Vertex Algebras and

the Landau-Ginzburg/Calabi-Yau Correspondence

Vassily Gorbounov, Fyodor Malikov

To Borya Feigin on his 50th birthday

Introduction

Introduced in [MSV] for any smooth (algebraic, analytic, etc.) manifold X there is a sheaf of vertex algebras Ω_X^{ch}. For example, the vertex algebra of global sections over $\mathbb{C}^N$, $\Omega_{\mathbb{C}^N}^{ch}(\mathbb{C}^N)$ or simply $\Omega^{ch}(\mathbb{C}^N)$, is well known as "$bc-\beta\gamma$-system", an apparently unsophisticated object. Despite various important contributions [B, BD, BL, KV1], however, very little is known about the cohomology vertex algebra $H^*(X,\Omega_X^{ch})$ even if X is toric – except perhaps the case of $\mathbb{P}^{2n}$ where at least the character of the space of global sections, $H^0(\mathbb{P}^{2n},\Omega_{\mathbb{P}^{2n}}^{ch})$, has been computed: it was shown to be equal to the elliptic genus of $\mathbb{P}^{2n}$ in [MS].

Let

$$\mathfrak{F}=\{f=0\}\subset\mathbb{P}^{N-1}. \qquad (1)$$

be a Calabi-Yau hypersurface. The present paper is devoted to an interplay between $\Omega^{ch}(\mathbb{C}^N)$ and closely related algebras on the one hand, and $H^*(\mathfrak{F},\Omega_{\mathfrak{F}}^{ch})$ on the other. Let us now formulate the main result.

Preparations. Set

$$\Lambda=\mathbb{Z}^N\oplus(\mathbb{Z}^N)^*.$$

Associated to this lattice in the standard manner there are the lattice vertex algebra V_Λ and the fermionic vertex algebra (bc-system), F_Λ, which is none other than the vacuum representation of the infinite-dimensional Clifford algebra $Cl(\mathbb{C}\otimes_{\mathbb{Z}^N}\Lambda)$. In the important paper [B] Borisov introduces the vertex algebra $\mathbb{B}_\Lambda=V_\Lambda\otimes F_\Lambda$ and the vertex algebra embedding

$$\Omega^{ch}(\mathbb{C}^N)\hookrightarrow\mathbb{B}_\Lambda.$$

Define the $\mathbb{Z}_N$-action

$$\mathbb{Z}_N\times\mathbb{C}^N\to\mathbb{C}^N,\ (m,\vec{x})\mapsto\exp{(2\pi i\frac{m}{N})}\vec{x}.$$

There arises the vertex subalgebra of $\mathbb{Z}_N$-invariants

$$(\Omega^{ch}(\mathbb{C}^N))^{\mathbb{Z}_N}\subset\Omega^{ch}(\mathbb{C}^N).$$

partially supported by National Science Foundation

Let us now warp the lattice Λ: denote by $\mathbb{Z}^N_{orb}$ the sublattice of $\mathbb{Z}^N$ consisting $(m_0, m_1, ..., m_{N-1})$ such that $\sum_j m_j$ is divisible by N and define

$$\Lambda_{orb} = \mathbb{Z}^N_{orb} \oplus (\mathbb{Z}^N_{orb})^*.$$

Just as above, there arises the vertex algebra $\mathbb{B}_{\Lambda_{orb}}$. Note that $\mathbb{B}_\Lambda$ and $\mathbb{B}_{\Lambda_{orb}}$ have a non-empty intersection, which contains $(\Omega^{ch}(\mathbb{C}^N))^{\mathbb{Z}_N}$; thus

$$(\Omega^{ch}(\mathbb{C}^N))^{\mathbb{Z}_N} \hookrightarrow \mathbb{B}_{\Lambda_{orb}}.$$

Now we extend $(\Omega^{ch}(\mathbb{C}^N))^{\mathbb{Z}_N}$ inside $\mathbb{B}_{\Lambda_{orb}}$ to a bi-differential vertex algebra.

Let $\{X_i\} \subset \mathbb{Z}^N$, $\{X_i^*\} \subset (\mathbb{Z}^N)^*$ be the standard dual bases. Associated to them inside V_Λ there are fields, such as, $X_i(z)$, $X_i^*(z)$, $e^{X_i^*}(z)$. Let the corresponding tilded letters denote their superpartners inside F_Λ, e.g. $\tilde{X}_i(z)$, $\tilde{X}_i^*(z)$.

Denote

$$X^*_{orb} = \frac{1}{N}(X^*_0 + X^*_1 + \cdots X^*_{N-1}) \in (\mathbb{Z}^N_{orb})^*.$$

Form

$$\widetilde{\mathrm{LG}} = \bigoplus_{n=0}^{\infty} \widetilde{\mathrm{LG}}^{(n)},\ \widetilde{\mathrm{LG}}^{(n)} = (\Omega^{ch}(\mathbb{C}^N))^{\mathbb{Z}_N} e^{nX^*_{orb}}.$$

This is clearly a $\mathbb{Z}_+$-graded subalgebra of $\mathbb{B}_{\Lambda_{orb}}$ (but not of $\mathbb{B}_\Lambda$).

Now define two operators

$$D_{orb} = (\sum_{j=0}^{N-1} e^{X^*_{orb}}(z)\tilde{X}^*_j(z))_{(0)}, d_{LG} = df(z)_{(0)} \in \mathrm{End}(\widetilde{\mathrm{LG}}),$$

where f is the polynomial appearing in (1) and $df(z)$ is computed by using the definition

$$d(x_0^{m_0} x_1^{m_1} \cdots x_{N-1}^{m_{N-1}})(z) = e^{\sum_j m_j X_j}(z) \sum_j m_j \tilde{X}_j(z).$$

These are commuting, square zero derivations of $\widetilde{\mathrm{LG}}$; thus we have obtained the bi-differential vertex algebra $(\widetilde{\mathrm{LG}}; D_{orb}, d_{LG})$. Note that it is filtered by the bi-differential vertex ideals

$$\widetilde{\mathrm{LG}}^{\geq n} = \bigoplus_{m=n}^{\infty} \widetilde{\mathrm{LG}}^{(m)};$$

hence there arises the projective system of bi-differential vertex algebras

$$\widetilde{\mathrm{LG}}^{<n} = \widetilde{\mathrm{LG}}/\widetilde{\mathrm{LG}}^{\geq n}.$$

Theorem 1. (cf. Theorems 4.7, 4.9) *There is a spectral sequence*

$$(E^{*,*}_*, d_*) \Rightarrow H^*(\mathfrak{F}, \Omega^{ch}_{\mathfrak{F}})$$

that satisfies:

(i)

$$(E_1^{*,i-*},d_1)\sim(H^i_{D_{orb}}(\widetilde{\mathrm{LG}}^{<N}),d_{LG});$$

(ii) at the conformal weight zero component this spectral system degenerates in the 2nd term so that

$$H^*(\mathfrak{F},\Lambda^*\mathcal{T}_{\mathfrak{F}})\xrightarrow{\sim}H^*(\mathfrak{F},\Omega^{ch}_{\mathfrak{F}})_0\sim(E_2^{*,*})_0=H_{d_{LG}}(\widetilde{\mathrm{LG}}^{<N})_0,$$

where $\Lambda^*\mathcal{T}_{\mathfrak{F}}$ *is the algebra of polyvector fields over* $\mathfrak{F}$. *Further,*

$$H_{d_{LG}}(\widetilde{\mathrm{LG}}^{(i)})_0=\begin{cases}\mathbb{C}\text{ if }1\leq i\leq N-1,\\ M_f^{\mathbb{Z}_N}\text{ if }i=0,\end{cases}\tag{2}$$

where $M_f^{\mathbb{Z}_N}=(\mathbb{C}[x_0,...,x_{N-1}]/<df>)^{\mathbb{Z}_N}$ *is the* $\mathbb{Z}_N$*-invariant part of the Milnor ring.* □

Remarks.

(i) The sign $\sim$ in item (i) means that rather than being isomorphic the complexes are filtered and the corresponding graded complexes are naturally isomorphic. This is not too serious a complication; in fact, $\sim$ is a genuine isomorphism if $i<N-1$ and there is a 1-step filtration if $i=N-1$.

(ii) The reader familiar with previous work might expect $H^*(\mathfrak{F},\Omega^*_{\mathfrak{F}})\xrightarrow{\sim}H^*(\mathfrak{F},\Omega^{ch}_{\mathfrak{F}})_0$ instead of the first isomorphism in the item (ii) of the theorem. We have indeed changed the conformal grading and find this important; so much so that in the main body of the text (see especially 2.3.3) we change the terminology and notation: we write $\Lambda^{ch}\mathcal{T}_{\mathfrak{F}}$ for $\Omega^{ch}_{\mathfrak{F}}$ with the changed grading and call it the *the algebra of chiral polyvector fields.* □

Now we would like to make two points. First, let us demonstrate how this result works.

Application: an elliptic genus formula. Let $\mathrm{Ell}_{\mathfrak{F}}(\tau,s)$ be the 2-variable elliptic genus of $\mathfrak{F}$ as defined, for example, in [BL] or [KYY].

Introduce

$$E(\tau,s)=\prod_{n=0}^{\infty}\frac{\left(1-e^{2\pi i((n+1)\tau+(1-1/N)s)}\right)^N\left(1-e^{2\pi i(n\tau+(-1+1/N)s)}\right)^N}{\left(1-e^{2\pi i((n+1)\tau+s/N)}\right)^N\left(1-e^{2\pi i(n\tau-s/N)}\right)^N}.$$

It follows easily from Theorem 1 (see 4.11- 4.12) that

$$\mathrm{Ell}_{\mathfrak{F}}(\tau,s)=\frac{1}{N}\sum_{l=0}^{N-1}\sum_{j=0}^{N-1}e^{\pi i(N-2)\{-js+(j^2-j)\tau+j^2\}}E(\tau,s-j\tau-l).\tag{3}$$

The structure of this formula is rather clear: the infinite product $E(\tau,s)$ reflects the polynomial nature of the space $\Omega^{ch}(\mathbb{C}^N)$ of which it is indeed an Euler character, the summation with respect to l extracts the $\mathbb{Z}_N$-invariants, and the summation w.r.t. j reminds of the summation over "twisted sectors" because the change of variable $s \mapsto s - j\tau$ is reminiscent of the spectral flow. All of this smacks of an orbifold and indeed formula (3) was proposed in [KYY] as the elliptic genus of the Landau-Ginzburg orbifold. This brings about the 2nd point we would like to make.

Landau-Ginzburg orbifold interpretation. The Landau-Ginzburg model is associated to an affine manifold and a function over it known as superpotential. In the case where the manifold is $\mathbb{C}^N$ and the function is a homogeneous polynomial f with a unique singularity at 0, Witten's discovery [W2] can perhaps be formulated in the language accessible to us as follows:

1) There is an action of the $N = 2$ superconformal algebra on $\Omega^{ch}(\mathbb{C}^N)$ such that it commutes with the differential $df(z)_{(0)}$.

2) The cohomology vertex algebra $H_{df(z)_{(0)}}(\Omega^{ch}(\mathbb{C}^N))$ with thus defined action of the $N = 2$ superconformal algebra is the chiral algebra attached to the Landau-Ginzburg model with superpotential f.

We find it convenient not to pass to the cohomology but to declare the Landau-Ginzburg model to be the differential vertex algebra $(\Omega^{ch}(\mathbb{C}^N), df(z)_{(0)})$ with the above fixed $N = 2$ superconformal algebra action.

Remark. We shall argue in 5.1.5 that alternatively one can think of $(\Omega^{ch}(\mathbb{C}^N), df(z)_{(0)})$ as the "right definition" of the chiral de Rham complex $\Omega^{ch}_{\mathrm{Spec} M_f}$ over the spectrum of the Milnor ring. □

Next consider the space $\Omega^{ch}(\mathbb{C}^N)e^{nX^*_{orb}}$. It does not belong to $\mathbb{B}_{\Lambda_{orb}}$ but carries a canonical structure of a *twisted* $\Omega^{ch}(\mathbb{C}^N)$-module and this is synonimous to being a twisted sector. Therefore, taking the direct sum of these, $0 \leq n \leq N-1$ and then extracting $\mathbb{Z}_N$-invariants corresponds accurately with what physicists call the Landau-Ginzburg orbifold, see e.g. [V]. In the notation we have adopted, the formula describing the outcome of this process is $(\widetilde{\mathrm{LG}}^{<N}; d_{LG})$; see sect. 5 for details. Thus item (i) of the above theorem can be interpreted as follows:

there is a spectral sequence abutting to $H^*(\mathfrak{F}, \Omega^{ch}_{\mathfrak{F}})$ whose 1st term is isomorphic to the D_{orb}-cohomology of the Landau-Ginzburg orbifold $H_{D_{orb}}(\widetilde{LG}^{<N})$.

One can say that the Landau-Ginzburg orbifold $(\widetilde{\mathrm{LG}}^{<N}; d_{LG})$ approximates the vertex algebra $H^*(\mathfrak{F}, \Omega^{ch}_{\mathfrak{F}})$. This approximation is consistent with the $N = 2$ supersymmetry. Indeed, on the one hand, the $N = 2$ superconformal algebra action on the Landau-Ginzburg model commutes with D_{orb} and all higher differentials and thus descends to an action on $H^*(\mathfrak{F}, \Omega^{ch}_{\mathfrak{F}})$. On the other hand, since $\mathfrak{F}$ is Calabi-Yau, $H^*(\mathfrak{F}, \Omega^{ch}_{\mathfrak{F}})$ carries a canonical $N = 2$ superconformal algebra action [MSV]. A direct computation (Lemmas 4.10.1, 5.1.1, 5.2.14) shows that

both the $N = 2$ superconformal algebra actions coincide.

Furthermore, not only one of the spaces involved in assertion (i) of the theorem, but both assertions (i, ii) themselves are reminiscent of some of the important developments in string theory. In order to explain this we shall have to pluck courage and discuss a little more of physics.

It seems that except for the torus case, see e.g. a mathematical exposition in [KO], the state space of the model describing the string propagation on a manifold is unknown even as a vector space to say nothing about its algebraic, vertex or otherwise, structure. One striking result towards understanding what this might be is Gepner's model proposed in [G]. Gepner's paper, a combination of guesswork and computational *tour de force*, is not an easy reading. More conceptual approach emerged soon afterwards, e.g. [V,VW], proclaiming that the Landau-Ginzburg orbifold is equivalent (in this or that sense) to the string theory on Calabi-Yau hypersurfaces in weighted projective spaces. (So far as we can tell, apart from both the theories carrying an $N=2$ symmetry with the same central charge and integral $U(1)$-charges, most of the supporting evidence amounted to the isomorphism of chiral rings – exactly as in Theorem 1 (ii).) This activity seems to have been crowned by Witten's paper [W1] where it is asserted, and we cite, "that rather than Landau-Ginzburg being "equivalent" to Calabi-Yau, they are two different phases of the same system."

Now, if one is allowed to think of a phase transition as a family where at certain values of the parameter something happens, then it seems that spectral sequences might be relevant. For example, the spectral sequence mentioned in Theorem 1 comes from a double complex equipped with two differentials. Let us denote them for the purposes of introduction by d_+ and d_-. The vertex algebra $H^*(\mathfrak{F},\Omega^{ch}_{\mathfrak{F}})$ is the cohomology of the total differential $d_+ + d_-$, and $H_{D_{orb}}(\widetilde{\mathrm{LG}}^{<N})$ arises as the d_--cohomology. Introduce a parameter, t, and form the differential $td_+ + d_-$. This defines a family of complexes over a line such that at $t=0$ the cohomology is $H_{D_{orb}}(\widetilde{\mathrm{LG}}^{<N})$, and elsewhere it is $H^*(\mathfrak{F},\Omega^{ch}_{\mathfrak{F}})$.

Furthermore, the geometric background as explained in sect. 4 of [W1] is very similar to that we use in the proof of Theorem 1.

Sketch of proof. The proof is based on the computation of two spectral sequences, both due to [B]. The first allows, in a sense, to replace $\mathfrak{F}$ with the canonical line bundle, $\mathcal{L}^*$, over $\mathbb{P}^{N-1}$. The 1st term of this sequence is $H^*(\mathcal{L}^*,\Omega^{ch}_{\mathcal{L}^*})$ and the corresponding differential d_1 has the meaning of the chiral Koszul differential[1]. Most of Theorem 1 is about identification of the complex $(H^*(\mathcal{L}^*,\Omega^{ch}_{\mathcal{L}^*}),d_1)$.

We identify this space in the following way. Along with $\mathcal{L}^*$ consider $\mathcal{L}^*-0$ obtained by deleting the zero section. The Čech complex that computes the desired $H^*(\mathcal{L}^*,\Omega^{ch}_{\mathcal{L}^*})$ naturally embeds into the analogous Čech complex over $\mathcal{L}^*-0$. Write this down schematically as

$$\check{C}(\mathcal{L}^*)\hookrightarrow\check{C}(\mathcal{L}^*-0).$$

Proposed in [B] there is a vertex algebra resolution that allows to extend the latter embedding to a resolution of complexes:

$$\check{C}(\mathcal{L}^*)\hookrightarrow\check{C}(\mathcal{L}^*-0)\to\check{C}(\mathcal{L}^*-0)^{(1)}\to\check{C}(\mathcal{L}^*-0)^{(2)}\to\cdots.$$

(As an aside let us note that this resolution serves the same purpose as the Cousin resolution but has a different flavor: its terms $\check{C}(\mathcal{L}^*-0)^{(j)}$ are identified with each

[1] this correspondence between $H^*(\mathcal{L}^*,\Omega^{ch}_{\mathcal{L}^*})$ and $H^*(\mathfrak{F},\Omega^{ch}_{\mathfrak{F}})$ reminds one of the main result in [S]

other as vector spaces, but are spectral flow transforms of each other as vertex modules.) There arises then the bi-complex

$$0 \to \check{C}(\mathcal{L}^* - 0) \to \check{C}(\mathcal{L}^* - 0)^{(1)} \to \check{C}(\mathcal{L}^* - 0)^{(2)} \to \cdots . \tag{4}$$

whose total cohomology is the desired $H^*(\mathcal{L}^*, \Omega^{ch}_{\mathcal{L}^*})$. Finally, and this is the geometry bit reminiscent of [W1], there is an isomorphism

$$(\mathbb{C}^N - 0)/\mathbb{Z}_N \stackrel{\sim}{\longrightarrow} \mathcal{L}^* - 0.$$

Pulling bi-complex (4) back onto $(\mathbb{C}^N - 0)/\mathbb{Z}_N$ and further writing it down in terms of the coordinates on the universal covering space $\mathbb{C}^N - 0$ allows to compute all the terms of the corresponding spectral sequence and thus identify $(H^*(\mathcal{L}^*, \Omega^{ch}_{\mathcal{L}^*}), d_1)$ with $(H_{D_{orb}}(\widetilde{\mathrm{LG}}^{<N}), d_{LG})$ as asserted in Theorem (i). (Note that these two pull-backs are made possible by the *naturality* property of Ω^{ch}_X, see 2.1.) $\square$

The "conformal weight zero component of this argument" gives a self-contained computation of the cohomology algebra of polyvector fields $H^*(\mathfrak{F}, \Lambda^*\mathcal{T}_{\mathfrak{F}})$ along with explicit "vertex" formulas for the cocycles representing the cohomology classes, see 4.13. Classically, the computation uses $H^*(\mathfrak{F}, \mathbb{C})$ obtained in [Gr], the Serre duality, and variations of the Hodge structure [Don, Theorem 2.2] – and even then powers of the class representing hyperplane sections require special, although simple, treatment. In our approach, it is the other way around: the fact that the Milnor ring is realized inside $H^*(\mathfrak{F}, \mathbb{C})$ is almost obvious, even its Koszul resolution arises naturally at an earlier term of the spectral sequence; on the other hand, the Serre duals of powers of hyperplane section are "interesting" because they are produced by the "twisted sectors" as indicated in Theorem 1 (ii), 1st line in (2). We restore the multiplicative structure of $H^*(\mathfrak{F}, \mathbb{C})$ by vertex algebra methods twice: first, in 4.13 in the context of the proof of Theorem 1; second, in 5.2.17 by way of testing our conjectural construction of the vertex algebra structure on the Landau-Ginzburg orbifold.

Acknowledgements. The indebtedness of this work to Borisov's constructions [B] should be clear to any reader. We gratefully acknowledge many illuminating conversations with V.Batyrev, A.Gerasimov, A.Givental, C.Hertling, M.Kapranov, R.Kaufmann, Yu.I.Manin, A.Semikhatov, A.Vaintrob. Special thanks go to V.Schechtman who participated at the early stages of this research and was the first enthusiast of the algebra of chiral polyvector fields. This work was started in 2002 when we were visiting the Max-Planck-Institut für Mathematik in Bonn and finished a year later at the same place. We are grateful to the institute for excellent working conditions.

1. Vertex algebras

This section is only a collection of well-known facts and examples that will be needed in the sequel and the reader may want to consult either [K] or [FB-Z] for more detail. We would like to single out sect. 1.12 on the spectral flow, the notion that seems to be not too popular in mathematics literature but is essential for understanding orbifolds and will reappear several times in sect. 2.3.5, 3.10, 4.6, 5.2.15. Our treatment of the spectral flow is greatly influenced by [LVW].

1.1. A vector space V is called a supervector space if it is $\mathbb{Z}_2$-graded, that is, $V = V^{(0)} \oplus V^{(1)}$. We define the parity par(a) of $a \in V$ so that par$(a) = \epsilon$ if and only if $a \in V^{(\epsilon)}$. If V and W are supervector spaces, then $V \otimes W$ is also with $\mathrm{par}(a \otimes b) = \mathrm{par}(a) + \mathrm{par}(b)$, and so is $\mathrm{Hom}_{\mathbb{C}}(V, W)$.

Given a supervector space V, let Field(V) be the subspace of $\mathrm{End}(V)[[z, z^{\pm 1}]]$ consisting of such formal series $x(z) = \sum_{n \in \mathbb{Z}} x_{(n)} z^{-n-1}$ that for any $a \in V$

$$x_{(n)} a = 0 \text{ if } n >> 0. \tag{1.1.1}$$

1.2. Definition. A vertex algebra is a supervector space V with a distinguished element $\mathbf{1} \in V$ called *vacuum* and a parity preserving map $Y(.,z): V \to \mathrm{Field}(V)$, $Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$, such that the following axioms hold:

(i) *vacuum*:

$$Y(\mathbf{1}, z) = Id_V,\ a_{(-1)}\mathbf{1} = a; \tag{1.2.1}$$

(ii) *Borcherds identity:* for any $a, b \in V$ and any rational function $F(z, w)$ in z, w with poles only at $z = 0$, $w = 0$, $z - w = 0$

$$\begin{aligned} &\mathrm{Res}_{z-w} Y(Y(a, z-w)b, w) i_{w,z-w} F(z, w) \\ &= \mathrm{Res}_z \left(Y(a, z)Y(b, w) i_{z,w} F(z, w) - (-1)^{\mathrm{par}(a)\mathrm{par}(b)} Y(b, w) Y(a, z) i_{w,z} F(z, w) \right). \end{aligned} \tag{1.2.2}$$

□

In the latter formula the standard notation is used: Res_t means the coefficient of t^{-1} in the indicated formal Laurent expansion; $i_{\bullet,\bullet}$ specifies exactly which Laurent expansion is to be used, e.g. $i_{z,w}$ stands for the expansion in the domain $|w| < |z|$, $i_{w,z-w}$ for that in the domain $|z - w| < |w|$, etc.

One thinks of $a_{(n)}$ as the " n-th multiplication by a", so there arises a family of multiplication

$$_{(n)} : V \otimes V \to V. \tag{1.2.3}$$

1.3. W- and cohomology vertex algebras.

If $F = 1$, then (1.2.2) gives

$$(a_{(0)}b)_{(n)} = [a_{(0)}, b_{(n)}],\ n \in \mathbb{Z}. \tag{1.3.1}$$

In other words, for any $a \in V$, $a_{(0)} \in \mathrm{End}V$ is a derivation of all the multiplications. Hence,

$$\mathrm{Ker} a_{(0)} \subset V \tag{1.3.2}$$

is a vertex subalgebra known as a W-algebra.

Furthermore, suppose $a \in V$ is odd and $a_{(0)}a = 0$. Then (1.3.1) implies that $a_{(0)}^2 = 0$ and $\mathrm{Im} a_{(0)} \subset \mathrm{Ker} a_{(0)}$ is an ideal. Therefore, the cohomology

$$H_{a_{(0)}}(V) \stackrel{\text{def}}{=} \mathrm{Ker} a_{(0)} / \mathrm{Im} a_{(0)} \tag{1.3.3}$$

carries a canonical vertex algebra structure. The vertex algebras to be used in this text will mostly be either W- or cohomology vertex algebras.

1.4. Chiral rings. Suppose that V is graded so that

$$V = \oplus_{n=0}^{\infty} V_n \text{ and } V_n\,{}_{(r)}V_m \subset V_{n+m-r-1}. \tag{1.4.1}$$

The grading satisfying this condition will be called *conformal.*

One can show that if (1.4.1) is valid, then

$$_{(-1)}: V_0 \otimes V_0 \to V_0$$

is associative and supercommutative. In the context of the unitary $N = 2$ supersymmetry, supercommutative associative algebras attached to graded vertex algebras in this way are often called *chiral rings* [LVW] – not to be confused with chiral algebras although these rings are indeed algebras. We shall take the liberty to call these rings chiral in any case.

1.5. Remarks.

(i) Let $\delta(z-w) = \sum_{n\in\mathbb{Z}} z^n w^{-n-1}$. It follows from (1.2.2) that $[Y(a,z), Y(b,w)]$ is local, that is, equals a linear combination of the delta-function derivatives, $\partial_w^n \delta(z-w)$, over fields in w.

(ii) A vertex algebra V is said to be generated by a collection of fields $Y(a_\alpha, z)$, $\{a_\alpha\} \subset V$ if V is the linear span of (non-commutative) monomials in $(a_\alpha)_{(j)}$ applied to vacuum $\mathbf{1} \in V$. The important *reconstruction theorem*, e.g. [K, Theorem 4.5], says, and we are omitting some details, that this can be reversed: if there is a collection of mutually local fields $v_\alpha(z)$ which generate V from a fixed vector, then V carries a unique vertex algebra structure such that $v_\alpha(z) = Y(v_\alpha, z)$ for some $v_\alpha \in V$. Because of this we will allow ourselves in our list of well-known examples, which we are about to begin, to fix only a space V and a collection of mutually local fields that generate this space. Typically, we shall have a Lie algebra, a collection of fields with values in this algebra, and a representation of this algebra such that nilpotency condition (1.1.1) is satisfied by the fields.

(iii) It should be clear what a vertex algebra homomorphism is. As in (ii), speaking of homomorphisms we shall often specify only images of generating fields.

Now to some basic excamples.

1.6. *bc*-system. Let Cl be the Lie superalgebra with basis $b_{(i)}, c_{(i)}$, $i \in \mathbb{Z}$ (all odd) and C (even) and commutation relations

$$[b_{(i)}, c_{(j)}] = \delta_{i,-j-1} C,\ [C, c_i] = [C, b_i] = 0,\ i, j \in \mathbb{Z}. \tag{1.6.1}$$

Let

$$F = \mathrm{Ind}_{Cl}^{Cl_+} \mathbb{C}, \tag{1.6.2}$$

where Cl_+ is the Lie subalgebra spanned by $x_{(i)}$, C, $i \geq 0$, and $\mathbb{C}$ is an Cl_+-module where $x_{(i)}$'s act by 0 and C as multiplication by 1. In terms of fields $x(z) \sum_{i\in\mathbb{Z}} x_{(i)} z^{-i-1}$, $x = b$ or c, (1.6.1) becomes:

$$[b(z), c(z)] = \delta(z-w). \tag{1.6.3}$$

Hence the vertex algebra structure on F, see 1.5.

A little more generally, to any purely odd $\mathbb{C}$-vector W with a non-degenerate symmetric form $(.,.)$ one can attach the Lie superalgebra $Cl(W) = W \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}C$ with the following bracket: if $x(z) = \sum_{i\in\mathbb{Z}}(x\otimes t^i)z^{-i-1}$, then

$$[x(z), y(z)] = \delta(z-w)(x,y)C, \ [x(z), C] = 0. \tag{1.6.4}$$

The corresponding vertex algebra is

$$F_W = \mathrm{Ind}_{Cl(W)_+}^{Cl(W)}\mathbb{C}, \tag{1.6.5}$$

where $Cl(W)_+ = W\otimes\mathbb{C}[t] \oplus \mathbb{C}C$, $W\otimes\mathbb{C}[t]$ operates on $\mathbb{C}$ by 0, and C by 1.

1.7. $\beta\gamma$**- and** $bc - \beta\gamma$**-system.** The $\beta\gamma$-system is obtained by the "parity change" functor applied to the beginning of 1.6: the even Lie algebra $\mathfrak{a}$ is spanned by $\beta_{(i)}$, $\gamma_{(i)}$, C, the bracket is

$$[\beta(z), \gamma(w)] = -[\gamma(z), \beta(w)] = \delta(z-w)C, \ [C, x(z)] = 0. \tag{1.7.1}$$

The vertex algebra, B, is defined in the same way as F, see (1.6.2).

If V and W are vertex algebras, then $V\otimes W$ carries the standard vertex algebra structure. Denote

$$FB = F\otimes B. \tag{1.7.2}$$

This algebra and its modifications are local models for the chiral de Rham complex.

1.8. Heisenberg algebra. Let, as in the end of 1.6, $\mathfrak{h}$ be a purely even vector space with a non-degenerate symmetric form $(.,.)$. There arises then the Heisenberg Lie algebra

$$\hat{\mathfrak{h}} = \mathfrak{h}\otimes\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\cdot C$$

with bracket defined by letting fields be $x(z) = \sum_{i\in\mathbb{Z}}(x\otimes t^i)z^{-i-1}$ and then setting

$$[a(z), b(w)] = (a,b)\partial_w\delta(z-w), \ [C, a(z)] = 0. \tag{1.8.1}$$

The vertex algebra attached to this Lie algebra is

$$V(\mathfrak{h}) = \mathrm{Ind}_{\hat{\mathfrak{h}}_+}^{\hat{\mathfrak{h}}}\mathbb{C}, \tag{1.8.2}$$

where $\hat{\mathfrak{h}}_+ = \mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}\cdot C$, $\hat{\mathfrak{h}}_+ = \mathfrak{h}\otimes\mathbb{C}[t]$ perates on $\mathbb{C}$ by zero, C by 1.

1.9. Lattice vertex algebras. We shall neeed a lattice L, that is, a free abelian group with integral bilinear form $(.,.)$ and a 2-cocycle

$$\epsilon: \ L\times L\to\mathbb{C}^*. \tag{1.9.1}$$

There arise the group algebra $\mathbb{C}[L]$ with multiplication $e^\alpha\cdot e^\beta = e^{\alpha+\beta}$, $\alpha,\beta\in L$, and the twisted group algebra, $\mathbb{C}_\epsilon[L]$, equal to $\mathbb{C}[L]$ as a vector space but with twisted multiplication:

$$e^\alpha\cdot_\epsilon e^\beta = \epsilon(\alpha,\beta)e^{\alpha+\beta}. \tag{1.9.2}$$

Let $\mathfrak{h}_L = \mathbb{C} \otimes_{\mathbb{Z}} L$. There arises the Heisenberg vertex algebra $V(\mathfrak{h}_L)$, see (1.8.2).

As a vector space, the lattice vertex algebra is defined by

$$V_L = V(\mathfrak{h}_L) \otimes \mathbb{C}_\epsilon[L], \ \mathrm{par}(V(\mathfrak{h}_L) \otimes e^\alpha) \equiv (\alpha, \alpha) \bmod 2, \tag{1.9.3}$$

where par means the parity, see 1.1.

This vertex algebra is generated by the familiar fields $x(z) = \sum_{i\in\mathbb{Z}} (x \otimes t^i) z^{-i-1}$ attached to $x_{(-1)} \otimes 1$ and the celebrated vertex operators

$$e^\alpha(z) = e^\alpha \exp\left(\sum_{j<0} \frac{\alpha_{(j)}}{-j} z^{-j}\right) \exp\left(\sum_{j>0} \frac{\alpha_{(j)}}{-j} z^{-j}\right) z^{\alpha_{(0)}}, \tag{1.9.4}$$

attached to $1 \otimes e^\alpha$.

The action of $x_{(i)}$, $i \neq 0$ ignores $\mathbb{C}_\epsilon(L)$, the action of $x_{(0)}$ is uniquely determined by

$$x_{(0)}(1 \otimes e^\alpha) = (\alpha, x) \otimes e^\alpha. \tag{1.9.5}$$

The following commutator and OPE formulas are valid:

$$[x(z), e^\alpha(w)] = \delta(z-w)(x, \alpha) e^\alpha(w), \tag{1.9.6}$$

$$e^\alpha(z) e^\beta(w) = (z-w)^{(\alpha,\beta)} : e^\alpha(z) e^\beta(w) :, \tag{1.9.7}$$

and the reader is advised to consult [K, (5.4.5b)] for the meaning of the two-variable field $: e^\alpha(z) e^\beta(w) :$. Note that (1.9.7) allows to compute all operations $(e^\alpha)_{(n)} e^\beta$. We shall need the following particular cases:

$$(e^\alpha)_{(-1)} e^\beta = \lim_{z\to w} (z-w)^{(\alpha,\beta)} : e^\alpha(z) e^\beta(w) := \begin{cases} 0 \text{ if } (\alpha,\beta) > 0 \\ \epsilon(\alpha,\beta) e^{\alpha+\beta} \text{ if } (\alpha,\beta) = 0, \end{cases} \tag{1.9.8}$$

$$[e^\alpha(z), e^\beta(w)] = 0 \text{ if } (\alpha, \beta) \geq 0. \tag{1.9.9}$$

For the future use let us mention that for any sub-semigroup $M \subset L$ there arises the vertex subalgebra

$$V_{M,L} \stackrel{\text{def}}{=} V(\mathfrak{h}_L) \otimes \mathbb{C}_\epsilon[M] \subset V_L. \tag{1.9.10}$$

naturally graded by M.

1.10. $N = 2$ super-Virasoro algebra. The celebrated $N = 2$ super-Virasoro algebra, to be denoted $N2$ following [K], is a supervector space with basis $G_{(n)}$, $Q_{(n)}$, $n \in \mathbb{Z}$ (all odd), $L_{(n)}$, $J_{(n)}$, $n \in \mathbb{Z}$, C (all even), and bracket

$$\begin{aligned} &[L(z), L(w)] = 2\partial_w \delta(z-w) L(w) + \delta(z-w) L(w)' \\ &[J(z), J(w)] = \partial_w \delta(z-w) C/3, \end{aligned} \tag{1.10.1a}$$

$$[L(z),G(w)] = 2\partial_w\delta(z-w)G(w) + \delta(z-w)G(w)',$$
$$[J(z),G(w)] = \delta(z-w)G(w), \tag{1.10.1b}$$

$$[L(z),Q(w)] = \partial_w\delta(z-w)Q(w) + \delta(z-w)Q(w)',$$
$$[J(z),Q(w)] = -\delta(z-w)Q(w), \tag{1.10.1c}$$

$$[L(z),J(w)] = \partial_w^2\delta(z-w)\frac{C}{6} + \partial_w\delta(z-w)J(w) + \delta(z-w)J(w)', \tag{1.10.1d}$$

$$[Q(z),G(w)] = \partial_w^2\delta(z-w)\frac{C}{6} - \partial_w\delta(z-w)J(w) + \delta(z-w)L(w). \tag{1.10.1e}$$

The vertex algebra structure is carried by the following $N2$-module:

$$V(N2)_c = \mathrm{Ind}_{N2_\geq}^{N2}\mathbb{C}_c. \tag{1.10.2}$$

where $N2_\geq$ is the subalgebra linearly spanned by $G_{(n)}, L_{(n)}, Q_{(n)}, J_{(n)}, C$, $n \geq 0$, and on $\mathbb{C}_c$ $G_{(n)}, L_{(n)}, Q_{(n)}, J_{(n)}$ operate by 0, and C as multiplication by c.

1.10.1. *Definition.* An $N2$-structure on a vertex algebra W is a vertex algebra homomorphism $V(N2)_c \to W$. □

Note that the field $L(z)$ generates the Virasoro algebra; thus an $N2$-structure on a vertex algebra induces a conformal structure, and the grading by eigenvalues of $L_{(1)}$ is conformal, cf. (1.4.1).

1.11. Automorphisms. It is obvious that if W is a vector space with a symmetric non-degenerate bilinear form, then there is an embedding

$$\mathrm{O}(W) \hookrightarrow \mathrm{Aut}F_W, \tag{1.11.1}$$

where $\mathrm{O}(W)$ is the orthogonal group and F_W a vertex algebra defined in (1.7.2); this comes from the standard action of $\mathrm{O}(W)$ on the Clifford Lie algebra $Cl(W)$.

The analogous construction with $\mathrm{O}(W)$ replaced with $\mathrm{Aut}(L)$ and F_W with V_L does not quite work because of the cocycle (1.9.1). Here is one trivial observation: if we let $\mathrm{Aut}_\epsilon(L)$ be the subgroup of $\mathrm{Aut}(L)$ stabilizing ϵ, then there is an (obvious) embedding:

$$\mathrm{Aut}_\epsilon(L) \hookrightarrow \mathrm{Aut}V_L. \tag{1.11.2}$$

The Lie algebra $N2$ affords an exceptional, mirror symmetry automorphism

$$Q(z) \mapsto G(z), G(z) \mapsto Q(z), J(z) \mapsto -J(z), L(z) \mapsto L(z) + J(z)'. \tag{1.11.3}$$

1.12. Spectral flow. In all our examples except V_L vertex algebras came from infinite dimensional Lie algebras. One feature these Lie algebras have in common is that they admit a *spectral flow.*

Define for any $n \in \mathbb{Z}$ a linear transformation:

$$S_n: \quad \begin{aligned} Cl \to Cl, &\text{ s.t. } b(z) \to b(z)z^{-n}, c(z) \to c(z)z^n,\\ \mathfrak{a} \to \mathfrak{a}, &\text{ s.t. } \beta(z) \to \beta(z)z^{-n}, \gamma(z) \to \gamma(z)z^n,\\ N2 \to N2, &\text{ s.t. } Q(z) \to Q(z)z^n, G(z) \to G(z)z^{-n},\\ &J(z) \mapsto J(z) - \frac{1}{z}\frac{nC}{3},\\ &L(z) \mapsto L(z) - \frac{1}{z}nJ(z) + \frac{1}{z^2}n(n-1)\frac{C}{6}, \end{aligned} \tag{1.12.1}$$

where we abused the notation by letting the same letter stand for the maps of different spaces, Cl, $\mathfrak{a}$, $N2$, defined in 1.6,7,10 resp. We hope this will not lead to confusion. An untiring reader will check that in each of the cases, S_n is an automorphism of the Lie algebra in question.

Maps (1.12.1) generate a $\mathbb{Z}$-action on each of the algebras known as the spectral flow. In each of the cases, therefore, there arises a family of functors on the category of modules

$$S_n : \text{Mod} \to \text{Mod},\ M \mapsto S_n(M), \tag{1.12.2}$$

action on $S_n(M)$ being defined by precomposing that on M with S_n of (1.12.1).

The origin of spectral flows (1.12.1) belongs to lattice vertex algebras. Let V_L be a lattice vertex algebra and $\text{Lie}V_L$ the linear span inside $\text{End}V_L$ of the coefficients of the fields $v(z)$, $v \in V$. It is well known [K, F-BZ] that $\text{Lie}V_L$ is a Lie subalgebra of $\text{End}V_L$.

If $M \subset L$ is a sub-semigroup, then we have, cf. (1.9.5),

$$V_{M,L} \subset V_L,\ \text{Lie}V_{M,L} \subset \text{Lie}V_L, \tag{1.12.3}$$

Let

$$e^\alpha : V_L \to V_L \tag{1.12.4}$$

be multiplication by e^α. If the restriction of the cocycle $\epsilon(.,.)$ to $M \subset L$ is trivial, that is,

$$\epsilon(M,.) = 1, \tag{1.12.5}$$

then the conjugation by map (1.12.4) defines an automorphism

$$S_\alpha : \text{Lie}V_{M,L} \to \text{Lie}V_{M,L},\ X \mapsto (e^\alpha)^{-1} \circ X \circ e^\alpha;\ \alpha \in L. \tag{1.12.6}$$

For example, under this map

$$e^\beta(z) \mapsto e^\beta(z) z^{(\alpha,\beta)},\ \beta \in M \tag{1.12.7}$$

cf. (1.12.1); the desired power of z ows its appearance to the factor $z^{\alpha(0)}$ in (1.9.4). Likewise,

$$x(z) \mapsto x(z) + \frac{(\alpha, x)}{z}. \tag{1.12.8}$$

Spectral flows (1.12.1) are all obtained as follows: embed the corresponding vertex algebra into an appropriate V_L, thus obtain a morphism of the corresponding Lie algebra $\text{Lie}(\bullet) \to \text{Lie}V_L$, and then restrict “spectral flow in the direction α” (1.12.6) to the image. This operation will be of importance for us in 3.10, 4.6, 5.2.15.

1.13. Boson-fermion correspondence. Let $\mathbb{Z}$ be the standard 1-dimensional lattice: this means that if we let χ be the generator, then $(\chi, \chi) = 1$. There arise $V_{\mathbb{Z}}$, where $\epsilon(.,.) = 1$, and the famous vertex algebra isomorphism:

$$\begin{aligned} &F \xrightarrow{\sim} V_{\mathbb{Z}},\ b(z) \mapsto e^\chi(z), \\ &c(z) \mapsto e^{-\chi}(z), :b(z)c(z): \mapsto \chi(z) \end{aligned} \tag{1.13.1}$$

The interested reader will check that $S_N|_{Cl}$ of (1.12.1) is indeed implemented by conjugation with $e^{-n\chi}$.

Likewise

$$F^{\otimes n} \xrightarrow{\sim} V_{\mathbb{Z}^n},\ b_i(z) \mapsto e^{\chi_i}(z), \\ c_i(z) \mapsto e^{-\chi_i}(z), : b_i(z)c_i(z) :\mapsto \chi_i(z), \tag{1.13.2}$$

where the cocycle on $V_{\mathbb{Z}^n}$ is chosen so as to ensure that $V_{\mathbb{Z}^n} \xrightarrow{\sim} V_{\mathbb{Z}}^{\otimes n}$.

2. The algebra of chiral polyvector fields

This section is a reminder on the chiral de Rham complex. Our exposition is close to [MSV] but has been influenced by [GMS]. We would like to single out 2.3.3, where the title is clarified, and 2.3.5, where a simple cohomology computation is carried out; the results of this computation will play an important role in the proof of Theorem 4.7. Sect. 2.4 is an exposition of a result of [B].

2.1. Suppose we have a family of sheaves of vector spaces $\mathcal{A}_X$, one for each smooth algebraic manifold X. We shall call $\mathcal{A}_X$ *natural* if for any étale morphism $\phi : Y \to X$ there is a sheaf embedding $\mathcal{A}(\phi) : \phi^{-1}\mathcal{A}_X \hookrightarrow \mathcal{A}_Y$ such that given a diagram:

$$X \stackrel{\phi}{\to} Y \stackrel{\psi}{\to} Z$$

the following associativity condition holds:

$$\mathcal{A}(\psi \circ \phi) = \mathcal{A}(\phi) \circ \phi^{-1}(\mathcal{A}(\psi)),$$

where ϕ^{-1} is understood as the inverse image functor on the category of sheaves of vector spaces.

It should be clear what a natural sheaf morphism $\mathcal{A}_X \to \mathcal{B}_X$ means.

Here are some obvious examples: $\mathcal{O}_X$, Ω^1_X, $\mathcal{T}_X$, and sheaves obtained as a result of all sorts of tensor operation performed on these. Note that all these sheaves are sheaves of $\mathcal{O}_X$-modules and our definition ignores this extra structure. However, one talks about natural sheaves $\mathcal{A}_X$ of different classes of algebras, such as, commutative, associative, Lie, vertex, etc., by requiring that $\mathcal{A}_\phi$ preserve this structure.

Constructed in [MSV] for any smooth algebraic manifold X there is a sheaf of vertex algebras, Ω^{ch}_X. It satisfies the following conditions.

(i) Ω^{ch}_X is natural as a sheaf of vertex algebras and it carries a bi-grading $\Omega^{ch}_X = \oplus_{m,n}\Omega^{ch,m}_{X,n}$ such that each homogeneous component $\Omega^{ch,m}_{X,n}$ is a natural sheaf of vector spaces.

(ii) There are natural morphisms:

$$\Omega^*_X \hookrightarrow \Omega^{ch}_X \hookleftarrow \Lambda^*\mathcal{T}_X; \tag{2.1.1}$$

(iii) Ω^{ch}_X is not a sheaf of $\mathcal{O}_X$-modules, but it carries a filtration such that there is the following family of natural sheaf isomorphisms

$$\mathrm{Gr}\Omega^{ch}_X \xrightarrow{\sim} \bigotimes_{n\geq 0}\left(S^*_{q^{n+1}}(\mathcal{T}_X) \otimes S^*_{q^n}(\mathcal{T}^*_X) \otimes \Lambda^*_{q^{n+1}y}(\mathcal{T}_X) \otimes \Lambda^*_{q^n y^{-1}}(\mathcal{T}^*_X)\right). \tag{2.1.2}$$

where we habitually use the following " generating functions of families of sheaves":

$$\mathrm{Gr}\Omega^{ch}_X = \bigoplus_{m,n} q^n y^m \mathrm{Gr}\Omega^{ch,m}_{X,n}, S^*_t(\mathcal{A}) = \bigoplus_{n=0}^{\infty} t^n S^n(\mathcal{A}), \Lambda^*_t(\mathcal{A}) = \bigoplus_{n=0}^{\infty} t^n \Lambda^n(\mathcal{A}).$$

(iv) it follows from (i) that for any X there is a canonical group embedding

$$\rho_X : \mathrm{Aut}X \to \mathrm{Aut}\Omega^{ch}_X(X). \tag{2.1.3}$$

Explicit formulas for the latter appeared in [MSV, (3.1.6)] as a result of guesswork and were used to define Ω^{ch}_X satisfying (i-iii).

In 2.2 we shall look at some examples that serve as a local model and are needed later; in 2.3 we shall very briefly discuss how these local models are glued together and what effect the gluing has on $N2$-structures and chiral rings.

2.2. A local model.

It is easiest to begin with a local situation in the presence of a coordinate system.

2.2.1. Let U be a smooth affine manifold with a coordinate system $\vec{x}$ by which we mean a collection of functions $x_1, ..., x_n \in \mathcal{O}(U)$, $n = \dim U$, such that the differential forms $dx_1, ..., dx_n$ form a basis of the space of 1-forms $\mathcal{T}^*(U)$ over $\mathcal{O}(U)$. A coordinate system determines a collection of vector fields $\partial_{x_1}, ..., \partial_{x_n}$ such that

$$\partial_{x_i} x_j =< \partial_{x_i}, dx_j >= \delta_{ij}.$$

It follows that $[\partial_{x_i}, \partial_{x_j}] = 0$ for all i, j, and $\partial_{x_1}, ..., \partial_{x_n}$ form a basis of the space of vector fields $\mathcal{T}^*(U)$.

Let $\Omega^{ch}(U, \vec{x})$ be the following superpolynomial ring over $\mathcal{O}(U)$.

$$\Omega^{ch}(U, \vec{x}) = \mathcal{O}(U)[x_{i,(-j-1)},\ \partial_{x_i,(-j)};\ dx_{i,(-j)} \partial_{dx_i,(-j)},\ 1 \leq i \leq n,\ j \geq 1], \tag{2.2.1}$$

the generators $x_{i,(-j-1)}, \partial_{x_i,(-j)}$ being even, $dx_{i,(-j)}\partial_{dx_i,(-j)}$ odd. Identifying $x_{i,(-j-1)}$, $\partial_{x_i,(-j)}$ with different even copies of dx_i and ∂_{x_i} resp., and $dx_{i,(-j)}, \partial_{dx_i,(-j)}$ with different odd copies thereof, we obtain an identification of superalgebras

$$\Omega^{ch}(U, \vec{x}) \xrightarrow{\sim} \bigotimes_{n \geq 0} (S^*(\mathcal{T}(U)) \otimes S^*(\mathcal{T}^*(U)) \otimes \Lambda^*(\mathcal{T}(U)) \otimes \Lambda^*(\mathcal{T}^*(U))). \tag{2.2.2}$$

This is a local version of (2.1.2) and the images of embeddings (2.1.1) are generated over $\mathcal{O}(U)$ by $dx_{i,(-1)}, \partial_{dx_i,(-1)}$.

The ring $\Omega^{ch}(U, \vec{x})$ carries a canonical vertex algebra structure [MSV]. To formulate the result introduce an even derivation $T \in \mathrm{End}\Omega^{ch}(U, \vec{x})$ determined by the conditions

$$T(f) = \sum_i x_{i,(-2)} \partial_{x_i} f,\ T(a_{(-n)}) = n a_{(-n-1)}, \tag{2.2.3}$$

where $f \in \mathcal{O}(U)$, $a = x_i$, ∂_{x_i}, or ∂_{dx_i}, and $n \geq 1$. Note that under identification (2.2.2) the first of these conditions says that $T(\mathcal{O}(U)) \subset \mathcal{T}^*(U)$ and the restriction $T|_{\mathcal{O}(U)}$ equals the de Rham differential.

2.2.2. Lemma. *There is a unique vertex algebra structure on* $\Omega^{ch}(U,\vec{x})$

$$Y:\ \Omega^{ch}(U,\vec{x})\rightarrow \text{Field}(\Omega^{ch}(U,\vec{x}))$$

determined by the conditions:

(i) $\Omega^{ch}(U,\vec{x})$ *is generated by the fields* $Y(f,z)$, $f\in\mathcal{O}(U)$, $Y(\partial_{x_i,(-1)},z)$, $Y(\partial_{dx_i,(-1)},z)$, $Y(dx_{j,(-1)},w)$, *the list of non-zero brackets amongst them being as follows:*

$$[Y(\partial_{x_i,(-1)},z),Y(f,z)]=\delta(z-w)Y(\partial_{x_i}f,w), \tag{2.2.4a}$$

$$[Y(\partial_{dx_i,(-1)},z),Y(dx_{j,(-1)},w)]=\delta_{ij}\delta(z-w); \tag{2.2.4b}$$

(ii) T*-covariance:*

$$\begin{gathered}[T,a(z)]=a(z)',\\ a(z)=Y(f,z),Y(\partial_{x_i,(-1)},z),Y(\partial_{dx_i,(-1)},z),Y(dx_{j,(-2)},w);\end{gathered} \tag{2.2.5}$$

(iii) vacuum:

$$Y(1,z)=\text{Id},\ Y(f,z)g|_{z=0}=fg, Y(a,z)1|_{z=0}=a, \tag{2.2.6}$$

where $1,f,g\in\mathcal{O}(U)$, $a=\partial_{x_i,(-1)}$, $dx_{i,(-1)}$ *or* $\partial_{dx_i,(-1)}$.

The uniqueness assertion of this lemma is an immediate consequence of Theorem 4.5 in [K]. While proving the existence assertion in general is something of a problem, in many examples, sufficient for our present purposes, this is easy. Before we begin discussing these examples, let us unburden the notation by setting:

$$\begin{gathered}f(z)=Y(f,z),\, dx_i(z)=Y(dx_{i,(-1)},z),\\ \partial_{x_i}(z)=Y(\partial_{x_i,(-1)},z),\,\partial_{dx_i}(z)=Y(\partial_{dx_i,(-1)},z).\end{gathered} \tag{2.2.7}$$

2.2.3. *Example: an affine space.* If $U=\mathbb{C}$ with the canonical coordinate $x=\vec{x}$, then $\Omega^{ch}(\mathbb{C},x)$ is nothing but FB of (1.7.2). Indeed, since in this case $\mathcal{O}(U)=\mathbb{C}[x]$, $\Omega^{ch}(\mathbb{C},x)$ is generated by the fields $x(z)$, $\partial_x(z)$, $dx(z)$, $\partial_{dx}(z)$, which according to (2.2.4a,b) satisfy

$$[\partial_x(z),x(w)]=\delta(z-w),\ [\partial_{dx}(z),dx(w)]=\delta(z-w).$$

A quick glance at (1.6.3, 1.7.1) shows that

$$b(z)\mapsto\partial_{dx}(z),c(z)\mapsto dx(z),\gamma(z)\mapsto x(z),\beta(z)\mapsto\partial_x(z)$$

identifies $\Omega^{ch}(\mathbb{C},x)$ with FB.

Likewise,

$$\Omega^{ch}(\mathbb{C}^N,\vec{x})=FB^{\otimes n}. \tag{2.2.8}$$

Incidentally, the same formulas define a vertex algebra morphism

$$FB^{\otimes N}\hookrightarrow\Omega^{ch}(U,\vec{x}). \tag{2.2.9}$$

The nature of this morphism is this: a coordinate system $\vec{x}$ determines an étale map $U \to \mathbb{C}^N$; hence (2.2.9) is a manifestation of the naturality of Ω^{ch}_X, see 2.1.

2.2.4. *Example: localization of an affine space.* Let $f \in \mathbb{C}[x_1, ..., x_n]$, $U_f = \mathbb{C}^N - \{\vec{x} : f(\vec{x}) = 0\}$, and $\mathbb{C}[x_1, ..., x_n]_f$ the corresponding localization. To extend the vertex algebra structure from $\Omega^{ch}(\mathbb{C}^N, \vec{x})$ to

$$\Omega^{ch}(U_f, \vec{x}) = \mathbb{C}[x_1, ..., x_n]_f \otimes_{\mathbb{C}[x_1,...,x_n]} \Omega^{ch}(\mathbb{C}^N, \vec{x})$$

it suffices to define the field $f^{-1}(z)$. In [MSV] an explicit formula for this field was written down using Feigin's insight. Lemma 2.2.2 is a convenient alternative tool to compute the action of this (and similar) fields. Indeed, in view of the commutation relations (2.2.4a-b) it suffices to know $f^{-1}(z)_{(n)}g$, $g \in \mathcal{O}(U)$. Due to (2.2.6) we have

$$f^{-1}(z)_{(n)}g = \begin{cases} 0 \text{ if } n \geq 0 \\ \dfrac{g}{f} \text{ if } n = -1. \end{cases}$$

The values $f^{-1}(z)_{(n)}g$, $n \leq -2$, are determined by using (2.2.5). The case where $g = 1$ suffices and the repeated application of (2.2.5) gives

$$f^{-1}(z)1 = e^{zT} f^{-1}.$$

For example,

$$f^{-1}(z)_{(-2)}1 = T(\frac{1}{f}) = -\sum_i x_{i,(-2)} \frac{\partial_{x_i} f}{f^2}.$$

We shall mostly need localization to the complements of hyperplanes. The corresponding vertex algebras can be realized, thanks to [B], inside lattice vertex algebras; this will be reviewed in some detail in sect. 3.

2.2.5. *Two $N2$-structures and two chiral rings.* We shall need two morphisms of vertex algebras

$$\rho_1, \rho_2 : \ V(N2)_{3n} \to \Omega^{ch}(U, \vec{x}), \ n = \dim U.$$

The first was used in [MSV] and in terms of fields is defined by

$$Q^{(1)}(z) = \sum_i dx_i(z)\partial_{x_i}(z), \ G^{(1)}(z) = \sum_i : x_i(z)'\partial_{dx_i}(z),$$
$$J^{(1)}(z) = -\sum_i : dx_i(z)\partial_{dx_i}(z) :, L^{(1)}(z) = \sum_i : x_i(z)'\partial_{x_i}(z) : + : dx_i(z)'\partial_{dx_i}(z) :, \quad (2.2.10)$$

where we let $A^{(1)} = \rho_1(A)$, A=Q, G, J, or L.

The second is obtained by composing the first with automorphism (1.11.3); the result is this:

$$Q^{(2)}(z) = \sum_i : x_i(z)'\partial_{dx_i}(z), \ G^{(2)}(z) = \sum_i dx_i(z)\partial_{x_i}(z),$$
$$J^{(2)}(z) = \sum_i : dx_i(z)\partial_{dx_i}(z) :, L^{(2)}(z) = \sum_i : x_i(z)'\partial_{x_i}(z) : - : dx_i(z)\partial_{dx_i}(z)' : . \quad (2.2.11)$$

As was noted in 1.10.1, the operators $L^{(i)}_{(1)}$ give two conformal gradings and a simple computation shows that the corresponding chiral rings, 1.4, are as follows:

$$\mathrm{Ker}L^{(1)}_{(1)} = \mathcal{O}(U)[dx_{1,(-1)}, ..., dx_{n,(-1)}], \mathrm{Ker}L^{(2)}_{(1)} = \mathcal{O}(U)[\partial_{dx_1,(-1)}, ..., \partial_{dx_n,(-1)}]. \tag{2.2.12}$$

2.3. *Gluing the local models.*

2.3.1. Localisation procedure explained in 2.2.4 carries over to any $\Omega^{ch}(U,\vec{x})$, see 2.2.1, and defines, in the presence of a coordinate system, a sheaf of vertex algebras

$$U \supset V \mapsto \Omega^{ch}_{U,\vec{x}}(V) \stackrel{\text{def}}{=} \Omega^{ch}(V,\vec{x})$$

over U. By using the action of the group of coordinate changes [MSV, (3.1.6)] one obtains canonical identifications

$$\Omega^{ch}_{U,\vec{x}} \stackrel{\sim}{\longrightarrow} \Omega^{ch}_{U,\vec{y}}$$

for any two coordinate systems $\vec{x}$, $\vec{y}$. This defines a family of sheaves $U \mapsto \Omega^{ch}_U$, where U is étale over $\mathbb{C}^N$, natural w.r.t. to étale morphisms.

Finally covering a smooth manifold X by charts $\{U_\alpha\}$ étale over $\mathbb{C}^N$ one defines a sheaf Ω^{ch}_X by gluing over intersections according to the diagram

$$\Omega^{ch}_{U_\alpha} \hookrightarrow \Omega^{ch}_{U_\alpha \cap U_\beta} \hookleftarrow \Omega^{ch}_{U_\beta}.$$

Let us recall, briefly but in some more detail, the effect of this procedure on the $N2$-structure.

2.3.2. *Two $N2$-structures, the chiral de Rham complex and algebra of chiral polyvector fields.* It was computed in [MSV] that a coordinate change $\vec{x} \mapsto \vec{y}$, via (2.1.3), induces the following transformation of fields (2.2.10):

$$\begin{aligned} Q^{(1)}(z) &\mapsto Q^{(1)}(z) + (d_{DR}(\mathrm{Tr}\log\{(\partial_{x_i} y_j)\}))(z)' \\ G^{(1)}(z) &\mapsto G^{(1)}(z) \\ J^{(1)}(z) &\mapsto J^{(1)}(z) + (\mathrm{Tr}\log\{(\partial_{x_i} y_j)\})(z)' \\ L^{(1)}(z) &\mapsto L^{(1)}(z), \end{aligned} \tag{2.3.1}$$

and of course similar transformation formulas can be written for fields (2.2.11). It follows that

$$L^{(i)}(z)_{(1)}, J^{(i)}(z)_{(0)},\ i = 1, 2$$

is a well-defined quadruple of operators acting on Ω^{ch}_X. Since $[L^{(i)}(z)_{(1)}, J^{(i)}(z)_{(0)}] = 0$, there arise two competing bi-gradings by "conformal weight, fermionic charge":

$$\begin{aligned} \Omega^{ch}_X &= \oplus_{n\geq 0, m\in\mathbb{Z}}\, {}^{(i)}\Omega^{ch,m}_{X,n},\ i = 1, 2, \\ {}^{(i)}\Omega^{ch,m}_{X,n} &= \mathrm{Ker}(L^{(i)}(z)_{(1)} - n\mathrm{Id}) \cap \mathrm{Ker}(J^{(i)}(z)_{(0)} - m\mathrm{Id}). \end{aligned} \tag{2.3.2}$$

Formula (2.2.12) shows that the chiral ring, 1.4, now technically a sheaf of chiral rings, associated to the first is the algebra of differential forms:

$$\mathcal{C}^{(1)} = \Omega_X^* : \; U \mapsto \mathcal{O}(U)[dx_{1,(-1)}, ..., dx_{n,(-1)}]; \tag{2.3.3a}$$

the chiral ring associated to the second is the algebra of polyvector fields

$$\mathcal{C}^{(2)} = \Lambda^* \mathcal{T}_X, \; U \mapsto \mathcal{O}(U)[\partial_{dx_1,(-1)}, ..., \partial_{dx_n,(-1)}]. \tag{2.3.3b}$$

This is how morphisms (2.1.1) come about.

2.3.3. *Terminology.* From now on we shall call Ω_X^{ch} equipped with grading (2.3.2) where $i=2$ the algebra of chiral polyvector fields and re-denote it by $\Lambda^{ch}\mathcal{T}_X$. The sheaf Ω_X^{ch} equipped with grading (2.3.2) where $i=1$ will retain the name of the chiral de Rham complex.

Equations (2.3.3a,b) is one justification of this terminology. Note that (2.1.2) uses bi-grading (2.3.2,i=1); the $i=2$ analogue is as follows:

$$\mathrm{Gr}\Lambda^{ch}\mathcal{T}_X \xrightarrow{\sim} \bigotimes_{n\geq 0} \left(S^*_{q^n}(\mathcal{T}_X) \otimes S^*_{q^{n+1}}(\mathcal{T}^*_X) \otimes \Lambda^*_{q^n y^{-1}}(\mathcal{T}_X) \otimes \Lambda^*_{q^{n+1} y}(\mathcal{T}^*_X) \right). \tag{2.3.5}$$

Transformation formulas (2.3.1) imply that $L^{(1)}(z)$ is preserved; hence Ω_X^{ch} always carries a conformal structure, see 1.10.1 for the definition. The situation is different with $\Lambda^{ch}\mathcal{T}_X$: it does not carry a conformal structure compatible with its conformal grading unless X is Calabi-Yau. Indeed, as follows from the last of formulas (2.2.11), $L^{(2)}(z) = L^{(1)}(z) + J^{(1)}(z)'$ and the latter picks the 1st Chern class as a result of transformation (2.3.1).

If, however, X is a projective Calabi-Yau manifold, then it can be derived from (2.3.1), [MSV], that the quadruple of fields $Q^{(i)}(z)$, $G^{(i)}(z)$, $J^{(i)}(z)$, $Q^{(i)}(z)$, $i=1,2$, can be made sense of globally, and both Ω_X^{ch}, $\Lambda^{ch}\mathcal{T}_X$ acquire an $N2$-structure, see 1.10.1 for the definition. What is especially clear is

2.3.4. Lemma. *If ω is a non-vanishing holomorphic form over X, and X admits an atlas consisting of charts $\{(U,\vec{x})\}$ such that locally $\omega = dx_1 \wedge \cdots \wedge dx_n$, then formulas (2.2.10,11) define an $N2$-structure on Ω_X^{ch} and $\Lambda^{ch}\mathcal{T}_X$ resp.*

Indeed, in this case the jacobian $\det(\partial_{x_i} y_j)$ equals 1, and the correction terms in (2.3.1) vanish.

2.3.5. *An example: $X = \mathbb{C}^N - 0$.* As an illsutration, let us compute the cohomology vertex algebra $H^*(\mathbb{C}^N - 0, \Lambda^{ch}\mathcal{T}_{\mathbb{C}^N - 0})$, an example that will prove important later on.

The manifold $\mathbb{C}^N - 0$ is quasiaffine and, therefore, it has the standard global coordinate system x_i, ∂_{x_i}, $0 \leq i \leq N-1$ inherited from $\mathbb{C}^N$. This places us in the situation of Lemma 2.2.2 and we obtain a morphism of bi-graded sheaves

$$\Omega^{ch}_{\mathbb{C}^N - 0} \xrightarrow{\sim} \bigoplus_{n\geq 0} (S^*_{q^n}(\mathcal{T}_{\mathbb{C}^N - 0}) \otimes S^*_{q^{n+1}}(\mathcal{T}^*_{\mathbb{C}^N - 0}) \otimes \Lambda^*_{q^n y^{-1}}(\mathcal{T}_{\mathbb{C}^N - 0}) \otimes \Lambda^*_{q^n y}(\mathcal{T}^*_{\mathbb{C}^N - 0})), \tag{2.3.6}$$

cf. (2.3.5). For the same reason, the sheaf on the R.H.S. of (2.3.6) is free, hence it suffices to compute $H^*(\mathbb{C}^N - 0, \mathcal{O}_{\mathbb{C}^N - 0})$. It is a nice excersise in Čech cohomology to prove that

$$H^n(\mathbb{C}^N - 0, \mathcal{O}_{\mathbb{C}^N - 0}) = \begin{cases} \mathbb{C}[x_0, ..., x_{N-1}] \text{ if } n = 0 \\ \bigotimes_{i=0}^{N-1} \mathbb{C}[x_i^{\pm 1}]/\mathbb{C}[x_i] \text{ if } n = N-1 \\ 0 \text{ otherwise.} \end{cases} \qquad (2.3.7)$$

The first line of (2.3.7) says that all the global sections of $\Lambda^{ch}\mathcal{T}_{\mathbb{C}^N - 0}$ are restrictions from $\mathbb{C}^N$. Therefore,

$$H^0(\mathbb{C}^N - 0, \Lambda^{ch}\mathcal{T}_{\mathbb{C}^N - 0}) = \Lambda^{ch}\mathcal{T}(\mathbb{C}^N) = FB^{\otimes N}, \qquad (2.3.8)$$

cf. (2.2.8).

Similarly, it follows from the 2nd line of (2.3.7) that

$$H^{N-1}(\mathbb{C}^N - 0, \Lambda^{ch}\mathcal{T}_{\mathbb{C}^N - 0}) = \left(\bigotimes_{i=0}^{N-1} \mathbb{C}[x_i^{\pm 1}]/\mathbb{C}[x_i]\right) \otimes_{\mathbb{C}[\vec{x}]} \Lambda^{ch}\mathcal{T}(\mathbb{C}^N), \qquad (2.3.9)$$

where we use the notation of (2.2.1) with $(U, \vec{x}) = (\mathbb{C}^N, \vec{x})$.

A moment's thought shows that as an $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)$-module, $H^{N-1}(\mathbb{C}^N - 0, \Lambda^{ch}\mathcal{T}_{\mathbb{C}^N - 0})$ is obtained from $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)$ by spectral flow (1.12.2):

$$H^{N-1}(\mathbb{C}^N - 0, \Lambda^{ch}\mathcal{T}_{\mathbb{C}^N - 0}) = S_1(\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)) = S_1(FB^{\otimes N}). \qquad (2.3.10)$$

Indeed, by definition 1.7, $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N) = FB^{\otimes N}$ is generated by a vector $\mathbf{1}$ annihilated by $x_{i,(j)}$, $\partial_{x_i,(j)}$, $j \geq 0$, (and we identify $x_i = x_{i,(-1)}$); according to (2.3.9), $H^{N-1}(\mathbb{C}^N - 0, \Lambda^{ch}\mathcal{T}_{\mathbb{C}^N - 0})$ is generated by a vector annihilated by $x_{i,(j-1)}$, $\partial_{x_i,(j+1)}$, $j \geq 0$; the latter annihilating subalgebra is mapped onto the former by S_1 of (1.12.1). The odd variables are treated similarly; but notice also that the Clifford algebra has only one irreducible module and so the spectral flow on it is inessential.

Of course, the 3rd line of (2.3.7) implies

$$H^i(\mathbb{C}^N - 0, \Lambda^{ch}\mathcal{T}_{\mathbb{C}^N - 0}) = 0 \text{ if } i \neq 0, N-1. \qquad (2.3.11)$$

2.4. The algebra of chiral polyvector fields over hypersurfaces.

This is an exposition of a result of [B].

2.4.1. Let

$$\mathcal{L} \to X \qquad (2.4.1)$$

be a line bundle,

$$\mathcal{L}^* \to X \qquad (2.4.2)$$

its dual,

$$t : X \to \mathcal{L} \tag{2.4.3}$$

its section with smooth zero locus $Z(t)$. Following [B] we shall relate $\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}$ and $\Lambda^{ch}\mathcal{T}_{Z(t)}$ as follows.

Identify t with a fiberwise linear function on $\mathcal{L}^*$. We have the de Rham differential of t, $dt \in H^0(\mathcal{L}^*, \Omega^1_{\mathcal{L}^*})$; via (2.1.1), $dt \in H^0(\mathcal{L}^*, \Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})$. It is clear that

$$dt_{(0)} : \Lambda^{ch}\mathcal{T}_{\mathcal{L}^*} \to \Lambda^{ch}\mathcal{T}_{\mathcal{L}^*} \tag{2.4.4}$$

is a derivation with zero square, cf. 1.3.

$\mathcal{L}^*$ carries an action of $\mathbb{C}^*$ defined by fiberwise multiplication. By the naturality, 2.1 (i), this action lifts to an action on $\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}$. Hence there arises the grading

$$\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*} = \bigoplus_{n\in\mathbb{Z}} R^n(\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}).$$

The operator $dt_{(0)}$ has degree 1 with respect to this grading and, therefore, (2.4.4) is actually a complex of sheaves $R^*\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*} = (\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}, dt_{(0)})$ such that

$$\cdots \xrightarrow{dt_{(0)}} R^{-1}(\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}) \xrightarrow{dt_{(0)}} R^0(\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}) \xrightarrow{dt_{(0)}} R^1(\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}) \xrightarrow{dt_{(0)}} \cdots \tag{2.4.5}$$

2.4.2. Lemma ([B]) *The cohomology sheaf $\mathcal{H}^n_{dt_{(0)}}(\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})$ of complex (2.4.5) is zero unless $n = 0$. The sheaf $\mathcal{H}^0_{dt_{(0)}}(\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})$ is supported on $Z(t)$ and naturally isomorphic to $\Lambda^{ch}\mathcal{T}_{Z(t)}$.*

2.4.3. Observe that $dt \in H^0(\mathcal{L}^*, \Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})$ is of conformal weight 1, as follows e.g. from (2.3.5). Therefore, the differential of complex (2.4.5) preserves conformal weight, and the conformal weight 0 component of (2.4.5) is the following classical complex:

$$\cdots \xrightarrow{dt} \Lambda^{i+1}\mathcal{T}_{\mathcal{L}^*} \xrightarrow{dt} \Lambda^{i}\mathcal{T}_{\mathcal{L}^*} \xrightarrow{dt} \Lambda^{i-1}\mathcal{T}_{\mathcal{L}^*} \xrightarrow{dt} \cdots, \tag{2.4.6}$$

with differential equal to the contraction with the 1-form $d_{DR}t$. Lemma 2.4.2 says, in particular, that this complex computes the algebra of polyvector fields on $Z(t)$, a well-known result perhaps. Note that in the chiral de Rham complex setting, cf. [B], this classical construction is somewhat harder to discern because there conformal weight is not preserved by $dt_{(0)}$.

2.4.4. *The $N2$-structure.* Let $\mathcal{L}^*$ be the canonical line bundle. Then both $\mathcal{L}^*$ and $Z(t)$ are Calabi-Yau – both have a nowhere zero global holomorphic volume form – and both $\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}$ and $\Lambda^{ch}\mathcal{T}_{Z(t)}$ carry an $N2$-structure, see the end of 2.3.3. Let us write down some explicit formulas.

Suppose there is a nowhere zero global holomorphic volume form ω and X can be covered by charts $s, y_1, ..., y_{N-1}$, s being the coordinate along the fiber, such that $\omega = ds \wedge y_1 \wedge \cdots \wedge y_{N-1}$. Then, as follows from Lemma 2.3.4,

$$Q(z) \mapsto s(z)'\partial_{ds}(z) + \sum_{j=1}^{N-1} y_j(z)'\partial_{dy_j}(z),\ G(z) \mapsto ds(z)\partial_s(z) + \sum_{j=1}^{N-1} dy_j(z)\partial_{y_j}(z),$$
$$J(z) \mapsto -ds(z)\partial_{ds}(z) - \sum_{j=1}^{N-1} dy_j(z)\partial_{dy_j}(z),\ L(z) \mapsto s(z)'\partial_s(z) + \sum_{j=1}^{N-1} y_j(z)'\partial_{y_j}(z)-$$
$$- ds(z)\partial_{ds}(z)' - \sum_{j=1}^{N-1} dy_j(z)\partial_{dy_j}(z)' \tag{2.4.7}$$

well defines an $N2$-structure on $\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}$.

[B, Proposition 5.8] says that, via Lemma 2.4.2, the $N2$-structure on $\Lambda^{ch}\mathcal{T}_{Z(t)}$ is determined by

$$\begin{aligned} G(z) &\mapsto ds(z)\partial_s(z) + \sum_{j=1}^{N-1} dy_j(z)\partial_{y_j}(z),\\ Q(z) &\mapsto s(z)'\partial_{ds}(z) + \sum_{j=1}^{N-1} y_j(z)'\partial_{dy_j}(z) - (s(z)\partial_{ds}(z))'. \end{aligned} \tag{2.4.8}$$

3. The lattice vertex algebra realization and applications to toric varieties.

This section is an exposition of part of Borisov's free field realization [B]. It does not contain any new results except perhaps Lemma 3.8, and in order to construct the spectral sequence appearing in the latter the entire section had to be written up.

3.1. Let M be a rank N free abelian group, $M^* = Hom_{\mathbb{Z}}(M,\mathbb{Z})$ its dual. Give $\Lambda = M \oplus M^*$ a lattice structure by defining the symmetric bilinear form

$$\Lambda \times \Lambda \to \mathbb{Z}, \tag{3.1.1}$$

induced by the natural pairing $M^* \times M \to \mathbb{Z}$, $(X^*, X) \mapsto X^*(X)$. There arises the lattice vertex algebra V_Λ, 1.9, where we fix the following cocycle, cf.(1.9.1),

$$\epsilon(X + X^*, Y + Y^*) = (-1)^{X^*(Y)},\ X, Y \in M, X^*, Y^* \in M^*. \tag{3.1.2}$$

Next, consider the complexification $\Lambda_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \Lambda$ onto which form (3.1.2) carries over. There arises the fermionic vertex algebra $F_{\Lambda_{\mathbb{C}}}$ which we re-denote by F_Λ, see (1.6.5).

Finally, following [B] introduce Borisov's vertex algebra

$$\mathbb{B}_\Lambda = V_\Lambda \otimes F_\Lambda. \tag{3.1.3}$$

Notation. The notational problem one faces here is that the lattice Λ twice manifests itself inside $\mathbb{B}_\Lambda$: first, as an ingredient of V_Λ; second, as that of F_Λ. We attempt to resolve this issue by letting capital latin letters, $X, Y, Z, ..$ ($X^*, Y^*, Z^*, ..$ resp.), denote elements of M (M^* resp.) in the context of V_Λ; and let the tilded letters, $\tilde{X}, \tilde{Y}, \tilde{Z}, ..$ or $\tilde{X}^*, \tilde{Y}^*, \tilde{Z}^*, ..$ denote their respective copies in the context of F_Λ. Later on this will be related to geometry and then we shall let the lowercase letters denote the respective coordinates. □

Note that the assignment $M^* \mapsto \mathbb{B}_\Lambda$ is functorial. Indeed, if $g \in \mathrm{Hom}(M_1^*, M_2^*)$ is an isomorphism of abelian groups, then

$$(g^{-1}, g^*) \in \mathrm{Hom}(M_2^*, M_1^*) \times \mathrm{Hom}(M_2, M_1) \hookrightarrow \mathrm{Hom}(\Lambda_2, \Lambda_1)$$

is an isomorphism of lattices preserving form (3.1.1) and cocycle (3.1.2). According to (1.11.1,2), this isomorphism induces the following isomorphism of vertex algebras

$$\hat{g} : \mathbb{B}_{\Lambda_2} \to \mathbb{B}_{\Lambda_1} \tag{3.1.4a}$$

$$\begin{aligned} &X^*(z) \mapsto g^{-1}X^*(z), \tilde{X}^*(z) \mapsto g^{-1}\tilde{X}^*(z), e^{X^*}(z) \mapsto e^{g^{-1}X^*}(z) \\ &X(z) \mapsto (g^*)X(z), \tilde{X}(z) \mapsto (g^*)\tilde{X}(z), e^{X}(z) \mapsto e^{(g^*)X}(z). \end{aligned} \tag{3.1.4b}$$

Therefore, if we introduce the category of lattices Λ morphisms being the described isomorphisms, then

$$M^* \mapsto \mathbb{B}_\Lambda, g \mapsto \hat{g} \tag{3.1.5}$$

is a contravariant functor.

Later we shall have to work with g such that $g(M_1^*) \subset M_2^*$ but after the extension of scalars to $\mathbb{Q}$ the induced $g \in \mathrm{Hom}_{\mathbb{Q}}((M_1^*)_{\mathbb{Q}}, (M_2^*)_{\mathbb{Q}})$ is an isomorphism. In this case the functorial nature of $M^* \mapsto \mathbb{B}_\Lambda$ is a little more subtle because g^{-1} may have non-integer entries. There are two ways around.

Consider a vertex subalgebra $\mathbb{B}_{M,\Lambda} \subset \mathbb{B}_\Lambda$, see definition (1.9.10). (This simply means that all the fields $e^{X^*}(z)$ are not allowed.) It is clear that

$$\mathbb{B}_{.,.} : M^* \mapsto \mathbb{B}_{M,\Lambda}, g \mapsto \hat{g}|_{\mathbb{B}_{M,\Lambda}} \tag{3.1.6}$$

is a contravariant functor because the indicated restriction of (3.1.4a) makes sense for any lattice embedding g.

Second, naturally associated to g there is a map

$$\hat{g} : \mathbb{B}_{\Lambda_2} \hookrightarrow \mathbb{B}_{g^{-1}\Lambda_2}, \tag{3.1.7}$$

still defined by (3.1.4a), where the lattice $g^{-1}\Lambda_2$ is defined to be $g^*M_2 \oplus g^{-1}M_2^* \subset (\Lambda_1)_{\mathbb{Q}}$.

Note that (3.1.6) is a "subfunctor" of (3.1.7).

3.2. Let us introduce the following terminology and notation pertaining to toric variety theory: by a basic cone $\sigma \subset M^*$ we shall mean a sub-semigroup spanned over $\mathbb{Z}_+$ by part of a basis (over $\mathbb{Z}$) of M^*. Let $< \sigma >$ denote the (uniquely determined) spanning set of σ . Given a basic cone $\sigma \subset M^*$, let $\check{\sigma} \subset M$ be its dual cone defined by $\check{\sigma} = \{X \in M \text{ s.t. } \sigma(X) \geq 0\}$.

A smooth toric variety will always be defined by fixing a lattice Λ as in 3.1 and a regular fan Σ. (Regular means that Σ is a collection of basic cones in M^*.) If we define

$$U_\sigma = \mathrm{Spec}\mathbb{C}[\check{\sigma}],\ \sigma \in \Sigma, \tag{3.2.1}$$

where $\mathbb{C}[\check{\sigma}]$ is the semigroup algebra of $\check{\sigma}$, then there arises a canonical embedding

$$U_{\sigma'} \subset U_\sigma,\ \sigma' \subset \sigma. \tag{3.2.2}$$

The toric variety X_Σ attached to Σ is defined by declaring that

$$\mathcal{U}_\Sigma = \{U_\sigma, \sigma \in \Sigma\} \tag{3.2.3}$$

is its covering and by gluing the charts over intersections

$$U_\alpha \hookleftarrow U_{\alpha\cap\beta} \hookrightarrow U_\beta$$

according to (3.2.2).

Note that the assignment $(\sigma, M^*) \mapsto U_\sigma$ is functorial. Indeed, if we introduce the category whose objects are pairs (σ, M^*) and morphisms $(\sigma_1, M_1^*) \to (\sigma_2, M_2^*)$ are abelian group morphisms $g : M_1^* \to M_2^*$ such that $g(\sigma_1) \subset \sigma_2$, then $g^*(\check{\sigma}_2) \subset \check{\sigma}_1$. Hence g^* induces a ring homomorphism $\mathbb{C}[\check{\sigma}_2] \to \mathbb{C}[\check{\sigma}_1]$ and thus a morphism $\tilde{g} : U_{\sigma_1} \to U_{\sigma_2}$. Of course,

$$(\sigma, M^*) \mapsto U_\sigma,\ g \mapsto \tilde{g} \tag{3.2.4}$$

is a covariant functor.

This can be globalized: given $(\Sigma_1, M_1^*)$and (Σ_2, M_2^*) with a lattice morphism $g : M_1^* \to M_2^*$ such that for each $\sigma_1 \in \Sigma_1$ there is $\sigma_2 \in \Sigma_2$ containing $g(\sigma_1)$, there arises a morphism

$$\tilde{g} : X_{\Sigma_1} \to X_{\Sigma_2}. \tag{3.2.5}$$

3.3. We would like to define Borisov's realization [B], that is, a vertex algebra embedding $\Lambda^{ch}\mathcal{T}(U_\sigma) \hookrightarrow \mathbb{B}_{M,\Lambda}$ for each basic cone $\sigma \in M^*$. To write down an explicit formula for this map, let us choose a basis of M^*, $X_0^*, ..., X_{N-1}^*$, such that σ is spanned by $X_0^*, ..., X_{m-1}^*$. This fixes the dual basis $X_0, ..., X_{N-1}$ of M.

In order to conform to the notation of sect.2, let $x_j \stackrel{\text{def}}{=} e^{X_j}$, ∂_{x_i}, $0 \leq i \leq m-1$, be a coordinate system on U_σ. Thus

$$U_\sigma = \mathrm{Spec}\mathbb{C}[x_0, ..., x_{m-1}, x_m^{\pm 1}, ..., x_{N-1}^{\pm 1}].$$

Borisov proves that there is a vertex algebra homomorphism

$$\mathcal{B}(\sigma) :\ \Lambda^{ch}\mathcal{T}(U_\sigma) \to \mathbb{B}_{M,\Lambda} \tag{3.3.1}$$

determined by the assignment

$$\begin{aligned} &x_i^{\pm 1}(z) \mapsto e^{\pm X_i}(z), x_j(z) \mapsto e^{X_j}(z),\ m \leq i \leq N-1, j \leq m-1,\\ &dx_i(z) \mapsto :e^{X_i}(z)\tilde{X}_i(z) : . \end{aligned} \tag{3.3.2a}$$

$$\partial_{x_i}(z) \mapsto: (X_i^*(z) - :\tilde{X}_i(z)\tilde{X}_i^*(z):)e^{-X_i}(z):,\ \partial_{dx_i}(z) \mapsto: e^{-X_i}(z)\tilde{X}_i^*(z):, \quad (3.3.2b)$$

cf. 3.1, Notation. (Note that (3.3.2a-b) are exactly (2.1.3) specialized to the exponential change of variables $x_i \to e^{X_i}$; this remark is also borrowed from [B].) Formulas (3.3.2a) are manifestly independent of the choice of variables; it is easy then to infer that so are (3.3.2b).

This embedding naturally depends on σ. To make a precise statement, give

3.3.1. Definition. Fix a number N. $\mathcal{C}$ is a category whose objects are pairs (σ, M^*), $\dim M^* = N$, all and morphisms $(\sigma_1, M_1^*) \to (\sigma_2, M_2^*)$ are abelian group embeddings $g : M_1^* \hookrightarrow M_2^*$ such that $g(\sigma_1) = \sigma_2$. $\square$

Note that the conditions imposed on morphisms in this definition strengthen those used in (3.2.4). In fact, a short computation shows that the map $\tilde{g} : U_{\sigma_1} \to U_{\sigma_2}$ associated with a morphism $g \in \mathrm{Mor}_{\mathcal{C}}((\sigma_1, M_1^*), (\sigma_1, M_1^*))$ in (3.2.4) is étale. Hence, by virtue of the naturality of $\Lambda^{ch}\mathcal{T}_X$, see 2.1(i), the composition

$$(\sigma, M^*) \mapsto U_\sigma \mapsto \Lambda^{ch}\mathcal{T}(U_\sigma)$$

defines a contravariant functor

$$\Lambda^{ch}\mathcal{T}(.) : \ (\sigma, M^*) \mapsto \Lambda^{ch}\mathcal{T}(U_\sigma). \quad (3.3.3)$$

By forgetting σ, we regard contravariant functor (3.1.6) as defined on $\mathcal{C}$. The main property of (3.3.1) is formulated as follows.

3.4. Lemma. *The assignment $(\sigma, M^*) \mapsto \mathcal{B}(\sigma)$, see (3.3.1), is a functor morphism*

$$\mathcal{B}(\sigma) : \ \Lambda^{ch}\mathcal{T}(.) \to \mathbb{B}_{.,.},$$

where $\mathbb{B}_{.,.}$ is the functor defined in (3.1.6).

Notational convention. Using this fact we shall not distinguish between $\Lambda^{ch}\mathcal{T}(U_\sigma)$ and $\mathcal{B}(\sigma)(\Lambda^{ch}\mathcal{T}(U_\sigma)) \subset \mathbb{B}_{M,\Lambda}$.

3.5. Let $S^* \in <\sigma>$ be one of the generators of σ and let $\sigma \setminus S^*$ denote the cone spanned by $<\sigma> \setminus S^*$. There arises then the restrtiction morphism

$$res(\sigma, S^*) : \Lambda^{ch}\mathcal{T}(U_\sigma) \hookrightarrow \Lambda^{ch}\mathcal{T}(U_{\sigma\setminus S^*}),$$

and one would like to extend it to a resolution.

Let

$$\mathcal{J}^*(\sigma, S^*) = \oplus_{n=0}^{\infty}\mathcal{J}^n(\sigma, S^*),\ \mathcal{J}^n(\sigma, S^*) = \Lambda^{ch}\mathcal{T}(U_{\sigma\setminus S^*})e^{nS^*}, \quad (3.5.1)$$

where $\Lambda^{ch}\mathcal{T}(U_{\sigma\setminus S^*})$ is thought of as a vertex subalgebra of $\mathbb{B}_{M,\Lambda}$, see 3.4, Notational convention, and this makes sense out of $\Lambda^{ch}\mathcal{T}(U_{\sigma\setminus S^*})e^{nS^*}$ as a subspace of $\mathbb{B}_\Lambda$.

$\mathcal{J}^*(\sigma, S^*)$ is evidently a $\mathbb{Z}_+$-graded vertex subalgebra of $\mathbb{B}_\Lambda$.

Let

$$D(\sigma, S^*) = (e^{S^*}(z)\tilde{S}^*(z))_{(0)}. \tag{3.5.2}$$

It is evidently a square zero derivation of $\mathcal{J}^*(\sigma, S^*)$, see 1.3. Thus $(\mathcal{J}^*(\sigma, S^*), D(\sigma, S^*))$ is a differential graded vertex algebra.

Now look upon $\Lambda^{ch}\mathcal{T}(U_\sigma)$ as a differential graded vertex algebra with $\Lambda^{ch}\mathcal{T}(U_\sigma)$ placed in degree 0, zero spaces placed everywhere else, and zero differential. Then, by taking the composition

$$\Lambda^{ch}\mathcal{T}(U_\sigma) \stackrel{res(\sigma,S^*)}{\to} \Lambda^{ch}\mathcal{T}(U_{\sigma\setminus S^*}) = \mathcal{J}^0(\sigma, S^*) \hookrightarrow \mathcal{J}^*(\sigma, S^*),$$

$res(\sigma, S^*)$ can be interpreted as a morphism of differential graded vertex algebras

$$res(\sigma, S^*) : (\Lambda^{ch}\mathcal{T}(U_\sigma), 0) \hookrightarrow (\mathcal{J}^*(\sigma, S^*), D(\sigma, S^*)). \tag{3.5.3}$$

This construction is natural. To explain this, let us give the following definition, cf. 3.3.1.

3.5.1. Definition. $\mathcal{C}_{pnt}$ is a category whose objects are triples (S^*, σ, M^*) with $S^* \in < \sigma >$, $(\sigma, M^*) \in \mathrm{Ob}(\mathcal{C})$, and morphisms $(S_1^*, \sigma_1, M_1^*) \to (S_2^*, \sigma_2, M_2^*)$ are abelian group morphisms $g : M_1^* \to M_2^*$ such that $g(\sigma_1) = \sigma_2$ and $g(S_1^*) = S_2^*$. $\square$

It is clear that both

$$(\Lambda^{ch}\mathcal{T}(.), 0) : \ (S^*, \sigma, M^*) \mapsto (\Lambda^{ch}\mathcal{T}(U_\sigma), 0) \tag{3.5.4}$$

and

$$(\mathcal{J}^*(.), D(.)) : \ (S^*, \sigma, M^*) \mapsto (\mathcal{J}^*(\sigma, S^*), D(\sigma, S^*)) \tag{3.5.5}$$

are contravariant functors from $\mathcal{C}_{pnt}$ to the category of differential graded vertex algebras. Indeed, the former is essentially (3.3.3), as to the latter, one has to apply map (3.1.7) restricted to $\mathcal{J}^*(\sigma, S^*)$.

3.6. Lemma. *(i) The assignment $(S^*, \sigma) \mapsto res(\sigma, S^*)$, see (3.5.3), is a functor morphism*

$$res(.) : \ (\Lambda^{ch}\mathcal{T}(.), 0) \mapsto (\mathcal{J}^*(.), D(.)).$$

(ii) For each (S^*, σ, M^*) map (3.5.3) is a quasiisomorphism.

3.7. Let us apply Lemma 3.6 to the situation where the fan Σ satisfies the following condition: there is $S^* \in M^*$ such that $S^* \in < \sigma >$ for all highest dimension cones $\sigma \in \Sigma$. Let $\Sigma \setminus S^* = \{\sigma \setminus S^* : \ \sigma \in \Sigma\}$, cf. the beginning of 3.5. This means that the morphism

$$X_\Sigma \to X_{\Sigma/\mathbb{Z}S^*} \tag{3.7.1}$$

induced by the canonical projection $M^* \to M^*/\mathbb{Z}S^*$ is a line bundle, and the map

$$X_{\Sigma\setminus S^*} \hookrightarrow X_\Sigma \tag{3.7.2}$$

induced by the tautological inclusion $\Sigma \setminus S^* \subset \Sigma$ is the embedding of the total space of the line bundle without the zero section. We would like to relate the cohomology groups $H^*(X_\Sigma, \Lambda^{ch}\mathcal{T}_{X_\Sigma})$ and $H^*(X_{\Sigma\setminus S^*}, \Lambda^{ch}\mathcal{T}_{X_{\Sigma\setminus S^*}})$.

The nerve of the covering $\mathcal{U}_\Sigma$ is a simplicial object in the category $\mathcal{C}_{pnt}$. Applying to it functor (3.5.4) one gets the complex $\Lambda^{ch}\mathcal{T}(\mathcal{U}_\Sigma)$ commonly known as the Čech complex. (A complex, not a bi-complex, because we ignore the trivial differential on $(\Lambda^{ch}\mathcal{T}(.),0)$.) We denote it by $\check{C}^*(\mathcal{U}_\Sigma,\Lambda^{ch}\mathcal{T}_{X_\Sigma};d_{\check{C}})$, where $d_{\check{C}}$ is the Čech differential.

Likewise applying functor (3.5.5) to the nerve of $\mathcal{U}_\Sigma$, we obtain the bi-complex $\check{C}^*(\mathcal{U}_\Sigma,\mathcal{J}^*;d_{\check{C}},D(S^*))$. By definition

$$\check{C}^p(\mathcal{U}_\Sigma,\mathcal{J}^q;d_{\check{C}},D(S^*))=\check{C}^p(\mathcal{U}_{\Sigma\setminus S^*},\Lambda^{ch}\mathcal{T}_{X_{\Sigma\setminus S^*}}e^{qS^*};d_{\check{C}},D(S^*)). \tag{3.7.3}$$

According to Lemma 3.6,

$$res(\mathcal{U}_\Sigma):\check{C}^*(\mathcal{U}_\Sigma,\Lambda^{ch}\mathcal{T}_{X_\Sigma};d_{\check{C}})\to\check{C}^*(\mathcal{U}_\Sigma,\mathcal{J}^*;d_{\check{C}},D(S^*)) \tag{3.7.4}$$

is a quasiisomorphism. More precisely, the bi-complex $\check{C}^*(\mathcal{U}_\Sigma,\mathcal{J}^*;d_{\check{C}},D)$ gives rise to two spectral sequences both converging to its total cohomology. The one where the vertex differential $D(.)$ is used first degenerates in the first term to the Čech complex $\check{C}^*(\mathcal{U}_\Sigma,\Lambda^{ch}\mathcal{T}_{X_\Sigma};d_{\check{C}})$ – this follows at once from Lemma 3.6 (ii). Hence both the sequences abut to $H^*(X_\Sigma,\Lambda^{ch}\mathcal{T}_{X_\Sigma})$. By definition, the first and the second terms of the second spectral sequence are as follows:

$$\begin{aligned}&(E_1^{p,q},d_1)=(H^p(X_{\Sigma\setminus S^*},\Lambda^{ch}\mathcal{T}_{X_{\Sigma\setminus S^*}}e^{qS^*}),D(S^*)),\\&D(S^*)=(e^{S^*}(z)\tilde{S}^*(z))_{(0)}.\end{aligned} \tag{3.7.5}$$

$$E_2^{p,q}=H^q_{D(S^*)}(H^p(X_{\Sigma\setminus S^*},\oplus_{n=0}^\infty\Lambda^{ch}\mathcal{T}_{X_{\Sigma\setminus S^*}}e^{nS^*})). \tag{3.7.6}$$

Let us summarize our discussion.

3.8. Lemma. *There is a spectral sequence $\{E_r^{p,q},d_r\}\Rightarrow H^*(X_\Sigma,\Lambda^{ch}\mathcal{T}_{X_\Sigma})$ that satisfies (3.7.5,6).*

3.9. The formation of bi-complex (3.7.3), and hence of the corresponding spectral sequences, is functorial in X_Σ. To make this precise, let a triple S^*, Σ_2 and M_2^* satisfy the conditions imposed on S^*, Σ and M^* in 3.7, and let us give ourselves another pair Σ_1 and M_1^*, $\dim M_1^*=\dim M_2^*$ along with a lattice embedding

$$g:\ M_2^*\to M_1^*\ \text{ s.t. } g(\Sigma_1)=\Sigma_2\setminus S^*. \tag{3.9.1}$$

According to (3.2.5) this induces a map

$$\tilde{g}:\ X_{\Sigma_1}\mapsto X_{\Sigma_2}, \tag{3.9.2}$$

which is étale, cf. 3.3. Map (3.9.1) gives rise to the lattice $g^{-1}\Lambda_2^*$ and the embedding of Borisov's algebras

$$\hat{g}:\ \mathbb{B}_{\Lambda_2}\hookrightarrow\mathbb{B}_{g^{-1}\Lambda_2} \tag{3.9.3}$$

due to (3.1.7). It is rather obvious that maps (3.9.2,3) allow to pull the bi-complex

$$\check{C}^p(\mathcal{U}_{\Sigma_2\setminus S^*},\Lambda^{ch}\mathcal{T}_{X_{\Sigma_2\setminus S^*}}e^{qS^*};d_{\check{C}},D(S^*)) \tag{3.9.4}$$

back onto X_{Σ_1}. Let us write down the relevant formula. The bi-complex (3.9.4) consists of the family of elements

$$f_\sigma \in \{\mathcal{T}_{X_{\Sigma_2\setminus S^*}}(U_\sigma)\}e^{qS^*},\ \sigma\in\Sigma_2.$$

Define

$$\check{C}^p(\mathcal{U}_{\Sigma_1},\hat{g}\Lambda^{ch}\mathcal{T}_{X_{\Sigma_1}}e^{qg^{-1}S^*};d_{\check{C}},g^{-1}D(S^*)) \tag{3.9.5}$$

to consist of the family of elements

$$f_\sigma \in \hat{g}\{\mathcal{T}_{X_{\Sigma_1}}(U_{g\sigma})\}e^{qg^{-1}S^*},\ \sigma\in\Sigma_1.$$

By construction, the map

$$f_\sigma \mapsto \hat{g}f(g^{-1}\sigma),\ \sigma\in\Sigma_2$$

delivers an isomorphism of (3.9.5) and (3.9.4):

$$\begin{aligned}&\check{C}^p(\mathcal{U}_{\Sigma_2\setminus S^*},\Lambda^{ch}\mathcal{T}_{X_{\Sigma_2\setminus S^*}}e^{qS^*};d_{\check{C}},D(S^*)) \xrightarrow{\sim}\\ &\check{C}^p(\mathcal{U}_{\Sigma_1},\hat{g}\Lambda^{ch}\mathcal{T}_{X_{\Sigma_1}}e^{qg^{-1}S^*};d_{\check{C}},g^{-1}D(S^*)).\end{aligned} \tag{3.9.6}$$

3.10. *Digression: Borisov's realization and the spectral flow.* We are now making good on our promise to show how spectral flow (1.12.1) is realized via lattice vertex algebras in the case of the $bc-\beta\gamma$-system. We shall show in 5.2.15 that a simple version of this construction does the same for $N2$.

According to our conventions $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)\subset\mathbb{B}_{M,\Lambda}$, and by definition (3.1.2) $M\subset L$ satisfies (1.12.1). Therefore any $\alpha\in M^*$ generates on $\mathrm{Lie}\mathbb{B}_{M,\Lambda}$ the spectral flow in the direction of α, see (1.12.6). A glance at (3.3.2a,b) shows that

$$S_{\sum_j X_j^*}:\ \begin{aligned}&x_i(z)\mapsto x_i(z)z,\ dx_i(z)\mapsto dx_i(z)z,\\ &\partial_{x_i}(z)\mapsto\partial_{x_i}(z)z^{-1},\partial_{dx_i}(z)\mapsto\partial_{dx_i}(z)z^{-1},\end{aligned} \tag{3.10.1}$$

cf. 1.12.7-8, and this does coincide with S_1 of (1.12.1).

It follows that the map

$$e^{-\sum_j X_j^*}:\ e^{\sum_j X_j^*}\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)\to\Lambda^{ch}\mathcal{T}(\mathbb{C}^N);v\mapsto e^{-\sum_j X_j^*}v \tag{3.10.2}$$

identifies $e^{\sum_j X_j^*}\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)$ with the spectral flow transform of $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)$:

$$e^{\sum_j X_j^*}\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)\xrightarrow{\sim}S_1(\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)), \tag{3.10.3}$$

see definition of the spectral flow transform (1.12.2).

Therefore, results of 2.3 can be rewritten as follows:

$$H^n(\mathbb{C}^N-0,\Lambda^{ch}\mathcal{T}_{\mathbb{C}^N-0})=\begin{cases}\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)\text{ if }n=0\\ \Lambda^{ch}\mathcal{T}(\mathbb{C}^N)e^{\sum_j X_j^*}\text{ if }n=N-1\\ 0\text{ otherwise.}\end{cases} \tag{3.10.4}$$

4. Chiral polyvector fields over hypersurfaces in projective spaces

Having put all the preliminaries out of the way we can tackle our main problem – computation of $H^*(F, \Lambda^{ch}\mathcal{T}_F)$ for a smooth hypersurface $F \subset \mathbb{P}^{N-1}$.

4.1. Let $\mathcal{L} \to \mathbb{P}^{N-1}$ be the degree N line bundle over $\mathbb{P}^{N-1}$ and

$$\pi : \mathcal{L}^* \to \mathbb{P}^{N-1} \tag{4.1.0}$$

its dual. Let us give the spaces $\mathbb{C}_{-N} = \mathbb{C}$ and $\mathbb{C}^N - 0$ a $\mathbb{C}^*$-space structure as follows:

$$\begin{gathered} \mathbb{C}_{-N} \times \mathbb{C}^* \to \mathbb{C}_{-N},\ u \cdot t = st^{-N}, \\ \mathbb{C}^* \times (\mathbb{C}^N - 0) \to \mathbb{C}^N - 0,\ t \cdot (x_0, ..., x_{N-1}) = (tx_0, ..., tx_{N-1}). \end{gathered} \tag{4.1.1}$$

One has the quotient realization

$$\mathcal{L}^* = \mathbb{C}_{-N} \times_{\mathbb{C}^*} (\mathbb{C}^N - 0), \tag{4.1.2}$$

where we impose the relation

$$(u; x_0, ..., x_{N-1}) \sim (ut^{-N}; tx_0, ..., tx_{N-1}),\ t \neq 0. \tag{4.1.3}$$

Let $\mathbb{Z}_N$ act on $\mathbb{C}^N$ as follows

$$\begin{gathered} \mathbb{Z}^N \times (\mathbb{C}^N - 0) \to (\mathbb{C}^N - 0), \\ \bar{m} \cdot (x_0, ..., x_{N-1}) = (e^{2\pi\sqrt{-1}m/N}x_0, ..., e^{2\pi\sqrt{-1}m/N}x_{N-1}). \end{gathered} \tag{4.1.4}$$

Crucial for our purposes is the following isomorphism of smooth algebraic varieties

$$\begin{gathered} p:\ (\mathbb{C}^N - 0)/\mathbb{Z}_N \xrightarrow{\sim} \mathcal{L}^* - 0, \\ \text{class of } (x_0, ..., x_{N-1}) \mapsto \text{class of } (1; x_0, ..., x_{N-1}), \end{gathered} \tag{4.1.5}$$

where $\mathcal{L}^* - 0$ denotes $\mathcal{L}^*$ without the zero section. (This is well known and obvious: deleting the zero section means requiring that $u \neq 0$; then one uses the $\mathbb{C}^*$-action to make $u = 1$; this leaves only the classes of $(1; x_0, ..., x_{N-1})$ and simultaneously breaks the $\mathbb{C}^*$-action down to the $\mathbb{Z}_N$-action defined in (4.1.4).)

We wish to study Calabi-Yau hypersurfaces in $\mathbb{P}^{N-1}$. To define any such hypersurface, take $x_0, ..., x_{N-1}$ to be, in accordance with (4.1.1), the homogeneous coordinates on $\mathbb{P}^{N-1}$. Let

$$\mathfrak{F} = \{(x_0 : \cdots : x_{N-1}) \text{ s.t. } f(x_0, ..., x_{N-1}) = 0\}, \tag{4.1.6}$$

where f is a degree N homogeneous polynomial with a unique singularity at 0. Such an f can be regarded as a section of $\mathcal{L}$. The corresponding fiberwise linear function $t = t(u; x_0, ..., x_{N-1})$ on $\mathcal{L}^*$is, in terms of coordinates on $\mathbb{C}_{-p} \times (\mathbb{C}^p - 0)$,

$$t = uf(x_0, ..., x_{N-1}), \tag{4.1.7}$$

cf. (4.1.2,3). The pull-back of this function onto $(\mathbb{C}^N - 0)/\mathbb{Z}_N$ under (4.1.5) is literally $f(x_0, ..., x_{N-1})$:

$$p^*(t) = f(x_0, ..., x_{N-1}) \tag{4.1.8}$$

as follows from (4.1.5).

4.2. $\mathcal{L}^*$ carries the standard affine covering $\mathcal{U} = \{U_j,\ 0 \leq j \leq N-1\}$ defined by

$$U_j = \{\text{class of } (u; x_0, ..., x_{N-1}) \text{ s.t. } x_j \neq 0\}. \tag{4.2.0}$$

Then $\pi\mathcal{U} = \{\pi(U_j),\ 0 \leq j \leq N-1\}$ is (also standard) affine covering of $\mathbb{P}^{N-1}$ and $\pi\mathcal{U} \cap \mathfrak{F} = \{\pi(U_j) \cap \mathfrak{F}\}$ is an affine covering of $\mathfrak{F}$.

By definition, the cohomology of the Čech complex $\check{C}^*(\pi\mathcal{U} \cap \mathfrak{F}, \Lambda^{ch}\mathcal{T}_{\mathfrak{F}})$ equals the cohomology $H^*(\mathfrak{F}, \Lambda^{ch}\mathcal{T}_{\mathfrak{F}})$. A more practical way to compute the latter is provided by Lemma 2.4.2. Indeed, being currently in the situation of this lemma we obtain the bi-complex $\check{C}^*(\mathcal{U}, R^*\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}; d_{\check{C}}, dt_{(0)})$, that is, the Čech complex with coefficients in the complex $(\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}, dt_{(0)})$ defined in (2.4.5). Associated to this bi-complex there are two standard spectral sequences, $'E_r^{p,q}$ and $''E_r^{p,q}$, such that

$$\begin{aligned} ('E_1^{p,q}, 'd_1) &= (H^p(\mathcal{L}^*, R^q(\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})), dt_{(0)}), \\ 'E_2^{p,q} &= H^q_{dt_{(0)}}(H^p(\mathcal{L}^*, \Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})), \end{aligned} \tag{4.2.1a}$$

$$(''E_1^{p,q}, ''d_1) = \check{C}^p(\mathcal{U}, \mathcal{H}^q_{dt_{(0)}}(R^q(\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})). \tag{4.2.1b}$$

In the latter $\mathcal{H}^q_{dt_{(0)}}(R^*\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})$ denotes the q-th cohomology sheaf of complex (2.3.5).

4.3. Lemma. *Both* $'E_r^{p,q}$ *and* $''E_r^{p,q}$ *abut to* $H^*(\mathfrak{F}, \Lambda^{ch}\mathcal{T}_{\mathfrak{F}})$.

4.4. *Proof.* Observe that even though sheaf complex (2.4.5) appears to be infnite in both directions, its differential preserves conformal weight (cf. 2.4.3) and it is easy to see that each fixed conformal weight component of (2.4.5) is finite. (Indeed, the differential $dt_{(0)}$ changes fermionic charge by one and it follows from (2.3.5) that for a fixed conformal weight fermionic charge may acquire only a finite number of values.) This implies, in a standard manner, that both the spectral sequences abut to the cohomology of the total complex $C^*(\mathcal{U}, R^*\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}; d_{\check{C}} + dt_{(0)})$. Lemma 2.4.2 implies that

$$''E_1^{p,q} \xrightarrow{\sim} \begin{cases} 0 \text{ if } q \neq 0 \\ C^p(\pi\mathcal{U} \cap \mathfrak{F}, \Lambda^{ch}\mathcal{T}_{\mathfrak{F}}) \text{ otherwise }. \end{cases}$$

Hence the second spectral sequence degenerates to the Čech complex over $\mathfrak{F}$ and the lemma follows. □

4.5. We now wish to compute the 1st term of the 1st spectral sequence recorded in (4.2.1a). Ignoring the double grading we rewrite (4.2.1a) as

$$('E_1^{**}, 'd_1) = (H^*(\mathcal{L}^*, \Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}), dt_{(0)}), \tag{4.5.1}$$

We have $\mathcal{L}^* - 0 \hookrightarrow \mathcal{L}^*$; this places us in the set-up of 3.7, see e.g. (3.7.1,2), and in order to compute $(H^*(\mathcal{L}^*, \Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}))$ we employ the spectral sequence of Lemma 3.8. – in this particular case we shall be able to compute all its terms. Begin with

4.5.1. *Toric description of $\mathcal{L}^*$.* Choose

$$s = x_0^N, y_j = \frac{x_j}{x_0},\ 1 \le j \le N \tag{4.5.2}$$

to be coordinates of $\mathcal{L}^*$ over the "big cell" U_0, see (4.2.0). Let

$$\begin{aligned} M_{\mathcal{L}^*} &= \mathbb{Z}S \bigoplus \{\bigoplus_{i=1}^{N-1} \mathbb{Z}Y_i\}, \\ M_{\mathcal{L}^*}^* &= \mathbb{Z}S^* \bigoplus \{\bigoplus_{i=1}^{N-1} \mathbb{Z}Y_i^*\}, \\ \Lambda_{\mathcal{L}^*} &= M_{\mathcal{L}^*} \oplus M_{\mathcal{L}^*}^*, \end{aligned} \tag{4.5.3}$$

and the bases $\{S, Y_1, ..., Y_{N-1}\}$ and $\{S^*, Y_1^*, ..., Y_{N-1}^*\}$ be dual to each other.

It follows from (4.5.2) and the identifications $s = e^S$, $y_j = e^{Y_j}$ that a fan $\Sigma_{\mathcal{L}^*}$ that defines $\mathcal{L}^*$ can be chosen as follows: the set of its 1-dimensional generators consists of

$$S^*, NS^* - \sum_{i=1}^{N-1} Y_i^*, Y_1^*, Y_2^*, ..., Y_{N-1}^*. \tag{4.5.4}$$

The set of the highest dimension cones consists of N cones, each generated by S^* and the rest of the vectors in (4.5.4) except one of them. These data determine $\Sigma_{\mathcal{L}^*}$ uniquely.

4.5.2. *Computation of* (E_1^{**}, d_1), (E_2^{**}, d_2),..., $(E_\infty^{**}, d_\infty)$.

Proceeding along the lines of 3.7 we obtain

$$(E(\mathcal{L}^*)_*^{**}, d_*) \Rightarrow H^*(\mathcal{L}^*, \Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}). \tag{4.5.41/2}$$

(3.7.5) reads (we skip $\mathcal{L}^*$):

$$(E_1^{p,q}, d_1) = (H^p(\mathcal{L}^* - 0, \Lambda^{ch}\mathcal{T}_{\mathcal{L}^*-0})e^{qS^*}, (e^{qS^*}(z)\tilde{S}^*(z))_{(0)}). \tag{4.5.5}$$

Thanks to (4.1.5) and the naturality of $\Lambda^{ch}\mathcal{T}_X$, 2.1 (i), there are canonical ismorphisms

$$\begin{aligned} &H^p(\mathcal{L}^* - 0, \Lambda^{ch}\mathcal{T}_{\mathcal{L}^*-0}) \xrightarrow{\sim} H^p(\mathbb{C}^N - 0/\mathbb{Z}^N, \Lambda^{ch}\mathcal{T}_{\mathbb{C}^N\backslash 0/\mathbb{Z}^N}) \\ &\xrightarrow{\sim} H^p(\mathbb{C}^N - 0, \Lambda^{ch}\mathcal{T}_{\mathbb{C}^N-0})^{\mathbb{Z}_N}. \end{aligned} \tag{4.5.6}$$

The latter has been already computed, see 2.3.5, formulas (2.3.8,10,11). Therefore, $(E_1^{p,q}, d_1)$ is as follows:

$$E_1^{p,q} = \begin{cases} \Lambda^{ch}\mathcal{T}(\mathbb{C}^N)^{\mathbb{Z}_N}e^{qS^*} \text{ if } p = 0 \\ S_1(\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)^{\mathbb{Z}_N})e^{qS^*} \text{ if } p = N-1 \\ 0 \text{ otherwise} \end{cases} \tag{4.5.7a}$$

$$d_1 = (e^{S^*}(z)\tilde{S}^*(z))_{(0)}. \tag{4.5.7b}$$

Note that $(E_1^{0,*}, d_1)$ is canonically a differential graded vertex algebra and $(E_1^{N-1,*}, d_1)$ is canonically a differential graded $(E_1^{0,*}, d_1)$-module. Therefore, as a vertex algebra,

$$(E_1^{*,*}, d_1) = (E_1^{0,*}, d_1) \oplus (E_1^{N-1,*}, d_1) \tag{4.5.8}$$

is the abelian extension of the former by the latter. Therefore, (3.7.6) reads as follows:

$$E_2^{*,*} = H_{d_1}(E_1^{0,*}, d_1) \oplus H_{d_1}(E_1^{N-1,*}, d_1), \tag{4.5.9}$$

and again $H_{d_1}(E_1^{0,*}, d_1)$ is a vertex algebra, $H_{d_1}(E_1^{N-1,*}, d_1)$ its module, and $E_2^{*,*}$ is the abelian extension of the former by the latter.

Thanks to (4.5.7a), the dimension consideration shows that

$$\begin{aligned} &d_2 = d_3 = \cdots d_{N-1} = 0, \\ &E_2^{*,*} = E_3^{*,*} = \cdots = E_{N-3}^{*,*} = E_N^{*,*}. \end{aligned} \tag{4.5.10}$$

The same argument shows that the non-zero components of the last non-zero differential are the following maps:

$$d_N^{(i)} : E_2^{N-1,i} \to E_2^{0,i+N}. \tag{4.5.11}$$

As follows from 4.5.1, $\mathcal{L}^*$ is covered by N open affine sets. Therefore, $H^n(\mathcal{L}^*, \Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}) = 0$ if $n > N-1$. This implies that the maps $d_N^{(i)}$ are isomorphisms if $i > 0$ and $d_N^{(0)}$ is an epimorphism. Hence

$$E_\infty^{*,*} = E_2^{0,0} \oplus E_2^{0,1} \oplus \cdots \oplus E_2^{0,N-2} \oplus E_2^{0,N-1} \oplus \underbrace{\mathrm{Ker} d_N^{(0)}}_{E_\infty^{N-1,0}}. \tag{4.5.12}$$

By the spectral sequence definition, (4.5.12) implies

$$\begin{aligned} &H^i(\mathcal{L}^*, \Lambda^{ch}\mathcal{T}_{\mathfrak{F}}) \xrightarrow{\sim} H_{d_1}(E_1^{0,i}, d_1),\ 0 \le i \le N-2, \\ 0 \to H_{d_1}(E_1^{0,N-1}, d_1) \to &H^{N-1}(\mathcal{L}^*, \Lambda^{ch}\mathcal{T}_{\mathfrak{F}}) \to \mathrm{Ker} d_N^{(0)} \to 0. \end{aligned} \tag{4.5.13}$$

Now let us perform a change of coordinates that will reveal so far invisible structure of result (4.5.7ab, 4.5.13).

4.6. We have used the coordinates attached to $\mathcal{L}^*$ by definition. Now let us employ the map

$$\mathbb{C}^N - 0 \to \mathcal{L}^* - 0, \tag{4.6.1}$$

that is, the composition of (4.1.5) and the natural projection $\mathbb{C}^N - 0 \to (\mathbb{C}^N - 0)/\mathbb{Z}_N$. We would like to recast the argument of 4.5 in terms inherent in $\mathbb{C}^N - 0$.

By invoking (4.6.1) we have placed ourselves in the situation of 3.9. Let us make this explicit.

$\mathbb{C}^N - 0$ has the standard coordinate system x_i, ∂_{x_i} and has, therefore, the following toric description:

$$M = \oplus_{i=0}^{N-1}\mathbb{Z}X_i,\ M^* = \oplus_{i=0}^{N-1}\mathbb{Z}X_i^*,\ \Lambda = M \oplus M^*, \tag{4.6.2a}$$

so that

$$X_i^*(X_j) = \delta_{ij}. \tag{4.6.2b}$$

$$\Sigma = \{\sigma_0, ..., \sigma_{N-1}\},\ \sigma_j = \{x_j \neq 0\}. \tag{4.6.2c}$$

Map (4.6.1) is induced by the lattice (cf. 3.2) embedding

$$g : M^* \to M^*_{\mathcal{L}^*}\ (= M^*_{\mathcal{L}^*-0}) \tag{4.6.3}$$

dual to the lattice embedding

$$\begin{aligned} &g^* : M_{\mathcal{L}^*} \to M, \\ &S \mapsto NX_0,\ Y_j \mapsto X_j - X_0,\ 1 \le j \le N-1. \end{aligned} \tag{4.6.4a}$$

Indeed, comparing (4.1.5) and (4.5.2) one obtains: under (4.6.1), $s \mapsto x_0^N$, $y_j \mapsto x_j/x_0$ and it remains to use $x_i = e^{X_i}$, $s = e^S$, $y_j = e^{Y_j}$ in order to obtain (4.6.4a).

It is immediate to see that

$$g(X_0^*) = NS^* - \sum_{j=1}^{N-1} Y_j^*,\ g(X_j^*) = Y_j^*,\ j \ge 1; \tag{4.6.4b}$$

thus $g(\Sigma) = \Sigma_{\mathcal{L}^*}$. Therefore, we are indeed in the situation of 3.9, and if we write down isomorphism (3.9.6) explicitly, we will have all the assertions of 4.5 recast in terms pertaining to $\mathbb{C}^N - 0$.

According to (3.1.7), there arises an isomorphism

$$\hat{g} : \mathbb{B}_{\Lambda_{\mathcal{L}^*}} \to \mathbb{B}_{g^{-1}\Lambda_{\mathcal{L}^*}}, \tag{4.6.5}$$

where

$$g^{-1}\Lambda_{\mathcal{L}^*} = g^* M_{\Lambda_{\mathcal{L}^*}} \oplus g^{-1} M^*_{\Lambda_{\mathcal{L}^*}} \subset \Lambda_{\mathbb{Q}}.$$

It follows from (4.6.4a) that

$$g^* M_{\Lambda_{\mathcal{L}^*}} = \{\sum_{i=0}^{N-1} m_i X_i \text{ s.t. } N | \sum_{i=0}^{N-1}\} \subset M. \tag{4.6.6}$$

Inverting (4.6.4b) we obtain that $g^{-1}M^*_{\Lambda_{\mathcal{L}^*}}$ is spanned over $\mathbb{Z}$ by

$$g^{-1}(S^*) = \frac{1}{N}(X_0^* + X_1^* + \cdots + X_{N-1}^*), g^{-1}(Y_j^*) = X_j^*, 1 \le j \le N-1. \tag{4.6.7}$$

Let us introduce the notation

$$X^*_{orb} = \frac{1}{N}(X_0^* + X_1^* + \cdots + X_{N-1}^*). \tag{4.6.9}$$

The first line of (4.5.7a) and (4.5.7b) rewrites as follows:

$$\begin{aligned} &(E_1^{0,*}, d_1) = (\oplus_{n=0}^{\infty} \Lambda^{ch} \mathcal{T}(\mathbb{C}^N)^{\mathbb{Z}_N} e^{nX^*_{orb}}, D_{orb}) \\ &D_{orb} = \frac{1}{N} \sum_{i=0}^{N-1} (\tilde{X}_i(z) e^{X^*_{orb}}(z))_{(0)}. \end{aligned} \tag{4.6.10}$$

(For the latter (3.1.4b) and (4.6.7) were used.)

This is a differential (w.r.t. D_{orb}) graded vertex algebra and because of its importance and its relation to Landau-Ginzburg models to be discovered later on, we make a digression.

4.6.1. *Vertex algebra* $\widetilde{\mathrm{LG}}_{orb}$. Introduce the following notation:

$$\widetilde{\mathrm{LG}}_{orb} = \oplus_{n=0}^{\infty}\widetilde{\mathrm{LG}}_{orb}^{(n)},\ \widetilde{\mathrm{LG}}_{orb}^{(n)} = \Lambda^{ch}\mathcal{T}(\mathbb{C}^N)^{\mathbb{Z}_N}e^{nX^*_{orb}} \subset \mathbb{B}_{g^{-1}\Lambda_{\mathcal{L}^*}}. \tag{4.6.11}$$

This vertex algebra is filtered by the system of differential vertex ideals

$$\widetilde{\mathrm{LG}}_{orb}^{\geq m} = \oplus_{n\geq m}\widetilde{\mathrm{LG}}_{orb}^{(n)}. \tag{4.6.12}$$

Hence there arises a projective system of differential vertex algebras

$$\widetilde{\mathrm{LG}}_{orb}^{<m} = \widetilde{\mathrm{LG}}_{orb}/\widetilde{\mathrm{LG}}_{orb}^{\geq m}. \tag{4.6.13}$$

It is obvious that the natural projection

$$\widetilde{\mathrm{LG}}_{orb} \to \widetilde{\mathrm{LG}}_{orb}^{<m} \tag{4.6.13}$$

induces the isomorphisms

$$H^i_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb}) \xrightarrow{\sim} H^i_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb}^{<m}) \text{ if } i < m-1. \tag{4.6.14}$$

If $i = m-1$, then (4.6.13) induces the embedding

$$H^{m-1}_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb}) \hookrightarrow H^{m-1}_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb}^{<m}), \tag{4.6.15}$$

which is included in the following short exact sequence

$$0 \to H^{m-1}_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb}) \to H^{m-1}_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb}^{<m}) \to B^m \to 0, \tag{4.6.16}$$

where $B^m \subset \widetilde{\mathrm{LG}}_{orb}^{(m)}$ is the group of m-couboundaries of $\widetilde{\mathrm{LG}}_{orb}$ w.r.t. D_{orb}. If we think of (4.6.15) as a 1-step filtration on $H^{m-1}_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb}^{<m})$, then we get from (4.6.16)

$$\mathrm{Gr}H^{m-1}_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb}^{<m}) = H^{m-1}_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb}) \oplus B^m. \tag{4.6.17}$$

□

Now, to the second line of (4.5.7a). The spectral flow transform appearing there had been earlier interpreted in (3.10.4) inside $\mathbb{B}_\Lambda$, currently in use, by employing exactly X^*_{orb} although not its name. According to (3.10.4) we have

$$E_1^{N-1,q} = \Lambda^{ch}\mathcal{T}(\mathbb{C}^N)^{\mathbb{Z}^N}e^{(q+N)X^*_{orb}}.$$

The complete translation of (4.5.7a) is then this:

$$(E_1^{*,*},d_1)=(E_1^{0,*}\oplus E_1^{N-1,*},d_1)=(\widetilde{\mathrm{LG}}\oplus\widetilde{\mathrm{LG}}^{\geq N},D_{orb}). \qquad (4.6.18)$$

The rest of 4.5 carries over in a straightforward manner: (4.5.9) is translated as

$$(E_2^{*,*},d_2)=(H^*_{D_{orb}}(\widetilde{\mathrm{LG}})\oplus H^*_{D_{orb}}(\widetilde{\mathrm{LG}}^{\geq N}),0), \qquad (4.6.19)$$

the 1st summand standing for $E_2^{0,*}$, the 2nd for $E_2^{N-1,*}$; (4.5.13) is translated as

$$\begin{aligned} H^i(\mathcal{L}^*,\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}) &\stackrel{\sim}{\longrightarrow} H^i_{D_{orb}}(\widetilde{\mathrm{LG}}),\ 0\leq i\leq N-2\\ 0\to H^{N-1}_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb})\to &H^{N-1}(\mathcal{L}^*,\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})\to B^N\to 0,\end{aligned} \qquad (4.6.20)$$

where we freely use the notation of digression 4.6.1. The first line of (4.6.20) is obvious, and in the second only the identification of B^N with $\mathrm{Ker}d_N^{(0)}$ needs an explanation. It follows from (4.5.12) that the latter includes in the short exact sequence

$$0\to\mathrm{Ker}d_N^{(0)}\to E_2^{N-1,0}\to E_2^{0,N}\to 0. \qquad (4.6.21)$$

According to (4.6.19), we have $E_2^{N-1,0}=Z^N$, $E_2^{0,N}=Z^N/B^N$, where B^N, Z^N are the groups of N-cocycles (coboundaries resp.) of the complex $(\tilde{\mathrm{LG}}_{orb},D_{orb})$. Therefore (4.6.21) can be identified with

$$0\to B^N\to Z^N\to Z^N/B^N\to 0. \qquad (4.6.22)$$

We have proved

4.7. Theorem. *The cohomology $H^*(\mathcal{L}^*,\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})$ satisfies:*

$$H^i(\mathcal{L}^*,\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})\stackrel{\sim}{\longrightarrow}H^i_{D_{orb}}(\widetilde{\mathrm{LG}}^{<N}_{orb})\text{ if } i<N-1, \qquad (4.7.1)$$

and if $i=N-1$, then there arises the short exact sequence

$$0\to H^{N-1}_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb})\to H^{N-1}(\mathcal{L}^*,\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})\to B^N\to 0. \qquad (4.7.2)$$

In particular,

$$\mathrm{Gr}H^*(\mathcal{L}^*,\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})\stackrel{\sim}{\longrightarrow}\mathrm{Gr}H^*_{D_{orb}}(\widetilde{\mathrm{LG}}^{<N}_{orb}), \qquad (4.7.3)$$

see (4.6.17) with $m=N$ for the definition of $\mathrm{Gr}H^*_{D_{orb}}(\widetilde{\mathrm{LG}}^{<N}_{orb})$.

4.7.1. Remark. Having put (4.16.14-16) on the table next to (4.6.20) one observes that (4.7.1) has a good chance of being valid for $i=N-1$ as well.

4.8. Recall that our ultimate goal is the cohomology vertex algebra $H^*(\mathfrak{F},\Lambda^{ch}\mathcal{T}_{\mathfrak{F}})$ and, according to (4.2.1a), Theorem 4.7 only computes the 1st term of the spectral sequence converging to $H^*(\mathfrak{F},\Lambda^{ch}\mathcal{T}_{\mathfrak{F}})$. It remains to write down explicitly its 2nd term, also see (4.2.1a), and for this we need the differential $dt_{(0)}$ and the grading $R^q(.)$ expressed in terms of $\widetilde{\mathrm{LG}}_{orb}$.

Thanks to (4.1.8) the differential is as follows:

$$\hat{g}(dt) = df(x_0, ..., x_{N-1}). \tag{4.8.1}$$

A quick computation using e.g. (1.9.9) shows that

$$[D_{orb}, df(z)_{(0)}] = 0. \tag{4.8.2}$$

The grading $R^q(\widetilde{\mathrm{LG}}_{orb})$, $q \in \mathbb{Z}$ is nicely described as follows:

$$R^q(\widetilde{\mathrm{LG}}_{orb}) = \mathrm{Ker}(X^*_{orb,(0)} - qI). \tag{4.8.3}$$

Indeed, $R^q(.)$ was defined in 2.4 as the eigenspace of the $\mathbb{C}^*$-action. In terms of S, Y_j, the infinitesimal generator of this action is $S^*_{(0)}$; for example, $S^*_{(0)}s = S^*_{(0)}e^S = e^S = s$, as it should because according to (3.3.2a) $s = e^S$ is the coordinate along the fiber. It remains to use (4.6.7).

Incidentally, the $\mathbb{Z}$-grading built into definition (4.6.11) of $\widetilde{\mathrm{LG}}_{orb}$ is likewise given by the eigenvalues of the operator $\sum_j X_{j,(0)}$. The two gradings are compatible because $[X^*_{orb}, \sum_j X_{j,(0)}] = 0$ as follows from (1.8.1).

Thus $\widetilde{\mathrm{LG}}$ is a bi-differential bi-graded vertex algebra, to be denoted in this capacity by $(\widetilde{\mathrm{LG}}; D_{orb}, df(z)_{(0)})$, and so is $(\widetilde{\mathrm{LG}}^{<N}; D_{orb}, df(z)_{(0)})$. Essentially it remains to summarize Lemma 4.3 and Theorem 4.7.

4.9. Theorem. *(i) Spectral sequence (4.2.1a) abuts to $H^*(\mathfrak{F}, \Lambda^{ch}\mathcal{T}_{\mathfrak{F}})$ so that*

$${}'E_2^{p,q} = H^q_{df(z)_{(0)}}(H^p_{D_{orb}}(\widetilde{\mathrm{LG}}^{<N}_{orb})), \quad \text{if } 0 \leq p \leq N-2, \tag{4.9.1}$$

and if $p = N-1$, then ${}'E_2^{N-1,q}$ is included into the long exact sequence stemming from the short exact sequence of complexes, cf. (4.7.2)

$$0 \to (H^{N-1}_{D_{orb}}(\widetilde{\mathrm{LG}}_{orb}), df(z)_{(0)}) \to ({}'E_1^{*,N-1-*}, d_1) \to (B^N, df(z)_{(0)}) \to 0. \tag{4.9.2}$$

(ii) In the conformal weight zero component the spectral sequence degenerates in the 2nd term and gives the isomorphism

$$\mathrm{Gr}H^*(\mathfrak{F}, \Lambda^*\mathcal{T}_{\mathfrak{F}}) \xrightarrow{\sim} \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{N-1} \oplus (\mathbb{C}[x_0, ..., x_{N-1}]/ < df >)^{\mathbb{Z}_N}. \tag{4.9.3}$$

4.9.1. Remark. For the same reason that was indicated in Remark 4.7.1, isomorphisms (4.9.1) have a good chance of being valid for $p = N-1$ as well. Were this the case, (4.9.2) would be unnecessary. □

Beginning of the proof. Item (i) is indeed simply (4.2.1a), Lemma 4.3, and (4.7.1-2) of Theorem 4.7 put together. The isomorphism in (ii) is well known classically and the degeneration assertion follows very easily. We prefer, however, to emphasize some additional structure hidden in the spectral sequence and then use it to give, among other things, a self-contained proof of (ii), see 4.13.

4.10. Addendum: $N2$-structure. Condition (4.1.0) ensures that $\mathcal{L}^*$ is the canonical line bundle and this places us in the situation of 2.4.4: coordinates $s, y_1, ..., y_{N-1}$ of (4.5.2) satisfy the conditions imposed in 2.4.4, formulas (2.4.7) define an $N2$-structure on $\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*}$ and (2.4.8) define that on $\Lambda^{ch}\mathcal{T}_{\mathfrak{F}}$. In order to compute this structure in terms of $\widetilde{\mathrm{LG}}_{orb}$ one has to do the following: first, compute the images of (2.4.8) in $\mathbb{B}_{\Lambda_{\mathcal{L}^*}}$ under Borisov's map (3.3.2a,b); second, apply map (4.6.5) to the result. This is straightforward and tedious but rewarding, the reward being the coincidence of the result with Witten's description of the Landau-Ginzburg model as we shall see in the next section, Lemma 5.2.14.

In order to record the result it is convenient to use the boson-fermion correspondence, 1.13. Let us introduce the standard lattice $\mathbb{Z}^N$ so that the standard basis $\chi_0, ..., \chi_{N-1}$ is orthonormal. Then, see 1.13, one can make identifications

$$\tilde{X}_i(z) = e^{\chi_i}(z),\ \tilde{X}_i^*(z) = e^{-\chi_i}(z). \tag{4.10.1}$$

4.10.1. Lemma. *The $N2$-structure on $H^*(\mathfrak{F}, \Lambda^{ch}\mathcal{T}_{\mathfrak{F}})$ comes from the following $N2$-structure on $\widetilde{\mathrm{LG}}_{orb}$:*

$$G(z) \mapsto \sum_{j=0}^{N-1} :\left(X_j^*(z) - \chi_j(z)\right) e^{\chi_j}(z):,\ Q(z) \mapsto \sum_{j=0}^{N-1} :\left(X_j(z) + \frac{1}{N}\chi_j(z)\right) e^{-\chi_j}(z):,$$

$$J(z) \mapsto \sum_{j=0}^{N-1} \left(-\frac{1}{N}X_j^*(z) + X_j(z) + \chi_j(z):\right),$$

$$L(z) \mapsto \sum_{j=0}^{N-1} \left(: X_j(z)X_j^*(z) : + \frac{1}{2} : \chi_j(z)^2 : - \frac{1}{2}\chi_j(z)' - X_j(z)'\right).$$

Proof. $N2$ is generated as a Lie algebra by 2 fields, $G(z)$ and $Q(z)$. Let us do $Q(z)$, the field that acquires the geometrically mysterious factor $1/N$. We have starting with (2.4.8)

$$\begin{aligned}
&Q(z) \mapsto s(z)'\partial_{ds}(z) + \sum_{j=1}^{N-1} y_j(z)'\partial_{dy_j}(z) - (s(z)\partial_{ds}(z))' = \\
&\sum_{j=1}^{N-1} : y_j(z)'\partial_{dy_j}(z) : - : s(z)(\partial_{ds}(z))' := \\
&\sum_{j=1}^{N-1} : e^{Y_j}(z)'(: e^{-Y_j}(z)\tilde{Y}_j^*(z) :) - : e^{S}(z)(: e^{-S}(z)\tilde{S}_j^*(z) :)' := \\
&\sum_{j=1}^{N-1} : Y_j(z)\tilde{Y}_j^*(z) : + : S(z)\tilde{S}^*(z) : - \tilde{S}^*(z)' = \\
&\sum_{j=1}^{N-1} : (X_j(z) - X_0(z))\tilde{X}_j^*(z) : + : NX_0(z)\frac{1}{N}\sum_{j=0}^{N-1}\tilde{X}_j^*(z) : - \frac{1}{N}\sum_{j=0}^{N-1}\tilde{X}_j^*(z)' = \\
&\sum_{j=0}^{N-1} : X_j(z)\tilde{X}_j^*(z) : - \frac{1}{N}\sum_{j=0}^{N-1}\tilde{X}_j^*(z)' = \\
&\sum_{j=0}^{N-1} : \left(X_j(z) + \frac{1}{N}\chi_j(z)\right) e^{-\chi_j}(z) : .
\end{aligned}$$

A brief guide to this computation is as follows: the 3rd line is (3.3.2a-b) applied to the 1st line, in the 5th line transformation formulas (4.6.7) are used, and boson-fermion correspondence (4.10.1) is used in the 7th. In addition, the well-known differentiation formula $e^{\alpha}(z)' =: \alpha(z)e^{\alpha}(z) :$ has been repeatedly employed. □

4.11. Character and Euler character formulas. Whenever one has a bi-graded vector space

$$V = \bigoplus_{m,n} V_n^m, \ \dim V_n^m < \infty,$$

one can define its character:

$$chV(s,\tau) = \sum_{m,n} e^{2\pi i(ms+n\tau)}\dim V_n^m, \tag{4.11.1}$$

and if in addition V is a supervector space, one can define its Euler character

$$\mathrm{Eu}V(s,\tau) = \sum_{m,n} e^{2\pi i(ms+n\tau)} s\dim V_n^m, \tag{4.11.2}$$

where the super-dimension $s\dim V_n^m$ is defined in the standard manner to be the dimension of the even component of V_n^m minus the dimension of its odd component. Note that if V carries an odd differential preserving the bi-grading, then

$$\mathrm{Eu}H_d(V)(s,\tau) = \mathrm{Eu}V(s,\tau). \tag{4.11.3}$$

As an example, we can consider $\widetilde{\mathrm{LG}}^{<N}_{orb}$ bi-graded by the eigenvalues of $L_{(1)}$, $J_{(0)}$, see Lemma 4.10.1. A straightforward computation (which, however, we postpone until 5.2.16) shows that if we introduce

$$E(\tau,s)=\prod_{n=0}^{\infty}\frac{\left(1-e^{2\pi i((n+1)\tau+(1-1/N)s)}\right)^N\left(1-e^{2\pi i(n\tau+(-1+1/N)s)}\right)^N}{\left(1-e^{2\pi i((n+1)\tau+s/N)}\right)^N\left(1-e^{2\pi i(n\tau-s/N)}\right)^N},$$

then

$$\mathrm{Eu}\widetilde{\mathrm{LG}}^{<N}_{orb}(\tau,s)=\frac{1}{N}\sum_{l=0}^{N-1}\sum_{j=0}^{N-1}e^{\pi i(N-2)\left\{-2js+\left(j^2-j\right)\tau+j^2\right\}}E(\tau,s-j\tau-l). \quad (4.11.4)$$

and if we introduce

$$\tilde{E}(\tau,s)=\prod_{n=0}^{\infty}\frac{\left(1+e^{2\pi i((n+1)\tau+(1-1/N)s)}\right)^N\left(1+e^{2\pi i(n\tau+(-1+1/N)s)}\right)^N}{\left(1-e^{2\pi i((n+1)\tau+s/N)}\right)^N\left(1-e^{2\pi i(n\tau-s/N)}\right)^N},$$

then

$$ch\widetilde{\mathrm{LG}}^{<N}_{orb}(\tau,s)=\frac{1}{N}\sum_{l=0}^{N-1}\sum_{j=0}^{N-1}e^{\pi i(N-2)\left\{-2js+\left(j^2-j\right)\tau\right\}}\tilde{E}(\tau,s-j\tau-l). \quad (4.11.5)$$

The repeated application of (4.11.3) to (4.7.3) shows that result (4.11.4) is valid for $H^*(\mathfrak{F},\Lambda^{ch}\mathcal{T}_{\mathfrak{F}})$ as well:

$$\mathrm{Eu}H^*(\mathfrak{F},\Lambda^{ch}\mathcal{T}_{\mathfrak{F}})=\frac{1}{N}\sum_{l=0}^{N-1}\sum_{j=0}^{N-1}e^{\pi i(N-2)\left\{-2js+\left(j^2-j\right)\tau+j^2\right\}}E(\tau,s-j\tau-l). \quad (4.11.6)$$

The importance of the latter is that, as was observed in [BL], the elliptic genus of $\mathfrak{F}$, $\mathrm{Ell}_{\mathfrak{F}}(\tau,s)$, satisfies

$$\mathrm{Ell}_{\mathfrak{F}}(\tau,s)=e^{\pi i(N-2)s}\mathrm{Eu}H^*(\mathfrak{F},\Lambda^{ch}\mathcal{T}_{\mathfrak{F}}). \quad (4.11.7)$$

We have proved

4.12. Corollary.

$$\mathrm{Ell}_{\mathfrak{F}}(\tau,s)=\frac{1}{N}\sum_{l=0}^{N-1}\sum_{j=0}^{N-1}e^{\pi i(N-2)\left\{-js+\left(j^2-j\right)\tau+j^2\right\}}E(\tau,s-j\tau-l). \quad (4.11.8)$$

4.13. Chiral rings and $H^*(\mathfrak{F},\Lambda^*\mathcal{T}_{\mathfrak{F}})$.

4.13.1. End of proof of Theorem 4.9. Arguments that led to Theorems 4.7, 9 consist of computations of two spectral sequences. Both the sequences are graded by $L_{(1)}$, see e.g. Lemma 4.10.1, and all the differentials preserve this grading. The classical story is about the events unfolding in the conformal weight zero component. Let us re-tell this story.

Therefore, we adopt that point of view according to which our task is to compute the cohomology algebra $H^*(\mathfrak{F},\Lambda^*\mathcal{T}_{\mathfrak{F}})$ of polyvector fields. Investigated in Theorem 4.9 is spectral sequence (4.2.1a); the origin of its conformal weight zero component is the classical Koszul sheaf complex (2.4.6) and its 1st term is the algebra $H^*(\mathcal{L}^*,\Lambda^*\mathcal{T}_{\mathcal{L}^*})$. The computation of this alegbra is accomplished in the classical part of Theorem 4.7 and that is where an important simplification occurs: (4.7.3) restricted to the conformal weight zero component is valid without the passage to the graded objects:

$$H^*(\mathcal{L}^*,\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})_0 \xrightarrow{\sim} H^*_{D_{orb}}(\widetilde{\mathrm{LG}}^{<N}_{orb})_0, \tag{4.13.1}$$

In fact, even taking the D_{orb}-cohomology is not necessary:

$$H^*(\mathcal{L}^*,\Lambda^{ch}\mathcal{T}_{\mathcal{L}^*})_0 \xrightarrow{\sim} (\widetilde{\mathrm{LG}}^{<N}_{orb})_0. \tag{4.13.2}$$

Indeed, by definition

$$(\widetilde{\mathrm{LG}}^{(0)}_{orb})_0 = \Lambda^*\mathcal{T}(\mathbb{C}^N)^{\mathbb{Z}_N}. \tag{4.13.3}$$

Further,

$$\dim(\widetilde{\mathrm{LG}}^{(i)}_{orb})_0 = 1,\ 1\le i\le N-1, \tag{4.13.4}$$

as easily follows from character formula (4.11.5). The corresponding generator is

$$(\widetilde{\mathrm{LG}}^{(i)}_{orb})_0 = \mathbb{C}e^{iX^*_{orb}-\sum_j(X_i+\chi_i)},\ 1\le i\le N-1, \tag{4.13.5}$$

where we again use boson-fermion correspondence (4.10.1). Now observe that spaces (4.13.3) and that spanned by elements (4.13.5) are both annihilated by D_{orb}. Thus we have

$$(('E_1^{*,*})_0,d_1) = \left(\underbrace{\mathbb{C}\oplus\cdots\oplus\mathbb{C}}_{N-1}\oplus\left(\Lambda^*\mathcal{T}(\mathbb{C}^N)\right)^{\mathbb{Z}_N}, df(z)_{(0)}\right). \tag{4.13.6}$$

Note that in view of (4.13.2), (4.9.1) is valid for $p=N-1$ as well. A quick computation shows that the elements (4.13.5) are annihilated by $df(z)_{(0)}$. As to (4.13.3), we have:

$(\Lambda^*\mathcal{T}(\mathbb{C}^N), df(z)_{(0)})$ *is the standard Koszul resolution of the Milnor ring.*

By virtue of (4.13.6),

$$('E_2^{*,*})_0 = \underbrace{\mathbb{C}\oplus\cdots\oplus\mathbb{C}}_{N-1}\oplus(\mathbb{C}[x_0,...,x_{N-1}]/<df>)^{\mathbb{Z}_N}. \tag{4.13.7}$$

Finally, all the higher differentials vanish simply because the elements listed in (4.13.5,7) are genuine cocycles. This completes the proof of Theorem 4.9. □

It is amusing to note that we have obtained "vertex" representatives of all the classes of the cohomology $H^*(\mathfrak{F},\Lambda^*\mathcal{T}_{\mathfrak{F}})$. Since, as was explained in 4.8, the eigenvalues of $X^*_{orb}+\sum_j X_{j,(0)}$ give the cohomological grading, we have:

$$(\text{class of})e^{iX^*_{orb}-\sum_j(X_j+\chi_j)}\in H^{i-1}(\mathfrak{F},\Lambda^{N-i-1}\mathcal{T}_{\mathfrak{F}}),\ 1\le i\le N-1, \tag{4.13.8}$$

$$\text{(class of)} \prod_j x_j^{m_j} = e^{\sum_j m_j X_j} \in H^m(\mathfrak{F}, \Lambda^m \mathcal{T}_{\mathfrak{F}}),\ m = \frac{1}{N}\sum_j m_j, 0 \le m_j \le N-2. \tag{4.13.9}$$

4.13.2. Multiplicative structure. The multiplicative structure of the ring $H^*(\mathfrak{F}, \Lambda^* \mathcal{T}_{\mathfrak{F}})$ is well known, of course. But let us restore it by the "vertex" methods.

According to (2.3.3b), the chiral ring of $H^*(\mathfrak{F}, \Lambda^{ch} \mathcal{T}_{\mathfrak{F}})$ is isomorphic to $H^*(\mathfrak{F}, \Lambda^* \mathcal{T}_{\mathfrak{F}})$. However, what we have at our disposal is the chiral ring $(\widetilde{\mathrm{LG}}^{<N}_{orb})_0$ and as a ring it only gives $\mathrm{Gr} H^*(\mathfrak{F}, \Lambda^{ch} \mathcal{T}_{\mathfrak{F}})$, as we have just proved – this is a common problem with spectral sequences.

Nevertheless, having re-examined the way in which spectral sequence (4.2.1a) was defined one concludes easily that the 0-th component, $(\widetilde{\mathrm{LG}}^{(0)}_{orb})_0$, remains unaffected by the passage to the graded object and thus carries the "right" multiplication; therefore, the Milnor ring, $\mathbb{C}[x]/ < df >$, embeds into $H^*(\mathfrak{F}, \Lambda^* \mathcal{T}_{\mathfrak{F}})$ as a ring, cf. (4.9.3).

Let us now look at elements (4.13.8). The computation

$$\left(e^{iX^*_{orb} - \sum_j (X_i + \chi_i)}\right)_{(-1)} e^{\sum_j m_j X_j} = \lim_{z \to w} (z-w)^{i \sum_j \frac{m_j}{N}} e^{iX^*_{orb} - \sum_j (X_j + \chi_j)} = \begin{cases} 0 \text{ if } \sum_j m_j > 0 \\ e^{iX^*_{orb} - \sum_j (X_i + \chi_i)} \text{ if } \sum_j m_j = 0, \end{cases} \tag{4.13.10}$$

as follows from (1.9.8), is valid even in $(\widetilde{\mathrm{LG}}^{(0)}_{orb})_0$; hence in $H^*(\mathfrak{F}, \Lambda^* \mathcal{T}_{\mathfrak{F}})$ as well.

Likewise, if $s + t < N$, one obtains

$$\begin{aligned} &\left(e^{sX^*_{orb} - \sum_j (X_j + \chi_j)}\right)_{(-1)} e^{tX^*_{orb} - \sum_j (X_j + \chi_j)} = \\ &(-1)^s \lim_{z \to w} (z-w)^{N-s-t} e^{(s+t)X^*_{orb} - 2\sum_j (X_j + \chi_j)} = 0 \end{aligned} \tag{4.13.11}$$

inside $(\widetilde{\mathrm{LG}}^{(0)}_{orb})_0$, hence inside $H^*(\mathfrak{F}, \Lambda^* \mathcal{T}_{\mathfrak{F}})$ as well.

Finally, the same computation shows that

$$\left(e^{sX^*_{orb} - \sum_j (X_j + \chi_j)}\right)_{(-1)} e^{(N-s)X^*_{orb} - \sum_j (X_j + \chi_j)} = (-1)^s e^{NX^*_{orb} - 2\sum_j (X_j + \chi_j)} \tag{4.13.12}$$

In $(\widetilde{\mathrm{LG}}^{<N}_{orb})_0$ the class of this element is zero because it belongs to $\widetilde{\mathrm{LG}}^{(N)}_{orb}$ and this component was cut off in definition (4.6.13). However, an amusing diagram search shows (we skip this computation) that in reality this element is cohomologous to $\pm \prod_j x_j^{N-2}$, which is a generator of $H^{N-2}(\mathfrak{F}, \Lambda^{N-2} \mathcal{T}_{\mathfrak{F}})$, see (4.13.9). Therefore,

$$\left(\text{class of } e^{sX^*_{orb} - \sum_j (X_j + \chi_j)}\right)_{(-1)} \left(\text{class of } e^{(N-s)X^*_{orb} - \sum_j (X_j + \chi_j)}\right) = \pm \prod_j x_j^{N-2} \tag{4.13.13}$$

and gives a non-degenerate pairing

$$H^{s-1}(\mathfrak{F},\Lambda^{N-s-1}\mathcal{T}_{\mathfrak{F}})\times H^{N-s-1}(\mathfrak{F},\Lambda^{s-1}\mathcal{T}_{\mathfrak{F}})\to H^{N-1}(\mathfrak{F},\Lambda^{N-1}\mathcal{T}_{\mathfrak{F}}), \qquad (4.13.14)$$

as it should. This completes the description of multiplication on $H^*(\mathfrak{F},\Lambda^*\mathcal{T}_{\mathfrak{F}})$.

5. Landau-Ginzburg orbifolds.

In this section we shall provide the necessary definitions and constructions so as to identify the complex $(\widetilde{\mathrm{LG}}^{<N}, df(z)_{(0)})$, which played an important role in Theorem 4.7 and 9, with the Landau-Ginzburg orbifold.

5.1. Landau-Ginzburg model. Let

$$f\in\mathbb{C}[x_0,...,x_{N-1}],\ \deg f=p \qquad (5.1.1)$$

be a homogeneous polynomial such that its partials $\partial_{x_i}f$, $0\leq i\leq N-1$, have only one common zero occurring at $\vec{x}=0$. For the time being this f need not be identified with f of (4.1.6), that is, p need not be N, but in the main application this assumption will be made.

By the Landau-Ginzburg model associated to $\mathbb{C}^N$ and f as in (5.1.1) we understand the differential vertex algebra

$$\mathrm{LG}_f=(\Lambda^{ch}\mathcal{T}(\mathbb{C}^N), df(z)_{(0)}). \qquad (5.1.2)$$

Note that the chiral ring of LG_f is the standard Koszul resolution:

$$K_f^*:\ (\mathrm{LG}_f)_0=(\Lambda^*\mathcal{T}(\mathbb{C}^N), df), \qquad (5.1.3)$$

the differential being the contraction with the 1-form df, cf. (4.13.6).

The following lemma is an important ingredient in Witten's approach [W2] to the Landau-Ginzburg model; exactly this form of the result appears as formula (3.1.1) in [KYY].

5.1.1. Lemma. *The assignment*

$$G(z)\mapsto\sum_{i=0}^{N-1}\partial_{x_i}(z)dx_i(z),\ Q(z)\mapsto\sum_{i=0}^{N-1}-\frac{1}{p}x_i(z)\partial_{dx_i}(z)'-(\frac{1}{p}-1)x_i(z)'\partial_{dx_i}(z),$$

$$\rho:\ J(z)\mapsto\sum_{i=0}^{N-1}-\frac{1}{p}:\partial_{x_i}(z)x_i(z):+(\frac{1}{p}-1):\partial_{dx_i}(z)dx_i(z):,$$

$$L(z)\mapsto\sum_{i=0}^{N-1}:\partial_{x_i}(z)x_i(z)':+:\partial_{dx_i}(z)'dx_i(z):$$

determines a vertex algebra morphism

$$\rho:\ V(N2)_{N\frac{p-2}{p}}\to\mathrm{LG}_f \qquad (5.1.4)$$

such that

$$df(z)_{(0)}\rho\left(V(N2)_{N\frac{p-2}{p}}\right) = 0. \tag{5.1.5}$$

5.1.2. Corollary. *The vertex algebra $H_{df(z)_{(0)}}(LG_f)$ carries an $N2$-structure inherited from that on LG_f.*

This follows at once from Lemma 5.1.1 and (1.3.1). Let us look at some basic examples.

5.1.3. Theorem ([FS]) *If $N = 1$, $f = x^p$, then $H_{df(z)_{(0)}}(\mathrm{LG}_f)$ is the direct sum of $p-1$ unitary $N2$-modules generated by $1, x, x^2, ..., x^{p-2}$.*

Denote by $U_{i,p}$ the unitary $N2$-module generated according to Theorem 5.1.3 by x^i, $0 \leq i \leq p-2$; here p keeps track of the "central charge", that is, the value by which the central element C operates on the module; in this case $C \mapsto (p-2)/p$. These modules are pairwise non-isomorphic.

5.1.4. Corollary. *If $f = \sum_j x_j^p$, then*

$$H_{df(z)_{(0)}}(\mathrm{LG}_f) = \bigoplus_{0 \leq j_0,...,j_{N-1} \leq p-2} \bigotimes_{t=0}^{N-1} U_{j_t,p}$$

which is a unitary $N2$-module w.r.t. the diagonal action of central charge $N(p-2)/p$. $\square$

Proof follows at once from Theorem 5.1.3 because in the case of the "diagonal" f the complex LG_f is the tensor product of the complexes of Theorem 5.1.3 – hence so is its cohomology as follows from the Künneth formula.

5.1.5. Remark. The reader will notice that complex (5.1.2) is nothing but (2.4.5) computed in the purely local situation with the function t replaced with f. Furthermore, the cohomology of (2.4.5) gives $\Lambda^{ch}\mathcal{T}_{Z(t)}$, see Lemma 2.4.2, and $Z(t)$ is exactly the singular locus of t: for any affine $U \subset \mathcal{L}^*$

$$Z(t) \cap U = \mathrm{Spec}\,(\mathcal{O}_{\mathcal{L}^*}(U)/ < dt >)\,.$$

Thus one is tempted to set

$$\Lambda^{ch}\mathcal{T}_{\mathrm{Spec}M_f}(\mathrm{Spec}M_f) = H_{df(z)_{(0)}}(\mathrm{LG}_f),$$

thereby defining the sheaf $\Lambda^{ch}\mathcal{T}_{\mathrm{Spec}M_f}$, where $M_f = \mathbb{C}[x_0, ..., x_{N-1}]/ < df >$.

To put this somewhat differently, we have resolved the singularity of M_f by passing to the DGA K_f^*, see (5.1.3), and then chiralized the latter so as to obtain (5.1.2). There seems to be a natural construction [KV2] allowing to chiralize in a similar manner other free DGA's thereby extending algebras of chiral polyvector fields from smooth varieties to spectra of Milnor rings to a wider class of schemes.

5.2. Landau-Ginzburg orbifold.

5.2.1. Familiar in vertex algebra theory is the following pattern: V is a vertex algebra; g is its order N automorphism; $V^{(i)}$ is a "naturally defined" g^i-twisted V-module, $1 \leq i \leq N-1$; assuming that the group $\{1, g, ..., g^{N-1}\}$ also acts on each

$V^{(i)}$, naturally w.r.t to the V-action, one forms the vertex algebra of g-invariants, V^g, and its (untwisted) modules $(V^{(i)})^g$, $1 \leq i \leq N-1$. It sometimes so happens that the space

$$V^g \oplus (V^{(1)})^g \oplus \cdots (V^{(N-1)})^g \tag{5.2.1}$$

carries an "interesting" vertex algebra structure compatible with the described V^g-module structure. Vertex algebra (5.2.1) is often referred to as an orbifold or a V orbifold.

The most famous example of an orbifold is undoubtedly the Monster vertex algebra V^{Mnstr} [FLM]. Indeed,

$$V^{\mathrm{Mnstr}} = V_L^g \oplus V_{L,1}^g,$$

where L is the Leech lattice, g its involution, and $V_{L,1}$ is an irreducible g-twisted V_L-module (unique by Dong's theorem [D2]).

As suggested in a number of physics papers, an incomplete list including [V,VW,W2] and references therein, realization of this pattern in the case where $V = \mathrm{LG}_f$, $\deg f = N$, and $g = \exp(2\pi i \rho J_0)$, $\rho J(z)$ defined in Lemma 5.1.1, should be closely related to "string vacua". We shall construct the candidates for $\mathrm{LG}_f^{(i)}$ and in order to do so we shall need a recollection on vertex algebra twisted modules. This notion was introduced in [FFR]. In our presentation we shall mostly follow [KR].

5.2.2. Twisting data.

Let G be an additive subgroup of $\mathbb{C}$ containing $\mathbb{Z}$. A vertex algebra V is called $G/\mathbb{Z}$-graded if

$$V = \oplus_{\bar{m} \in G/\mathbb{Z}} V[\bar{m}], \tag{5.2.2a}$$

so that

$$V[\bar{m}]_{(n)} V[\bar{l}] \subseteq V[\bar{m} + \bar{l}]. \tag{5.2.2b}$$

It is clear that $V[0] \subset V$ is a vertex subalgebra.

To give an example, let g be an order N automorphism of a vertex algebra V. Let $G = \frac{1}{N}\mathbb{Z}$. We have $G/\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}_N$.

Then

$$V = \oplus_{m=0}^{N-1} V[m] = \{v \in V : gv = e^{2\pi i m/N} v\} \tag{5.2.3}$$

is a $\mathbb{Z}_N$-grading. By definition, in this case $V[0]$ is the vertex subalgebra of g-invariants, V^g.

Let W be a vector space and $\bar{m} \in G/\mathbb{Z}$, where G is as in 2.1.1. An $\bar{m}$-twisted $\mathrm{End}M$-valued field is a series

$$a(z) = \sum_{m \in \bar{m}} a_{(m)} z^{-m-1},$$

where $a_{(m)} \in \mathrm{End}M$ is such that for any $v \in W$, $a_{(m)} v = 0$ if $\mathrm{Re}\, m >> 0$. Let $\mathrm{Field}_G(W)$ be the linear space of m-twisted $\mathrm{End}W$-valued fields for all $m \in G/\mathbb{Z}$.

5.2.3. Definition. (cf. Definition 1.2 and [KR, sect.5) A G-twisted V-module W is a parity preserving linear map

$$\rho : V \to \mathrm{Field}_G(W),\ (\rho a)(z) = \sum_m \rho a_{(m)} z^{-m-1}$$

satisfying the following axioms:

(i) if $a \in V[\bar{m}]$, then $(\rho a)(z)$ is $\bar{m}$-twisted;

(ii) $\rho(\mathbf{1}) = \mathrm{id}$;

(iii) (twisted Borcherds identity) for any $a \in V[\bar{m}]$, $b \in V$ and $F(z,w) = z^m(z-w)^l$ such that $m \in \bar{m}$, $l \in \mathbb{Z}$

$$\begin{aligned} &\mathrm{Res}_{z-w}\rho(a(z-w)b,w)i_{w,z-w}F(z,w) \\ &= \mathrm{Res}_z\left((\rho a)(z)(\rho b)(w)i_{z,w}F(z,w) - (-1)^{\mathrm{par}(a)\mathrm{par}(b)}(\rho b)(w)(\rho a)(z)i_{w,z}F(z,w)\right). \end{aligned} \tag{5.2.4}$$

5.2.4. Remarks.

(i) Note that the $l = 0$ case of the twisted Borcherds identity is the following commutator formula

$$[\rho a_{(m)}, \rho b_{(k)}] = \sum_{j=0}^{\infty} \binom{m}{j} \rho(a_{(j)}b)_{(m+k-j)}. \tag{5.2.5}$$

This shows that the coefficients of the fields $(\rho a)(z)$ form a Lie algebra.

(ii) A $\mathbb{Z}$-twisted vertex algebra module is called simply a vertex algebra module. In particular, the restriction of a twisted V-module W to the vertex subalgebra $V[0] \subset V$ is a $V[0]$-module. If G arises from an order N automorphism g as in (5.2.3), then a G-twisted module is called g-twisted or twisted by g. The restriction of a g-twisted V-module to the vertex subalgebra V^g is a V^g-module.

(iii) It should be clear what the phrases "W is an irreducible twisted V-module" and " W is a twisted V-module generated by a collection of fields $\{(\rho a_\alpha)(z)$ from a given vector $m \in M$".

(iv) A vertex algebra is canonically a module over itself. Given an arbitrary $G/\mathbb{Z}$-graded vertex algebra, there is no obvious way to construct a G-twisted module, but let us consider some concrete examples.

5.2.5. *The twisted module* $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)_{\vec{\lambda},\vec{\mu}}$. Given 2 n-tuples $\vec{\lambda} = (\lambda_0, ..., \lambda_{N-1}) \in \mathbb{C}^N$, $\vec{\mu} = (\mu_0, ..., \mu_{N-1}) \in \mathbb{C}^N$, let G be the $\mathbb{Z}$-span of λ_i, μ_i, and 1, $0 \le i \le N-1$. Define the $G/\mathbb{Z}$ grading on $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)$ by declaring that

$$\begin{aligned} &x_i \in \Lambda^{ch}\mathcal{T}(\mathbb{C}^N)[-\bar{\lambda_i}],\ \partial_{x_i} \in \Lambda^{ch}\mathcal{T}(\mathbb{C}^N)[\bar{\lambda_i}], \\ &dx_i \in \Lambda^{ch}\mathcal{T}(\mathbb{C}^N)[-\bar{\mu_i}],\ \partial_{dx_i} \in \Lambda^{ch}\mathcal{T}(\mathbb{C}^N)[-\bar{\mu_i}]. \end{aligned}$$

5.2.6. Lemma.

(i) There is a unique up to isomorphism structure of a G-twisted $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)$-module

$$\rho_{\vec{\lambda},\vec{\mu}} : \Lambda^{ch}\mathcal{T}(\mathbb{C}^N) \to \mathrm{Field}_G W$$

generated by $vac \in W$ such that

$$\begin{aligned}(\rho_{\vec\lambda,\vec\mu}x_i)_{(-\lambda_i+j)}vac = (\rho_{\vec\lambda,\vec\mu}\partial_{x_i})_{(\lambda_i+j)}vac = \\ (\rho_{\vec\lambda,\vec\mu}(dx_i))_{(-\mu_i+j)}vac = (\rho_{\vec\lambda,\vec\mu}\partial_{dx_i})_{(\mu_i+j)}vac = 0.\end{aligned} \tag{5.2.6}$$

(ii) This module can be constructed by letting $W = \Lambda^{ch}\mathcal{T}(\mathbb{C}^N)$ *as a vector space and*

$$\begin{aligned}(\rho_{\vec\lambda,\vec\mu}x_i)(z) = x_i(z)z^{\lambda_i},\ (\rho_{\vec\lambda,\vec\mu}\partial_{x_i})(z) = \partial_{x_i}(z)z^{-\lambda_i}, \\ (\rho_{\vec\lambda,\vec\mu}dx_i)(z) = dx_i(z)z^{\mu_i},\ (\rho_{\vec\lambda,\vec\mu}\partial_{dx_i})(z) = \partial_{dx_i}(z)z^{-\mu_i}.\end{aligned} \tag{5.2.7}$$

Proof. The commutation relations (5.2.5) applied to the quadruple of fields $(\rho_{\vec\lambda,\vec\mu}x_i)(z)$ $(\rho_{\vec\lambda,\vec\mu}\partial_{x_i})(z)$, $(\rho_{\vec\lambda,\vec\mu}dx_i)(z)$, $(\rho_{\vec\lambda,\vec\mu}\partial_{x_i})(z)$ imply that their coefficients span the Lie algebra isomorphic to $Cl \oplus \mathfrak{a}$, see 1.6, 1.7. For example,

$$[(\rho_{\vec\lambda,\vec\mu}\partial_{x_i})_{(\alpha)}, (\rho_{\vec\lambda,\vec\mu}x_i)_{(\beta)}] = \delta_{\alpha,-\beta-1}.$$

Conditions (5.2.6) become then the vacuum vector conditions for $Cl \oplus \mathfrak{a}$ which of course determine W uniquely even as $Cl \oplus \mathfrak{a}$-module, and uniqueness follows. This fixes recipe (5.2.7), and a standard argument lucidly explained e.g. in [KR, sect. 5] allows one to extend (5.2.7) naturally to a twisted module structure $\rho_{\vec\lambda,\vec\mu}$: $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N) \to \text{Field}_G W$. $\square$

Notation. We shall let $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)_{\vec\lambda,\vec\mu}$ denote the twisted $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)$-module occurring in Lemma 5.2.6.

5.2.7. *The twisted module* $V_{\Lambda+\eta}$. While constructing $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)_{\vec\lambda,\vec\mu}$ requires a little effort, a family of twisted modules over a lattice vertex algebra seems to be built into the definition of the latter. For simplicity we shall only consider the case where the lattice Λ is that defined in 3.1. (The following should be regarded as known even though we failed to find the needed results in the literature, but see e.g. a similar and more involved discussion in [D1].)

Fix $\eta \in \mathbb{C}\otimes_{\mathbb{Z}} M^*$ and consider the abelian group $L_\eta = \Lambda + \mathbb{C}\eta \subset \mathbb{C}\otimes\Lambda$. Let $\mathbb{C}[\Lambda_\eta]$ be its group algebra. By analogy with the lattice vertex algebra $V_\Lambda = V(\mathfrak{h}_L)\otimes\mathbb{C}_\epsilon[\Lambda]$ introduce $V_{\Lambda_\eta} = V(\mathfrak{h}_L)\otimes\mathbb{C}[\Lambda_\eta]$.

Let

$$\rho_\Lambda : V_L \to \text{Field}V_\Lambda \tag{5.2.8}$$

be the vertex algebra structure map. Note that $V_{\Lambda_\eta} = V(\mathfrak{h}_L)\otimes\mathbb{C}[\Lambda_\eta]$ is naturally a $V(\mathfrak{h}_L)$-module: indeed $x(z)1\otimes e^\beta$, $x \in \mathfrak{h}_L$, makes perfect sense if $\beta \in L_\eta$, or indeed if $\beta \in \mathbb{C}\otimes_{\mathbb{Z}} L$, see (1.9.5). Similarly, formula (1.9.2) may be extended without any changes to define a $\mathbb{C}_\epsilon[\Lambda]$-action on $V_{\Lambda_\eta} = V(\mathfrak{h}_L)\otimes\mathbb{C}[\Lambda_\eta]$; indeed, cocycle $\epsilon(\alpha,\beta)$ defined in (3.1.2) makes perfect sense for any $\beta \in L + \mathbb{C}\eta$ – it, so to say, ignores η. Since any field $\rho_\Lambda a(z)$, $a \in V_L$, is written in terms of the operators we have just described, cf (1.9.4-5), the very formula for $\rho_\Lambda a(z)$ defines it as a "field" operating on $V_{\Lambda_\eta} = V(\mathfrak{h}_L)\otimes\mathbb{C}[\Lambda_\eta]$. For example,

$$x(z)e^\eta = \left(x(z) + \frac{1}{z}(x,\eta)\mathbf{1}\right)e^\eta,\ x \in \mathfrak{h}_L \tag{5.2.9a}$$

$$e^{\alpha}(z)e^{\eta}=\left(e^{\alpha}(z)z^{(\alpha,\eta)}\mathbf{1}\right)e^{\eta}, \tag{5.2.9b}$$

where $\mathbf{1}=e^0$ is the vacuum vector of V_Λ. (Note, by the way, that (5.2.9a-b) are analogous to spectral flow formulas (1.12.7-8).) In order to fix the twist, see 5.2.2, let us restrict such fields to the subspace $V_{\Lambda+\eta}\subset V_{\Lambda_\eta}$ where only $e^{\alpha+\eta}$, $\alpha\in L$, are allowed. Formulas (5.2.9a-b) allow us to conclude that there is a tautological embedding

$$\rho_\Lambda V_\Lambda\hookrightarrow \mathrm{Field}_{G_\eta}V_{\Lambda+\eta}, \tag{5.2.10}$$

where G_η is the grading on V_Λ that assigns degree $-(\alpha,\eta)$ to e^α, $\alpha\in\Lambda$. Hence there arises the composition

$$\rho_{\Lambda+\eta}: V_\Lambda\stackrel{\rho_\Lambda}{\to}\rho_\Lambda V_\Lambda\hookrightarrow \mathrm{Field}_{G_\eta}V_{\Lambda+\eta}, \tag{5.2.11}$$

5.2.8. Lemma. *Map (5.2.11) endows $V_{\Lambda+\eta}$ with a G_η-twisted V_Λ-module structure.*

Proof. Recall that the intuition behind twisted Borcherds identity (5.2.4) – as well as its untwisted version (1.2.2) – is that the 3 expressions appearing in it, $\rho(a(z-w),w)$, $(\rho a)(z)(\rho b)(w)$, and $(-1)^{\mathrm{par}(a)\mathrm{par}(b)}(\rho b)(w)(\rho a)(z)$ are Laurent series expansions of the same function in 3 respective domains, see the short discussion after Definition 1.2. This can be made precise: if W is a vector space graded by finite dimensional subspaces, then one can define matrix elements of fields and their products, such as, $< v^*,\rho(a(z-w),w)v >$, $< v^*,(\rho a)(z)(\rho b)(w)v >$, and $(-1)^{\mathrm{par}(a)\mathrm{par}(b)} < v^*,(\rho b)(w)(\rho a)(z)v >$. If these are indeed Laurent series expansions of the same rational function twisted by z^{-m}, w^k, m, k not necessarily integral, with poles on $z=w$, $z=0$, $w=0$, then (5.2.4) holds; cf. [K, Remark 4.9a]. In our case a suitable grading is easy to exhibit and a familiar argument along the lines of [K, sect.5.4] shows that matrix elements of the products of fields from $\rho_{\Lambda+\eta}(V_\Lambda)$ are indeed such rational functions. Furthermore, it is easy to see that if we identify

$$V_{\Lambda+\eta}\stackrel{\sim}{\longrightarrow}V_\Lambda,\ v\mapsto ve^{-\eta},$$

then these matrix elements, with fixed $v^*,v\in V_\Lambda$, become analytic functions of η, see (5.2.9a-b). But if $\eta\in M^*$, then $V_{\Lambda+\eta}=V_\Lambda$ by definition and the matrix elements are indeed equal to each other rational functions. Thanks to analyticity, this must hold for all η and the lemma follows.

5.2.9. The constructions of 5.2.5 and 5.2.6 are related to each other. Invoke Borisov's algebra $\mathbb{B}_\Lambda$, see (3.1.3). Since $\mathbb{B}_\Lambda=V_\Lambda\otimes F_\Lambda$, the space

$$\mathbb{B}_{\Lambda+\eta}\stackrel{\mathrm{def}}{=}V_{\Lambda+\eta}\otimes F_\Lambda \tag{5.2.12}$$

is naturally a twisted $\mathbb{B}_\Lambda$-module. Hence its pull-back w.r.t. Borisov's embedding

$$\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)\to\mathbb{B}_\Lambda \tag{5.2.13}$$

is a twisted $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)$-module.

5.2.10. Lemma. *If*

$$\eta=\sum_{j=0}^{N-1}\lambda_jX_j^*, \tag{5.2.13}$$

then the twisted $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)$*-module generated by* $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)$ *from the vector* $e^\eta \otimes \mathbf{1} \in V_{\Lambda+\eta}$ *is isomorphic to* $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)_{\vec{\lambda},\vec{\lambda}}$*, where* $\vec{\lambda} = (\lambda_0, ..., \lambda_{N-1})$*.*

Proof. According to Lemma 5.2.5 in order to prove the lemma one only has to check that the vector e^η satifies relations (5.2.6). But this is obvious. For example, (5.2.9b) gives

$$\rho_{L+\eta} x_i(z) e^\eta = \left(e^{X_i}(z) z^{(X_i,\eta)} \mathbf{1}\right) e^\eta = \left(e^{X_i}(z) z^{\lambda_i} \mathbf{1}\right) e^\eta,$$

which is exactly the first of conditions (5.2.6). The remaining 3 fields are dealt with in exactly the same manner.

5.2.11. Notation. It is natural to denote the twisted module $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)_{\vec{\lambda},\vec{\lambda}}$ realized as in Lemma 5.2.10 by $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N) e^{\sum_j \lambda_j X_j^*}$.

5.2.12. Landau-Ginzburg orbifold. We now wish to orbifoldize the differential graded vertex algebra LG_f, (5.1.2), with respect to the $\mathbb{Z}_p$-action generated by the operator $g = \exp(\rho J(z)_{(0)})$, where $\rho J(z)_{(0)}$ is defined in Lemma 5.1.1. According to the pattern reviewed in 5.2.1, in order to do so one has to exhibit a g^i-twisted differential LG_f-module for each $1 \leq i \leq N-1$. (Remark 5.2.4 (ii) explains what "g^i-twisted" means)

As follows from (5.2.3), the ith twisting gradation is determined by the action of $\rho J(z)_{(0)}$ on the generators. Formulas of Lemma 5.1.1 imply

$$\begin{aligned} &\rho J(z)_{(0)} x_i = -\frac{1}{p} x_i, \; \rho J(z)_{(0)} \partial x_i = \frac{1}{p} \partial_{x_i}, \\ &\rho J(z)_{(0)} dx_i = (1 - \frac{1}{p}) dx_i, \; \rho J(z)_{(0)} \partial dx_i = (\frac{1}{p} - 1) \partial_{x_i}. \end{aligned} \tag{5.2.14}$$

Therefore, $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)_{i\frac{\vec{1}}{p}, i\frac{\vec{1}}{p}}$, where $\frac{\vec{1}}{p} = (1/p, ..., 1/p)$, is a natural candidate for the $\exp i(\rho J(z)_{(0)})$-twisted module. So we define, cf. 5.2.11,

$$\mathrm{LG}_f^{(i)} \stackrel{\text{def}}{=} \Lambda^{ch}\mathcal{T}(\mathbb{C}^N) e^{\frac{1}{p}\sum_j X_j^*}. \tag{5.2.15}$$

As mentioned in 5.2.4 (ii), $\mathrm{LG}_f^{(i)}$ is a (untwisted) $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N)^g$-module. (Recall that $\Lambda^{ch}\mathcal{T}(\mathbb{C}^N) = \mathrm{LG}_f = \mathrm{LG}_f^{(0)}$ as vertex algebras.)

Since the Landau-Ginzburg differential $df(z)_{(0)}$ comes from $df \in \Lambda^{ch}\mathcal{T}(\mathbb{C}^N)^g$, it operates naturally on $\mathrm{LG}_f^{(i)}$ thus making it a differential LG_f^g-module.

Since (5.1.4) maps $V(N2)_{N\frac{p-2}{p}}$ into LG_f^g, each $\mathrm{LG}_f^{(i)}$ acquires an $N2$-structure. In particular, g operates on $\mathrm{LG}_f^{(i)}$ and $\left(\mathrm{LG}_f^{(i)}\right)^g$ is also a differential $\left(\mathrm{LG}_f^{(0)}\right)^g$-module.

5.2.13. Definition. Define the Landau-Ginzburg orbifold to be the following differential LG_f^g-module:

$$\mathrm{LG}_{f,\mathrm{orb}} = \bigoplus_{j=0}^{p-1} \left(\mathrm{LG}_f^{(j)}\right)^g.$$

□

Note that if $\deg f = p = N$, then $\mathrm{LG}_{f,\mathrm{orb}}$ simply coincides with $(\widetilde{\mathrm{LG}}_{orb}, df)$ appearing in Theorems 4.7, 4.9. This space, however, carries two *a priori* different $N2$-structures: one computed in Lemma 4.10.1 and having purely geometric origin and another, Landau-Ginzburg structure, recorded in Lemma 5.1.1.

5.2.14. Lemma. *The two $N2$-structures coincide with each other.*

Proof is, of course, a routine computation consisting in applying Borisov's formulas (3.3.2a-b) to the 4 fields of Lemma 5.1.1. Let us consider $Q(z)$ and leave the rest as an exercise for the interested reader. We have (and recall that we are using boson-fermion correspondence (4.10.1)):

$$
\begin{aligned}
&Q(z) \mapsto \sum_{i=0}^{N-1} -\frac{1}{N} x_i(z)\partial_{dx_i}(z)' - (\frac{1}{N}-1)x_i(z)'\partial_{dx_i}(z) \mapsto \\
&\sum_{i=0}^{N-1} -\frac{1}{N} : e^{X_i}(z) \left(: (-X_i(z) - \chi_i(z))\, e^{-X_i-\chi_i}(z) :\right) : - \\
&\left(\frac{1}{N}-1\right) : \left(: X_i(z) e^{X_i}(z) :\right) e^{-X_i-\chi_i}(z) := \\
&\sum_{i=0}^{N-1} \frac{1}{N} : (X_i\,(z) + \chi_i\,(z))\, e^{-\chi_i}\,(z) : + \left(1-\frac{1}{N}\right) : X_i\,(z)\, e^{-\chi_i}\,(z) := \\
&\sum_{i=0}^{N-1} \frac{1}{N} : \chi_i(z) e^{-\chi_i}(z) : + : X_i(z) e^{-\chi_i}(z) :,
\end{aligned}
$$

as it should, cf. Lemma 4.10.1. Note that this computation is parallel to that we performed in the proof of Lemma 4.10.1. □

Lemma 5.2.14 represents the last step in the identification of $\widetilde{\mathrm{LG}}_{orb}^{<N}$ that played an important role in Theorems 4.7, 4.9 with the Landau-Ginzburg orbifold[2]. The next lemma says that as $N2$-modules the twisted sectors are spectral flow transforms of the untwisted one, see (1.12.1-2) for the definition of the spectral flow.

5.2.15. Lemma. *The map*

$$
e^{-j\sum_i(\frac{1}{p}X_i^* - X_i - \chi_i)} : \mathrm{LG}_f^{(j)} \to \mathrm{LG}_f;\ \text{s.t. } x \mapsto x e^{-j\sum_i(\frac{1}{p}X_i^* - X_i - \chi_i)}. \qquad (5.2.16)
$$

delivers the following isomorphisms:

(i)

$$
\mathrm{LG}_f^{(j)} \xrightarrow{\sim} S_j\left(\mathrm{LG}_f\right), H_{df(z)_{(0)}}(\mathrm{LG}_f^{(j)}) \xrightarrow{\sim} S_j\left(H_{df(z)_{(0)}}(\mathrm{LG}_f)\right).
$$

[2]the tensor product decomposition of $\widetilde{\mathrm{LG}}_{orb}^{<N}$ arising in the case of a diagonal equation, see Corollary 5.1.4, is perhaps a bridge to Gepner's models [G] also cooked up of the tensor products of $N2$ unitary representations

(ii) If $p = N$, then

$$\left(\mathrm{LG}_f^{(j)}\right)^g \xrightarrow{\sim} S_j\left(\mathrm{LG}_f^g\right),\ H_{df(z)_{(0)}}\left(\mathrm{LG}_f^{(j)}\right)^g \xrightarrow{\sim} S_j\left(H_{df(z)_{(0)}}(\mathrm{LG}_f^g)\right)$$

if $p = N$.

Proof is parallel to the discussion in 3.10, and we will be brief: a glance at the formulas in Lemma 4.10.1 shows that under map (5.2.16) $Q(z)$ gets multiplied by z^j, $G(z)$ by z^{-j} thereby recovering (1.12.1). This proves the first of ismorphisms (i); the second follows from an equally obvious observation that the differential $df(z)_{(0)}$ is invariant under (5.2.16).

As to (ii), this argument does not quite apply if $p \neq N$ because then $e^{-j\sum_i(\frac{1}{p}X_i^*-X_i-\chi_i)}$ is not Z_N-invariant and does not give a well-defined map. If, however, $p = N$, then (ii) is proved in the same way as (i).

5.2.16. Application: the character formulas. Lemma 5.2.15 is an intelligent way to obtain (4.11.4-5). Focus on the Euler character formulas. First of all, LG_f being bi-graded by the eigenvalues of $\rho J_{(0)}$ and $\rho L_{(1)}$, definition (4.11.2) is equivalent to

$$\mathrm{Eu}(\mathrm{LG}_f)(s,\tau) = \mathrm{Tr}|_{\mathrm{LG}_f}(-1)^{\mathrm{par}(.)}e^{2\pi i(sJ_{(0)}+\tau L_{(1)})}, \tag{5.2.17}$$

where par(.) is the parity function, see 1.1 – and we are skipping the ρ. It is quite standard to deduce that

$$\mathrm{Eu}(\mathrm{LG}_f)(s,\tau) = E(\tau,s), \tag{5.2.18}$$

where $E(\tau,s)$ is the function appearing right before (4.11.4), after all LG_f is a superpolynomial ring in infinitely many variables. In view of (5.2.16), to obtain the Euler character of a spectral flow transformed module, one has to perform the linear change of variables, determined by (1.12.1), in the Euler character of the original module and then take care of the parity. By virtue of Lemma 5.2.15, in the case of $\mathrm{LG}_f^{(j)}$ one has to replace

$$\begin{aligned} &J_{(0)} \text{ with } J_{(0)} - (N-2)j,\\ &L_{(1)} \text{ with } L_{(1)} - jJ_{(0)} + j(j-1)(N-2)/2,\\ &\text{par with par} + \|j\sum_i \frac{1}{N}X_i^* - X_i - \chi_i\|^2 \bmod 2. \end{aligned} \tag{5.2.19}$$

The 1st two of these follow from (1.12.1), the last from (5.2.16) and definition of parity (1.9.3). Plugging (5.2.19) in (5.2.17) we obtain

$$\begin{aligned}\mathrm{Eu}(\mathrm{LG}_f^{(j)})(s,\tau) =&\mathrm{Tr}|_{\mathrm{LG}_f}(-1)^{\mathrm{par}(.)+j^2(N-2)}\times\\ &e^{2\pi i(s(J_{(0)}-(N-2)j)+\tau(L_{(1)}-jJ_{(0)}+j(j-1)(N-2)/2))}.\end{aligned} \tag{5.2.20}$$

Then (5.2.18) can be rewritten as follows

$$\mathrm{Eu}(\mathrm{LG}_f^{(j)})(s,\tau) = e^{\pi i(N-2)\{-2js+(j^2-j)\tau+j^2\}}E(\tau,s-j\tau). \tag{5.2.21}$$

Finally, in order to extract g-invariants, one applies the averaging operator $\frac{1}{N}\sum_l e^{2\pi i l J_{(0)}}$ which results in shifts $s \mapsto s+l$ and gives (4.11.4).

5.2.17. Vertex algebra structure. Let us focus on the case where $\deg f = p = N$. *A priori* the orbifold $\mathrm{LG}_{f,\mathrm{orb}}$ carries only uninteresting vertex algebra structure: by definition it can be regarded as an abelian extension of LG_f^g by the sum of vertex modules $\oplus_i \left(\mathrm{LG}_f^{(i)}\right)^g$. Cut-off definition (4.6.13) does somewhat better: it endows $\mathrm{LG}_{f,\mathrm{orb}}$ with a filtered vertex algebra structure such that the corresponding graded object is the mentioned above uninteresting vertex algebra. And yet this vertex algebra structure does not seem to be quite right either. Formally, one would expect from an orbifold multiplicative structure to be $\mathbb{Z}_N$-graded. Instead, our vertex algebra is graded by the semigroup $\mathbb{Z}_+/(N+\mathbb{Z}_+)$. Almost the same problem has already manifested itself: by virtue of (4.9.3) the chiral ring of $\mathrm{LG}_{f,orb}$ is isomorphic to the cohomology algebra $H_{\mathfrak{F}}(\mathfrak{F},\Lambda^*\mathcal{T}_{\mathfrak{F}})$ as a vector space but not as a ring, see (4.13.12) and the sentence that follows. Let us propose a conjectural way out in the case where $f=\sum_i x_i^N$ and $\mathrm{LG}_{f,orb}$ is replaced with the Landau-Ginzburg cohomology: $H_{df(z)_{(0)}}(\mathrm{LG}_{f,orb})$.

In this case, Corollary 5.1.4 combined with Lemma 5.2.15 provides us with a very explicit description of the cohomology:

$$H_{df(z)_{(0)}}(\mathrm{LG}_{f,orb}) = \bigoplus_{n=0}^{N-1} \bigoplus_{j_0,\ldots,j_{N-1}\leq p-2} \bigotimes_{t=0}^{N-1} S_n\left(U_{j_t,N}\right). \tag{5.2.22}$$

The point is, the category of unitary $N2$-modules has a remarkable periodicity property: there are ismorphisms

$$T_m : S_{mN}(U_{j,N}) \xrightarrow{\sim} U_{j,N},\ m\in\mathbb{Z}. \tag{5.2.23}$$

Here is what this "Bott periodicity" suggests: since $\mathrm{LG}_f^{(n)}$ is naturally defined for any $n\in\mathbb{Z}$, instead of taking $\oplus_{n\geq 0} H_{df(z)_{(0)}}(\mathrm{LG}_{f,orb}^{(n)})$ and then quotienting out by the ideal $\oplus_{n\geq N} H_{df(z)_{(0)}}(\mathrm{LG}_{f,orb}^{(n)})$, consider $\oplus_{n\in\mathbb{Z}} H_{df(z)_{(0)}}(\mathrm{LG}_{f,orb}^{(n)})$ and identify the components $H_{df(z)_{(0)}}(\mathrm{LG}_{f,orb}^{(\alpha)})$ and $H_{df(z)_{(0)}}(\mathrm{LG}_{f,orb}^{(\beta)})$ with $\alpha-\beta\in N\mathbb{Z}$ by using (5.2.23). Note that (5.2.23) is determined up to proportionality and herein lies the difficulty. But we believe that the coefficients can be adjusted so that the vertex algebra structure on $\oplus_{n\in\mathbb{Z}} H_{df(z)_{(0)}}(\mathrm{LG}_{f,orb}^{(n)})$ descends on the quotient. The latter is of course $H_{df(z)_{(0)}}(\mathrm{LG}_{f,orb})$ as a vector space.

By way of supporting this conjecture, let us show that this prescription does seem to restore multiplication on $H_{\mathfrak{F}}(\mathfrak{F},\Lambda^*\mathcal{T}_{\mathfrak{F}})$. For the reader's convenience let us copy (4.13.12):

$$\begin{gathered}\left(e^{sX^*_{orb}-\sum_j(X_i+\chi_i)}\right)_{(-1)} e^{(N-s)X^*_{orb}-\sum_j(X_i+\chi_i)} = \\ (-1)^s e^{NX^*_{orb}-2\sum_j(X_i+\chi_i)} \in \mathrm{LG}_{f,orb}^{(N)}.\end{gathered} \tag{5.2.24}$$

We assert that up to proportionality

$$T_1\left(\text{class of } e^{\sum_j(X_j^*-2(X_j+\chi_j))}\right) = \text{class of } e^{(N-2)\sum_j X_j} \in H_{df(z)_{(0)}}(\mathrm{LG}_{f,orb}^{(N)}). \tag{5.2.25}$$

If proven, (5.2.25) will recover the desired (4.13.13).

In order to prove (5.2.25), we shall obtain a representation theoretic interpretation of both the elements that determines them up to proprtionality.

Since in the case at hand everything is a tensor product of 1-dimensional Landau-Ginzburg models and their cohomology groups, we can drop the subindex j, and our task is to show that

$$T_1(\text{ class of } e^{X^*-2(X+\chi)}) = \text{class of } e^{(N-2)X} \in H_{df(z)_{(0)}}(\mathrm{LG}_f). \tag{5.2.26}$$

According to Theorem 5.1.3, $e^{(N-2)X}$ generates in the cohomology the unitary $N2$-module $U_{N-2,N}$. Actually, $e^{(N-2)X}$ is a *highest weight vector* of $U_{N-2,N}$ meaning that it is annihilated by the Lie algebra span of

$$\begin{aligned} G_{(2)}, G_{(3)}, \ldots, \\ Q_{(1)}, Q_{(2)}, Q_{(3)}. \end{aligned} \tag{5.2.27}$$

It follows from Lemma 5.2.15 (i) that $S_N(U_{N-2,N}) \subset H_{df(z)_{(0)}}(\mathrm{LG}_f^{(N)})$ is generated by $e^{X^*-2X-N\chi}$. The same lemma implies that the latter is rather a *twisted highest weight vector* meaning that it is annihilated by the Lie algebra span of

$$\begin{aligned} G_{(N+2)}, G_{(N+3)}, \ldots, \\ Q_{(-N+1)}, Q_{(-N+2)}, \ldots, Q_{(N+2)}, Q_{(N+3)}, \ldots \end{aligned} \tag{5.2.28}$$

Now we would like to find a highest weight vector of $S_N(U_{N-2,N}) \subset H_{df(z)_{(0)}}(\mathrm{LG}_f^{(N)})$. A direct computation shows that $e^{X^*-2X-N\chi}$ is annihilated by $G_{(N+1)}$ in addition to subalgebra (5.2.28). Commutation relations (1.10.1) imply then that the vector

$$G_{(3)}G_{(4)}\cdots G_{(N-1)}G_{(N)}e^{X^*-2X-N\chi} \tag{5.2.29}$$

is annihilated by the untwisted annihilating subalgebra, (5.2.27), except perhaps the element $G_{(2)}$. It also follows from [FS, (A.5)] that the class of (5.2.29) is non-zero in $S_N(U_{N-2,N})$. Now we compute

$$\begin{aligned} &G_{(N)}e^{X^*-2X-N\chi} = (:(X^*(z)-\chi(z))e^{\chi}(z):)_{(N)}\, e^{X^*-2X-N\chi} = \\ &e^{\chi}(X^*_{(0)}-\chi_{(0)})e^{X^*-2X-N\chi} = (N-2)e^{X^*-2X-(N-1)\chi}. \end{aligned}$$

Likewise,

$$G_{(N-1)}G_{(N)}e^{X^*-2X-N\chi} = (N-3)(N-2)e^{X^*-2X-(N-2)\chi}.$$

Continuing in the same vein we obtain

$$G_{(3)}G_{(4)}\cdots G_{(N-1)}G_{(N)}e^{X^*-2X-N\chi} = (N-2)!e^{X^*-2X-2\chi} \tag{5.2.30}$$

The same computation shows that an application of $G_{(2)}$ annihilates vector (5.2.30). Hence (5.2.30) is a highest weight vector and since it is determined by this condition up to proportionality, (5.2.26) follows. The Nth tensor power of the latter gives the desired (5.2.25).

References

[BD] A. Beilinson, V. Drinfeld, Chiral algebras, Preprint.

[B] L.Borisov, Vertex algebras and mirror symmetry, *Comm. Math. Phys.* **215** (2001), no. 3, 517-557

[BL] L.Borisov, A.Libgober, Elliptic genera of toric varieties and applications to mirror symmetry, *Inv. Math.* **140** (2000) no. 2, 453-485,

[D1] C.Dong, Vertex algebras associated with even lattices, *J. Algebra* **161** (1993), 245-265

[D2] C.Dong, Twisted modules for vertex algebras associated with even lattices, *J. Algebra* **165** (1994), 91-112

[Don] R.Donagi, Generic Torelli for projective hypersurfaces, *Comp. Math.* **50** (1983) 325-353

[FS] B.L.Feigin, A.M.Semikhatov, Free-field resolutions of the unitary N=2 super-Virasoro representations, preprint, hep-th/9810059

[FFR] A.Feingold, I.Frenkel, J.Ries, Spinor constructions of vertex operator algebras, tiality, and $E_8^{(1)}$, *Contemp. Math.*, **121**, AMS, 1991

[FB-Z] E.Frenkel, D.Ben-Zvi, Vertex algebras and algebraic curves, *Mathematical Surveys and Monographs* **88**, 2001,

[FLM] I.Frenkel, J.Lepowski, A.Meurman, Vertex operator algebras and the Monster, *Academic Press*, 1988

[G] D.Gepner, Exactly solvable string compactifications on manifolds of $SU(N)$ holonomy, *Phys.Lett.* **B199** (1987)380-388,

[GMS] V.Gorbounov, F.Malikov, V.Schechtman, Gerbes of chiral differential operators. II, to appear in *Inv. Math.*, AG/0003170

[Gr] P.Griffiths, On the periods of certain rational integrals, *Ann. Math.* **90** (1969) 460-541,

[K] V.Kac, Vertex algebras for beginners, 2nd edition, *AMS*, 1998

[KR] V.Kac, A.Radul, Representation theory of the vertex algebra $W_{1+\infty}$, *Transform. groups.* **1** (1996), no. 1-2, 41-70

[KV1] M.Kapranov, E.Vasserot, Vertex algebras and the formal loop space, preprint, math.AG/0107143,

[KV2] M.Kapranov, E.Vasserot, private communication

[KO] A.Kapustin, D.Orlov, Vertex algebras, mirror symmetry, and D-branes: the case of complex tori, *Comm. Math. Phys.* **233** (2003) 79-163,

[KYY] T.Kawai, Y.Yamada, S-K. Yang, Elliptic genera and N=2 superconformal field theory, *Nucl.Phys.* **B414** (1994) 191-212

[LVW] W.Lerche, C.Vafa, N.Warner, Chiral rings in N=2 superconformal theory, *Nuclear Physics* **B324** (1989), 427,

[MS] F.Malikov, V.Schechtman, Deformations of vertex algebras, quantum cohomology of toric varieties, and elliptic genus, *Comm. Math. Phys.* **234** (2003), no.1 77-100,

[MSV] F. Malikov, V. Schechtman, A. Vaintrob, Chiral de Rham complex, *Comm. Math. Phys.*, **204** (1999), 439 - 473,

[S] A.Schwarz, Sigma-models having supermanifolds as target spaces, *Lett. Math. Phys.* **38** (1996), no.4, 349-353

[V] C.Vafa, String vacua and orbifoldized Landau-Ginzburg model, *Mod.Phys.Lett.* **A4** (1989) 1169

[VW] C.Vafa, N.P.Warner, Catastrophes and the classification of conformal theories, *Phys.Lett.* **B218** (1989) 51

[W1] E. Witten, Phases of N=2 theories in two-dimensions, *Nucl.Phys.* **B403**, 159-222,1993, hep-th/9301042

[W2] E. Witten, On the Landau-Ginzburg description of N=2 minimal models, *Int.J.Mod.Phys.* **A9** (1994), 4783-4800, hep-th/9304026

V.G.: Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA; vgorb@ms.uky.edu

F.M.: Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA; fmalikov@mathj.usc.edu