,\mathbb{F}_3)$.

As the order of S_n and A_n is respectively $n!$ and $n!/2$, these groups are never p-groups for $n \geq 2$, and hence their classifying spaces cannot be $\mathrm{B}\mathbb{Z}/p$-cellular.

5.4. ***p*-groups.** We finish by describing the effect of the $\mathrm{B}\mathbb{Z}/p$-cellularization functor on the classifying spaces of three family of p-groups, namely the quaternionic groups, semidihedral groups and $M_m(p)$-groups.

Consider the quaternion group $Q_{m+1} = \{H, K;\ H^{2^{m-1}} = K^2 = 1,\ HK = KH^{-1}\}$, with $m \geq 3$. This group has order 2^{m+1}, and the $\mathbb{Z}/2$-socle is the center, which is the subgroup generated by $H^{2^{m-1}}$. This subgroup is isomorphic to $\mathbb{Z}/2$, and hence $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}Q_m \simeq \mathrm{B}\mathbb{Z}/2$ for every m.

We turn now our attention to the semidihedral groups of order 2^m, that admit a presentation $SD_m = \{X, Y;\ X^{2^{m-1}} = Y^2 = 1,\ YXY^{-1} = X^{-1+2^{m-2}}\}$. The $\mathbb{Z}/2$-socle of SD_m is generated by $X^{2^{m-2}}$ and Y, and it is isomorphic to $D_{2^{m-2}}$. So, $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}SD_m$ is homotopy equivalent to $\mathrm{B}D_{2^{m-2}}$.

Finally, if $p = 2$ and $m > 3$, or if p is odd and $m > 2$, we define the group $M_m(p) = \{X, Y;\ X^{p^{m-1}} = Y^p = 1,\ YXY^{-1} = X^{1+p^{m-2}}\}$. The $\mathbb{Z}/p$-socle of

$M_m(p)$ is generated by $X^{p^{m-2}}$ and Y and it is isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Thus, the $B\mathbb{Z}/p$-cellularization of $BM_m(p)$ is $B\mathbb{Z}/p \times B\mathbb{Z}/p$.

These last computations have been very simple, but they give us an explicit expression for the $B\mathbb{Z}/p$-cellularization of a very wide class of p-groups, among which we quote the following families:

- The p-groups of order p^m which contain a cyclic subgroup of order p^{m-1}.
- The p-groups with no noncyclic abelian normal subgroups.
- The groups of order p^3.

The proof of this relies on the classifying results that can be found, for example, in ([Gor80], 5.4).

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