AN ELEMENTARY CONSTRUCTION OF ANICK'S FIBRATION

BRAYTON GRAY AND STEPHEN THERIAULT

Abstract. Cohen, Moore, and Neisendorfer's work on the odd primary homotopy theory of spheres and Moore spaces, as well as the first author's work on the secondary suspension, predicted the existence of a p-local fibration $S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}$ whose connecting map is degree p^r. In a long and complex monograph, Anick constructed such a fibration for $p \geq 5$ and $r \geq 1$. Using new methods we give a much more conceptual construction which is also valid for $p = 3$ and $r \geq 1$. We go on to establish several properties of the space T.

1. Introduction

In [CMN1, CMN2, N1] Cohen, Moore, and Neisendorfer proved a landmark result concerning the exponent of the homotopy groups of spheres localized at an odd prime p. When $p \geq 3$ and $r \geq 1$ they constructed a map $\pi_n : \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ such that the composition with the double suspension

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is homotopic to the p^r-power map. The existence of such a map for $r = 1$ was used to show that p^n annihilates the p-torsion in $\pi_*(S^{2n+1}) = 0$.

In [CMN3], the authors raised the question of whether the map π_n occurs in a fibration sequence

$$\text{(A)} \qquad \Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}.$$

The first construction of such a fibration was accomplished for $p \geq 5$ by Anick [A] and was the subject of a 270 page book. There has been much interest in finding a simpler construction. It is the purpose of this paper to give an elementary construction of the space T and the fibration (A) which is valid for all odd primes. The methods are new and have the advantage of being straightforward and accessible to nonexperts. It is anticipated that they should be of use for other problems as well. A comparison of our methods and Anick's will be given once we state our results.

The question of the existence of a fibration as in (A) appeared in another context at about the same time. In trying to understand the secondary suspension [C, M], the first author [G4, G5] was led to conjecture the existence of $(i-1)$-connected spaces T_i which fit into secondary EHP sequences

$$T_{2n-1} \xrightarrow{E} \Omega T_{2n} \xrightarrow{H} BW_n$$

$$T_{2n} \xrightarrow{E'} \Omega T_{2n+1} \xrightarrow{H'} BW_{n+1}$$

2000 *Mathematics Subject Classification.* Primary 55P45, 55P40, 55P35.

where BW_n is the classifying space of the fiber of the double suspension constructed in [G3]. These EHP fibrations should fit together in such a way that the resulting spectrum $\{T_i\}$ is equivalent to the Moore spectrum $S^0 \cup_{p^r} e^1$. The T_i's would then give a refinement of the secondary suspension into $2p$ stages. The analysis indicated that T_{2n} is homotopy equivalent to $S^{2n+1}\{p^r\}$, the fiber of the map of degree p^r on S^{2n+1}, and that T_{2n-1} would sit in the fibration sequence (A).

Our first objective is to construct a secondary Hopf invariant $H : \Omega S^{2n+1}\{p^r\} \longrightarrow BW_n$ for $p \geq 3$. This lets us define T as the homotopy fiber of H. It follows easily that T satisfies the fibration in (A) and the second EHP fibration. We also show that the space we construct is homotopy equivalent to Anick's when $p \geq 5$.

The EHP viewpoint also predicted that the T_i's should have a rich structure. They should be homotopy associative and homotopy commutative H-spaces enjoying a certain universal property. Together, these properties would imply that the mod-p^r homotopy classes of the T_i's could be represented by multiplicative maps. That is, letting $P^i(p^r)$ be the mod-p^r Moore space of dimension i, there should be a one-to-one correspondence

$$[P^i(p^r), T_j] \leftrightarrow \{\text{homotopy classes of } H\text{-maps from } T_i \text{ to } T_j\}.$$

The properties were easy to establish when i is even [G4]. Subsequent to Anick's work, Anick and the first author [AG] constructed an H-space structure on T by showing that, for each n, there is a $(2n-2)$-connected co-H space G with the property that T is a retract of ΩG and G is a retract of ΣT. They also proved a semi-universal property for T. The other properties were later established by the second author [T2].

Our second objective is to take advantage of our construction of the space T to give a new, simpler construction of the space G, and prove all the properties in [AG] for $p \geq 3$. Collectively, our results are as follows.

Theorem 1.1. *Suppose $p \geq 3$ and $r \geq 1$. Then the following hold:*

(a) *there is an H-fibration sequence*

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}$$

where the composition

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is the p^r-power map;

(b) *there is a fibration sequence*

$$\Omega G \xrightarrow{h} T \longrightarrow R \longrightarrow G$$

where h has a right homotopy inverse $g : T \longrightarrow \Omega G$ so that

$$\Omega G \simeq T \times \Omega R$$

with R a wedge of mod-p^s Moore spaces for $s \geq r$;

(c) *the adjoint of g,*

$$\tilde{g} : \Sigma T \longrightarrow G,$$

has a right homotopy inverse $f : G \longrightarrow \Sigma T$ and there is a homotopy equivalence

$$\Sigma T \simeq G \vee W$$

where W is a wedge of mod-p^s Moore spaces for $s \geq r$;

(d) *there are "EHP fibrations"*

$$W_n \xrightarrow{P} T_{2n-1} \xrightarrow{E} \Omega T_{2n} \xrightarrow{H} BW_n$$

$$W_{n+1} \xrightarrow{P'} T_{2n} \xrightarrow{E'} \Omega T_{2n+1} \xrightarrow{H'} BW_{n+1}$$

where $T_{2n} = S^{2n+1}\{p^r\}$, $T_{2n-1} = T$, and there is an equivalence of spectra $\{T_i\} \simeq S^0 \cup_{p^r} e^1$.

Our methods are simpler and more direct than those of Anick. He constructed T as a retract of a loop space ΩD, where D is an infinite dimensional CW-complex whose bottom two cells are the mod-p^r Moore space $P^{2n+1}(p^r)$ and whose other cells come from iteratively attaching certain Moore spaces in a delicately prescribed fashion. A great deal of his effort was directed towards constructing the attaching maps, and this necessitated the introduction of many new techniques. The restriction to primes strictly larger than 3 was due to a heavy reliance on differential graded Lie algebras which require that the primes 2 and 3 be inverted in order for the Lie identities to be satisfied. By contrast, we construct the space T directly for all $p \geq 3$ without reference to the space D and without reference to differential graded Lie algebras. The main ingredient in this new construction is an extension theorem (presented as Theorem 2.3). This allows for a straightforward extension of the map $\Omega^2 S^{2n+1} \longrightarrow BW_n$ constructed in [G3] to an EHP map $H : \Omega S^{2n+1}\{p\} \longrightarrow BW_n$.

The new methods may be useful in positively resolving a long-standing conjecture that the fiber W_n of the double suspension is a double loop space at odd primes. Including dimension and torsion parameters, the space $T_{2np-1}(p)$ gives a candidate for a double delooping: potentially $W_n \simeq \Omega^2 T_{2np-1}(p)$. Such a homotopy equivalence would have deep implications in homotopy theory, one of which being a much better understanding of the differentials in the EHP spectral sequence calculating the homotopy groups of spheres.

This paper is the result of combining separate efforts by the two authors. The second author discovered the extension theorem and obtained part (a) of Theorem 1.1 without the H-space structure, as well as part (d). The first author later found a different application of the extension theorem to obtain a factorization of the map H, as well as a further application of the extension theorem to obtain parts (b), (c), and the H-space structure.

2. The extension theorem

We begin by restating a theorem of the first author [G3] which identifies certain homotopy pullbacks as homotopy pushouts. A homotopy fibration $X \longrightarrow Q \longrightarrow A$ has a *trivialization* if there is a homotopy equivalence $Q \simeq A \times X$ in which the map $Q \longrightarrow A$ becomes the projection $A \times X \xrightarrow{\pi_1} A$.

Theorem 2.1. *Suppose $X \longrightarrow F' \longrightarrow E'$ is a homotopy fibration and there is a map $A \longrightarrow E'$. Let Q be the homotopy pullback*

$$\begin{array}{ccc} Q & \longrightarrow & F' \\ \downarrow & & \downarrow \\ A & \longrightarrow & E' \end{array}$$

and let E be the homotopy cofiber of $A \longrightarrow E'$. Then the homotopy fibration $X \longrightarrow Q \longrightarrow A$ has a trivialization if and only if there is a homotopy pullback

$$\begin{array}{ccc} F' & \longrightarrow & F \\ \downarrow & & \downarrow \\ E' & \longrightarrow & E \end{array}$$

for some space F. Further, if the trivialization exists, there is a homotopy pushout

$$\begin{array}{ccc} Q \simeq A \times X & \longrightarrow & F' \\ \downarrow{\scriptstyle \pi_2} & & \downarrow \\ X & \longrightarrow & F \end{array}$$

where π_2 the projection onto the second factor. □

There is a special case of Theorem 2.1 in the context of principal fibrations which is the key tool used to construct T and prove Theorem 1.1. In general, suppose there is a homotopy fibration sequence

$$\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B.$$

Then there is a canonical homotopy action $\theta : F \times \Omega B \longrightarrow F$ satisfying homotopy commutative diagrams

$$\begin{array}{ccc} \Omega B \times \Omega B & \xrightarrow{\mu} & \Omega B \\ \downarrow{\scriptstyle \partial \times 1} & & \downarrow{\scriptstyle \partial} \\ F \times \Omega B & \xrightarrow{\theta} & F \end{array} \qquad\qquad \begin{array}{ccc} F \times \Omega B & \xrightarrow{\theta} & F \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ F & \longrightarrow & E \end{array}$$

where μ is the loop multiplication and π_1 is the projection. Note that both squares are homotopy pullbacks. Now suppose there is a homotopy cofibration $A \xrightarrow{b} E' \longrightarrow E$. Define spaces Q and F' by the iterated homotopy pullback diagram

$$\begin{array}{ccccc}
Q & \longrightarrow & F' & \longrightarrow & F \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{b} & E' & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
B & = & B & = & B.
\end{array}$$

In particular, the map $A \longrightarrow B$ is null homotopic as it factors through the middle row which consists of two consecutive maps in a homotopy cofibration. So $Q \simeq A \times \Omega B$. This lets us apply Theorem 2.1 to see that there is a homotopy pushout

$$\begin{array}{ccc}
A \times \Omega B & \longrightarrow & F' \\
\downarrow{\scriptstyle \pi_2} & & \downarrow \\
\Omega B & \longrightarrow & F.
\end{array}$$

What we wish to do is choose a particular trivialization of Q which lets us identify the map $A \times \Omega B \longrightarrow F'$.

The fact that there is some decomposition $Q \simeq A \times \Omega B$ implies that we can choose a lift

$$a : A \longrightarrow F'$$

of b. There may be many choices of a lift, but for the moment any choice suffices. The definition of F' as a homotopy pullback results in a homotopy fibration sequence $\Omega B \longrightarrow F' \longrightarrow E' \longrightarrow B$. This determines a homotopy action $\theta : F' \times \Omega B \longrightarrow F'$. Let $\overline{\theta}$ be the composite

$$\overline{\theta} : A \times \Omega B \xrightarrow{a \times 1} F' \times \Omega B \xrightarrow{\theta} F'.$$

Proposition 2.2. *Let $F \longrightarrow E \longrightarrow B$ be a homotopy fibration and suppose there is a homotopy cofibration $A \xrightarrow{b} E' \longrightarrow E$. Define the space F' and the map $\overline{\theta}$ as above. Then there is a homotopy pushout*

$$\begin{array}{ccc}
A \times \Omega B & \xrightarrow{\overline{\theta}} & F' \\
\downarrow{\scriptstyle \pi_2} & & \downarrow \\
\Omega B & \longrightarrow & F.
\end{array}$$

Proof. Consider the diagram

$$\begin{array}{ccccc}
A \times \Omega B & \xrightarrow{a \times 1} & F' \times \Omega B & \xrightarrow{\theta} & F' \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} & & \downarrow \\
A & \xrightarrow{a} & F' & \longrightarrow & E'.
\end{array}$$

The right square is a homotopy pullback as it is one of the canonical properties of the homotopy action θ. The left square is a homotopy pullback by the naturality of the projection. So the outer rectangle is also a homotopy pullback. Observe that the top row of the rectangle is the definition of $\overline{\theta}$ while the bottom row is the given map b by the definition of a as a lift. Thus if Q is the homotopy pullback

$$\begin{array}{ccc} Q & \xrightarrow{f} & F' \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow \\ A & \longrightarrow & E' \end{array}$$

then there is a homotopy equivalence $e : A \times \Omega B \longrightarrow Q$ such that $g \circ e \sim \pi_1$ – so the homotopy fibration $\Omega B \longrightarrow Q \longrightarrow A$ has been trivialized – and $g \circ e \sim \overline{\theta}$. Therefore Theorem 2.1 implies the existence of the asserted homotopy pushout. □

We now state Theorem 2.3, which uses Proposition 2.2 to construct an extension under certain conditions. The conditions involve exponent information, so we first make two definitions. If A is a co-H space, let $\underline{p}^r : A \longrightarrow A$ be the map of degree p^r. If Z is an H-space, let $p^r : Z \longrightarrow Z$ be the p^r-power map.

Theorem 2.3. *Let*

$$\begin{array}{ccccccc} \Omega B & \longrightarrow & F' & \longrightarrow & E' & \longrightarrow & B \\ \Big\| & & \Big\downarrow & & \Big\downarrow & & \Big\| \\ \Omega B & \longrightarrow & F & \longrightarrow & E & \longrightarrow & B \end{array}$$

be a homotopy fibration diagram and suppose there is a homotopy cofibration $A \xrightarrow{b} E' \longrightarrow E$ where A is a suspension. Observe that the map b lifts to F'; suppose there is a choice of lift $a : A \longrightarrow F'$ with the property that $\Sigma a \sim t \circ \underline{p}^r$ for some map t. Suppose there is a map $f' : F' \longrightarrow Z$ where Z is a homotopy associative H-space whose p^r-power map is null homtopic. Then there is an extension

$$\begin{array}{ccc} F' & \xrightarrow{f'} & Z \\ \Big\downarrow & & \Big\| \\ F & \xrightarrow{f} & Z \end{array}$$

for some map f.

Before beginning the proof, we state a Theorem of James [J] and prove two preliminary Lemmas. If X is a space, let $E : X \longrightarrow \Omega\Sigma X$ be the suspension.

Theorem 2.4. *Let X be a path-connected space and Z be a homotopy associative H-space. Let $f : X \longrightarrow Z$ be a map. Then there is a unique H-map $\overline{f} : \Omega\Sigma X \longrightarrow Z$ such that $\overline{f} \circ E \sim f$.* □

We say that $\overline{f}$ is the *multiplicative extension* of f.

To prepare for Lemmas 2.5 and 2.6 we establish some notation. Let X and Y be spaces. Let $i_1 : X \longrightarrow X \times Y$ and $i_2 : X \longrightarrow X \times Y$ be the inclusions, and let $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$ be the projections. It is well known that there is a natural homotopy equivalence

$$e : \Sigma X \vee \Sigma Y \vee (\Sigma X \wedge Y) \longrightarrow \Sigma(X \times Y)$$

such that the restrictions of e to ΣX and ΣY are Σi_1 and Σi_2, and $\Sigma\pi_1 \circ e$ and $\Sigma\pi_2 \circ e$ are homotopic to the pinch maps onto ΣX and ΣY. There may be many choices of such a homotopy equivalence; any fixed choice will do. Let j be the restriction

$$j : \Sigma X \vee (\Sigma X \wedge Y) \hookrightarrow \Sigma X \vee \Sigma Y \vee (\Sigma X \wedge Y) \xrightarrow{e} \Sigma(X \times Y).$$

Lemma 2.5. *Let Z be a homotopy associative H-space. Suppose there is a map $f : X \times Y \longrightarrow Z$ whose multiplicative extension $\overline{f} : \Omega\Sigma(X \times Y) \longrightarrow Z$ has the property that the composite*

$$X \vee (X \wedge Y) \xrightarrow{E} \Omega\Sigma(X \vee (X \wedge Y)) \xrightarrow{\Omega j} \Omega\Sigma(X \times Y) \xrightarrow{\overline{f}} Z$$

is null homotopic. Then there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega\Sigma(X \times Y) & \xrightarrow{\overline{f}} & Z \\ \downarrow{\scriptstyle \Omega\Sigma\pi_2} & & \| \\ \Omega\Sigma Y & \xrightarrow{\overline{f}_Y} & Z \end{array}$$

where $\overline{f}_Y$ is the multiplicative extension of $f_Y : Y \xrightarrow{i_2} X \times Y \xrightarrow{f} Y$.

Proof. Consider the diagram

$$\begin{array}{ccccc} \Omega\Sigma(X \vee Y \vee (X \wedge Y)) & \xrightarrow{\Omega e} & \Omega\Sigma(X \times Y) & \xrightarrow{\overline{f}} & Z \\ \downarrow{\scriptstyle \Omega\Sigma q} & & \downarrow{\scriptstyle \Omega\Sigma\pi_2} & & \| \\ \Omega\Sigma Y & = & \Omega\Sigma Y & \xrightarrow{\overline{f}_Y} & Z \end{array}$$

where q is the pinch map. By definition of e, we have $\Sigma\pi_2 \circ e \sim \Sigma q$, so the left square homotopy commutes. The assertion of the Lemma is that the right square homotopy commutes as well. As e is a homotopy equivalence, it is equivalent to show that the outer rectangle homotopy commutes. Since all maps are multiplicative and Z is homotopy associative, Theorem 2.4 implies that it is equivalent to show that $\overline{f} \circ \Omega e \circ E \sim \overline{f}_Y \circ \Omega\Sigma q \circ E$, where $E : X \vee Y \vee (X \wedge Y) \longrightarrow \Omega\Sigma(X \vee Y \vee (X \wedge Y))$ is the suspension. By hypothesis and the naturality of E, the restriction of $\overline{f} \circ \Omega e \circ E$ to $X \vee (X \wedge Y)$ is null homotopic, so $\overline{f} \circ \Omega e \circ E$ factors as the composite $X \vee Y \vee (X \wedge Y) \xrightarrow{q} Y \xrightarrow{f_Y} Z$. On the other hand, as $\overline{f}_Y$ is the multiplicative extension of f_Y, we have $\overline{f}_Y \circ E \sim f_Y$. The naturality of the suspension therefore implies that $\overline{f}_Y \circ \Omega\Sigma q \circ E \sim \overline{f}_Y \circ E \circ q \sim f_Y \circ q$. Hence $\overline{f} \circ \Omega e \circ E \sim f_Y \circ q \sim \overline{f}_Y \circ \Omega\Sigma q \circ E$, as required. $\square$

Lemma 2.6. *Let $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$ be a homotopy fibration sequence and let $\theta : F \times \Omega B \longrightarrow F$ be the associated homotopy action. Suppose A is a suspension and there is a map $a : A \longrightarrow F$ such that $\Sigma a \sim t \circ \underline{p}^r$ for some map t. Let $\overline{\theta}$ be the composite*

$$\overline{\theta} : A \times \Omega B \xrightarrow{a \times 1} F \times \Omega B \xrightarrow{\theta} F.$$

Suppose there is a map $f : F \longrightarrow Z$ where Z is a homotopy associative H-space whose p^r-power map is null homotopic. Then there is a homotopy commutative diagram

$$\begin{array}{ccc} A \times \Omega B & \xrightarrow{\overline{\theta}} & F \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} \\ \Omega B & \xrightarrow{f \circ \partial} & Z. \end{array}$$

Proof. First suspend and look at $\Sigma(a \times 1)$. Consider the diagram

$$\begin{array}{ccc} \Sigma A \vee (\Sigma A \wedge \Omega B) & \xrightarrow{\Sigma \underline{p}^r \vee (\Sigma \underline{p}^r \wedge 1)} & \Sigma A \vee (\Sigma A \wedge \Omega B) \\ \| & & \downarrow{\scriptstyle t \vee (t \wedge 1)} \\ \Sigma A \vee (\Sigma A \wedge \Omega B) & \xrightarrow{\Sigma a \vee (\Sigma a \wedge 1)} & \Sigma F \vee (\Sigma F \wedge \Omega B) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\ \Sigma(A \times \Omega B) & \xrightarrow{\Sigma(a \times 1)} & \Sigma(F \times \Omega B). \end{array}$$

The top square homotopy commutes by the hypothesis that $\Sigma a \sim t \circ \underline{p}^r$. The bottom square homotopy commutes by the naturality of the map j.

Looping this diagram and using the naturality of the suspension, we obtain a homotopy commutative diagram

$$\begin{array}{ccc} A \vee (A \wedge \Omega B) & \xrightarrow{\underline{p}^r \vee (\underline{p}^r \wedge 1)} & A \vee (A \wedge \Omega B) \\ \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E} \\ \Omega\Sigma(A \vee (A \wedge \Omega B)) & \xrightarrow{\Omega\Sigma(\underline{p}^r \vee (\underline{p}^r \wedge 1))} & \Omega\Sigma(A \vee (A \wedge \Omega B)) \\ \downarrow{\scriptstyle \Omega j} & & \downarrow{\scriptstyle \Omega(j \circ (t \vee (t \wedge 1)))} \\ \Omega\Sigma(A \times \Omega B) & \xrightarrow{\Omega\Sigma(a \times 1)} & \Omega\Sigma(F \times \Omega B). \end{array} \tag{1}$$

Let $\phi = \Omega(j \circ (t \vee (t \wedge 1))) \circ E \circ (\underline{p}^r \vee (\underline{p}^r \wedge 1))$ be the upper direction around (1), and let $\varphi = \Omega\Sigma(a \times 1) \circ \Omega j \circ E$ be the lower direction around (1). So $\phi \sim \varphi$. Now compose to Z as follows. By hypothesis, Z is homotopy associative, so by Theorem 2.4 the identity map on Z extends to an H-map $r : \Omega\Sigma Z \longrightarrow Z$ such that $r \circ E \sim 1$. Define γ by the composite

$$\gamma : \Omega\Sigma(F \times \Omega B) \xrightarrow{\Omega\Sigma\theta_{k-1}} \Omega\Sigma F \xrightarrow{\Omega\Sigma f} \Omega\Sigma Z \xrightarrow{r} Z.$$

Observe that $\gamma \circ \phi$ is an element of

$$N = [A \vee (A \wedge \Omega B), Z]$$

which is divisible by p^r. Here, the group structure on N is determined by $A \vee (A \wedge \Omega B)$ being a suspension. As Z is homotopy associative, this group structure on N is equivalent to the one determined by the H-structure on Z. By hypothesis, the p^r-power map on Z is null homotopic, and so N has exponent p^r. Thus $\gamma \circ \phi$ is null homotopic, and so $\gamma \circ \varphi$ is null homotopic.

Now we set up to apply Lemma 2.5. Consider the diagram

$$\begin{array}{ccccccccc}
A \times \Omega B & \xrightarrow{a \times 1} & F \times \Omega B & \xrightarrow{\theta} & F & \xrightarrow{f} & Z & & \\
\downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E} & \searrow\!\!\!= & \\
\Omega\Sigma(A \times \Omega B) & \xrightarrow{\Omega\Sigma(a\times 1)} & \Omega\Sigma(F \times \Omega B) & \xrightarrow{\Omega\Sigma\theta} & \Omega\Sigma F & \xrightarrow{\Omega\Sigma f} & \Omega\Sigma Z & \xrightarrow{r} & Z.
\end{array}$$

The three squares homotopy commute by the naturality of E. The right triangle homotopy commutes by the definition of r. So the entire diagram homotopy commutes. Let $g = f \circ \theta \circ (a \times 1)$ be the composite along the top row, and let $\overline{g} = r \circ \Omega\Sigma f \circ \Omega\Sigma\theta \circ \Omega\Sigma(a \times 1)$ be the composite along the bottom row. Observe that $\overline{g}$ is an H-map as it is the composite of H-maps, and $\overline{g} \circ E \sim g$. Thus $\overline{g}$ is the multiplicative extension of g. Further, by their definitions, $\overline{g} = \gamma \circ \Omega\Sigma(s \times 1)$ and $\varphi = \Omega\Sigma(s \times 1) \circ \Omega j \circ E$, so the null homotopy for $\gamma \circ \varphi$ implies that the composite

$$A \vee (A \wedge \Omega B) \xrightarrow{\Omega j \circ E} \Omega\Sigma(A \times \Omega B) \xrightarrow{\overline{g}} Z$$

is null homotopic. Therefore, by Lemma 2.5, there is a homotopy commutative square

$$\begin{array}{ccc}
\Omega\Sigma(A \times \Omega B) & \xrightarrow{\overline{g}} & Z \\
\downarrow{\scriptstyle \Omega\Sigma\pi_2} & & \| \\
\Omega\Sigma\Omega B & \xrightarrow{\overline{h}} & Z
\end{array}$$

where $\overline{h}$ is the multiplicative extension of

$$h : \Omega B \xrightarrow{i_2} A \times \Omega B \xrightarrow{g} Z.$$

Finally, the previous square and the naturality of the suspension give a homotopy commutative diagram

$$\begin{array}{ccccc}
A \times \Omega B & \xrightarrow{E} & \Omega\Sigma(A \times \Omega B) & \xrightarrow{\overline{g}} & Z \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \Omega\Sigma\pi_2} & & \| \\
B & \xrightarrow{E} & \Omega\Sigma\Omega B & \xrightarrow{\overline{h}} & Z.
\end{array}$$

Since $\overline{g}$ and $\overline{h}$ are the multiplicative extensions of g and h respectively, we have $\overline{g} \circ E \sim g$ and $\overline{h} \circ E \sim h$, and so the homotopy commutativity of the diagram implies that $g \sim h \circ \pi_2$. Now we untangle definitions. By their definitions, $g = f \circ \theta \circ (a \times 1)$ and $\overline{\theta} = \theta \circ (a \times 1)$. So $g \sim f \circ \overline{\theta}$. By their definitions, $h = g \circ i_2$ and $g = f \circ \theta \circ (a \times 1)$. The definition of θ as a homotopy action implies that $\theta \circ (a \times 1) \circ i_2 \sim \partial$. Thus $h \sim f \circ \partial$. Hence $f \circ \overline{\theta} \sim f \circ \partial \circ \pi_2$, precisely as asserted by the Lemma. $\square$

Proof of Theorem 2.3: The given diagram of principal fibrations and homotopy cofibration let us apply Proposition 2.2 to obtain a homotopy pushout

$$\begin{array}{ccc} A \times \Omega B & \xrightarrow{\overline{\theta}} & F' \\ \big\downarrow{\scriptstyle \pi_2} & & \big\downarrow \\ \Omega B & \longrightarrow & F. \end{array}$$

Since A is a suspension, the lift $A \xrightarrow{a} F'$ has the property that $\Sigma a \sim t \circ \underline{p}^r$, and Z is a homotopy associative H-space whose p^r-power map is null homotopic, we can apply Lemma 2.6 to the homotopy fibration sequence $\Omega B \longrightarrow F' \longrightarrow E' \longrightarrow B$ and the given map $F' \xrightarrow{f'} Z$ in order to obtain a homotopy commutative diagram

$$\begin{array}{ccc} A \times \Omega B & \xrightarrow{\overline{\theta}} & F' \\ \big\downarrow{\scriptstyle \pi_2} & & \big\downarrow{\scriptstyle f'} \\ \Omega B & \longrightarrow & Z. \end{array}$$

Therefore there is a pushout map $f : F \longrightarrow Z$ with the property that the composite $F' \longrightarrow F \xrightarrow{f} Z$ is homotopic to f', as required. $\square$

3. The construction of the space T

The purpose of this section is to construct the spaces T and produce several fibration sequences. We begin our discussion with the Moore space

$$P^k(p^r) = S^{k-1} \cup_{p^r} e^k$$

which we will abbreviate as P^k. Let us fix some notation by defining a diagram of fibration sequences induced by the lower right hand corner

$$\text{(B)} \qquad \begin{array}{ccccccc} \Omega^2 S^{2n+1} & \xrightarrow{\partial} & E & \xrightarrow{\pi} & F & \longrightarrow & \Omega S^{2n+1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \sigma} & & \big\downarrow & & \big\downarrow \\ * & \longrightarrow & P^{2n+1} & = & P^{2n+1} & \longrightarrow & * \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \Omega S^{2n+1} & \longrightarrow & S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1} & \xrightarrow{p^r} & S^{2n+1}. \end{array}$$

The spaces E and F were first introduced in [CMN2, CMN1]. It is easy to see that

$$H^i(F) = \begin{cases} \mathbb{Z} & i = 2kn \\ 0 & i \neq 2kn \end{cases}$$

and the fibration connecting map $\Omega S^{2n+1} \longrightarrow F$ is divisible by p^r in each nonzero degree in integral cohomology. In their work [CMN2], the authors introduced certain maps $x_i : P^{2ni-1} \longrightarrow \Omega F$ whose adjoints $\tilde{x}_i : P^{2ni} \longrightarrow F$ induce epimorphisms in integral cohomology. For each $i > 1$, x_i is a relative Samelson product, so the composition

$$P^{2ni} \xrightarrow{\tilde{x}_i} F \longrightarrow P^{2n+1}$$

is an iterated Whitehead product. Since $S^{2n+1}\{p^r\}$ is an H space, these classes lift to E, giving diagrams

$$\begin{array}{ccc} P^{2ni} & \xrightarrow{y_i} & E \\ \big\downarrow{\scriptstyle \tilde{x}_i} & & \big\downarrow{\scriptstyle \sigma} \\ F & \longrightarrow & P^{2n+1} \end{array}$$

for some maps y_i. In particular, $\pi y_i - \tilde{x}_i : P^{2ni} \longrightarrow F$ composes trivially to P^{2n+1} and so factors through ΩS^{2n+1}. Thus the induced homomorphism in $2ni$ dimensional cohomology is divisible by p^r and hence trivial. Consequently, we obtain the following.

Lemma 3.1. *The composite $H^{2ni}(F) \xrightarrow{\pi^*} H^{2ni}(E) \xrightarrow{y_i^*} H^{2ni}(P^{2ni})$ is an epimorphism for each $i > 1$.* □

We require one more lemma to apply the results of Section 2.

Lemma 3.2. *Suppose X is 2-connected and M is either a sphere or a Moore space. Let $f : \Sigma M \longrightarrow X$ be given. Define A by the cofibration sequence*

$$M \xrightarrow{p^s} M \xrightarrow{i} A \xrightarrow{j} \Sigma M.$$

Suppose there is a commutative diagram

$$\begin{array}{ccc} \Sigma A & \xrightarrow{\Sigma j} & \Sigma^2 M \\ \big\downarrow{\scriptstyle x} & & \big\| \\ X \cup_f C\Sigma M & \xrightarrow{\rho} & \Sigma^2 M \end{array}$$

for some map x, where ρ is the quotient map. Then there is a commutative diagram

$$\begin{array}{ccc} \Sigma M & \xrightarrow{\Sigma i} & \Sigma A \\ \big\downarrow{\scriptstyle x'} & & \big\downarrow{\scriptstyle x} \\ X & \longrightarrow & X \cup_f C\Sigma M \end{array}$$

for some map x', and f is homotopic to $p^s \cdot x'$.

Proof. Consider the standard map from a cofibration sequence to a fibration sequence defined by the right hand square

$$\begin{array}{ccccccc} \Sigma M & \xrightarrow{p^s} & \Sigma M & \xrightarrow{\Sigma i} & \Sigma A & \xrightarrow{\Sigma j} & \Sigma^2 M \\ \downarrow{\scriptstyle x''} & & \downarrow{\scriptstyle x'} & & \downarrow{\scriptstyle x} & & \| \\ J(\Sigma M) & \longrightarrow & J(X, \Sigma M) & \longrightarrow & X \cup_f C\Sigma M & \xrightarrow{\rho} & \Sigma^2 M. \end{array}$$

Here $J(\Sigma M)$ is the James construction and $J(X, SM)$ is the fiber of ρ [G2]. The map x'' is the adjoint to the identity for an appropriate choice of x'. Suppose ΣM has dimension k. Since X is 2-connected, the $k+1$ skeleton of $J(X, \Sigma M)$ is X and x' factors through X up to homotopy. Since x'' factors through ΣM as well we have a homotopy commutative square

$$\begin{array}{ccc} \Sigma M & \xrightarrow{p^s} & \Sigma M \\ \| & & \downarrow{\scriptstyle x'} \\ \Sigma M & \xrightarrow{f} & X \end{array}$$

which proves the Lemma. □

We apply these results as follows. Let $F_{(i)}$ be the $2ni$ skeleton of F, so

$$F_{(i)} = F_{(i-1)} \cup_{\gamma_i} e^{2ni}$$

where γ_i is the attaching map. Now combining 3.1 and 3.2 with $M = S^{2ni-2}$, $X = F_{(i-1)}$, $f = \gamma_i$, $s = r$, and $x = \pi y_i$ we obtain the following.

Corollary 3.3. *For each $i > 1$ we have a homotopy commutative diagram*

$$\begin{array}{ccc} S^{2ni-1} & \xrightarrow{p^r} & S^{2ni-1} \\ \| & & \downarrow{\scriptstyle \delta_i} \\ S^{2ni-1} & \xrightarrow{\gamma_i} & F_{(i-1)} \end{array}$$

where δ_i satisfies a homotopy commtutative diagram

$$\begin{array}{ccccc} S^{2ni-1} & \longrightarrow & P^{2ni} & \longrightarrow & E \\ \downarrow{\scriptstyle \delta_i} & & & & \downarrow{\scriptstyle \pi} \\ F_{(i-1)} & & \longrightarrow & & F. \end{array}$$

□

We now set up to apply Theorem 2.3. Define the space $E_{(i)}$ as the homotopy pullback

$$\begin{array}{ccc} E_{(i)} & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ F_{(i)} & \longrightarrow & F. \end{array}$$

Observe that there is a homotopy pullback diagram

$$\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \longrightarrow & E_{(i-1)}) & \longrightarrow & F_{(i-1)} & \longrightarrow & \Omega S^{2n+1} \\
\Vert & & \downarrow & & \downarrow & & \Vert \\
\Omega^2 S^{2n+1} & \longrightarrow & E_{(i)} & \longrightarrow & F_{(i)} & \longrightarrow & \Omega S^{2n+1}.
\end{array}$$

By the definition of $F_{(i)}$ there is a homotopy cofibration $S^{2ni-1} \xrightarrow{\gamma_i} F_{(i-1)} \longrightarrow F_{(i)}$. Thus γ_i lifts to $E_{(i-1)}$. A lift can be chosen which is divisible by p^r. Specifically, by Corollary 3.3, $\gamma_i \sim \delta_i \circ p^r$. Moreover, $S^{2ni-1} \xrightarrow{\delta_i} F_{(i-1)}$ composed to F factors through $E \xrightarrow{\pi} F$. Thus there is a pullback map $\bar{y}_i : S^{2ni-1} \longrightarrow E_{(i-1)}$ such that the composite $S^{2ni-1} \xrightarrow{\bar{y}_i} E_{(i-1)} \longrightarrow F_{(i-1)}$ is homotopic to δ_i. Hence $a = \bar{y}_i \circ p^r$ is a lift of γ_i. Theorem 2.3 now immediately implies the following.

Theorem 3.4. *If Z is a homotopy associative H space whose p^r-power map is null homotopic, then for $i > 1$ any map $E_{(i-1)} \xrightarrow{\phi} Z$ extends to a map $\overline{\phi} : E_{(i)} \longrightarrow Z$.* □

In [G3], a classifying space BW_n of the fiber of the double suspension was constructed, along with a fibration sequence

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n.$$

Corollary 3.5. *There is a map $\nu^E : E \longrightarrow BW_n$ such that the composition*

$$\Omega^2 S^{2n+1} \xrightarrow{\partial} E \xrightarrow{\nu^E} BW_n$$

is homotopic to ν.

Proof. Since $F_{(1)} = S^{2n}$, we have the fibration

$$\Omega^2 S^{2n+1} \longrightarrow E_{(1)} \longrightarrow S^{2n} \longrightarrow \Omega S^{2n+1}.$$

This fibration was analyzed in [G3] and it was shown that $E_{(1)} \simeq S^{4n-1} \times BW_n$ in such a way that the composition

$$\Omega^2 S^{2n+1} \xrightarrow{\partial} S^{4n-1} \times BW_n \xrightarrow{\pi_2} BW_n$$

is homotopic to ν. It was also shown that for $p \geq 5$ BW_n is a homotopy associative H space. The H space structure on BW_n was shown to be homotopy associative for $p = 3$ and that the p^{th}-power map on BW_n is null homotopic in [T5]. Thus for $i > 1$ we can apply Theorem 3.4 to construct maps $\nu_i : E_{(i)} \longrightarrow BW_n$ by induction such that $\nu_i \partial_i \sim \nu$. Since $E = \cup E_{(i)}$, we define $\nu^E : E \longrightarrow BW_n$ by $\nu^E \mid E_i = \nu_i$. □

Theorem 3.6. *There is a diagram of fibrations*

$$\begin{array}{ccccc} S^{2n-1} & \longrightarrow & \Omega^2 S^{2n+1} & \xrightarrow{\nu} & BW_n \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \partial} & & \| \\ R_0 & \longrightarrow & E & \xrightarrow{\nu^E} & BW_n \\ \downarrow & & \downarrow & & \\ F & = & F & & \end{array}$$

with i null homotopic and so $\Omega F \simeq S^{2n-1} \times \Omega R_0$.

Proof. The space R_0 is defined as the fiber of ν^E. Since the fibration

$$\Omega^2 S^{2n+1} \xrightarrow{\partial} E \longrightarrow F$$

is induced by a map to ΩS^{2n+1} which induces an isomorphism in $H_{2n}(\ )$, the map $\Omega F \to S^{2n-1}$ induces an isomorphism in $H_{2n-1}(\ )$ and hence has a right homotopy inverse. □

It is worth noting at this point that the space ΩR_0 is split in [CMN1]; there is a homotopy decomposition

$$\Omega R_0 \simeq \prod_{i \geq 1} S^{2np^i-1}\{p^{r+1}\} \times \Omega P(n,r)$$

where $P(n,r)$ is a complicated wedge of mod-p^r Moore spaces. The fact that the product on the right is a loop space and is mapped to ΩF by a loop map is not obvious from their analysis. The structure of R_0 is rather simple.

Proposition 3.7. *We have*

$$H^m(R_0) = \begin{cases} \mathbb{Z}/p^r i & \textit{if } m = 2ni \\ 0 & \textit{otherwise.} \end{cases}$$

Furthermore, there is a choice of generators $e_i \in H^{2mi}(X)$ such that $e_i e_j = p^r \binom{i+j}{i} e_{i+j}$.

Proof. Apply the Serre spectral sequence to the fibration $S^{2n-1} \longrightarrow R_0 \longrightarrow F$ in Theorem 3.6. □

We now construct the space T in Theorem 1.1 and prove the existence of the fibrations in parts (a) and (d), leaving the H-structure to the next section. By Diagram (B) there is a fibration sequence $\Omega S^{2n+1}\{p^r\} \xrightarrow{\tau} E \xrightarrow{\sigma} P^{2n+1} \longrightarrow S^{2n+1}\{p^r\}$. Define H by the composition

$$H : \Omega S^{2n+1}\{p^r\} \xrightarrow{\tau} E \xrightarrow{\nu^E} BW_n.$$

Note that H can be regarded as a secondary Hopf invariant. Define T as the homotopy fiber of H. Then Theorem 3.6 implies the following.

Theorem 3.8. *There is a diagram of fibrations*

$$\begin{array}{ccccc} T & \longrightarrow & \Omega S^{2n+1}\{p^r\} & \xrightarrow{H} & BW_n \\ \downarrow & & \downarrow{\scriptstyle \tau} & & \| \\ R_0 & \longrightarrow & E & \xrightarrow{\nu^E} & BW_n \\ \downarrow & & \downarrow{\scriptstyle \sigma} & & \\ P^{2n+1} & = & P^{2n+1}. & & \end{array}$$

□

The connecting maps for the vertical fibrations in Theorem 3.8 immediately give the following.

Corollary 3.9. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega P^{2n+1} & = & \Omega P^{2n+1} \\ \downarrow & & \downarrow \\ T & \longrightarrow & \Omega S^{2n+1}\{p^r\} \end{array}$$

where the right map is the loop of the inclusion of the bottom Moore space. □

Continuing the diagram in (B), we have

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{\rho} & \Omega S^{2n+1}\{p^r\} \\ \| & & \downarrow{\scriptstyle \tau} \\ \Omega^2 S^{2n+1} & \xrightarrow{\partial} & E. \end{array}$$

Observe that $H\rho \sim \nu^E \tau\rho \sim \nu^E \partial \sim \nu$. Theorem 3.8 therefore implies the following.

Theorem 3.10. *There is a diagram of fibrations*

$$\begin{array}{ccccc} \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} & & \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle p^r} & & \\ S^{2n-1} & \xrightarrow{E^2} & \Omega^2 S^{2n+1} & \xrightarrow{\nu} & BW_n \\ \downarrow & & \downarrow{\scriptstyle \rho} & & \| \\ T & \longrightarrow & \Omega S^{2n+1}\{p^r\} & \xrightarrow{H} & BW_n \\ \downarrow & & \downarrow & & \\ \Omega S^{2n+1} & = & \Omega S^{2n+1}. & & \end{array}$$

□

In particular, the top square in Theorem 3.10 is Cohen, Moore, and Neisendorfer's factorization of the p^r-power map on $\Omega^2 S^{2n+1}$. Since π_n has degree p^r, we have the following corollary.

Corollary 3.11. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{p^r} & S^{2n-1} \\ \downarrow & \overset{\pi_n}{\nearrow} & \downarrow \\ \Omega^2 S^{2n+1} & \xrightarrow{p^r} & \Omega^2 S^{2n+1} \end{array}$$

for each $r \geq 1$. □

4. The construction of G and the H-space structure on T

In this section we construct an H-space structure on T. In fact we do more than that. We construct a corresponding co-H space G in the sense of [G7]; i.e., we construct a $(2n-2)$-connected space G and maps

$$f : G \longrightarrow \Sigma T$$

$$g : T \longrightarrow \Omega G$$

$$h : \Omega G \longrightarrow T$$

such that the compositions

$$G \xrightarrow{f} \Sigma T \xrightarrow{\tilde{g}} G$$

$$T \xrightarrow{g} \Omega G \xrightarrow{h} T$$

are homotopic to the identity, where $\tilde{g}$ is the adjoint of g. We go on to derive several interesting results from this structure.

We will write T^m for the m-skeleton of T. We will also reintroduce the torsion parameter for Moore spaces as we will need to consider mod-p^s Moore spaces $P^m(p^s)$ for $s \neq r$. The space G will be filtered by subcomplexes G_k which will be constructed inductively starting with $G_{-1} = *$. We will construct a map

$$\alpha_k : P^{2np^k}(p^{r+k}) \longrightarrow G_{k-1}$$

and define G_k as the mapping cone of α_k.

The induction proceeds through 14 steps for each k, and we collect some information outside of the induction first.

Proposition 4.1. *As an algebra,* $H^*(T; \mathbb{Z}/p)$ *is generated by classes* u *of dimension* $2n-1$ *and* v_i *of dimension* $2np^i$ *for each* $i \geq 0$ *subject to the relations* $v_i^p = 0$ *and* $u^2 = 0$*. For each* i *define*

$$u_i = uv_0^{p-1}v_1^{p-1}\dots v_{i-1}^{p-1}$$

Then $\beta^{(r+i)}u_i = v_i$*. As a vector space* $\tilde{H}^*(T;\mathbb{Z}/p)$ *is generated by classes* $v(m)$ *of dimension* $2mn$ *and* $u(m)$ *of dimension* $2mn-1$ *for each* $m \geq 1$ *where*

$$v(m) = v_s^{e_s} \dots v_t^{e_t} = \beta^{(r+s)}u(s)$$
$$u(m) = u_s v_s^{e_s} v_{s+1}^{e_{s+1}} \dots v_t^{e_t}$$

and $m = \sum_{i=s}^{t} e_i p^i$*,* $0 \leq e_i < p$*,* $e_s \neq 0$*.*

Proof. We apply the Serre spectral sequence for the cohomology of the fibration

$$S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}$$

Using $\mathbb{Z}/p$ coefficients we see that

$$H^*(T;\mathbb{Z}/p) \cong H^*\left(S^{2n-1};\mathbb{Z}/p\right) \otimes H^*\left(\Omega S^{2n+1};\mathbb{Z}/p\right)$$

as algebras. Using integer coefficients we see that $\nu(m)$ is the reduction of a class of order p^{r+s} so $v(m) = \beta^{(r+s)}u(s) \neq 0$. We define $v_s = \beta^{(r+s)}u_s$. □

Note that dually the homology of T has a very simple description. There is a Hopf algebra isomorphism

$$H_*(T) \cong \Lambda(\bar{u}) \otimes \mathbb{Z}/p\mathbb{Z}[\bar{v}]$$

where $\bar{u}$ and $\bar{v}$ are dual to u and v respectively, and the dual Bocksteins are determined by $\beta^{(r+i)}\bar{v}^{p^i} = \bar{u}\bar{v}^{p^i-1}$ for $i \geq 0$.

Anick [A] introduced the notation $\mathcal{W}_a^b$ for the class of all spaces that are locally finite wedges of mod-p^s Moore spaces for $a \leq s \leq b$. Note that any simply connected Moore space is a suspension, so any simply connected space in $\mathcal{W}_a^b$ is a suspension. Recall that the smash of two Moore spaces is homotopy equivalent to a wedge of Moore spaces: if $s \leq t$ then there is a homotopy equivalence

$$P^m(p^s) \wedge P^n(p^t) \simeq P^{m+n}(p^s) \wedge P^{m+n-1}(p^s).$$

In particular, $\mathcal{W}_a^b$ is closed under smash products. Recall also that any retract of a wedge of Moore spaces is homotopy equivalent to a wedge of Moore spaces, so $\mathcal{W}_a^b$ is closed under retracts.

Lemma 4.2. *Suppose* $W \in \mathcal{W}_a^b$ *is simply connected and* $f : P^k(p^t) \longrightarrow W$ *is divisible by* p^b*.*

(a) *Write* $W = W_1 \vee W_2$ *with* $W_1 \in \mathcal{W}_a^{b-1}$ *and* $W_2 \in \mathcal{W}_b^b$*. Then* f *factors through* W_2 *up to homotopy.*

(b) *Suppose in addition that* W_2 *is* $(d-1)$ *connected and* $k < pd$*. Then* $f \sim *$*.*

Proof. Since W is a wedge, there is a homotopy equivalence $\Omega W = \Omega W_2 \times \Omega(W_1 \rtimes \Omega W_2)$ (see, for example, [G1]). Since $W_1, W_2 \in \mathcal{W}_a^b$, both spaces are suspensions, and we can write $W_1 = \Sigma\overline{W}_1$ and

$W_2 = \Sigma\overline{W}_2$. Since W_1 is a suspension, we have $W_1 \rtimes \Omega W_2 \simeq W_1 \vee (W_1 \wedge \Omega W_2)$. For the right wedge summand, the James splitting of $\Sigma\Omega\Sigma X$ as $\bigvee \Sigma X^{(i)}$ gives

$$W_1 \wedge \Omega W_2 \simeq \Sigma\overline{W}_1 \wedge \Omega\Sigma\overline{W}_2 \simeq \overline{W}_1 \wedge \left(\bigvee \Sigma\overline{W}_2^{(i)}\right).$$

Combining, we have

$$W_1 \rtimes \Omega W_2 \simeq W_1 \vee \left(W_1 \wedge \left(\bigvee W_2^{(i)}\right)\right).$$

In particular, since $\mathcal{W}_a^b$ is closed under smash products, we have $W_1 \rtimes \Omega W_2 \in \mathcal{W}_a^b$. Applying the Hilton-Milnor theorem therefore implies that $\Omega(W_1 \rtimes \Omega W_2) \simeq \prod_i \Omega P^{n_i}(p^{s_i})$ with $a \leq s \leq b-1$.

By [N3], the p^{r+1}-power map on $\Omega^2 P^m(p^r)$ is null homotopic for any $r \geq 1$ and $m \geq 3$. Thus $P^m(p^r)$ admits no nontrivial maps which are divisible by p^{r+1}. In our case, this implies that $\prod_i \Omega P^{n_i}(p^{s_i})$ admits no nontrivial maps which are divisible by p^b. Thus the adjoint of f, which is divisible by p^b, is trivial on $\Omega(W_1 \rtimes \Omega W_2)$ and so factors through the inclusion $\Omega W_2 \longrightarrow \Omega W$. Hence, adjointing, f factors through the inclusion $W_2 \longrightarrow W$, proving part (a).

For part (b), since $W_2 \in \mathcal{W}_b^b$ and W_2 is $(d-1)$-connected, the Hilton-Milnor theorem implies that $\Omega W_2 = \prod \Omega P^{n_i}(p^b)$ where $n_i > d$. By [CMN1, N3], $P^{2m+1}(p^r)$ admits no nontrivial maps which are divisible by p^r from a CW-complex of dimension $t < 2mp$, and $P^{2m}(p^b))$ admits no nontrivial maps which are divisible by p^r from a CW-complex of dimension $t < 2(2m-1)p$. In our case, the CW-complex is $P^k(p^t)$, the domain of f, and the target Moore spaces are the $P^{n_i}(p^b)$ in the decomposition of ΩW_2. Since $n_i > d$ for each i, the hypothesis $k < pd$ guarantees that the component of f on $P^{n_i}(p^b)$, being divisible by p^b, is null homotopic. Hence f is null homotopic. □

Theorem 4.3. *For each $k \geq 0$ there are spaces G_k and $W_k \in \mathcal{W}_r^{r+k-1}$ satisfying the following conditions:*

(a) $\Sigma T^{2np^k-2} \simeq G_{k-1} \vee W_k$;

(b) *there are maps $g_k : T^{2np^k-2} \longrightarrow \Omega G_{k-1}$ and $h_{k-1} : \Omega G_{k-1} \longrightarrow T$ such that $h_{k-1}g_k$ is homotopic to the inclusion of T^{2np^k-2} into T;*

(c) *there is a homotopy commutative diagram of cofibration sequences which defines G_k*

$$\begin{array}{ccccc}
P^{2np^k}(p^{r+k}) & \xrightarrow{m_k} & \Sigma T^{2np^k-2} & \longrightarrow & \Sigma T^{2np^k} \\
\Big\| & & \Big\downarrow{\scriptstyle \tilde{g}_k} & & \Big\downarrow{\scriptstyle g'_k} \\
P^{2np^k}(p^{r+k}) & \xrightarrow{\alpha_k} & G_{k-1} & \longrightarrow & G_k
\end{array}$$

where $\tilde{g}_k$ is the adjoint of g_k;

(d) *there is a map $e : P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \longrightarrow \Sigma T^{2np^k}$ which induces an epimorphism in mod-p cohomology;*

(e) *the map $m_k : P^{2np^k}(p^{r+k}) \longrightarrow \Sigma T^{2np^k-2}$ is divisible by p^{r+k-1};*

(f) *there is a map $\varphi_k : G_k \longrightarrow S^{2n+1}\{p^r\}$ extending φ_{k-1};*

(g) $\Sigma G_k \in \mathcal{W}_r^{r+k}$;

(h) *there is a homotopy commutative diagram of fibration sequences*

$$\begin{array}{ccccccc}
\Omega G_k & \xrightarrow{h_k} & T & \longrightarrow & R_k & \longrightarrow & G_k & & \\
& & \downarrow & & \downarrow & & \| & & \\
& & \Omega S^{2n+1}\{p^r\} & \longrightarrow & E_k & \longrightarrow & G_k & \longrightarrow & S^{2n+1}\{p^r\} \\
& & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle \nu_k} & & & & \\
& & BW_n & = & BW_n; & & & &
\end{array}$$

(i) $\Sigma^2 \Omega G_{k-1} \in \mathcal{W}_r^{r+k-1}$;

(j) *the equivalence in* (a) *extends to an equivalence* $\Sigma T^{2np^k} \simeq G_k \vee W_k$;

(k) $\Sigma^2 T^{2np^k} \in \mathcal{W}_r^{r+k}$;

(l) $G_k \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$;

(m) $\Sigma T^{2np^k} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$;

(n) *there is a map* $\mu_k : T^{2np^k} \times T \longrightarrow T$ *which is the inclusion on the first axis and the identity on the second. Furthermore there is a homotopy commutative square*

$$\begin{array}{ccc}
T^{2np^k} \times T & \xrightarrow{\mu_k} & T \\
\downarrow & & \downarrow \\
\Omega S^{2n+1} \times \Omega S^{2n+1} & \longrightarrow & \Omega S^{2n+1}.
\end{array}$$

Proof. With $G_{-1} = *$ and $G_0 = P^{2n+1}$ these statements are all immediate for $k = 0$ with $\varphi_0 : P^{2n+1} \longrightarrow S^{2n+1}\{p^r\}$ the inclusion, $E_0 = E$ from Theorem 3.6, $\nu_0 = \nu^E$, $\mu_0 : P^{2n} \times T \longrightarrow T$ obtained from the action of ΩP^{2n+1} on T defined by the fibration in Theorem 3.6. We now supposed that (a)–(n) are all valid with $k-1$ in the place of k and we proceed to prove them for k.

Proof of (a). We will construct a map

$$f_m : P^{2mn+1}(p^{r+s}) \longrightarrow \Sigma T^{2np^k-2}$$

which induces a monomorphism in mod-p homology for each m satisfying $p^{k-1} < m < p^k$, where $s = \nu_p(m)$. We then assemble these into a map

$$\Sigma T^{2np^{k-1}} \vee \left(\bigvee_{m=p^{k-1}+1}^{p^k-1} P^{2mn+1}(p^{r+s}) \right) \longrightarrow \Sigma T^{2np^k-2}$$

which induces an isomorphism in mod-p homology. By applying (j) in the case $k-1$ we are done.

To construct the maps f_m we appeal to (n) in the case $k-1$ and iterate this to produce a diagram with p factors

$$\begin{array}{ccc} T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}} & \longrightarrow & T^{2np^k} \\ \downarrow & & \downarrow \\ J(S^{2n})_{p^{k-1}} \times \cdots \times J(S^{2n})_{p^{k-1}} & \longrightarrow & J(S^{2n})_{p^k} \end{array}$$

where $J(S^{2n})_j$ is the $2nj$ skeleton of ΩS^{2n+1}. Since $p^{k-1} < m < p^k$,we can write $m = a_s p^s + \cdots + a_{k-1}p^{k-1}$ with $a_s > 0$ and $a_{k-1} > 0$. Write $l = a_s p^s + \cdots + a_{k-2}p^{k-2}$ so that $m = l + a_{k-1}p^{k-1}$ and further restrict the above diagram to one with $a_{k-1} + 1$ factors

$$\begin{array}{ccc} T^{2nl} \times T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}} & \xrightarrow{\overline{\mu}} & T^{2nm} \\ \downarrow & & \downarrow \\ J(S^{2n})_l \times J(S^{2n})_{p^{k-1}} \times \cdots \times J(S^{2n})_{p^{k-1}} & \longrightarrow & J(S^{2n})_m. \end{array}$$

By applying the maps in this diagram to a generator of $H^{2mn}(J(S^{2n})_m; \mathbb{Z}/p)$ we see that

$$(\overline{\mu})^* (v(m)) = v(l) \otimes v_{k-1} \otimes \cdots \otimes v_{k-1}.$$

Now $v(m) = \beta^{(r+s)}u(m)$ and $v(l) = \beta^{(r+s)}u(l)$, so

$$v(l) \otimes v_{k-1} \otimes \cdots \otimes v_{k-1} = \beta^{(r+s)} (u(l) \otimes v_{k-1} \otimes \cdots \otimes v_{k-1}).$$

Applying (k) and (l) in case $k-1$ we see that

$$\Sigma \left(T^{2nl} \times T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}}\right) \in \mathcal{W}_r^{r+k-1}.$$

Now given any space $W \in \mathcal{W}_r^{r+k-1}$ and any class $\xi \in H^i(W; \mathbb{Z}/p)$ with $\beta^{(j)}\xi \neq 0$, there is a map

$$f_\xi : P^{i+1}(p^j) \longrightarrow W$$

with f_ξ^* an epimorphism. Thus for each m satisfying $p^{k-1} < m < p^k$ we may choose such a map corresponding to $\xi = u(l) \otimes v_{k-1} \otimes \cdots \otimes v_k$. The composition

$$P^m(p^{r+s}) \xrightarrow{f_\xi} \Sigma \left(T^{2nl} \times T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}}\right) \xrightarrow{\Sigma\overline{\mu}} \Sigma T^{2mn}$$

therefore gives the desired map f_m.

Proof of (b). From part (a) we obtain a map $T^{2np^k-2} \longrightarrow \Omega G_{k-1}$ which induces an isomorphism in $\pi_{2n-1}(\ )$. The composition

$$T^{2np^k-2} \longrightarrow \Omega G_{k-1} \xrightarrow{h_{k-1}} T$$

factors through T^{2np^k-2} and provides a self map of T^{2np^k-2} which induces an isomorphism on $\pi_{2n-1}(\ )$. Calculations with cup products and Bocksteins show that this map is a homotopy equivalence, so composing with the inverse provides a possibly different map

$$g_k : T^{2np^k-2} \longrightarrow \Omega G_{k-1}$$

such that $h_{k-1}g_k$ is homotopic to the inclusion.

Proof of (c). Using the map g_k from (b) we construct a commutative diagram where the bottom row is the fibration sequence from (h) in case $k-1$ and the middle row is a cofibration sequence

$$\begin{array}{ccccccc}
 & & & & T^{2np^k}/T^{2np^k-2} & = & P^{2np^k}(p^{r+k}) \\
 & & & & \downarrow & & \downarrow \\
T^{2np^k-2} & \longrightarrow & T & \longrightarrow & T/T^{2np^k-2} & \longrightarrow & \Sigma T^{2np^k-2} \\
\downarrow g_k & & \| & & \downarrow & & \downarrow \tilde{g}_k \\
\Omega G_{k-1} & \xrightarrow{h_{k-1}} & T & \longrightarrow & R_{k-1} & \longrightarrow & G_{k-1}.
\end{array}$$

Define $\alpha_k : P^{2np^k}(p^{r+k}) \longrightarrow G_{k-1}$ as the vertical composition on the right. Define g'_k by the diagram of cofibration sequences

$$\begin{array}{ccccc}
P^{2np^k}(p^{r+k}) & \xrightarrow{m_k} & \Sigma T^{2np^k-2} & \longrightarrow & \Sigma T^{2np^k} \\
\| & & \downarrow \tilde{g}_k & & \downarrow g'_k \\
P^{2np^k}(p^{r+k}) & \xrightarrow{\alpha_k} & G_{k-1} & \longrightarrow & G_k.
\end{array}$$

Proof of (d). As in part (a), we consider the diagram:

$$\begin{array}{ccc}
T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}} & \xrightarrow{\tilde{\mu}} & T^{2np^k} \\
\downarrow & & \downarrow \\
J(S^{2n})_{p^{k-1}} \times \cdots \times J(S^{2n})_{p^{k-1}} & \longrightarrow & J(S^{2n})_{p^k}
\end{array}$$

with p factors on the left. This is defined by iterated application of part (n) in case $k-1$. Clearly

$$\tilde{\mu}^*(v_k) = v_{k-1} \otimes \cdots \otimes v_{k-1} = \beta^{(r+k-1)}(uv_1^{p-1} \ldots v_{k-2}^{p-1} \otimes v_{k-1} \otimes \cdots \otimes v_{k-1}).$$

As before there is a map

$$q : P^{2np^k+1}(p^{r+k-1}) \longrightarrow \Sigma\left(T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}}\right)$$

such that $(\Sigma(\tilde{\mu})q)^*$ is an epimorphism in $\mathbb{Z}/p$ cohomology obtained by applying (k) and (m) in case $k-1$. Similarly, $v_k = \beta^{(r+k)}u_k$ and

$$(\tilde{\mu})^* u_k = \sum_{p \text{ terms}} v_{k-1} \otimes \cdots \otimes v_{k-1} \otimes u_{k-1} \otimes v_{k-1} \cdots \otimes v_{k-1}.$$

In particular, the map

$$T^{2np^{k-1}-1} \times T^{2np^{k-1}} \times \cdots \times T^{2np^{k-1}} \xrightarrow{\tilde{\mu}'} T^{2np^k}$$

has the property that

$$\begin{aligned}(\tilde{\mu}')^*(u_k) &= u_{k-1}\otimes v_{k-1}\otimes\cdots\otimes v_{k-1}\\ &= \beta^{(r+k-1)}\left(u_{k-1}\otimes u_{k-1}\otimes v_k\otimes\cdots\otimes v_k\right).\end{aligned}$$

It follows, as before, that there is a map

$$r: P^{2np^k}(p^{r+k-1}) \longrightarrow \Sigma\left(T^{2np^{k-1}-1}\times T^{2np^{k-1}}\times\cdots\times T^{2np^{k-1}}\right)$$

such that $(\Sigma(\tilde{\mu}')r)^*$ is an epimorphism in $\mathbb{Z}/p$ cohomology. We construct e as the wedge sum

$$e = (\Sigma\tilde{\mu}')r\vee(\Sigma\tilde{\mu})q : P^{2np^k}(p^{r+k-1})\vee P^{2np^k+1}(p^{r+k-1})\longrightarrow \Sigma T^{2np^k}.$$

Proof of (e). We apply Lemma 3.2 with $x=e$, $s=r+k-1$, $X=\Sigma T^{2np^k-2}$, $M=P^{2np^k-1}(p^{r+k})$, and $f=m_k:\Sigma M\longrightarrow X$. In this case $A=P^{2np^k-1}(p^{r+k-1})\vee P^{2np^k}(p^{r+k-1})$, which is the cofiber of p^{r+k-1} on $M=P^{2np^k-1}\left(p^{r+k}\right)$. It follows that m_k is divisible by p^{r+k-1}.

Proof of (f). To show that there is an extension of φ_{k-1} to φ_k,

$$\begin{array}{ccccc} P^{2np^k}(p^{r+k}) & \xrightarrow{\alpha_k} & G_{k-1} & \longrightarrow & G_k = G_{k-1}\cup_{\alpha_k} CP^{2np^k}(p^{r+k}) \\ & & \downarrow{\scriptstyle\varphi_{k-1}} & \swarrow{\scriptstyle\varphi_k} & \\ & & S^{2n+1}\{p^r\} & & \end{array}$$

it suffices to show that α_k is divisible by p^r. This holds by (e) since $r+k-1\geq r$.

Proof of (g). By part (g) for $k-1$, there is a homotopy equivalence $SG_{k-1}\simeq\bigvee_{i=0}^{k-1}P^{2np^i+2}\left(p^{r+i}\right)$. Also, by definition, $SG_k = SG_{k-1}\cup_{S\alpha_k} CP^{2np^k}\left(p^{r+k}\right)$. By part (e) $\alpha_k=\tilde{\alpha}\circ\left(p^{r+k-1}\iota\right)$ so $S\alpha_k = \tilde{\alpha}\circ\left(p^{r+k-1}\iota\right)\sim\left(p^{r+k-1}\iota\right)\circ\tilde{\alpha}$. However

$$p^{r+k-1}\iota\colon \bigvee_{i=0}^{k-1}P^{2np^i+2}\left(p^{r+i}\right)\to\bigvee_{i=0}^{k-1}P^{2np^i+2}\left(p^{r+i}\right)$$

is null homotopic since the order of the identity map on a mod-p^r Moore space is p^r.

Proof of (h). As we have constructed $\varphi_k : G_k \longrightarrow S^{2n+1}\{p^r\}$ in part (f), we have a pullback diagram of principal fibrations

$$\begin{array}{ccc}
\Omega S^{2n+1}\{p^r\} & = & \Omega S^{2n+1}\{p^r\} \\
\downarrow & & \downarrow \\
E_{k-1} & \longrightarrow & E_k \\
\downarrow & & \downarrow \\
G_{k-1} & \longrightarrow & G_k \\
\downarrow{\scriptstyle \varphi_{k-1}} & & \downarrow{\scriptstyle \varphi_k} \\
S^{2n+1}\{p^r\} & = & S^{2n+1}\{p^r\}.
\end{array}$$

We wish to apply Theorem 2.3 to extend $\nu_{k-1} : E_{k-1} \longrightarrow BW_n$ to E_k. It suffices to show that there is a lifting δ of α_k,

$$\begin{array}{ccc}
 & & E_{k-1} \\
 & \overset{\delta}{\nearrow} & \downarrow \\
P^{2np^k}(p^{r+k}) & \xrightarrow{\alpha_k} & G_{k-1}
\end{array}$$

which is divisible by p. Since $\alpha_k \sim p^{r+k-1}\tilde{\alpha}$, it follows that $p^r\tilde{\alpha}$ lifts to a map $\delta' : P^{2np^k}(p^{r+k}) \longrightarrow E_{k-1}$ with $p^{k-1}\delta' = \delta$ a lifting of α_k. Thus as long as $k > 1$ we can construct δ with the requisite property. When $k = 1$, we appeal to [CMN1] where it is shown that $\alpha_1 = p\delta_1$ with $\delta_1 : P^{2np}(p^{r+1}) \longrightarrow P^{2n+1}(p^r)$ lifting to E_0.

Proof of (i). By part (j) in case $k-1$, $\Sigma^2\Omega G_{k-1}$ is a retract of $\Sigma^2\Omega\Sigma T^{2np^{k-1}}$. The latter space splits since the loop space can be approximated by the James construction [J], giving

$$\Sigma^2\Omega\Sigma T^{2np^{k-1}} \simeq \Sigma^2\left(\bigvee_{i\geq 1}(T^{2np^{k-1}})^{(i)}\right)$$

which is in $\mathcal{W}_r^{r+k-1}$ by (k) and (l) in case $k-1$. Since $\mathcal{W}_r^{r+k-1}$ is closed under retracts we are done.

Proof of (j). By part (a), we have $\Sigma T^{2np^k-2} \simeq G_{k-1} \vee W_k$ and by (e), we have

$$\Sigma T^{2np^k} \simeq \left(\Sigma T^{2np^k-2}\right) \cup_{m_k} CP^{2np^k}(p^{r+k})$$

with m_k divisible by p^{r+k-1}. It suffices to show that the map

$$m_k : P^{2np^k}(p^{r+k}) \longrightarrow \Sigma T^{2np^k-2} \simeq G_{k-1} \vee W_k$$

factors though G_{k-1}. To this end, observe that there is a homotopy decomposition

$$\Omega\left(G_{k-1} \vee W_k\right) \simeq \Omega G_{k-1} \times \Omega\left(W_k \rtimes \Omega G_{k-1}\right).$$

We will show that any map $P^{2np^k}(p^{r+k}) \longrightarrow W_k \rtimes \Omega G_{k-1}$ which is divisible by p^{r+k-1} is null homotopic. Since W_k is $(4n-1)$-connected, the Moore spaces in W_k are double suspensions, so $W_k \rtimes \Omega G_{k-1} \in \mathcal{W}_r^{r+k-1}$. In fact, $W_k \rtimes \Omega G_{k-1} \simeq W_1 \vee W_2$ with $W_1 \in \mathcal{W}_r^{r+k-2}$ and W_2 a retract of

$$\bigvee_{r=2}^{p-1} P^{2np^{k-1}+1}(p^{r+k-1}) \rtimes \Omega G_{k-1}$$

which is $4np^{k-1}-1$ connected. The result follows from Lemma 4.2.

Proof of (k). This follows immediately from (g) and (j).

Proof of (l). This follows from 3 steps based on an analysis which first appeared in [T1].

Step 1: $G_k \wedge T^{2np^{k-1}} \in \mathcal{W}_r^{r+k-1}$.

Consider the cofibration sequence

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^{k-1}} \xrightarrow{\alpha_k \wedge 1} G_{k-1} \wedge T^{2np^{k-1}} \longrightarrow G_k \wedge T^{2np^{k-1}}.$$

We have $P^{2np^k}(p^{r+k}) \wedge T^{2np^{k-1}} \in \mathcal{W}_r^{r+k-1}$ and $\alpha_k \wedge 1$ is divisible by p^{r+k-1}. Consequently, $\alpha_k \wedge 1 \sim *$ and so there is a homotopy decomposition

$$G_k \wedge T^{2np^{k-1}} \simeq (G_{k-1} \wedge T^{2np^{k-1}}) \vee (P^{2np^k+1}(p^{r+k}) \wedge T^{2np^{k-1}})$$

which is in $\mathcal{W}_r^{r+k-1}$ by (k) in case $k-1$.

Step 2: $G_{k-1} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k-1}$.

By (j) in case $k-1$, $G_{k-1} \wedge T^{2np^k}$ is a retract of $\Sigma T^{2np^{k-1}} \wedge T^{2np^k}$. But

$$\Sigma T^{2np^{k-1}} \wedge T^{2np^k} \simeq T^{2np^{k-1}} \wedge \Sigma T^{2np^k} \simeq T^{2np^{k-1}} \wedge (G_k \vee W_k)$$

by (j). By Step 1 and (k) in case $k-1$, the latter space is in $\mathcal{W}_r^{r+k-1}$. Since $\mathcal{W}_r^{r+k-1}$ is closed under retracts, we therefore have $G_{k-1} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k-1}$.

Step 3: $G_k \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$.

Consider here the cofibration sequence

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\alpha_k \wedge 1} G_{k-1} \wedge T^{2np^k} \longrightarrow G_k \wedge T^{2np^k}.$$

The first space is in $\mathcal{W}_r^{r+k}$ by (k) and the second is in $\mathcal{W}_r^{r+k-1}$ by Step 2. In fact, $G_{k-1} \wedge T^{2np^k} \simeq (P^{2np^{k-1}+1}(p^{r+k-1}) \wedge T^{2np^k}) \vee W'$ with $W' \in \mathcal{W}_r^{r+k-2}$. Here, the projection onto the first factor is $\rho_{k-1} \wedge 1$, where ρ_{k-1} is obtained by collapsing G_{k-2} to a point. Applying Lemma 4.2(b), we see that if $\alpha_k \wedge 1$ is nontrivial, so is the composition

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\alpha_k \wedge 1} G_{k-1} \wedge T^{2np^k} \xrightarrow{\rho_{k-1} \wedge 1} P^{2np^{k-1}}(p^{r+k-1}) \wedge T^{2np^k}.$$

We will show that this composition is null homotopic. Let $\delta = \rho_{k-1}\alpha_k$, which is divisible by p^{r+k-1} because α_k is. According to [N1], the p^{r+k-1}-power map on $S^{2np^{k-1}+1}\{p^{r+k-1}\}$ is null homotopic.

Therefore the composition

$$P^{2np^k}(p^{r+k}) \xrightarrow{\delta} P^{2np^{k-1}+1}(p^{r+k-1}) \longrightarrow S^{2np^{k-1}+1}\{p^{r+k-1}\}$$

is null homotopic. It follows that the composition

$$P^{2np^k}(p^{r+k}) \xrightarrow{\delta} P^{2np^{k-1}+1}(p^{r+k-1}) \xrightarrow{\rho} S^{2np^{k-1}+1}$$

is null homotopic. Since the map

$$P^{2np^{k-1}+1}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\rho\wedge 1} S^{2np^{k-1}+1} \wedge T^{2np^k}$$

has a left homotopy inverse, the map

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\delta\wedge 1} P^{2np^{k-1}+1}(p^{r+k}) \wedge T^{2np^k}$$

is null homotopic. Since $\alpha_k \wedge 1$ is the composition

$$P^{2np^k}(p^{r+k}) \wedge T^{2np^k} \xrightarrow{\alpha\wedge 1} G_{k-1} \wedge T^{2np^k} \xrightarrow{\delta\wedge 1} P^{2np^{k-1}+1}(p^{r+k}) \wedge T^{2np^k},$$

it is null homotopic as well. Consequently, there is a homotopy decomposition

$$G_k \wedge T^{2np^k} \simeq (G_{k-1} \wedge T^{2np^k}) \vee (P^{2np^k+1}(p^{r+k}) \wedge T^{2np^k}).$$

Both terms on the right are in $\mathcal{W}_r^{r+k}$ by (k) and Step 2.

Proof of (m). By (j), $\Sigma T^{2np^k} \wedge T^{2np^k} \simeq (G_k \vee W_k) \wedge T^{2np^k}$. By (l), $G_k \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$, and as W_k is a wedge of Moore spaces which are at least $(4n-1)$-connected, it is a double suspension, so by (k) we have $W_k \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$. Thus $\Sigma T^{2np^k} \wedge T^{2np^k} \in \mathcal{W}_r^{r+k}$.

Proof of (n). Since the composite $R_k \longrightarrow E_k \longrightarrow G_k \longrightarrow S^{2n+1}\{p^r\} \longrightarrow S^{2n+1}$ is null homotopic by (h), there is a commutative diagram of principal fibrations:

$$\begin{array}{ccc}
\Omega G_k & \longrightarrow & \Omega S^{2n+1} \\
\downarrow{\scriptstyle h_k} & & \| \\
T & \longrightarrow & \Omega S^{2n+1} \\
\downarrow & & \downarrow \\
R_k & \longrightarrow & PS^{2n+1} \\
\downarrow & & \downarrow \\
G_k & \longrightarrow & S^{2n+1}
\end{array}$$

where PS^{2n+1} is the path space on S^{2n+1}. Consequently the actions are compatible

$$\begin{array}{ccc}
\Omega G_k \times T & \longrightarrow & \Omega S^{2n+1} \times \Omega S^{2n+1} \\
\downarrow & & \downarrow \\
T & \longrightarrow & \Omega S^{2n+1}.
\end{array}$$

Using (j) we construct a map $g_k : T^{2np^k} \longrightarrow \Omega G_k$ such that the composition

$$T^{2np^k} \xrightarrow{g_k} \Omega G_k \xrightarrow{h_k} T$$

is homotopic to the inclusion as in (b). This gives a homotopy commutative diagram

$$\begin{array}{ccc} T^{2np^k} \times T & \xrightarrow{g_k \times 1} & \Omega G_k \times T \\ \downarrow{\scriptstyle \mu_k} & & \downarrow{\scriptstyle a} \\ T & = & T. \end{array}$$

Combining the preceeding two diagrams gives the result and completes the induction. □

We now consider the limiting case. Write $G = \bigcup G_k$, $R = \bigcup R_k$ and $E_\infty = \bigcup E_k$.

Theorem 4.4. *There is a diagram of fibration sequences*

$$\begin{array}{ccccccccc} \Omega G & \xrightarrow{h} & T & \xrightarrow{i} & R & \longrightarrow & G & & \\ & & \downarrow{\scriptstyle E} & & \downarrow & & \| & & \\ & & \Omega S^{2n+1}\{p^r\} & \longrightarrow & E_\infty & \longrightarrow & G & \xrightarrow{\varphi} & S^{2n+1}\{p^r\} \\ & & \downarrow{\scriptstyle H} & & \downarrow & & & & \\ & & BW_n & = & BW_n & & & & \end{array}$$

and there are maps $\tilde{g} : T \longrightarrow \Omega G$ and $f : G \longrightarrow \Sigma T$ such that the composites

$$G \xrightarrow{f} \Sigma T \xrightarrow{g} G$$

$$T \xrightarrow{\tilde{g}} \Omega G \xrightarrow{h} T$$

are homotopic to the identity maps.

Proof. The diagram is the direct limit of the diagrams in Theorem 4.3 (h) with $h = \varinjlim h_k$, $g = \varinjlim g_k$ and $f = \varinjlim f_k$, where $f_k : G_k \longrightarrow \Sigma T^{2np^k}$ is a right inverse for g_k given by Theorem 4.3 (j). □

Theorem 4.5. *The following space belong to $\mathcal{W}_r^\infty$: $\Sigma^2\Omega G$, ΣG, $G \wedge T$, $\Sigma T \wedge T$, and W where $\Sigma T \simeq G \vee W$.*

Proof. This follows immediately from the results in Theorem 4.3 by taking limits. □

The retraction of T off ΩG in Theorem 4.4 induces an H-structure on T by the composite

$$m : T \times T \xrightarrow{\tilde{g} \times \tilde{g}} \Omega G \times \Omega G \longrightarrow \Omega G \xrightarrow{h} T.$$

The following proposition establishes the H-fibration property in Theorem 1.1 (a) as a consequence of a slightly stronger result.

Proposition 4.6. *The map $T \xrightarrow{E} \Omega S^{2n+1}\{p^r\}$ is an H map with respect to the H-space structure m on T. Consequently, there is an H-fibration sequence $S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}$.*

Proof. Filling in the fibration diagram in Theorem 4.4 on the right, we obtain a homotopy commutative square

$$\begin{array}{ccc} \Omega G & \xrightarrow{h} & T \\ \| & & \downarrow{\scriptstyle E} \\ \Omega G & \xrightarrow{\Omega\varphi} & \Omega S^{2n+1}\{p^r\}. \end{array}$$

Now consider the following diagram

$$\begin{array}{ccccccc} T \times T & \xrightarrow{\tilde{g}\times\tilde{g}} & \Omega G \times \Omega G & \longrightarrow & \Omega G & \xrightarrow{h} & T \\ & \searrow{\scriptstyle E\times E} & \downarrow{\scriptstyle \Omega\varphi\times\Omega\varphi} & & \downarrow{\scriptstyle \Omega\varphi} & \swarrow{\scriptstyle E} & \\ & & \Omega S^{2n+1}\{p^r\} \times \Omega S^{2n+1}\{p^r\} & \longrightarrow & \Omega S^{2n+1}\{p^r\}. & & \end{array}$$

The middle square commutes as $\Omega\varphi$ is an H-map and we have just seen that the right triangle commutes. The left triangle commutes since $\varphi \sim Eh$, so $\varphi\tilde{g} \sim E$. As the top row is the definition of the multiplication m on T, the commutativity of the diagram implies that E is an H-map.

Consequently, the composition $T \xrightarrow{E} \Omega S^{2n+1}\{p^r\} \longrightarrow \Omega S^{2n+1}$ is an H-map as it is a composite of H-maps, and so the homotopy fibration $S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}$ is of H-spaces and H-maps. □

The next proposition and the following corollary give structural properties of the spaces T, G, and R.

Proposition 4.7. *The spaces T and G are atomic.*

Proof. It is easy to see that T is atomic using the product structure and the Bockstein relations. The case of G is more difficult. We first show that if G is not atomic then the map $P^{2np^k}(p^{r+k}) \xrightarrow{\alpha_k} G_{k-1}$ is null homotopic for some k. Suppose $\gamma : G \longrightarrow G$ is a map with the property that $\gamma|_{G_{k-1}} : G_{k-1} \longrightarrow G_{k-1}$ is a homotopy equivalence and α_k has order p. Consider the diagram

$$\begin{array}{ccccc} P^{2np^k}(p^{r+k}) & \xrightarrow{\alpha_k} & G_{k-1} & \longrightarrow & G_k \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\ P^{2np^k}(p^{r+k}) & \xrightarrow{\alpha_k} & G_{k-1} & \longrightarrow & G_k. \end{array}$$

Since $\gamma|_{G_{k-1}} : G_{k-1} \longrightarrow G_{k-1}$ is an equivalence, $\gamma\alpha_k$ has the same order as α_k. Consequently $d \not\equiv 0 \pmod{p}$ and hence d is an equivalence. It follows that $\gamma|_{G_k} : G_k \longrightarrow G_k$ is an equivalence.

Suppose now that $\alpha_k \sim *$. Then we can construct a map

$$s : P^{2np^k+1}(p^{r+k}) \longrightarrow G_k$$

which induces an isomorphism in $H^{2np^k+1}(\ )$. We now show that the composite

$$H^{2np^k}(T) \xrightarrow{h_k^*} H^{2np^k}(\Omega G_k) \xrightarrow{(\Omega s)^*} H^{2np^k}(\Omega P^{2np^k+1}(p^{r+k}))$$

is an isomorphism. First we note that the image of $(h_k)^*$ is a direct summand since $g_{k+1} : T^{2np^{k+1}-2} \longrightarrow \Omega G_k$ induces a left inverse in cohomology. However, there is an isomorphism

$$H^{2np^k}(\Omega G_k) \cong H^{2np^k}(\Omega G_{k-1}) \oplus H^{2np^k}(\Omega P^{2np^k+1}(p^{r+k}))$$

since the inclusion $G_{k-1} \vee P^{2np^k+1}(p^{r+k}) \longrightarrow G_{k-1} \times P^{2np^k+1}(p^{r+k})$ is an equivalence in this range. However, $H^{2np^k}(\Omega G_{k-1})$ contains no elements of order p^{r+k} since $\Sigma^2\Omega G_{k-1} \in \mathcal{W}_r^{r+k-1}$ by Theorem 4.3 (i). We conclude that the composition is an isomorphism. Now apply cellular approximation to obtain a homotopy commutative diagram

$$\begin{array}{ccccc} P^{2np^k}(p^{r+k}) & & \xrightarrow{\quad s' \quad} & & T^{2np^k} \\ \downarrow E & & & & \downarrow \\ \Omega P^{2np^k+1}(p^{r+k}) & \xrightarrow{\Omega s} & \Omega G_k & \longrightarrow & T \end{array}$$

where E is the suspension and s' is a skeletal factorization. It follows that s' induces an isomorphism in $H^{2np^k}(\ ) = \mathbb{Z}/p^{r+k}$. From this we see that

$$H_{2np^k-1}(P^{2np^k}(p^{r+k}) : \mathbb{Z}/p) \xrightarrow{(s')_*} H_{2np^k-1}(T; \mathbb{Z}/p)$$

induces an isomorphism as well because of the Bockstein structure. However $H_{2np^k-1}(T; \mathbb{Z}/p)$ is generated by u_k which is decomposable if $k > 0$. This is a contradiction which implies that α_k is essential and G is atomic. The fact that G and T are atomic and the maps f, g, h exist implies that (G, T) is a corresponding pair in the sense of [G7]. □

Corollary 4.8. $R \in \mathcal{W}_r^\infty$.

Proof. According to [G7, Theorem 3.2], R is a retract of $\Sigma T \wedge T \in \mathcal{W}_r^\infty$. □

The next proposition implies that the space T constructed in this paper is homotopy equivalent to the space Anick constructed in [A] when $p \geq 5$ (the primes for which Anick's construction holds).

Proposition 4.9. *Suppose X is an H space and there is a fibration sequence:*

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \xrightarrow{i} X$$

such that the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is homotopic to the p^r power map. Then $X \simeq T$.

Proof. Consider the diagram of fibrations

$$\begin{array}{ccccc}
\Omega W_n & \longrightarrow & \Omega X & \longrightarrow & \Omega^2 S^{2n+1}\{p^r\} \\
\downarrow & & \downarrow & & \downarrow \\
PW_n & \longrightarrow & \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} \\
\downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle p^r} \\
W_n & \longrightarrow & S^{2n-1} & \xrightarrow{E^2} & \Omega^2 S^{2n+1} \\
 & & \downarrow & & \\
 & & X. & &
\end{array}$$

Since $p \cdot \pi_*(W_n) = 0$ and $p^r \cdot \pi_*\left(S^{2n+1}\{p^r\}\right) = 0$ we conclude that $p^{r+1} \cdot \pi_*(X) = 0$. Since $\pi_{2np-1}(W_n) = 0$, we also see that $p^r \cdot \pi_{2np-1}(X) = 0$. According to [AG, Corollary 4.2] this is sufficient to construct a map

$$\varphi : G \longrightarrow \Sigma X$$

which induces an isomorphism in π_{2n}. The construction given in [AG] depends only on the co-H space structure on G and the fact that α_k is divisible by p^{r+k-1}, so the proof works in this context as well. From this we construct the composition

$$T \xrightarrow{g} \Omega G \xrightarrow{\Omega\varphi} \Omega\Sigma X \longrightarrow X.$$

It is an easy calculation with the Serre spectral sequence that $H^*(X;\mathbb{Z}/p) \cong H^*(T;\mathbb{Z}/p)$, so this map is a homotopy equivalence. □

References

[A] David Anick, *Differential algebras in topology*, Research Notes in Mathematics, 3, A K Peters, Ltd., Wellesley, MA, 1993, xxvi+274 pp.

[AG] David Anick and Brayton Gray, *Small H spaces related to Moore spaces*, Topology **34** (1995), no. 4, 859–881.

[C] Frederick R. Cohen, *The unstable decomposition of* $\Omega^2\Sigma^2 X$ *and its applications*, Math. Z. **182** (1983), no. 4, 553–568.

[CMN1] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer, *Torsion in homotopy groups*, Ann. of Math. (2) **109** (1979), no. 1, 121–168.

[CMN2] ______, *The double suspension and exponents of the homotopy groups of spheres*, Ann. of Math. (2) **110** (1979), no. 3, 549–565.

[CMN3] ______, *Decompositions of loop spaces and applications to exponents, algebraic topology, aarhus, 1978*, Proc. Sympos., Univ. Aarhus, Aarhus, 1978 (Berlin), Lecture Notes in Math., vol. 763, Springer, 1979, pp. 1–12.

[G1] Brayton Gray, *A note on the Hilton-Milnor theorem*, Topology **10** (1971), 199–201.

[G2] ______, *On the homotopy groups of mapping cones*, Proc. London Math. Soc. (3) **26** (1973), 497–520.

[G3] ______, *On the iterated suspension*, Topology **27** (1988), no. 3, 301–310.

[G4] ______, *EHP spectra and periodicity. I. Geometric constructions*, Trans. Amer. Math. Soc. **340** (1993), no. 2, 595–616.

[G5] ______, *EHP spectra and periodicity. II.* Λ*-algebra models*, Trans. Amer. Math. Soc. **340** (1993), no. 2, 617–640.

[G6] ——, *Composition methods in the homotopy groups of ring spectra*, Progr. Math., vol. 196, pp. 131–148, Birkhäuser, Basel, 2001.

[G7] ——, *On decompositions in homotopy theory*, Trans. Amer. Math. Soc. **358** (2006), no. 8, 3305–3328.

[GT] Brayton Gray and Stephen Theriault, *On the double suspension and the mod-p Moore space*, An alpine anthology of homotopy theory, Contemp. Math., vol. 399, Amer. Math. Soc., Providence, RI, 2006, pp. 101–121.

[J] I. M. James, *Reduced product spaces*, Ann. of Math. (2) **62** (1955), 170–197.

[M] Mark Mahowald, *On the double suspension homomorphism*, vol. 214, Amer. Math. Soc., 1975.

[N1] Joseph A. Neisendorfer, 3-*primary exponents*, Math. Proc. Cambridge Philos. Soc. **90** (1981), no. 1, 63–83.

[N2] ——, *Properties of certain H-spaces*, Quart. J. Math. Oxford Ser. (2) **34** (1983), no. 134, 201–209.

[N3] ——, *The exponent of a Moore space*, Algebraic Topology and Algebraic K-theory, Ann. of Math. Study **113**, 1987, pp. 35-71.

[N4] ——, *James-Hopf invariants, Anick's spaces, and the double loops on odd primary Moore spaces*, Canad. Math. Bull. **43** (2000), no. 2, 226–235.

[N5] ——, *A survey of Anick-Gray-Theriault constructions and applications to the exponent theory of spheres and Moore spaces*, Une dégustation topologique [Topological morsels]: homotopy theory in the Swiss Alps (Arolla, 1999), Contemp. Math., vol. 265, Amer. Math. Soc., Providence, RI, 2000, pp. 159–174.

[S] Manfred Stelzer, *On certain co-H spaces related to Moore spaces*, Trans. Amer. Math. Soc. **354** (2002), no. 8, 3085–3093.

[T1] S. D. Theriault, PhD Thesis, University of Toronto, 1997.

[T2] ——, *Properties of Anick's spaces*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 1009–1037.

[T3] ——, *Anick's spaces and the double loops of odd primary Moore spaces*, Trans. Amer. Math. Soc. **353** (2001), no. 4, 1551–1566.

[T4] ——, *Proofs of two conjectures of Gray involving the double suspension*, Proc. Amer. Math. Soc. **131** (2003), no. 9, 2953–2962.

[T5] ——, *The 3-primary classifying space of the fiber of the double suspension*, accepted by Proc. Amer. Math. Soc.

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL, 60607-7045, USA

E-mail address: `brayton@uic.edu`

Department of Mathematical Sciences, University of Aberdeen, Aberdeen AB24 3UE, United Kingdom

E-mail address: `s.theriault@maths.abdn.ac.uk`