

# Filtering the fiber of the pinch map

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## 1 Introduction

In [CMN1], the authors analyzed the homotopy type of the mod  $p^r$  Moore space

$$P^{2n+1} = S^{2n} \cup_{p^r} e^{2n+1}$$

by constructing a fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\partial_n} \Omega F_n \longrightarrow \Omega P^{2n+1} \longrightarrow \Omega S^{2n+1}.$$

A key result is that both  $\Omega F_n$  and  $\Omega P^{2n+1}$  contain a factor which is the loop space on a one point union of Moore spaces that are at least  $4n - 2$  connected. Removing this factor (which can be inductively considered later) leaves a fibration sequence:

$$\dots \rightarrow \Omega^2 S^{2n+1} \xrightarrow{\bar{\partial}} \bar{F}_n \longrightarrow T^{2n+1} \longrightarrow \Omega S^{2n+1}$$

and they show that

$$\bar{F}_n \cong S^{2n-1} \times \prod_{k \geq 1} S^{2np^k-1} \{p^{r+1}\}.$$

An important remaining question is to understand the components of  $\bar{\partial}$ :

$$\begin{aligned} \bar{\partial}_0 : \Omega^2 S^{2n+1} &\longrightarrow S^{2n-1} \\ \bar{\partial}_k : \Omega^2 S^{2n+1} &\longrightarrow S^{2np^k-1} \{p^{r+1}\}. \end{aligned}$$

$\bar{\partial}_0$  is well understood and plays a key role in homotopy theory. There is much interest in understanding  $\bar{\partial}_k$  for  $k > 0$ ; in particular, it is not known whether they are all null

homotopic [AG], [CMN1], [G2], [G3], [GT], [N1], [N2]. In case  $r > 1$ , Neisendorfer ([N1], [N2]) has shown that  $\bar{\partial}_k = 0$  for  $k \geq 1$ , so our emphasis will be on the case  $r = 1$ .

A remarkable observation of [CMN2] is that there is an isomorphism of Hopf algebras (see appendix):

$$H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p) \cong H_*(\bar{F}_n; \mathbb{Z}/p).$$

However, the induced homomorphism  $(\bar{\partial})_* = 0$ .

The intention of this work is to consider constructions involving  $\bar{F}_n$  which are analogous to constructions involving  $\Omega^2 S^{2n+1}$ . We will be able to do this compatibly with the map  $\bar{\partial}$ . In particular, we will look at the filtration of Selick [S] and the classifying space of the double suspension [G2]. Our results will follow from the construction of a new ‘‘Hopf invariant’’ type map:

$$F_n \xrightarrow{h} F_{np}$$

which induces an epimorphism in  $p$ -local homology. Recall that there is a  $p$ -local fibration sequence:

$$S_{(p^s-1)}^{2n} \rightarrow S_\infty^{2n} \xrightarrow{H_{p^s}} S_\infty^{2np^s}.$$

By analogy, we compare  $F_{(p^s-1)}$  to the fiber  $\Delta_s$  of the Hopf invariant  $h^s$ :

$$\Delta_s \rightarrow F_n \xrightarrow{h^s} F_{np^s}$$

**Theorem 1.1**  $\Omega\Delta_s \simeq \Omega F_{(p^s-1)} \times \Omega X_s$  where  $X_s$  is a wedge of mod  $p^r$  Moore spaces.

This will appear as 7.1. These same ideas lead to:

**Theorem 1.2** If  $k > 1$ , there is a homotopy commutative diagram:

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{\bar{\partial}_k} & S^{2np^k-1}\{p^{r+1}\} \\ \gamma_k \downarrow \cong & & \uparrow \bar{\partial}_1 \\ \Omega^2 S^{2n+1} & \xrightarrow{\Omega H_{p^{k-1}}} & \Omega^2 S^{2np^{k-1}+1} \end{array}$$

where  $H_{p^k}$  is a James-Hopf invariant and  $\gamma_k$  is a homotopy equivalence.

**Theorem 1.3** The map  $\bar{\partial}_k : \Omega^2 S^{2n+1} \longrightarrow S^{2np^k-1}\{p^{r+1}\}$  is homotopic to a composition:

$$\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n \xrightarrow{\epsilon} S^{2np^k-1}\{p^{r+1}\}.$$

These appear as 8.2 and 9.3 in the sequel.

Throughout this paper all spaces will be assumed to be localized at a fixed prime  $p > 2$  and all homology will be  $p$ -local unless otherwise indicated.

## 2 Filtration of $\Omega^2 S^{2n+1}$

$H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p)$  is a free commutative algebra on generators of dimensions  $2np^i - 1$ ,  $2np^{i+1} - 2$  for  $i \geq 0$ . Selick [S] has described a sequence of  $H$ -spaces whose homology filters  $H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p)$  by successively adding one generator at each stage. Let us write  $S_{(k)}^{2n} = J_k(S^{2n})$  for the  $k$ th filtration of the James construction  $J(S^{2n}) = S_{\infty}^{2n}$ . Then the space capturing the first  $2s$  generators of  $H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p)$  is precisely  $\Omega S_{(p^s-1)}^{2n}$ . Selick [S] defines spaces  $G_s$  which contain the first  $2s+1$  generators. We define  $G_s$  by a diagram of fibrations:

$$\begin{array}{ccccccc}
 G_s & \longrightarrow & S^{2np^s-1} & \xrightarrow{\alpha} & S_{(p^s-1)}^{2n} & & \\
 \downarrow & & \downarrow & & \parallel & & \\
 \Omega^2 S^{2n+1} & \longrightarrow & \Omega^2 S^{2np^s+1} & \longrightarrow & S_{(p^s-1)}^{2n} & \longrightarrow & \Omega S^{2n+1} \xrightarrow{H_{np^s}} \Omega S^{2np^s+1} \\
 \downarrow & & \downarrow \nu & & & & \\
 BW_{np^s} & \xlongequal{\quad} & BW_{np^s} & & & & 
 \end{array}$$

where  $\alpha : S^{2np^s-1} \rightarrow S_{(p^s-1)}^{2n}$  is the attaching map for the  $2np^s$  cell of  $S_{\infty}^{2n}$ ,  $H_{np^s}$  is the appropriate James-Hopf invariant, and  $BW_{np^s}$  is the classifying space for the double suspension. Since  $G_s$  is the fiber of  $\nu \circ \Omega H_{np^s}$ , it is an  $H$  space if  $p > 3$  ([G2; Proposition 6]).  $G_s$  could also be described as the fiber of a Toda-Hopf invariant  $\Omega S_{(p^{s+1}-1)}^{2n} \xrightarrow{H'} \Omega S^{2np^{s+1}-1}$  using the techniques of ([G2], [MN]).

Constructing an analogous filtration for  $\overline{F}_n$  is not difficult. The key result is in the compatibility of the two filtrations.

## Proof of Theorem CMN

In this section we give a brief summary of the proof of the main result of [CMN1].

**Theorem CMN** *There is a diagram of fibration sequences:*

$$\begin{array}{ccccccc}
 \Omega F_n & \longrightarrow & \Omega P^{2n+1} & \longrightarrow & \Omega S^{2n+1} & & \\
 \downarrow & & \downarrow & & \parallel & = & \\
 S^{2n-1} \times V_n & \longrightarrow & T^{2n+1} & \longrightarrow & \Omega S^{2n+1} & & \\
 \downarrow i & & \downarrow i' & & & & \\
 P_n & \xrightarrow{\cong} & P_n & & & & \\
 \downarrow \bar{\phi} & & \downarrow & & & & \\
 F_n & \longrightarrow & P^{2n+1} & \xrightarrow{p} & S^{2n+1} & & 
 \end{array}$$

where the maps  $i$  and  $i'$  are null homotopic,  $p$  is the pinch map,  $P_n$  is a one point union of mod  $p^r$  Moore spaces of dimension  $\geq 4n$  and

$$V_n = \prod_{k \geq 1} S^{2np^k-1}\{p^{r+1}\}.$$

Here  $S^m\{d\}$  is the fiber of the degree  $d$  map on  $S^m$ . The inessentiality of  $i$  and  $i'$  implies that

$$\begin{aligned}\Omega F_n &\simeq S^{2n-1} \times V_n \times \Omega P_n \\ \Omega P^{2n+1} &\simeq T^{2n+1} \times \Omega P_n.\end{aligned}$$

*Sketch of the proof.* Both the mod  $p^r$  homotopy and the mod  $p$  homology of  $\Omega P^{2n+1}$  have differential Lie algebra structures, and the Hurewicz map is a Lie algebra homomorphism. Furthermore  $\Omega F_n$  has an extended ideal structure. These structures are obtained from the Samelson product and the  $r$ th Bockstein  $\beta^{(r)}$  ( $\beta^{(i)} = 0$  for  $i < r$ ).

$H_*(\Omega P^{2n+1}; \mathbb{Z}/p) = U(L)$  where  $L$  is a free Lie algebra on  $v \in H_{2n}(\Omega P^{2n+1}; \mathbb{Z}/p)$  and  $u = \beta^{(r)}(v)$  and  $U(L)$  is the universal enveloping algebra.  $L^{(0)} \subset L$  is the Lie ideal generated by  $x_i = ad^{i-1}(v)(u)$  for  $i > 1$  and  $H_*(\Omega F_n; \mathbb{Z}/p) = U(L^{(0)})$ .

Furthermore the suspensions of the  $x_i$ ,

$$\sigma(x_i) \in H_{2ni}(F_n; \mathbb{Z}/p) \cong \mathbb{Z}/p$$

are non zero.  $u$  and  $v$  lie in the image of the mod  $p^r$  Hurewicz homomorphism, so all of  $L$  and hence  $L^{(0)}$  does as well

$$L^{(0)} \subset im\{\pi_*(\Omega F_n; \mathbb{Z}/p^r) \longrightarrow H_*(\Omega F_n; \mathbb{Z}/p)\}$$

using the extended ideal structure in homotopy.

Both in homotopy and homology,  $\beta^{(r)}(x_{p^k}) = 0$ , so one may construct an extension:

$$P^{2np^k-1}(p^{r+1}) \xrightarrow{\delta_k} \Omega F_n$$

of the mod  $p^r$  homotopy class

$$P^{2np^k-1} \longrightarrow \Omega F_n$$

representing  $x_{p^k}$  in homology. The only property of the maps  $\delta_k$  that is needed is that their Hurewicz image is  $x_{p^k}$ . From these maps the authors construct

$$S^{2np^k-1}\{p^{r+1}\} \xrightarrow{e} \Omega P^{2np^k}(p^{r+1}) \xrightarrow{\Omega \bar{\delta}_k} \Omega F_n$$

where  $\bar{\delta}_k$  is the adjoint of  $\delta_k$  and  $e$  is a particular map described in the next section. These maps are assembled via loop multiplication to obtain:

$$\Theta : S^{2n-1} \times V_n \longrightarrow \Omega F_n$$

The Bockstein  $\beta^{(r)}$  is trivial in  $H_*(S^{2n-1} \times V_n; \mathbb{Z}/p)$  but not in  $H_*(\Omega F_n; \mathbb{Z}/p)$ . However  $\Theta$  induces an isomorphism in the Bockstein homology of these homology groups (see the appendix for a calculation of the Bockstein homology).

Next the authors construct a sequence of sub-Lie algebras  $L^{(k+1)} \subset L^{(k)} \subset L^{(0)}$  via short exact sequences of Lie algebras:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L^{(1)} & \longrightarrow & L^{(0)} & \longrightarrow & \langle \tau_0 \rangle \longrightarrow 0 \\ 0 & \longrightarrow & L^{(k+1)} & \longrightarrow & L^{(k)} & \longrightarrow & \langle \tau_k, \sigma_k \rangle \longrightarrow 0. \end{array}$$

Here  $\langle \tau_0 \rangle$  and  $\langle \tau_{k_1} \sigma_K \rangle$  are free commutative Lie algebras generated by  $\tau_k = x_{p^k}$  of dimension  $2np^k - 1$  and

$$\sigma_k = \frac{1}{2p} \sum_{i=1}^{p^k-1} \binom{p^k}{i} [x_i, x_{p^k-i}] \in L^{(0)}.$$

This is possible since  $\tau_k$  is a generator and  $\sigma_k$  is decomposable in  $L^{(k-1)}$  but not in  $L^{(k)}$  since  $\frac{1}{2p} \binom{p^k}{p^{k-1}}$  is a  $p$ -local unit and  $x_{p^{k-1}} = \tau_{k-1} \notin L^{(k)}$ .

It follows that there is a split short exact sequence of differential Hopf algebras:

$$0 \longrightarrow U(L^{(k+1)}) \longrightarrow U(L^{(k)}) \longrightarrow H_*(S^{2np^k-1}\{p^{r+1}\}; \mathbb{Z}/p) \rightarrow 0$$

and hence

$$U(L^{(\infty)}) \otimes H_p(S^{2n-1} \times V_n; \mathbb{Z}/p) \cong U(L^{(0)}) = H_*(\Omega F_n; \mathbb{Z}/p)$$

where  $L^{(\infty)} = \bigcap_{k \geq 0} L^{(k)}$ .

Consequently  $H_i(U(L^{(\infty)}); \beta^{(r)}) = 0$  for  $t > 0$  and hence  $L^{(\infty)}$  has a basis consisting of classes  $\{x_\alpha, \beta^{(r)} x_\alpha\}$ . Since  $x_\alpha \in L^{(\infty)} \subset L^{(0)}$  one can choose maps

$$\phi_\alpha : P^{n_\alpha} \longrightarrow \Omega F_n$$

whose homology image is  $x_\alpha$  and  $\beta^{(r)} x_\alpha$  using the extended ideal structure. Assembling these one produces a one point union of mod  $p^r$  Moore spaces,  $P_n$  and a map:

$$\bar{\phi} : P_n \longrightarrow F_n$$

such that the homology image of

$$(\Omega \bar{\phi})_* : H_*(\Omega P_n; \mathbb{Z}/p) \longrightarrow H_*(\Omega F_n; \mathbb{Z}/p)$$

is exactly  $U(L^{(\infty)})$ . From this the authors produce a homotopy equivalence

$$S^{2n-1} \times V_n \times \Omega P_n \longrightarrow \Omega F_n$$

by multiplying the maps  $\Theta$  and  $\bar{\phi}$  via the  $H$  space structure in  $\Omega F_n$ . Let  $\bar{F}_n$  be the fiber of  $\bar{\phi}$ . Then

$$S^{2n-1} \times V_n \xrightarrow{\Theta} \Omega F_n \xrightarrow{\partial} \bar{F}_n$$

is a homotopy equivalence and  $i$  is inessential. Thus completes the outline of the proof.

### 3 Combinatorial description of $F_n$

In [G1] a combinatorial description of the fiber of the pinch map:

$$F \longrightarrow X \cup CA \xrightarrow{\pi} SA$$

was described in the spirit of the James construction  $X_\infty$  for  $\Omega SX$ . The model, designated  $(X, A)_\infty$  consists of all words in  $X_\infty$  with the property that all letters after the first letter are required to lie in  $A$ , where  $A \subset X$ . Alternatively, this can be described by a push out diagram:

$$(3.0) \quad \begin{array}{ccc} X \times A_\infty & \longrightarrow & (X, A)_\infty \\ \uparrow & & \uparrow \\ A \times A_\infty & \longrightarrow & A_\infty \end{array}$$

**Proposition 3.1** ([G1]). *There is a map  $(X, A)_\infty \rightarrow F$  which is a homotopy equivalence when the inclusion  $A \subset X$  is a cofibration.*

There is an action of the monoid  $A_\infty$  on  $(X, A)_\infty$  and  $(X, A)_\infty$  can be thought of as a universal space in the following sense. If  $Y$  is any space on which  $A_\infty$  acts and  $g : X \rightarrow Y$  is any map such that  $g(a) = a \cdot *$  for some point  $*$   $\in Y$ , there is a unique  $A_\infty$  equivariant map

$$g_\infty : (X, A)_\infty \rightarrow Y.$$

(See [G; 3.2]). The map  $(X, A)_\infty \rightarrow F$  is constructed from the action  $A_\infty \times F \rightarrow \Omega SA \times F \rightarrow F$ .

The orbit space of  $(X, A)_\infty$  under the action of  $A_\infty$  is  $X/A$  and we may use the universal property to establish the following diagram

$$(3.2) \quad \begin{array}{ccc} (X, A)_\infty & \xrightarrow{e} & X_\infty \\ \rho \downarrow & & \downarrow \\ X/A & \longrightarrow & (X/A)_\infty \end{array}$$

Note that the inclusion  $e : (X/A)_\infty \rightarrow X_\infty$  is the unique  $A_\infty$  equivariant extensions of the inclusion of  $X$  in  $X_\infty$ .

**Example 3.3** Let  $A = S^{2n-2} \subset P^{2n-1}_{(p^t)} = X$ . Then  $(X, A)_\infty$  is the homotopy fiber of the map of degree  $p^t$

$$(X, A)_\infty = S^{2n-1}\{p^t\} \longrightarrow S^{2n-1} \xrightarrow{p^t} S^{2n-1}$$

and the map  $e : S^{2n-1}\{p^t\} \longrightarrow (P^{2n-1}(p^t))_\infty \cong \Omega P^{2n-1}(p^t)$  is uniquely determined as a  $\Omega S^{2n-1}$  equivariant map extending the inclusion of  $P^{2n-1}(p^t)$ .

One of the main features of the construction  $(X, A)_\infty$  is that we can define functorial Hopf invariants using the same formulas as in James [J]. In particular we have a pointwise commutative diagram:

$$\begin{array}{ccc}
A_\infty & \xrightarrow{H_k} & (A^{(k)})_\infty \\
\downarrow & & \downarrow \\
(X, A)_\infty & \xrightarrow{H_k} & (X \wedge A^{(k-1)})_\infty \\
\downarrow & & \downarrow \\
X_\infty & \xrightarrow{H_k} & (X^{(k)})_\infty
\end{array}$$

It would be desirable to construct functorial compressions of these maps:

$$h : (X, A)_\infty \longrightarrow (X \wedge A^{(k-1)}, A^{(k)})_\infty \subset (X \wedge A^{(k-1)})_\infty$$

but I have been unable to do this. In the next section we will construct a map  $h$  of this form, but we have no knowledge of how it relates to  $H_k$ .

## 4 Construction of $h : F_n \longrightarrow F_{np}$

In this section we will define a kind of Hopf invariant which is key for the results of this paper.

**Theorem 4.1** *Suppose  $(X, A)$  is a suspension pair and  $r \geq 1$ . Then there is a map:*

$$h : (X, A)_\infty \longrightarrow (X \wedge A^{(r-1)}, A^{(r)})_\infty$$

*such that the composition:*

$$X \times A_\infty \longrightarrow (X, A)_\infty \xrightarrow{h} (X \wedge A^{(r-1)}, A^{(r)})_\infty \xrightarrow{p} X \wedge A^{(r-1)}/A^{(r)}$$

*is homotopic to the composition:*

$$X \times A_\infty \xrightarrow{1 \times H_{r-1}} X \times A_\infty^{(r-1)} \longrightarrow X \wedge A_\infty^{(r-1)} \xrightarrow[1 \wedge \epsilon]{} X \wedge A^{(r-1)} \longrightarrow X \wedge A^{(r-1)}/A^{(r)}$$

*where  $H_{r-1} : A_\infty \rightarrow A_\infty^{(r-1)}$  is any map and  $\epsilon$  uses the suspension structure of  $X$  to collapse of  $X \wedge A_\infty^{(r-1)}$  to  $X \wedge A^{(r-1)}$ .*

**Corollary 4.2** *Let  $F_n$  be the fiber of the pinch map*

$$p : P^{2n-1}(p^r) \longrightarrow S^{2n+1}$$

*for each  $n$  and  $H_{r-1}$  be any choice of James-Hopf invariants. Then there is a map:*

$$h : F_n \longrightarrow F_{rn}$$

such that the composition:

$$S^{2n} \times \Omega S^{2n+1} \longrightarrow F_n \xrightarrow{h} F_{rn} \longrightarrow P^{2rn+1}$$

is homotopic to:

$$S^{2n} \times \Omega S^{2n+1} \xrightarrow{1 \times H_{r-1}} S^{2n} \times \Omega S^{2(r-1)n+1} \longrightarrow S^{2rn} \longrightarrow P^{2rn+1};$$

in particular,  $h_* : H_{2rn}(F_n) \longrightarrow H_{2rn}(F_{rn})$  is an isomorphism.

Note: It is an easy calculation to see that

$$H_i(F_n) = \begin{cases} \mathbb{Z}_{(p)} & n \mid i \\ 0 & n \nmid i \end{cases}$$

and the map  $F_n \xrightarrow{\pi} P^{2n+1}(p^r)$  is reduction mod  $p^r$  in homology. In particular we have defined a map:

$$F_n \longrightarrow P^{2nr+1}(p^r)$$

for each  $r \geq 1$  which is onto in  $p$ -local homology.

*Proof of 4.1* We begin by constructing a map

$$\mu : (X, A)_\infty \longrightarrow (X, A)_\infty / A_\infty \longrightarrow X \times A_\infty / A \times A_\infty \equiv X \rtimes A_\infty / A \rtimes A_\infty$$

which follows from the push out diagram (3.0). This map is functorial and we have a commutative diagram:

$$\begin{array}{ccc} (X, A)_\infty & \xrightarrow{\mu} & X \rtimes A_\infty / A \rtimes A_\infty \\ \uparrow & & \uparrow \\ X \times A_\infty & \longrightarrow & X \rtimes A_\infty \end{array}$$

using the functorial property, we have the commutative diagram:

$$\begin{array}{ccc} (X, A)_\infty & \longrightarrow & X \rtimes A_\infty / A \rtimes A_\infty \\ \downarrow & & \downarrow \\ (CX, A)_\infty & \longrightarrow & CX \rtimes A_\infty / A \rtimes A_\infty \simeq S(A \rtimes A_\infty) \end{array}$$

however  $(CX, A)_\infty$  is the fiber of  $CX/A \longrightarrow SA$  and hence is contractible. Consequently  $\mu$  lifts to  $(X \rtimes A_\infty, A \rtimes A_\infty)_\infty$  which is the homotopy fiber of

$$X \rtimes A_\infty / A \rtimes A_\infty \longrightarrow SA \rtimes A_\infty;$$

we now choose a map  $H_{r-1} : A_\infty \longrightarrow A_\infty^{(r-1)}$  and use the suspension structure of the pair  $(X, A)$  to obtain a map

$$\begin{aligned} (X \rtimes A_\infty, A \rtimes A_\infty)_\infty &\longrightarrow (X \wedge A_\infty^{(r-1)}, A \wedge A_\infty^{(r-1)})_\infty \\ &\longrightarrow (X \wedge A^{(r-1)}, A^{(r)})_\infty; \end{aligned}$$

composing with  $\mu'$  defines  $h$ .

**Proposition 4.3** *The map  $h : F_n \longrightarrow F_{np^s}$  induces an epimorphism in  $p$ -local homology.*

*Proof:* We use the map  $\Omega S^{2n+1} \xrightarrow{\partial} F_n$  which has degree  $p$  in  $H^{2nk}$  for each  $k > 0$  to calculate the cup product structure in  $H^*(F_n)$ . We choose generators  $e_i \in H^{2ni}(F_n)$  such that  $\partial^*(e_i)$  is  $p$  times the generator of  $H^{2ni}(\Omega S^{2n+1})$  dual to the  $i$ th power of a homology generator in  $H_{2n}(\Omega S^{2n+1})$ . Then it is easy to see that

$$e_i e_j = p \binom{i+j}{i} e_{i+j}.$$

Let us designate  $d_i \in H^{2nip^s}(F_{np^s})$  for the corresponding generator; then

$$h^*(d_1) = u_1 e_{p^s}$$

for some  $p$  local unit  $u_1$  by 4.2. We show that

$$h^*(d_i) = u_i e_{ip^s}$$

where  $u_i$  is a  $p$ -local unit for each  $i \geq 1$  by induction. Using the product structure we have

$$pid_i = d_1 d_{i-1},$$

so  $pih^*(d_i) = h^*(d_1 d_{i-1}) = h^*(d_1)h^*(d_{i-1})$

$$\begin{aligned} &= (u_1 e_{p^s})(u_{i-1} e_{(i-1)p^s}) \\ &= pu_1 u_{i-1} \binom{ip^s}{p^s} e_{ip^s}. \end{aligned}$$

It suffices to show that  $\frac{1}{i} \binom{ip^s}{p^s}$  is a  $p$ -local unit.

Now let  $v_p(m)$  be the exponent of  $p$  in  $m$  and  $[x]$  be the greatest integer less than or equal to  $x$ . Then

$$v_p(n!) = \sum_{i \geq 1} \left[ \frac{n}{p^i} \right].$$

Consequently  $v_p((p^s i)!) = p^{s-1}i + p^{s-2}i + \cdots + i + v_p(i!)$  so

$$\begin{aligned} v_p \left( \binom{p^s i}{p^s} \right) &= v_p((p^s i)!) - v_p((p^s(i-1))!) - v_p(p^s!) \\ &= v_p(i!) - v_p((i-1)!) \\ &= v_p(i). \end{aligned}$$

and we are done.

By 4.2, the composition

$$\Omega S^{2n+1} \xrightarrow{\partial} F_n \xrightarrow{h} F_{np^s} \longrightarrow P^{2np^2+1}$$

is null homotopic, so there is a lifting  $\tilde{H}_{p^s}$

$$\begin{array}{ccc} \Omega S^{2n+1} & \xrightarrow{\tilde{H}_{p^s}} & \Omega S^{2np^s+1} \\ \partial \downarrow & & \downarrow \partial \\ F_n & \xrightarrow{h} & F_{np^s}. \end{array}$$

**Proposition 4.4**  $(\tilde{H}_{p^s})_* : H_*(\Omega S^{2n+1}) \rightarrow H_*(\Omega S^{2np^s+1})$  is an epimorphism.

*Proof:* This follows since both maps labeled  $\partial$  have degree  $p$  in homology and  $h_*$  is an epimorphism.

In particular, the fiber of  $\tilde{H}_{p^s}$  is  $S_{(p^s-1)}^{2n}$  as if  $\tilde{H}_{p^s}$  were the James-Hopf invariant  $H_{p^s}$ . Both  $\Omega\tilde{H}_{p^s}$  and  $\Omega H_{p^s}$  can be placed in the fibration sequence induced by the inclusion  $S_{(p^s-1)}^{2n} \subset \Omega S^{2n+1}$ , so there is an equivalence  $\gamma_s : \Omega^2 S^{2n+1} \rightarrow \Omega^2 S^{2n+1}$  such that

$$\Omega\tilde{H}_{p^s} = \Omega H_{p^s} \circ \gamma_s$$

hence we have

**Corollary 4.5** *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{\partial_n} & \Omega F_n \\ \Omega\tilde{H}_{p^s} \downarrow & & \downarrow \Omega h^s \\ \Omega^2 S^{2np^s+1} & \xrightarrow{\partial_{np}} & \Omega F_{np^s} \end{array}$$

where  $\Omega\tilde{H}_{p^s} \sim \Omega H_{p^s} \circ \gamma_s$  for some homotopy equivalence  $\gamma_s : \Omega^2 S^{2n+1} \rightarrow \Omega^2 S^{2n+1}$ .

## 5 A Filtered Decomposition

Let  $\Delta_s$  be the homotopy fiber of  $h^s : F_n \rightarrow F_{np^s}$ . In this section we will compare the decompositions of  $\Omega F_n$  and  $\Omega F_{np^s}$  and prove

**Theorem 5.1**  $\Omega\Delta_s \cong S^{2n-1} \times \prod_{1 \leq k < s} S^{2np^k-1}\{p^{r+1}\} \times \Omega S^{2np^s-1} \times \Omega R_s$  where  $R_s$  is a wedge of mod  $p^r$  Moore spaces.

Let  $\Psi : S^{2n-1} \times V_n \rightarrow S^{2np-1} \times V_{np}$  be defined by first projecting onto  $V_n$  and then applying

$$V_n = S^{2np-1}\{p^2\} \times V_{np} \xrightarrow{\rho \times 1} S^{2np-1} \times V_{np}$$

where  $\rho : S^{2np-1}\{p^2\} \rightarrow S^{2np-1}$  is the projection.

**Lemma 5.2** *With appropriate choices there is a homotopy commutative diagram:*

$$\begin{array}{ccc} S^{2n-1} \times V_n & \xrightarrow{\Theta_n} & \Omega F_n \\ \downarrow \psi & & \downarrow \Omega h \\ S^{2np-1} \times V_{np} & \xrightarrow{\Theta_{np}} & \Omega F_{np}. \end{array}$$

*Proof:* Given choices  $\bar{\delta}_k : P^{2np^k}(p^{r+1}) \rightarrow F_n$  inducing epimorphisms in  $p$ -local cohomology, define  $\bar{\delta}_k(np)$  as the composition:

$$P^{2np^{k+1}}(p^{r+1}) \xrightarrow{\bar{\delta}_{k+1}} F_n \xrightarrow{h} F_{np}$$

From this we construct

$$\begin{array}{ccccc} S^{2np^{k+1}-1}\{p^{r+1}\} & \xrightarrow{e} & \Omega P^{2np^{k+1}}(p^{r+1}) & \xrightarrow{\Omega \bar{\delta}_{k+1}} & \Omega F_n \\ \downarrow = & & \downarrow = & & \downarrow \Omega h \\ S^{2np^{k+1}-1}\{p^{r+1}\} & \xrightarrow{e} & \Omega P^{2np^{k+1}}(p^{r+1}) & \xrightarrow{\Omega \bar{\delta}_k(np)} & \Omega F_{np} \end{array}$$

where  $k > 0$ , and

$$\begin{array}{ccccc} S^{2np-1}\{p^{r+1}\} & \longrightarrow & \Omega P^{2np}(p^{r+1}) & \xrightarrow{\Omega \bar{\delta}_1} & \Omega F_n \\ \downarrow & & \downarrow & & \downarrow \Omega h \\ S^{2np-1} & \longrightarrow & \Omega S^{2np} & \longrightarrow & \Omega F_{np} \end{array}$$

using diagram 3.2. Multiplying these together in order gives the result.

**Lemma 5.3** *The map  $\bar{\phi}_n : P_n \rightarrow F_n$  can be chosen so that  $W_n = P_{np} \vee Q_n$  and there is a homotopy commutative diagram:*

$$\begin{array}{ccc} P_{np} \vee Q_n & \xrightarrow{\bar{\phi}_n} & F_n \\ p \downarrow & & \downarrow h \\ P_{np} & \xrightarrow{\bar{\phi}_{np}} & F_{np} \end{array}$$

where  $p$  is the projection.

*Proof:* Since  $h : H_*(F_n; \mathbb{Z}/p) \rightarrow H_*(F_{np}; \mathbb{Z}/p)$  is onto the same holds for

$$U(L^{(0)}(n)) \cong H_*(\Omega F_n; \mathbb{Z}/p) \rightarrow H_*(\Omega F_{np}; \mathbb{Z}/p) \cong U(L^{(0)}(np))$$

in fact the generators  $x_{ip} \in L^{(0)}(n)$  satisfy

$$h_*(x_{ip}(n)) = u_i x_i(np)$$

where  $u_i$  is a  $p$ -local unit (see 4.3).

Now given a basis  $\{x_\alpha, \beta^{(r)}x_\alpha\}$  for  $L^{(\infty)}(np)$ , each  $x_\alpha$  is a Lie bracket in the  $x_i(np)$  and this element consequently lifts to the same bracket in  $x_{ip}(n)$ . Thus these liftings are linearly independent and can be chosen as part of a basis. They are all in the image of the mod  $p^r$  Hurewicz homomorphism. Thus we have chosen generators for  $L^{(\infty)}(n)$  which sort into those which are lifting of the generators for  $L^{(\infty)}(np)$  and the others. Realizing these via the Hurewicz homomorphism gives the maps  $\bar{\phi}_n$ . By a change in basis for  $Q_n$ , we can assume that the map  $p : P_n \rightarrow P_{np}$  is trivial on  $Q_n$ .

We now use 5.3 to construct the following ladder of fibrations:

$$(5.4) \quad \begin{array}{ccccccc} \Omega F_n & \xrightarrow{\bar{\partial}_n} & \bar{F}_n & \longrightarrow & P_n & \xrightarrow{\bar{\phi}_n} & F_n \\ \downarrow & & \downarrow \gamma^s & & \downarrow & & \downarrow h^s \\ \Omega F_{np^s} & \xrightarrow{\bar{\partial}_{np^s}} & \bar{F}_{np^s} & \longrightarrow & P_{np^s} & \xrightarrow{\bar{\phi}_{np^s}} & F_{np^s} \end{array}$$

and we use 5.2 to construct compatible equivalences:

$$(5.5) \quad \begin{array}{ccccc} S^{2n-1} \times V_n & \longrightarrow & \Omega F_n & \xrightarrow{\partial_n} & \bar{F}_n \\ \downarrow \phi^s & & \downarrow \Omega h^s & & \downarrow \gamma^s \\ S^{2np^s-1} \times V_{np^s} & \longrightarrow & \Omega F_{np^s} & \longrightarrow & \bar{F}_{np^s}. \end{array}$$

Taking fibers vertically in (5.4), we have a fibration sequence:

$$\Omega \Delta_s \rightarrow K_s \rightarrow R_s \rightarrow \Delta_s$$

where  $R_s = (Q_n \vee \cdots \vee Q_{pn^{s-1}}) \rtimes \Omega P_{np^s}$  is a wedge of mod  $p^r$  Moore spaces and

$$K_s = S^{2n-1} \times \prod_{1 \leq k < s} S^{2np^k-1} \{p^{r+1}\} \times \Omega S^{2np^s-1}.$$

We use the splitting in (5.5) to construct a splitting

$$K_s \rightarrow \Omega \Delta_s \rightarrow K_s.$$

This completes the proof.

## 6 Decomposition of $\Omega F_{(p^s-1)}$

Let  $F_{(p^s-1)}$  be the  $2np^s - 1$  skeleton of  $F$ , so  $H_*(\Omega F_{(p^s-1)} \cdot \mathbb{Z}/p)$  is a tensor algebra on classes  $x_i$  of dimension  $2ni - 1$  for  $i < p^s$ .

**Theorem 6.1**  $\Omega F_{(p^s-1)} \simeq S^{2n-1} \times \prod_{1 \leq k < s} S^{2np^k-1} \{p^{r+1}\} \times \Omega S^{2np^s-1} \times \Omega Q_s$  where  $Q_s \subset Q_{s+1} \subset \cdots \subset P_n$  is a wedge of mod  $p^r$  Moore spaces for some choice of  $P_n$  in [CMN1].

*Proof:* This proceeds along the same lines as [CMN1]. The maps

$$\bar{\delta}_k : P^{2np^k} \rightarrow F$$

factor through  $F_{(p^{s-1})}$  when  $k < s$  as does the restriction

$$\bar{\delta}_s \Big|_{S^{2np^s-1}} : S^{2np^s-1} \rightarrow F.$$

We thus construct a map

$$\Theta_s : K_s = S^{2n-1} \times \prod_{1 \leq k < s} S^{2np^k-1} \{p^{r+1}\} \times \Omega S^{2np^s-1} \rightarrow \Omega F_{(p^{s-1})}$$

as before. The Bocksteins  $\beta^{(i)}$  for  $i < r$  are trivial and  $\Theta_s$  induces an isomorphism in the homology under the  $r$ th Bockstein. To see this observe that the Bockstein homology spectral sequence of  $H_*(\Omega F_{(p^{s-1})}; \mathbb{Z}/p)$  is a restriction of the spectral sequence for  $H_*(\Omega F; \mathbb{Z}/p)$  (see appendix).  $H_*(\Omega F_{(p^{s-1})}; \mathbb{Z}/p) = U(L_s^{(0)})$  where  $L_s^{(0)}$  is a free Lie algebra generated by  $x_i$  of dimension  $2ni - 1$  for  $1 \leq i < p^s$ .

To complete the proof we construct Lie algebras  $L_s^{(k)} \subset L_s^{(0)}$  which are compatible with the subalgebras  $L^{(k)}$  of  $L^{(0)}$ . First we examine the construction of  $L^{(k)}$ . (See section 2). Recall that we have short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L^{(1)} & \longrightarrow & L^{(0)} & \longrightarrow & \langle \tau_0 \rangle \longrightarrow 0 \\ 0 & \longrightarrow & L^{(k+1)} & \longrightarrow & L^{(k)} & \xrightarrow{\tau_{k+1}} & \langle \tau_k, \sigma_k \rangle \longrightarrow 0 \end{array}$$

where  $\pi_{k+1}$  is any map of Lie algebras such that

$$\begin{aligned} \pi_{k+1}(\sigma_k) &= \sigma_k \\ \pi_{k+1}(\tau_k) &= \tau_k \end{aligned}$$

where  $\sigma_k, \tau_k \in L^{(k)} \subset L^{(0)}$  are given by the formulas:

$$\begin{aligned} \tau_k &= x_{p^k} \\ \sigma_k &= \frac{1}{2p} \sum_{i=1}^{p^k-1} \binom{p^k}{i} [x_i, x_{p^k-i}]. \end{aligned}$$

Nothing is said about the value of  $\pi_{k+1}$  on the other generators. We need to be more specific at this point. Let us define the weight of an element in a free Lie algebra to be the minimal number of brackets in any term; in particular,  $L^{(0)}$  is free on generators  $x_i$ , we define  $\omega(x_i) = 1, \omega[x, y] = \omega(x) + \omega(y)$  and  $\omega(\sum a_i) = \min \omega(a_i)$ . For an element  $z \in L^{(k)}$  we define the weight of  $z$  to be the weight considered as an element of  $L^{(0)}$ . Thus  $\omega(\tau_k) = 1$  and  $\omega(\sigma_k) = 2$ . We further specify the Lie algebra homomorphism  $\pi_{k+1}$  by demanding that

$\pi_{k+1}(z) = 0$  if  $\omega(z) > 2$ . Now define  $L_s^{(k)} = L_s \cap L^{(k)}$  for  $k \leq s$ . Since  $\sigma_k, \tau_k \in L_s^{(k)}$ , we have short exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & L_s^{(1)} & \longrightarrow & L_s^{(0)} & \longrightarrow & \langle \tau_0 \rangle \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & L^{(1)} & \longrightarrow & L^{(0)} & \longrightarrow & \langle \tau_0 \rangle \longrightarrow 0 \\
\\
0 & \longrightarrow & L_s^{(k+1)} & \longrightarrow & L_s^{(k)} & \longrightarrow & \langle \sigma_k, \tau_k \rangle \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & L^{(k+1)} & \longrightarrow & L^{(k)} & \longrightarrow & \langle \sigma_k, \tau_k \rangle \longrightarrow 0
\end{array}$$

for  $k < s$ .

Define  $L_s^{(s+1)}$  to be the kernel of  $L_s^{(s)} \rightarrow \langle \sigma_s \rangle \rightarrow 0$ . In fact we have  $L_s^{(s+1)} \subset L^{(\infty)}$ . To see this we need to show that  $\pi_{r+1}(L_s^{(s+1)}) = 0$  for  $r \geq s$ . The generators of  $L_s^{(s+1)}$  of filtration 1 are of the form  $x_i$  for  $i < p^s$  and those of filtration 2 are of the form  $[x_i, x_j]$  for  $i, j < p^s$ . None of these have dimension  $2np^s - 1$ , so  $L_s^{(s+1)}$  lies in the kernel of the composition  $L^{(s+1)} \rightarrow \langle \tau_s, \sigma_s \rangle \rightarrow \langle \tau_s \rangle$ . Consequently  $L_s^{(s+1)} \subset L^{(s+1)}$ . Similarly for  $r > s$  the generators of  $L_s^{(s+1)}$  of weight 1 and 2 have dimensions  $\leq 4n(p^s - 1)$  and consequently their images are 0 in  $\langle \sigma_r, \tau_r \rangle$  for  $r > s$ .

It follows that we may first choose a basis for  $L_s^{(s+1)}$  and then choose a basis for  $L^{(\infty)}$  containing these elements. This completes the proof.

**Corollary 6.2** *There is a commutative diagram of fibration sequences:*

$$\begin{array}{ccccccc}
\Omega F_n & \longrightarrow & V_n & \xrightarrow{*} & P_n & \longrightarrow & F_n \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega F_{(p^{s+1}-1)} & \longrightarrow & K_{s+1} & \xrightarrow{*} & Q_{s+1} & \longrightarrow & F_{(p^{s+1}-1)} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega F_{(p^s-1)} & \longrightarrow & K_s & \xrightarrow{*} & Q_s & \longrightarrow & F_{(p^s-1)}
\end{array}$$

where all the vertide maps are mod  $p$  homology monomorphisms and each  $Q_s$  is a wedge of mod  $p^r$  Moore spaces.

## 7 On the sequence $F_{(p^s-1)} \rightarrow F_n \rightarrow F_{np^s}$

Recall that there is a  $p$ -local fibration sequence

$$S_{(p^s-1)}^{2n} \rightarrow S_\infty^{2n} \xrightarrow{H_{p^s}} S_\infty^{2np^s}.$$

By analogy, we compare  $F_{(p^s-1)}$  to the fiber  $\Delta_s$  of the Hopf invariant  $h^s$ :

$$\Delta_s \rightarrow F_n \xrightarrow{h^s} F_{np^s}$$

**Theorem 7.1**  $\Omega\Delta_s \simeq \Omega F_{(p^s-1)} \times \Omega X_s$  where  $X_s$  is a wedge of mod  $p^r$  Moore spaces.

*Proof:* For dimensional reasons, the inclusion  $F_{(p^s-1)} \subset F_n$  lifts to  $\Delta_s$ . Consider the pull back diagram:

$$\begin{array}{ccccccccc} \Omega Y_s & \longrightarrow & \Omega F_{(p^s-1)} & \longrightarrow & K_s & \longrightarrow & Y_s & \longrightarrow & F_{(p^s-1)} \\ \downarrow & & \downarrow & & \downarrow = & & \downarrow & & \downarrow \\ \Omega R_s & \longrightarrow & \Omega\Delta_s & \longrightarrow & K_s & \xrightarrow{*} & R_s & \longrightarrow & \Delta_s. \end{array}$$

The map  $K_s \rightarrow \Omega\Delta_s \rightarrow \Omega F_n$  constructed in section 6 is obtained from maps:

$$\begin{array}{ccc} \bar{\delta}_k : P^{2np^k} & \longrightarrow & F_n & & k < s \\ S^{2np^{s-1}} & \longrightarrow & P^{2np^s} & \xrightarrow{\bar{\delta}_s} & F_n \end{array}$$

and all of these maps factor through  $F_{(p^s-1)}$  for dimensional reasons. Consequently the retraction  $K_s \rightarrow \Omega\Delta_s$  factors through  $\Omega F_{(p^s-1)}$  as well and hence the map  $K_s \rightarrow Y_s$  is null homotopic. Consequently  $\Omega Q_s \simeq \Omega Y_s$  and the map  $Q_s \rightarrow F_{(p^s-1)}$  lifts to an equivalence  $Q_s \simeq Y_s$ . Now the inclusion  $Q_s \rightarrow P_n$  factors through  $R_s$ , so  $R_s = Q_s \vee E_s$  and we have a diagram of fibrations:

$$\begin{array}{ccccc} K_s & \longrightarrow & Q_s & \longrightarrow & F_{(p^s-1)} \\ = \downarrow & & \downarrow & & \downarrow \\ K_s & \longrightarrow & Q_s \vee E_s & \longrightarrow & \Delta_s. \end{array}$$

Now the fiber of  $Q_s \rightarrow Q_s \vee E_s$  is  $\Omega(E_s \rtimes \Omega Q_s)$  which is a retract of  $\Omega(Q_s \vee E_s) = \Omega R_s$  and hence a retract of  $\Omega\Delta_s$ . This completes the proof with  $X_s = E_s \rtimes \Omega Q_s$ .

## 8 Factorization of $\bar{\partial}$

In this section we will consider the components of

$$\bar{\partial}_n : \Omega^2 S^{2n+1} \longrightarrow S^{2n-1} \times V_n = S^{2n-1} \times \prod_{k \geq 1} S^{2np^k-1} \{p^{r+1}\}$$

we will write  $\partial_n(k) : \Omega^2 S^{2n+1} \longrightarrow S^{2np^k-1} \{p^{r+1}\}$  for the  $k$ th component,  $k > 0$  and  $\bar{\partial}_n(0)$  for the projection  $\Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ .

We begin by combing the diagram from 4.5:

$$\begin{array}{ccccc} \Omega^2 S^{2n+1} & \xrightarrow{\partial_n} & \Omega F_n & \longrightarrow & \bar{F}_n \\ \Omega \tilde{H}_{p^s} \downarrow & & \downarrow \Omega h^s & & \downarrow h' \\ \Omega^2 S^{2np^s+1} & \longrightarrow & \Omega F_{np^s} & \longrightarrow & \bar{F}_{np} \end{array}$$

with the equivalences from 5.2

$$\begin{array}{ccccc} S^{2n-1} \times V_n & \xrightarrow{\Theta_n} & \Omega F_n & \longrightarrow & \overline{F}_n \\ \downarrow \psi & & \downarrow \Omega h & & \downarrow h' \\ S^{2np-1} \times V_{np} & \xrightarrow{\Theta_{np}} & \Omega F_{np} & \longrightarrow & \overline{F}_{np} \end{array}$$

where  $h'$  is defined in 5.3, to get a homotopy commutative diagram:

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{\overline{\partial}_n} & S^{2n-1} \times V_n \\ \Omega \tilde{H}_p \downarrow & & \downarrow \psi \\ \Omega^2 S^{2np^s+1} & \xrightarrow{\overline{\partial}_{np^s}} & S^{2np^s-1} \times V_{np^s} \end{array}$$

From this we see that

$$\overline{\partial}_n(k+1) = \overline{\partial}_{np^k}(1) \circ \Omega H_{p^k} \circ \gamma_k, \quad k \geq 0$$

while

$$(8.1) \quad \begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{\overline{\partial}_n(1)} & S^{2np-1}\{p^{r+1}\} \\ \Omega \tilde{H}_p \downarrow & & \downarrow P \\ \Omega^2 S^{2np+1} & \xrightarrow{\overline{\partial}_{np}(0)} & S^{2np-1}. \end{array}$$

Putting these together gives:

$$(8.2) \quad \overline{\partial}_n = \overline{\partial}_n(0) + \sum_{k=0}^{\infty} \overline{\partial}_{np^k}(1) \circ \Omega H_{p^k} \circ \gamma_k$$

where  $\gamma_k : \Omega^2 S^{2n+1} \rightarrow \Omega^2 S^{2n+1}$  is an equivalence.

**Remark:** If we had better control over the relationship between the Hopf invariants  $h^s$  and the James-Hopf invariant, we could remove the factor  $\gamma_k$ .

**Proposition 8.3** *Suppose  $\overline{\partial}_n(1) : \Omega^2 S^{2n+1} \rightarrow S^{2np-1}\{p^{r+1}\}$  is null homotopic for each  $n$ . Then*

- a)  $\overline{\partial}_n$  factors through  $S^{2n-1}$
- b) The loops on the  $p$ th James-Hopf invariant:

$$\Omega H_p : \Omega^2 S^{2n+1} \longrightarrow \Omega^2 S^{2np+1}$$

is homotopic to a composition:

$$\Omega^2 S^{2n+1} \longrightarrow \Omega^2 P^{2np+1} \xrightarrow{\Omega^2 P} \Omega^2 S^{2np+1}$$

- c) The fiber  $B$  of  $\overline{\partial}_{np}(0) : \Omega^2 S^{2np+1} \rightarrow S^{2np-1}$  is a retract of  $\Omega^2 P^{2np+1}$
- d)  $B \simeq BW_n$ .

*Proof:* a) follows directly from 7.2. To verify b), note that by hypothesis

$$\Omega H_p : \Omega^2 S^{2n+1} \longrightarrow \Omega^2 S^{2np+1}$$

lifts to the fiber  $B$  of  $\bar{\partial}_{np}(0)$ . The following diagram provides a map from  $B$  to  $\Omega^2 P^{2np+1}$ :

$$\begin{array}{ccc} B & \longrightarrow & \Omega^2 P^{2np+1} \\ \downarrow & & \downarrow \\ \Omega^2 S^{2np+1} & \xrightarrow{1} & \Omega^2 S^{2np+1} \\ \downarrow \bar{\partial}_{np}(0) & & \downarrow \\ S^{2np-1} & \longrightarrow & \Omega F_{np} \end{array}$$

c) follows since  $\Omega F_n \simeq S^{2np-1} \times V_{np} \times \Omega P_{np}$ , so

$$\Omega^2 P^{2np+1} \simeq B \times \Omega V_{np} \times \Omega^2 P_{np}.$$

To verify d), note that  $S^2 BW_n$  is a retract of  $S^2 \Omega^2 S^{2n+1}$  by [G2], so there is a map  $S^2 BW_n \rightarrow P^{2np+1}$  which is non zero in homology, It's adjoint provides an equivalence:

$$BW_n \rightarrow \Omega^2 P^{2np+1} \rightarrow B.$$

## 9 Relationship with $BW_n$

In this section we describe a factorization of  $\bar{\partial}$  using the classifying space  $BW_n$  for the double suspension. Recall from [G2]

**Proposition 9.1** *Suppose  $\alpha : S^{2n} \rightarrow Y$  and  $\beta : S^{2n-1}Y \rightarrow S^{4n-1}$  are maps such that  $\beta \circ S^{2n-1}\alpha : S^{4n-1} \rightarrow S^{2n-1}Y \rightarrow S^{4n-1}$  is homotopic to the identity. Then there is a fibration sequence:*

$$S^{2n-1} \xrightarrow{\bar{\alpha}} \Omega Y \longrightarrow B.$$

*This is natural in the following sense: Suppose we are also given*

$$\begin{aligned} \alpha' : S^{2n} &\longrightarrow \mathbb{Z} \\ \beta' : S^{2n-1}Z &\longrightarrow S^{4n-1} \\ f : Y &\longrightarrow Z \end{aligned}$$

*such that  $\beta \sim \beta' \circ S^{2n-1}f$  and  $\alpha' \sim f\alpha$ .*

*Then we have a homotopy commutative diagram;*

$$\begin{array}{ccccc} S^{2n-1} & \xrightarrow{\bar{\alpha}} & \Omega Y & \longrightarrow & B \\ 1 \downarrow & & \downarrow \Omega f & & \downarrow \psi \\ S^{2n-1} & \xrightarrow{\bar{\alpha}'} & \Omega Z & \longrightarrow & B' \end{array}$$

**Corollary 9.2** *There is a homotopy commutative diagram:*

$$\begin{array}{ccccc}
 S^{2n-1} & \longrightarrow & \Omega^2 S^{2n+1} & \longrightarrow & BW_n \\
 \downarrow & & \downarrow & & \downarrow \\
 S^{2n-1} & \longrightarrow & \Omega F_n & \longrightarrow & B
 \end{array}$$

Since  $S^{2n-1}$  is a retract of  $\Omega F_n$ ,  $B \simeq V_n \times \Omega W_n$  and we have

**Corollary 9.3** *There is a homotopy commutative diagram:*

$$\begin{array}{ccc}
 \Omega^2 S^{2n+1} & \longrightarrow & BW_n \\
 \downarrow \partial & & \downarrow \\
 \Omega F_n & \longrightarrow & \overline{F}_n.
 \end{array}$$

In particular,  $\overline{\partial}_n(k) : \Omega^2 S^{2n+1} \rightarrow S^{2np^k-1}\{p^{r+1}\}$  factors through  $BW_n$ , proving Theorem 1.2.

## A Appendix

The following observation appears in the work of Cohen, Moore and Neisenderfer. We repeat it here because it sheds light on their results and is easily extended to the case of the filtrations in section 6.

We begin by looking at the homology Serre spectral sequence for the fibering

$$\Omega F_n \longrightarrow \Omega P^{2n+1} \xrightarrow{\Omega\pi} \Omega S^{2n+1}.$$

This is a multiplicative spectral sequence with  $E_{p,q}^2 \cong H_p(\Omega S^{2n+1}; \mathbb{Z}/p) \otimes H_q(\Omega F_n; \mathbb{Z}/p)$ . Since all elements of  $E_{p,0}^2$  are permanent cycles,  $E^2 = E^\infty$ . Restricting this fibration to  $S_{(k)}^{2n}$  leads to fibration

$$\Omega F_n \longrightarrow E_k \longrightarrow S_{(k)}^{2n}$$

and  $H_*(E_k; \mathbb{Z}/p) \subset H_*(E_{k+1}; \mathbb{Z}/p) \subset \cdots \subset H_*(\Omega P^{2n+1}; \mathbb{Z}/p)$ .

Thus  $H_*(\Omega P^{2n+1}; \mathbb{Z}/p)$  is a filtered differential group using  $\beta^{(r)}$ . It has a spectral sequence:

$$E_{p,q}^2 = H_p(\Omega S^{2n+1}; \beta^{(r)}) \otimes H_q(H_*(\Omega F_n; \mathbb{Z}/p), \beta^{(r)})$$

converging to  $H_{p+q}(\Omega P^{2n+1}; \mathbb{Z}/p) = 0$  if  $p + q > 0$ . This gives a multiplicative spectral sequence:

$$E_{p,q}^2 = \mathbb{Z}_p[u_{2n}] \otimes H_q(H_*(\Omega F_n; \mathbb{Z}/p); \beta^{(r)}) \Rightarrow 0.$$

This spectral sequence has the same form as the Serre Spectral sequence for the homology of the fibration:

$$\Omega^2 S^{2n+1} \longrightarrow P\Omega S^{2n+1} \longrightarrow \Omega S^{2n+1}.$$

The differentials are forced by the multiplicative structure and consequently these spectral sequences are isomorphic. From this we conclude

**Proposition A.1** ([CMN1])  $H_*(H_*(\Omega F_n; \mathbb{Z}/p); \beta^{(r)}) \cong H^*(\Omega^2 S^{2n+1}; \mathbb{Z}/p)$  as algebras and co-algebras.

**Corollary A.2**  $H_*(H_*(\Omega F_{(p^s-1)}; \mathbb{Z}/p); \beta^{(r)}) \cong H_*(\Omega S_{(p^s-1)}^{2n}; \mathbb{Z}/p)$ .

The later follows by restricting the spectral sequence to the first  $p^s - 1$  columns.

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