RATIONAL S^1-EQUIVARIANT ELLIPTIC COHOMOLOGY.

J.P.C.GREENLEES, M.J.HOPKINS, AND I.ROSU

Abstract. We give a functorial construction of a rational S^1-equivariant cohomology theory from an elliptic curve equipped with suitable coordinate data. The elliptic curve may be recovered from the cohomology theory; indeed, the value of the cohomology theory on the compactification of an S^1-representation is given by the sheaf cohomology of a suitable line bundle on the curve. The construction is easy: by considering functions on the elliptic curve with specified poles one may write down the representing S^1-spectrum in the first author's algebraic model of rational S^1-spectra [6].

Contents

1. Introduction.

Two of the most important cohomology theories are associated to one dimensional group schemes in a way which is clearest in the equivariant context. Ordinary cohomology of the Borel construction is associated to the additive group and equivariant K theory is associated to the multiplicative group. It is therefore natural to hope for an equivariant cohomology theory associated to an elliptic curve A, and it is the purpose of the present note to construct such a theory over the rationals which is equivariant for the circle group. A programme to extend this work to higher dimensional abelian varieties and higher dimensional tori is underway [7, 8, 9].

Let $\mathbb{T}$ denote the circle group, and z denote its natural representation on the complex numbers. The main purpose of this paper is to construct a rational $\mathbb{T}$-equivariant cohomology

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theory $EA^*_{\mathbb{T}}(\cdot)$ associated to any elliptic curve A over a $\mathbb{Q}$-algebra. We write $A[n]$ for the points of order dividing n in A. The properties of the cohomology theory when we work over a field may be summarized as follows; we give full details in Section 8 below.

Theorem 1.1. *For any elliptic curve A over a field k of characteristic 0, there is a 2-periodic multiplicative rational $\mathbb{T}$-equivariant cohomology theory $EA^*_{\mathbb{T}}(\cdot)$. The coefficient ring in degrees 0 and 1 is related to cohomology of the structure sheaf $\mathcal{O}$ by*

$$EA^*_{\mathbb{T}} \cong H^*(A; \mathcal{O}),$$

and, more generally, the value on the one point compactification S^W of the representation $W = \sum_n a_n z^n$ gives the sheaf cohomology of an associated line bundle $\mathcal{O}(-D(W))$, where $D(W) = \sum_n a_n A[n]$:

$$\widetilde{EA}^*_{\mathbb{T}}(S^W) \cong H^*(A; \mathcal{O}(-D(W))$$

and in homology we have

$$\widetilde{EA}^{\mathbb{T}}_*(S^W) \cong H^*(A; \mathcal{O}(D(W)).$$

The construction and the isomorphisms in the statement are natural: for this it is necessary to specify suitable coordinate data on the elliptic curve.

The first version of $\mathbb{T}$-equivariant elliptic cohomology was constructed by Grojnowksi in 1994 [10]. He was interested in implications for the representation theory of certain elliptic algebras: these implications are the subject of the work of Ginzburg-Kapranov-Vasserot [5] and the context is explained further in [4]. For this purpose it was sufficient to construct a theory on finite complexes taking values in sheaves over the elliptic curve. Later Rosu [13] used this sheaf-valued theory to give a proof of Witten's rigidity theorem for the equivariant elliptic genus of a spin manifold with non-trivial $\mathbb{T}$-action. Ando [1] has related the sheaf valued theory to the representation theory of loop groups.

However, to exploit the theory fully, it is essential to have a theory defined on general $\mathbb{T}$-spaces and $\mathbb{T}$-spectra, and to have a conventional group-valued theory represented by a $\mathbb{T}$-spectrum. This allows one to use the full apparatus of equivariant stable homotopy theory. For example, twisted pushforward maps are immediate consequences of Atiyah duality; in more concrete terms, it allows one to calculate the theory on free loop spaces, and to describe algebras of operations. It is also likely to be useful in constructing an integral version of the theory, and we hope it may also prove useful in the continuing search for a geometric definition of elliptic cohomology.

The theory we construct has these desirable properties, whilst retaining a very close connection with the geometry of the underlying elliptic curve. Our construction directly models the representing spectrum EA in the first author's algebraic model $\mathcal{A}_s$ of rational $\mathbb{T}$-spectra [6]. Any object (such as that modelling $\mathbb{T}$-equivariant elliptic cohomology) in the algebraic model $\mathcal{A}_s$ of [6] can be viewed as a sheaf over the space of closed subgroups of $\mathbb{T}$ [7]. Moreover, the way a sheaf over the closed subgroups of $\mathbb{T}$ models a $\mathbb{T}$-equivariant cohomology theory gives a precise means by which the sheaf-valued cohomology can be recovered from a conventional theory with values in graded vector spaces. The construction of the Grojnowski sheaf on the elliptic curve from the sheaf on the space of closed subgroups of $\mathbb{T}$ helps put the earlier construction in a topological context. It is intended to give a full treatment elsewhere, giving an equivalence between a category of modules over the structure sheaf of A and a category of modules over the representing spectrum for the cohomology theory.

Returning to the geometry, a very appealing feature is that although our theory is group valued, the original curve can still be recovered from the cohomology theory. It is also notable that the earlier sheaf theoretic constructions work over larger rings and certainly require the coefficients to contain roots of unity: the loss of information can be illustrated by comparing the rationalized representation ring $R(C_n) = \mathbb{Q}[z]/(z^n - 1)$ (with components corresponding to *subgroups* of C_n) to the complexified representation ring, isomorphic to the character ring $\mathrm{map}(C_n, \mathbb{C})$ (with components corresponding to the *elements* of C_n).

Finally, the ingredients of the model are very natural invariants of the curve given by sheaves of functions with specified poles at points of finite order: Definition 8.4 simply writes down the representing object in terms of these,[1] and readers already familiar with elliptic curves and the model of [6] need read nothing else. In fact the algebraic model of [6] gives a generic de Rham model for all $\mathbb{T}$-equivariant theories, and the models of elliptic cohomology theories highlight this geometric structure. These higher de Rham models should allow applications in the same spirit as those made for de Rham models of ordinary cohomology and K-theory [11].

By way of motivation, we will discuss the way that a $\mathbb{T}$-equivariant cohomology theory is associated to several other geometric objects. Perhaps most familiar is the complete case discussed in Section 2, where the Borel theory for a complex oriented cohomology theory is associated to a formal group. Amongst global groups, the additive and multiplicative ones are the simplest, and in Section 4 we describe how they give rise to ordinary Borel cohomology and equivariant K-theory. This construction is notable in that it gives a construction of equivariant cohomology theories from oriented 1-dimensional group schemes which is *functorial for isomorphisms*. It is also functorial for certain isogenies as explained in 4.3.

2. Formal groups from complex oriented theories.

The purpose of this section is to recall that any complex orientable cohomology theory $E^*(\cdot)$ determines a one dimensional, commutative formal group $\widehat{\mathbb{G}}$ and to explain how the cohomology of various spaces can be described in terms of the geometry of $\widehat{\mathbb{G}}$. This is well known but it introduces the geometric language, and motivates our main construction, which uses this geometric data to construct the cohomology theory. Indeed, we will show that the machinery of [6] permits a functorial construction of a 2-periodic rational $\mathbb{T}$-equivariant cohomology theory $E\mathbb{G}^*_{\mathbb{T}}(\cdot)$ from a one dimensional group scheme $\mathbb{G}$ over a $\mathbb{Q}$-algebra. Furthermore, the construction is reversible in the sense that $\mathbb{G}$ can be recovered from $E\mathbb{G}^*_{\mathbb{T}}(\cdot)$. The most interesting case of this is when $\mathbb{G}$ is an elliptic curve.

Before introducing the cohomology theory into the picture, we introduce the geometric language. Whilst all schemes are affine, the geometric language is equivalent to the ring theoretic language, and all geometric statements can be given meaning by translating them to algebraic ones. This excuses us from setting up the geometric foundations of formal groups, and for the present the geometric language is purely suggestive: all notions are defined in terms of the algebra. The geometric language becomes essential later, since elliptic curves are not affine.

A one dimensional commutative formal group law over a ring k is a commutative and associative coproduct on the complete topological k-algebra $k[[y]]$. Equivalently, it is a complete topological Hopf k-algebra $\mathcal{O}$ together with an element $y \in \mathcal{O}$ so that $\mathcal{O} = k[[y]]$.

[1] This 3rd version of the paper is the first to make the model completely explicit.

A topological Hopf k-algebra $\mathcal{O}$ for which such a y exists is the ring of functions on a one dimensional commutative formal group $\widehat{\mathbb{G}}$. The counit $\mathcal{O} \longrightarrow k$, is viewed as evaluation of functions at the identity $e \in \widehat{\mathbb{G}}$, and the augmentation ideal I consists of functions vanishing at e. The element y generates the ideal I, and is known as a *coordinate* (at e).

We also need to discuss locally free sheaves $\mathcal{F}$ over $\widehat{\mathbb{G}}$, and in the present affine context these are specified by the $\mathcal{O}$-module $M = \Gamma\mathcal{F}$ of global sections. In particular, line bundles L over $\widehat{\mathbb{G}}$ correspond to modules M which are submodules of the ring of rational functions and free of rank 1. Line bundles can also be described in terms of the zeros and poles of their generating section: we only need this in special cases made explicit below. The generator f of the $\mathcal{O}$-module M is a section of L, and as such it defines a divisor $D = D_+ - D_-$, where D_+ is the subscheme of $\widehat{\mathbb{G}}$ where f vanishes (with multiplicities), and D_- is the subscheme of $\widehat{\mathbb{G}}$ where f has poles (with multiplicites). This divisor determines L, and we write $L = \mathcal{O}(-D)$. For example, $M = I = (y)$ corresponds to $\mathcal{O}(-e)$, and $M = I^a = (y^a)$ corresponds to $\mathcal{O}(-ae)$. Next we may consider the $[n]$-series map $[n] : \mathcal{O} \longrightarrow \mathcal{O}$, which corresponds to the n-fold sum map $n : \widehat{\mathbb{G}} \longrightarrow \widehat{\mathbb{G}}$. We write $\widehat{\mathbb{G}}[n]$ for the kernel of n, and its ring of functions is $\mathcal{O}/([n](y))$. Hence, since $n^*y = [n](y)$ by definition, $M = ([n](y))$ corresponds to $\mathcal{O}(-\widehat{\mathbb{G}}[n])$, and $M = (\ ([n](y))^a \ )$ corresponds to $\mathcal{O}(-a\widehat{\mathbb{G}}[n])$. Finally, if M corresponds to $\mathcal{O}(-D)$ and M' corresponds to $\mathcal{O}(-D')$ then $M^\vee := \mathrm{Hom}(M, \mathcal{O})$ corresponds to $\mathcal{O}(D)$ and $M \otimes M'$ corresponds to $\mathcal{O}(-D-D')$. This gives sense to enough line bundles for our purposes.

Now suppose that E is a 2-periodic ring valued theory with coefficients E^* concentrated in even degrees. The collapse of the Atiyah-Hirzebruch spectral sequence for $\mathbb{C}P^\infty$ shows that E is complex orientable. We may define the $\mathbb{T}$-equivariant Borel cohomology by $E^*_{\mathbb{T}}(X) = E^*(E\mathbb{T} \times_{\mathbb{T}} X)$. We work over the ring $k = E^0_{\mathbb{T}}(\mathbb{T}) = E^0$, and view $E^0_{\mathbb{T}} = E^0(\mathbb{C}P^\infty)$ as the ring of functions on a formal group $\widehat{\mathbb{G}}$ over k. The tensor product and duality of line bundles makes $\mathbb{C}P^\infty$ into a group object, so $E^0(\mathbb{C}P^\infty)$ is a Hopf algebra and $\widehat{\mathbb{G}}$ is a group. From this point of view, the augmentation ideal $I = \ker(E^0_{\mathbb{T}} \longrightarrow E^0)$ consists of functions vanishing at the identity $e \in \mathbb{G}$.

Now, if V is a complex representation of the circle group $\mathbb{T}$, we also let V denote the associated bundle over $\mathbb{C}P^\infty$ and the Thom isomorphism shows $\tilde{E}^0((\mathbb{C}P^\infty)^V) = \tilde{E}^0_{\mathbb{T}}(S^V)$ is a rank 1 free module over $E^0_{\mathbb{T}}$, and hence corresponds to a line bundle $\mathbb{L}(V)$ over $\widehat{\mathbb{G}}$, whose global sections are naturally isomorphic to the module

$$\Gamma\mathbb{L}(V) = \tilde{E}^0_{\mathbb{T}}(S^V).$$

From the fact that Thom isomorphisms are transitive we see that $\mathbb{L}(V \oplus W) = \mathbb{L}(V) \otimes \mathbb{L}(W)$. The values of all these line bundles can be deduced from those of powers of z.

Lemma 2.1. (1) $\mathbb{L}(0) = \mathcal{O}$ *is the trivial bundle.*

(2) $\mathbb{L}(z) = \mathcal{O}(-e)$ *is the sheaf of functions vanishing at e, and its module of sections I is generated by the coordinate y.*

(3) $\mathbb{L}(z^n) = \mathcal{O}(-\widehat{\mathbb{G}}[n])$ *is the sheaf of functions vanishing on $\widehat{\mathbb{G}}[n]$, and its module of sections is generated by the multiple $[n](y)$ of the coordinate y.*

(4) $\mathbb{L}(az^n) = \mathcal{O}(-a\widehat{\mathbb{G}}[n])$ *is the sheaf of functions vanishing on $\widehat{\mathbb{G}}[n]$ with multiplicity a, and its module of sections is generated $([n](y))^a$.*

Proof: The first statement is clear since $\tilde{E}^0_{\mathbb{T}}(S^0) = E^0_{\mathbb{T}}$. For the second we use the equivalence $(\mathbb{C}P^\infty)^z \simeq (\mathbb{C}P^\infty)^0/(\mathbb{C}P^0)^0$. The third statement follows from the Gysin sequence since z^k

is the pullback of z along the kth power map $\mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$. The final statement follows from the tensor product property. $\square$

This gives the fundamental connection between the equivariant cohomology of a sphere and sections of a line bundle.

Corollary 2.2. *For any $a \in \mathbb{Z}$, $n \neq 0$ we have*

$$\tilde{E}^0_{\mathbb{T}}(S^{az^n}) = \mathcal{O}(-a\widehat{\mathbb{G}}[n]). \quad \square$$

We want to finish this section by pointing out that if the formal group $\widehat{\mathbb{G}}$ (which is affine) is replaced by a group $\mathbb{G}$ with higher cohomology, we cannot expect a cohomology theory entirely in even degrees. Whenever the group is not affine, we write $\mathcal{O}$ for the structure sheaf of $\mathbb{G}$. This is reconciled by the above usage since in the affine case the structure sheaf is determined by its ring of global sections. In the non-affine case, the cofibre sequence

$$S^{az} \wedge \mathbb{T}_+ \longrightarrow S^{az} \longrightarrow S^{(a+1)z}$$

forces there to be odd cohomology. Indeed, there is a corresponding short exact sequence of sheaves

$$\mathcal{O}(-ae)/\mathcal{O}(-(a+1)e) \longleftarrow \mathcal{O}(-ae) \longleftarrow \mathcal{O}(-(a+1)e).$$

Any satisfactory cohomology theory will be functorial, and applying $\tilde{E}^0_{\mathbb{T}}(\cdot)$ will give sections of the associated sheaves. However the global sections functor on sheaves is not usually right exact, and the sequence of sections continues with the sheaf cohomology groups $H^1(\mathbb{G};\cdot)$. It is natural to hope that the long exact cohomology sequence induced by the sequence of spaces should be the long exact cohomology sequence induced by the sequence of sheaves. This gives a natural candidate for the odd cohomology:

$$\tilde{E}^i_{\mathbb{T}}(S^{az}) = H^i(\mathbb{G}; \mathcal{O}(-ae)) \text{ for } i = 0, 1.$$

This explains why it is possible for complex orientable cohomology theories to have coefficient rings in even degrees (formal groups are affine), and indeed how their values on all complex spheres can be the same. It also explains why we cannot expect either property for a theory associated to an elliptic curve.

3. The model for rational $\mathbb{T}$-spectra.

For most of the paper we work with the representing objects of these cohomology theories, namely $\mathbb{T}$-spectra [3]. Thus we prove results about the representing spectra, and deduce consequences about the cohomology theories. More precisely, any suitable $\mathbb{T}$-equivariant cohomology theory $E^*_{\mathbb{T}}(\cdot)$ is represented by a $\mathbb{T}$-spectrum E in the sense that

$$\tilde{E}^*_{\mathbb{T}}(X) = [X, E]^*_{\mathbb{T}}.$$

This enables us to define the associated homology theory

$$\widetilde{E}^{\mathbb{T}}_*(X) = [S^0, E \wedge X]^{\mathbb{T}}_*$$

in the usual way. We shall make use of the elementary fact that the Spanier-Whitehead dual of the sphere S^V is S^{-V}, as one sees by embedding S^V as the equator of $S^{V\oplus 1}$. Hence, for example

$$\tilde{E}^0_{\mathbb{T}}(S^V) = [S^V, E]^{\mathbb{T}} = [S^0, S^{-V} \wedge E]^{\mathbb{T}} = \pi^{\mathbb{T}}_0(S^{-V} \wedge E) = \tilde{E}^{\mathbb{T}}_0(S^{-V}).$$

We say that a cohomology theory is *rational* if its values are graded rational vector spaces. A spectrum is rational if the cohomology theory it represents is rational. It suffices to check the values on the homogeneous spaces $\mathbb{T}/H$ for closed subgroups $\mathbb{T}$, since all spaces are built from these.

Convention 3.1. Henceforth all spaces, groups and spectra are rationalized whether or not this is indicated in the notation.

Our results are made possible because there is a complete algebraic model of the category of *rational* $\mathbb{T}$-spectra, and hence of rational $\mathbb{T}$-equivariant cohomology theories [6]. There are two models for rational $\mathbb{T}$-spectra, as derived categories of abelian categories:

$$\mathbb{T}\text{-Spectra} \simeq D(\mathcal{A}_s) \simeq D(\mathcal{A}_t).$$

The *standard* abelian category $\mathcal{A}_s$ has injective dimension 1, and the *torsion* abelian category $\mathcal{A}_t$ is of injective dimension 2. It is usually easiest to identify the model for a $\mathbb{T}$-spectrum in $D(\mathcal{A}_t)$, at least providing its model has homology of injective dimension 1. This is then transported to the standard category, where calculations are sometimes easier. To describe the categories, we need to use the discrete set $\mathcal{F}$ of finite subgroups of $\mathbb{T}$. On this we consider the constant sheaf $\mathcal{R}$ of rings with stalks $\mathbb{Q}[c]$ where c has degree -2. We need to consider the ring $R = \mathrm{map}(\mathcal{F}, \mathbb{Q}[c])$ of global sections. For each subgroup H, we let $e_H \in R$ denote the idempotent with support H. If $w : \mathcal{F} \longrightarrow \mathbb{Z}_{\geq 0}$ is a function, we write c^w for the element of R with $c^w(H) = c^{w(H)}$. Now consider the multiplicative set $\mathcal{E}$ generated by the universal Euler classes $e(V)$ for the representations V of $\mathbb{T}$ with $V^{\mathbb{T}} = 0$. These are defined by $e(V) = c^v$, where $v(H) = \dim_{\mathbb{C}}(V^H)$. In particular for $V = z^n$ we have $e(z^n) = c^{\mathrm{sub}(n)}$ where $\mathrm{sub}(n)(H) = 1$ if $H \subseteq \mathbb{T}[n]$ and 0 otherwise. Equivalently,

$$\mathcal{E} = \{c^w \mid w : \mathcal{F} \longrightarrow \mathbb{Z}_{\geq 0} \text{ of finite support}\}.$$

We let $t_*^{\mathcal{F}} = \mathcal{E}^{-1}R$: as a graded vector space this is $\bigoplus_H \mathbb{Q}$ in positive degrees and $\prod_H \mathbb{Q}$ in degrees zero and below.

The objects of the standard model $\mathcal{A}_s$ are triples (N, β, V) where N is an R-module (called the *nub*), V is a graded rational vector space (called the *vertex*) and $\beta : N \longrightarrow t_*^{\mathcal{F}} \otimes V$ is a morphism of R-modules (called the *basing map*) which becomes an isomorphism when $\mathcal{E}$ is inverted. When no confusion is possible we simply say that $N \longrightarrow t_*^{\mathcal{F}} \otimes V$ is an object of the standard abelian category. An object of $\mathcal{A}_s$ should be viewed as the module N with the additional structure of a trivialization of $\mathcal{E}^{-1}N$. A morphism $(N, \beta, V) \longrightarrow (N', \beta', V')$ of objects is given by an R-map $\theta : N \longrightarrow N'$ and a $\mathbb{Q}$-map $\phi : V \longrightarrow V'$ compatible under the basing maps.

Since the standard abelian category has injective dimension 1, homotopy types of objects of the derived category $D(\mathcal{A}_s)$ are classified by their homology in $\mathcal{A}_s$, so that homotopy types correspond to isomorphism classes of objects of the abelian category $\mathcal{A}_s$. In the sheaf theoretic approach, N is the space of global sections of a sheaf on the space of closed subgroups $\mathbb{T}$, the vertex V is the value of the sheaf at the subgroup $\mathbb{T}$ and the fact that the basing map $\beta : N \longrightarrow t_*^{\mathcal{F}} \otimes V$ is an isomorphism away from $\mathcal{E}$ is the manifestation of the patching condition for sheaves.

The objects of the torsion abelian category $\mathcal{A}_t$ are triples (V, q, T) where V is a graded rational vector space T is an $\mathcal{E}$-torsion R-module and $q : t_*^{\mathcal{F}} \otimes V \longrightarrow T$ is a morphism of R-modules. The condition on T is equivalent to requiring (i) that T is the sum of its idempotent factors $T(H) = e_H T$ in the sense that $T = \bigoplus_H T(H)$ and (ii) that each $T(H)$ is a torsion

$\mathbb{Q}[c]$-module. When no confusion is possible we simply say that $t_*^{\mathcal{F}} \otimes V \longrightarrow T$ is an object of the torsion abelian category. In the sheaf theoretic approach, the module $T(H)$ is the cohomology of the structure sheaf with support at H. By contrast with the standard abelian category, the torsion abelian category has injective dimension 2. Thus not every object X of the derived category $D(\mathcal{A}_t)$ is determined up to equivalence by its homology $H_*(X)$ in the abelian category $\mathcal{A}_t$. We say that X is (intrinsically) *formal* if it is determined up to isomorphism by its homology. Evidently, X is formal if its homology has injective dimension 0 or 1 in $\mathcal{A}_t$. In general, if $H_*(X) = (t_*^{\mathcal{F}} \otimes V \longrightarrow T)$, the object X is equivalent to the fibre of a map $(t_*^{\mathcal{F}} \otimes V \longrightarrow 0) \longrightarrow (t_*^{\mathcal{F}} \otimes 0 \longrightarrow \Sigma T)$ (in the derived category) between objects in $\mathcal{A}_t$ of injective dimension 1. This map is classified by an element of $\mathrm{Ext}(t_*^{\mathcal{F}} \otimes V, \Sigma T)$, so that X is formal if the Ext group is zero in even degrees. Thus X is formal if both V and T are in even degrees or if T is injective in the sense that each $T(H)$ is an injective $\mathbb{Q}[c]$-module.

Definition 3.2. [6, 5.8.2] Suppose given a function $w : \mathcal{F} \longrightarrow \mathbb{Z}$ with finite support. The algebraic w-sphere is the object of $\mathcal{A}_s$ defined by

$$S^w = (R(c^{-w}) \longrightarrow t_*^{\mathcal{F}})$$

where $R(c^{-w})$ is the R-submodule of $t_*^{\mathcal{F}}$ generated by the Euler class c^{-w}.

Now for an object X of $\mathcal{A}_s$ there is an exact sequence

$$0 \longrightarrow \mathrm{Ext}_{\mathcal{A}_s}(S^{1+w}, M) \longrightarrow [S^w, M] \longrightarrow \mathrm{Hom}_{\mathcal{A}_s}(S^w, M) \longrightarrow 0,$$

so we shall need to calculate these Hom and Ext groups. For the present we restrict ourselves to the Hom groups.

Lemma 3.3. *For an object $M = (N \stackrel{\beta}{\longrightarrow} t_*^{\mathcal{F}} \otimes V)$ of the abelian category $\mathcal{A}_s$ we have*

$$\mathrm{Hom}_{\mathcal{A}_s}(S^w, (N \longrightarrow t_*^{\mathcal{F}} \otimes V)) = N(c^{-w}) := \{n \in N \mid \beta(n) \in c^{-w} \otimes V\}. \quad \square$$

We may now describe how to construct the counterparts $M_s(E) = (N \longrightarrow t_*^{\mathcal{F}} \otimes V)$ (in the standard abelian category $\mathcal{A}_s$) and $M_t(E) = (t_*^{\mathcal{F}} \otimes V \longrightarrow T)$ (in the torsion abelian category $\mathcal{A}_t$) of a rational $\mathbb{T}$-spectrum E. From the above discussion, the model $M_s(E)$ determines E itself, but $M_t(E)$ only determines E if $M_t(E)$ is formal. First, we may define the vertex V, the nub N and torsion module T by formulae and then turn to practical computations in terms of data easily accessible to us. To describe the answer, we need the universal $\mathcal{F}$-space $E\mathcal{F}$, and the basic cofibre sequence

$$E\mathcal{F}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{F}$$

where $\tilde{E}\mathcal{F}$ is the join $S^0 * E\mathcal{F}_+$. We also use functional duality on $\mathbb{T}$-spectra defined by $DX = F(X, S^0)$. The nub vertex and torsion modules associated to a $\mathbb{T}$-spectrum E are given by

- $N = \pi_*^{\mathbb{T}}(E \wedge DE\mathcal{F}_+)$
- $V = \pi_*^{\mathbb{T}}(E \wedge \tilde{E}\mathcal{F})$
- $T = \pi_*^{\mathbb{T}}(E \wedge \Sigma E\mathcal{F}_+)$

The vertex is straightforward to calculate in terms of available data:

$$V = \pi_*^{\mathbb{T}}(E \wedge \tilde{E}\mathcal{F}) = \lim_{\substack{\rightarrow \\ V^{\mathbb{T}}=0}} \pi_*^{\mathbb{T}}(E \wedge S^V).$$

An approach to the nub via limits is possible but not very illuminating.

The associated torsion sheaf T may be described by saying that its sections over the set $[\subseteq H]$ of subgroups of H is $\pi_*^{\mathbb{T}}(\Sigma E[\subseteq H]_+ \wedge E)$. Using idempotents from the Burnside ring of H this may be split up into stalks $\pi_*^{\mathbb{T}}(\Sigma E\langle K\rangle \wedge E)$ one for each subgroup $K \subseteq H$ (it turns out that these are in independent of H, as is required for consistency). Now if H has order n, the infinite sphere $S(\infty z^n)$ is a model for $E[\subseteq H]$, and hence there is a long exact sequence

$$\cdots \longrightarrow \pi_*^{\mathbb{T}}(E) \longrightarrow \pi_*^{\mathbb{T}}(S^{\infty z^n} \wedge E) \longrightarrow \pi_*^{\mathbb{T}}(E[\subseteq H]_+ \wedge E) \longrightarrow \cdots .$$

Since $\pi_*^{\mathbb{T}}(S^{\infty z^n} \wedge E) = \lim_{\substack{\to \\ a}} \pi_*^{\mathbb{T}}(S^{az^n} \wedge E)$ we may conclude there is a short exact sequence

$$0 \longrightarrow \Sigma E_{\mathbb{T}}^{*V(H)}/e(z^n)^\infty \longrightarrow \pi_*^{\mathbb{T}}(E[\subseteq H]_+ \wedge E) \longrightarrow e(z^n)\text{-power torsion}(E_{\mathbb{T}}^{*V(H)}) \longrightarrow 0$$

where $E_{\mathbb{T}}^{*V(H)}$ is the ring graded by multiples of z^n with az^n-th component $\pi_0^{\mathbb{T}}(S^{az^n} \wedge E)$ and $e(z^n)$ is the degree $-z^n$ Euler class.

In this account we have described the calculation of V and T in terms of available data. If this is to determine E we must show in addition that $M_t(E)$ is formal. In our case this will hold because V and T are in even degrees. It is convenient for calculation to deduce $M_s(E)$.

Lemma 3.4. *If $M_t(E) = (t_*^{\mathcal{F}} \otimes V \xrightarrow{q} T)$ has surjective structure map, then $M_t(E)$ is formal and*

$$M_s(E) = (N \longrightarrow t_*^{\mathcal{F}} \otimes V)$$

where

$$N = \ker(t_*^{\mathcal{F}} \otimes V \longrightarrow T),$$

and the basing map is the inclusion. Furthermore we have the explicit injective resolution

$$0 \longrightarrow M_s(E) = \begin{pmatrix} N \\ \downarrow \\ t_*^{\mathcal{F}} \otimes V \end{pmatrix} \longrightarrow \begin{pmatrix} t_*^{\mathcal{F}} \otimes V \\ \downarrow \\ t_*^{\mathcal{F}} \otimes V \end{pmatrix} \longrightarrow \begin{pmatrix} T \\ \downarrow \\ 0 \end{pmatrix} \longrightarrow 0$$

in $\mathcal{A}_s$.

Proof: To see that $M_t(E)$ is formal, it is only necessary to remark that T is the quotient of an $\mathcal{E}$-divisible group and therefore injective [6, 5.3.1]. □

Finally, we should record that spheres and suspensions in the algebraic and topological contexts correspond.

Lemma 3.5. [6, 5.8.3] *Suppose W is a virtual representation with $W^{\mathbb{T}} = 0$ and let $w = \dim_{\mathbb{C}}(W)$. The object modelling the sphere S^W with $V^{\mathbb{T}} = 0$ in $\mathcal{A}_s$ is the algebraic sphere S^w:*

$$M_s(S^W) = S^w = (R(c^{-w}) \longrightarrow t_*^{\mathcal{F}})$$

where $R(c^{-w})$ is the R-submodule of $t_*^{\mathcal{F}}$ generated by $c^{-w} = e(-W)$.

Convention 3.6. In the present paper we are interested in cohomology theories with a periodicity element u of degree 2. We may therefore shift even degree elements into degree zero. For example uc is the degree 0 counterpart of c. For the rest of the paper we use c to denote the degree 0 version.

4. The affine case: $\mathbb{T}$-equivariant cohomology theories from additive and multiplicative groups.

The algebraic models of equivariant K-theory and Borel cohomology are easily described [6]. In this section we show they are special cases of a general functorial construction of a cohomology theory $E\mathbb{G}^*_{\mathbb{T}}(\cdot)$ associated to a one dimensional affine group scheme $\mathbb{G}$. This will serve to illustrate the algebraic categories described in Section 3 and also complete the motivation of our construction for elliptic curves.

The additive group scheme $\mathbb{G}_a$ and the multiplicative group scheme $\mathbb{G}_m$ are affine, and therefore the construction of associated cohomology theories is considerably simpler than that for elliptic curves. Nonetheless the general features are the same, and it is useful to have seen the phenomena first in a familiar setting. It turns out that the associated 2-periodic $\mathbb{T}$-equivariant theories are concentrated in even degrees and

$$(E\mathbb{G}_a)^0_{\mathbb{T}}(X) = H^*(E\mathbb{T} \times_{\mathbb{T}} X)$$

and

$$(E\mathbb{G}_m)^0_{\mathbb{T}}(X) = K^0_{\mathbb{T}}(X),$$

and models for these theories were given in [6]. We will repeat the answer here in our present language.

We start by summarizing the properties we want of such a construction, and then observe that the algebraic categories of Section 3 immediately gives a unique construction.

- The subgroup $\mathbb{T}[n]$ of order n corresponds to the subgroup $\mathbb{G}[n]$ of elements of order dividing n
- The family $\mathcal{F}$ of finite subgroups corresponds to the set $\mathbb{G}[tors]$ of elements of torsion points.
- The suspension $S^{az^n} \wedge E\mathbb{G}$ corresponds to the sheaf $\mathcal{O}(a\mathbb{G}[n])$ and more generally, suspension by z^n corresponds to tensoring with $\mathcal{O}(\mathbb{G}[n])$.
- The inclusion $S^0 \longrightarrow S^{z^n}$ which induces multiplication by the Euler class (in the presence of a Thom isomorphism) corresponds to $\mathcal{O} \longrightarrow \mathcal{O}(\mathbb{G}[n])$.
- We extend the notation to allow
$$S^{\infty z^n} := \lim_{\rightarrow\ a} S^{az^n}$$
to correspond to the sheaf
$$\mathcal{O}(\infty\mathbb{G}[n]) := \lim_{\rightarrow\ a} \mathcal{O}(a\mathbb{G}[n])$$
and
$$\tilde{E}\mathcal{F} := \lim_{\rightarrow\ a,n} S^{az^n}$$
to correspond to
$$\mathcal{O}(\infty\mathbb{G}[tors]) := \lim_{\rightarrow\ a,n} \mathcal{O}(a\mathbb{G}[n]).$$
(This description of $\tilde{E}\mathcal{F}$ requires us to be working rationally; more generally one only has $\tilde{E}\mathcal{F} = \lim_{\rightarrow\ V^{\mathbb{T}}=0} S^V$.)

We need to say more about Euler classes. Consider the subgroup $\mathbb{T}[n]$ of order n. The natural geometric construction is the Euler class induced by $S^0 \longrightarrow S^{z^n}$. Pulling back a Thom class for S^{z^n} gives the function $e(z^n)$ in R, which vanishes at all subgroups of $\mathbb{T}[n]$.

Evidently, if we take c_d to be the function vanishing to the first order on the group of order d and taking the value 1 elsewhere, we have

$$e(z^n) = \prod_{d|n} c_d,$$

so that we may view c_d as a universal cyclotomic function.

We have already motivated the idea that $S^0 \longrightarrow S^{z^n}$ should correspond to $\mathcal{O} \longrightarrow \mathcal{O}(\mathbb{G}[n])$. The Thom class for S^{z^n} corresponds to a generating section of $\mathcal{O}(\mathbb{G}[n])$ and hence $e(z^n)$ should correspond to a function $\chi(z^n)$ defining $\mathbb{G}[n]$ in $\mathbb{G}$.

Now choose a coordinate $y =: \chi(z)$ at $e \in \mathbb{G}$. We may then take

$$\chi(z^n) := [n](y) := n^*(y).$$

so that $\chi(z^n)$ is a function vanishing to first order on $\mathbb{G}[n]$.

Next, we may a decompose the divisor $\mathbb{G}[n]$:

$$\mathbb{G}[n] = \sum_{d|n} \mathbb{G}\langle d\rangle$$

where $\mathbb{G}\langle d\rangle$ is the divisor of points of exact order d. Now we define a function $\phi\langle d\rangle := \phi\mathbb{G}\langle d\rangle$ vanishing to the first order on $\mathbb{G}\langle d\rangle$ recursively by the condition

$$\chi(z^n) = \prod_{d|n} \phi\langle d\rangle :$$

the formula defines $\phi\langle n\rangle$ directly for $n = 1$, and for larger values of n, it is defined by dividing $\chi(z^n)$ by the previously defined $\phi\langle d\rangle$.

Definition 4.1. Given a virtual complex representation $V = \sum_n a_n z^n$, we define a divisor by $D(V) = \sum_n a_n \mathbb{G}[n]$. We say that a 2-periodic $\mathbb{T}$-equivariant cohomology theory $E^*_{\mathbb{T}}(\cdot)$ is of *type* $\mathbb{G}$ if

$$\widetilde{E}^i_{\mathbb{T}}(S^V) \cong H^i(\mathbb{G}; \mathcal{O}(-D(V))$$

whenever V or $-V$ is a complex representation.

We also make a naturality requirement. For this it will be clearer if we insist V is an actual representation, and reformulate the other case as the isomorphism

$$\widetilde{E}_i^{\mathbb{T}}(S^V) \cong H^i(\mathbb{G}; \mathcal{O}(D(V)).$$

Now we require these isomorphisms to be natural for inclusions $j : V \longrightarrow V'$ of representations. First note that such a map induces a map $S^V \longrightarrow S^{V'}$ of $\mathbb{T}$-spaces and hence maps

$$j^* : \widetilde{E}^i_{\mathbb{T}}(S^{V'}) \longrightarrow \widetilde{E}^i_{\mathbb{T}}(S^V)$$

and

$$j_* : \widetilde{E}_i^{\mathbb{T}}(S^V) \longrightarrow \widetilde{E}^i_{\mathbb{T}}(S^{V'}).$$

On the other hand we have inclusion of divisors $D(V) \longrightarrow D(V')$, inducing maps

$$\mathcal{O}(-D(V')) \longrightarrow \mathcal{O}(-D(V))$$

and

$$\mathcal{O}(D(V)) \longrightarrow \mathcal{O}(D(V')).$$

The induced maps in sheaf cohomology are required to j^* and j_*.

Theorem 4.2. *Given a commutative 1-dimensional affine group scheme $\mathbb{G}$ over a ring containing $\mathbb{Q}$, and a coordinate y at $e \in \mathbb{G}$ there is a 2-periodic cohomology theory $E\mathbb{G}^*_{\mathbb{T}}(\cdot)$ of type $\mathbb{G}$. Furthermore, $E\mathbb{G}^*_{\mathbb{T}}$ is in even degrees and $\mathbb{G} = \mathrm{spec}(E\mathbb{G}^0_{\mathbb{T}})$. The construction is natural for isomorphisms.*

Remark 4.3. The construction is also natural for quotient maps $p : \mathbb{G} \longrightarrow \mathbb{G}/\mathbb{G}[n]$ in the sense that there is a map $p^* : E(\mathbb{G}/\mathbb{G}[n]) \longrightarrow \mathrm{infl}^{\mathbb{T}}_{\mathbb{T}/\mathbb{T}[n]} E\mathbb{G}$ of $\mathbb{T}$-spectra, where $E\mathbb{G}$ is viewed as a $\mathbb{T}/\mathbb{T}[n]$-spectrum and inflated to a $\mathbb{T}$-spectrum.

More precisely, if y is a coordinate on $\mathbb{G}$ then its norm $\Pi_{a\in\mathbb{G}[n]} T_a y$ is a coordinate on $\mathbb{G}/\mathbb{G}[n]$ (where T_a denotes translation by a). Using these coordinates, we obtain equivariant spectra $E\mathbb{G}/\mathbb{G}[n]$ and $E\mathbb{G}$. As a first step to maps between them, note that we have maps $p^*_V : V(\mathbb{G}/\mathbb{G}[n]) \longrightarrow V\mathbb{G}$ and $p^*_T : T(\mathbb{G}/\mathbb{G}[n]) \longrightarrow T\mathbb{G}$ corresponding to pullback of functions. However p^*_V and p^*_T do not give a map of $\mathbb{T}$-spectra $E(\mathbb{G}/\mathbb{G}[n]) \longrightarrow E\mathbb{G}$; for example the non-equivariant part of $E(\mathbb{G}/\mathbb{G}[n])$ corresponds to functions on $\mathbb{G}/\mathbb{G}[n]$ with support at the identity, and these pull back to functions on $\mathbb{G}$ supported on $\mathbb{G}[n]$, which correspond to the part of $E\mathbb{G}$ with isotropy contained in $\mathbb{T}[n]$. The answer is to view the circle of equivariance of $E\mathbb{G}$ as $\mathbb{T}/\mathbb{T}[n]$, and then to use the inflation functor studied in Chapters 10 and 24 of [6] to obtain a $\mathbb{T}$-spectrum.

Proof: The construction was motivated in Section 2. We take

$$V\mathbb{G} = \mathcal{O}(\infty tors),$$

$$T\mathbb{G} = \mathcal{O}(\infty tors)/\mathcal{O},$$

and use the map

$$q\mathbb{G} : t^{\mathcal{F}}_* \otimes \mathcal{O}(\infty tors) \longrightarrow \mathcal{O}(\infty tors)/\mathcal{O}$$

given by

$$s/e(W) \otimes f \longmapsto s \cdot \overline{f/\chi(W)}.$$

We must explain how $T\mathbb{G}$ is a module over R, and why β is a map of R-modules. We make $T\mathbb{G}$ into a module over R by letting c_d act as ϕ_d. Since any function only has finitely many poles, all but finitely many c_d act as the identity on any element of $T\mathbb{G}$, and since poles are of finite order, $T\mathbb{G}$ is a $\mathcal{E}$-torsion module. The definition of the map $q\mathbb{G}$ shows it is an R-map.

Finally, we must show that the homotopy groups of the resulting object are as required in 4.1. By 3.4 we have $M_s(E\mathbb{G}) = (\beta\mathbb{G} : N\mathbb{G} \longrightarrow t^{\mathcal{F}}_* \otimes V\mathbb{G})$, where $N\mathbb{G} = \ker(t^{\mathcal{F}}_* \otimes V\mathbb{G} \longrightarrow T\mathbb{G})$, and we need to calculate

$$[S^W, E\mathbb{G}]^{\mathbb{T}}_* = [S^w, M_s(E\mathbb{G})]_*.$$

Since $q\mathbb{G}$ is epimorphic, $\beta\mathbb{G}$ is monomorphic, and $T\mathbb{G}$ is injective. Thus by 3.4 we have the explicit injective resolution

$$0 \longrightarrow M_s(E\mathbb{G}) = \begin{pmatrix} N\mathbb{G} \\ \downarrow \\ t^{\mathcal{F}}_* \otimes V\mathbb{G} \end{pmatrix} \longrightarrow \begin{pmatrix} t^{\mathcal{F}}_* \otimes V\mathbb{G} \\ \downarrow \\ t^{\mathcal{F}}_* \otimes V\mathbb{G} \end{pmatrix} \longrightarrow \begin{pmatrix} T\mathbb{G} \\ \downarrow \\ 0 \end{pmatrix} \longrightarrow 0.$$

Now, applying 3.3 we see $\mathrm{Ext}(S^w, M_s(E\mathbb{G})) = 0$ since any torsion element $t \in T\mathbb{G}$ lifts to $f \in V\mathbb{G}$ and hence to $1/e(W) \otimes \chi(W) f$. It is immediate from the definition that

$$\mathrm{Hom}(S^w, M_s(E\mathbb{G})) = \{c^{-w} \otimes f \mid f/\chi(W) \text{ regular } \}.$$

By construction the divisor associated to the function $\chi(V)$ is $D(V)$, so $f/\chi(V)$ is regular if and only if $f \in \mathcal{O}(-D(V))$ as required. □

Remark 4.4. In the above proof we made use of the fact that the Euler class $\chi(W)$ exists as a function in $V\mathbb{G}$. The point of this comment will become apparent when we treat the elliptic case which behaves rather differently: there the Euler class is given by different functions at different points, corresponding to the fact that the cohomology theory is not complex orientable, so that the bundle specified by W is not trivializable.

We make the construction explicit in a few cases.

The ring of functions on $\mathbb{G}_a$ is $\mathbb{Q}[x]$, and the group structure is defined by the coproduct $x \longmapsto 1 \otimes x + x \otimes 1$. We choose x as a coordinate about the identity, zero. The group $\mathbb{G}_a[n]$ of points of order dividing n is defined by the vanishing of $\chi(z^n) = nx$, so the identity is the only element of finite order over $\mathbb{Q}$-algebras. This case becomes rather degenerate in that it only detects isotropy 1 and $\mathbb{T}$.

Proposition 4.5. *The model of 2-periodic Borel cohomology in the torsion model is formal, concentrated in even degrees and in each even degree is the map*

$$t_*^{\mathcal{F}} \otimes \mathcal{O}(\infty tors) = t_*^{\mathcal{F}} \otimes \mathbb{Q}[x, x^{-1}] \longrightarrow \mathbb{Q}[x, x^{-1}]/\mathbb{Q}[x] = \mathcal{O}(\infty tors)/\mathcal{O}$$

$$s/e(V) \otimes f \longmapsto s \cdot \overline{f/\chi(V)}.$$

Here $\mathcal{O} = \mathbb{Q}[x]$ *and* $\chi(z^n) = nx$. *The ring* $\mathcal{O}(\infty tors) = \mathbb{Q}[x, x^{-1}]$ *of functions with poles only at points of finite order is obtained by inverting the Euler class of* z. *Accordingly, 2-periodic Borel cohomology is the theory associated to the additive group in the sense of 4.2.* □

The ring of functions on $\mathbb{G}_m$ is $\mathcal{O} = R(\mathbb{T}) = \mathbb{Q}[z, z^{-1}]$, and the group structure is defined by the coproduct $z \longmapsto z \otimes z$. We choose $y = 1 - z$ as a coordinate about the identity element, 1. The coproduct then takes the more familiar form $y \longmapsto 1 \otimes y + y \otimes 1 - y \otimes y$. The group $\mathbb{G}_m[n]$ of points of order dividing n is defined by the vanishing of $\chi(z^n) = 1 - z^n$.

Proposition 4.6. [6, 13.4.4] *The model of equivariant K-theory in the torsion model is formal, concentrated in even degrees and in each even degree is the map*

$$t_*^{\mathcal{F}} \otimes \mathcal{O}(\infty tors) \longrightarrow \mathcal{O}(\infty tors)/\mathcal{O}$$

$$s/e(V) \otimes f \longmapsto s \cdot \overline{f/\chi(V)}.$$

Here $\mathcal{O} = \mathbb{Q}[z, z^{-1}]$ *and* $\chi(z^n) = 1 - z^n$. *The ring* $\mathcal{O}(\infty tors)$ *of functions with poles only at points of finite order is obtained by inverting all Euler classes. Accordingly, equivariant K theory is the theory associated to the multiplicative group in the sense of 4.2.* □

By way of completeness we also record the analogue for formal groups. This completes the circle by establishing the universality of the motivation described in Section 2. However, since we must work over $\mathbb{Q}$, there is little difference from the additive group above. Suppose given a commutative one dimensional formal group $\widehat{\mathbb{G}}$ over a ring k containing $\mathbb{Q}$, with a coordinate y. We may identify the ring of functions on $\widehat{\mathbb{G}}$ with $k[[x]]$, and the group structure is the coproduct $x \longmapsto F(x \otimes 1, 1 \otimes x)$. The group $\widehat{\mathbb{G}}[n]$ of points of order dividing n is defined by the vanishing of $\chi(z^n) = [n](x)$ so the identity is the only element of finite order over

$\mathbb{Q}$-algebras. We may now make the direct analogue of the construction in 4.2. This case becomes rather degenerate in that it only detects isotropy 1 and $\mathbb{T}$.

Proposition 4.7. *The model of the 2-periodic Borel cohomology associated to a complex orientable theory $E^*(\cdot)$ in the torsion model is formal, concentrated in even degrees and in each even degree is the map*

$$t_*^{\mathcal{F}} \otimes \mathcal{O}(\infty tors) = t_*^{\mathcal{F}} \otimes E^0((x)) \longrightarrow E^0((x))/E^0[[x]] = \mathcal{O}(\infty tors)/\mathcal{O}$$

$$s/e(V) \otimes f \longmapsto s \cdot \overline{f/\chi(V)}.$$

Here $\mathcal{O} = E^0[[x]]$ and $\chi(z^n) = [n](x)$. The ring $\mathcal{O}(\infty tors) = E^0[[x]][1/x] = E^0((x))$ of functions with poles only at points of finite order is obtained by inverting the Euler class of z. Accordingly, 2-periodic E-Borel cohomology is the theory associated to the formal group of E in the sense of 4.2. □

5. Elliptic curves.

In this section we record the well known facts about elliptic curves that will play a part in our construction. We use [15] as a basic reference for facts about elliptic curves, and [12] as background from algebraic geometry.

Let A be an elliptic curve (i.e. a smooth projective curve of genus 1 with a specified point e) over an algebraically closed field k of characteristic 0 and let $\mathcal{O} = \mathcal{O}_A$ be its sheaf of regular functions. Note that $\Gamma\mathcal{O} = k$, so the sheaf contains a great deal more information than its ring of global sections. A divisor on A is a finite $\mathbb{Z}$-linear combination of points, and associated to any rational function f on A we have the divisor $\mathrm{div}(f) = \Sigma_P \mathrm{ord}_P(f)(P)$, where $\mathrm{ord}_P(f) \in \mathbb{Z}$ is the order of vanishing of f at P. In the usual way, if D is a divisor on A, we write $\mathcal{O}(D)$ for the associated invertible sheaf. Its global sections are given by

$$\Gamma\mathcal{O}(D) = \{f \mid \mathrm{div}(f) \geq -D\} \cup \{0\},$$

so that for a point P, the global sections of $\mathcal{O}(-P)$ are the functions vanishing at P.

We also have $\mathcal{O}(D_1) \otimes \mathcal{O}(D_2) = \mathcal{O}(D_1 + D_2)$.

Since the global sections functor is not right exact, we are led to consider cohomology, but since A is one-dimensional this only involves $H^0(A;\cdot) = \Gamma(\cdot)$ and $H^1(A;\cdot)$, which are related by Serre duality. This takes a particularly simple form since the canonical divisor is zero on an elliptic curve:

$$H^0(A;\mathcal{O}(D)) = H^1(A;\mathcal{O}(-D))^\vee,$$

where $(\cdot)^\vee = \mathrm{Hom}_k(\cdot, k)$ denotes vector space duality.

From the Riemann-Roch theorem we deduce that the canonical divisor is 0 and the cohomology of each line bundle:

$$\dim(H^0(A;\mathcal{O}(D)) = \begin{cases} \deg D & \text{if } \deg(D) \geq 1 \\ 0 & \text{if } \deg(D) \leq -1 \end{cases}$$

and

$$\dim(H^1(A;\mathcal{O}(D)) = \begin{cases} |\deg D| & \text{if } \deg(D) \leq -1 \\ 0 & \text{if } \deg(D) \geq 1. \end{cases}$$

For the trivial divisor one has

$$\dim(H^0(A;\mathcal{O})) = \dim(H^1(A;\mathcal{O})) = 1.$$

Now if $D = \Sigma_P n_P(P)$ is a divisor of degree 0, we may form the sum $P(D) = \Sigma_P n_P P$ in A, and D is linearly equivalent to $(P(D)) - (e)$. If $P(D) = e$ then the sheaf $\mathcal{O}(D)$ has the same cohomology as $\mathcal{O}$. Otherwise, since no function vanishes to order exactly 1 at P, we find

$$H^0(A; \mathcal{O}(D)) = H^1(A; \mathcal{O}(D)) = 0.$$

We may recover A from the graded ring $\Gamma(\mathcal{O}(*e)) = \{\Gamma\mathcal{O}(ne)\}_{n\geq 0}$. Indeed, this is the basis of the proof in [15, III.3.1] that any elliptic curve is a subvariety of $\mathbb{P}^2$ defined by a Weierstrass equation. We choose a basis $\{1, x\}$ of $\Gamma\mathcal{O}(2e)$ and a extend it to a basis $\{1, x, y\}$ of $\Gamma\mathcal{O}(3e)$. Now observe that since $\Gamma\mathcal{O}(6e)$ is 6-dimensional, there is a relation between the seven elements $1, x, x^2, x^3, y, xy$ and y^2: this is the Weierstrass equation, and it may be verified that A is the closure in $\mathbb{P}^2$ of the plane curve it defines. The graded ring $\Gamma(\mathcal{O}(*e))$ has generator Z of degree 1 corresponding to the constant function 1 in $\Gamma\mathcal{O}(e)$, X of degree 2 corresponding to x, and Y of degree 3 corresponding to y. These three variables satisfy the homogeneous form of the Weierstrass equation. The statement that A is the projective closure of the plane curve defined by the Weierstrass equation may be restated in terms of Proj:

$$A = \mathrm{Proj}(\Gamma(\mathcal{O}(*e))).$$

6. Coordinate data

Our main theorem constructs a cohomology theory of type A from an elliptic curve together with suitable coordinate data. In this section we describe the data, and the choices of functions that they permit.

Definition 6.1. *Coordinate data* for an elliptic curve is a choice of two functions x_e with a pole of order 2 at the identity and nowhere else, and y_e with a pole of order 3 at the identity and nowhere else. We also require that x_e and y_e only vanish at torsion points. This coordinate data determines a local uniformizer $t_e = x_e/y_e$ of $\mathcal{O}_e$, and hence also a uniformizer t_P at P by translating t_e.

Remark 6.2. (i) Since t_e is a uniformizer, $t_e^2 x_e = u_e$ is a unit in $\mathcal{O}_e$.

However, we note that any global representative of t_e must have two poles Z, Z' away from e, so u_e cannot be a constant.

(iii) One popular choice of coordinate data involves choosing a point P of order 2. This determines a choice of x_e and y_e up to a constant multiple by the conditions

$$\mathrm{div}(x_e) = -2(e) + 2(P) \text{ and } \mathrm{div}(y_e) = -3(e) + (P) + (P') + (P'')$$

where $A[2] = \{e, P, P', P''\}$. Thus

$$\mathrm{div}(t_e) = (e) + (P) - (P') - (P''). \quad \square$$

The divisor $A\langle n\rangle$ of points of exact order n will play a central role. Note that

$$A[n] = \sum_{d|n} A\langle d\rangle,$$

and

$$A[tors] = \sum_{d\geq 1} A\langle d\rangle.$$

The coordinate data allow us to specify a function defining the points of exact order d.

Lemma 6.3. *Given a choice of coordinate data on the elliptic curve d, for each $d \geq 2$, there is a unique function t_d with the properties*

(1) *t_d vanishes exactly to the first order on $A\langle d\rangle$,*
(2) *t_d is regular except at the identity $e \in A$ where it has a pole of order $|A\langle d\rangle|$,*
(3) *$t_e^{|A\langle d\rangle|}t_d$ takes the value 1 at e*

Proof: Consider the divisor $A\langle d\rangle - |A\langle d\rangle|(e)$. Note that the sum of the points of $A\langle d\rangle$ in A is the identity: if $d \neq 2$ this is because points occur in inverse pairs, and if $d = 2$ it is because the $A[2]$ is isomorphic to $C_2 \times C_2$. It thus follows from the Riemann-Roch theorem that there is a function f with $A\langle d\rangle - |A\langle d\rangle|(e)$ as its divisor. This function (which satisfies the first two properties in the statement) is unique up to multiplication by a non-zero scalar. The third condition fixes the scalar. □

Remark 6.4. If we choose any finite collection $\pi = \{d_1, \ldots, d_s\}$ of orders ≥ 2, there is again a unique function $\phi\langle\pi\rangle$ with analogous properties. Indeed, the good multiplicative property of the normalization means we may take

$$\phi\langle\pi\rangle = \prod_i \phi\langle d_i\rangle.$$

This applies in particular to the set $A[n] \setminus \{e\}$. □

For some purposes, it is convenient to have a basis for functions with specified poles. We already have the basis $1, x, y, x^2, xy, \ldots$ if all the poles are at the identity. Multiplication by a function f induces an isomorphism

$$f\cdot : \Gamma\mathcal{O}(D) \longrightarrow \Gamma\mathcal{O}(D + (f))$$

so we can translate the basis we have.

Lemma 6.5. *For the divisor $D = \Sigma_{d\geq 1} n(d)A\langle d\rangle$ let $t^*(D) := \prod_{b\geq 2} t_b^{n(b)}$. Multiplication by $t^*(D)$ gives an isomorphism*

$$t^*(D)\cdot : H^0(A; \mathcal{O}(\deg(D)(e)) \xrightarrow{\cong} H^0(A; \mathcal{O}(D)).$$

A basis of $H^0(A; \mathcal{O}(D))$ is given by $t^*(D)$ if $\deg(D) = 0$, and by the first $\deg(D)$ terms in the sequence

$$t^*(D), t^*(D)x, t^*(D)y, t^*(D)x^2, t^*(D)xy, \ldots$$

otherwise. □

Remark 6.6. It is essential to be aware of the exceptional nature of the degree zero case.

7. Local cohomology sheaves on elliptic curves.

The basic ingredients of the torsion model of a the cohomology theory associated to an elliptic curve A are analogous to the affine case. The vertex

$$VA = \Gamma\mathcal{O}(\infty tors)$$

consists of rational functions whose poles are all at torsion points, however the torsion module is not simply the quotient of this by regular functions, but rather

$$TA = \Gamma(\mathcal{O}(\infty tors)/\mathcal{O}).$$

Before we work with this definition we need some basic tools.

Convention 7.1. Here and elsewhere, we only consider open sets obtained by deleting torsion points. Thus localization only permits poles at torsion points: for example $\mathcal{O}_P$ is the subsheaf of $\mathcal{O}(\infty tors)$ consisting of functions regular at P.

For any effective divisor D we may use the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(aD) \longrightarrow Q(aD) \longrightarrow 0$$

of sheaves to define the quotient sheaf $Q(aD)$ for $0 \leq a \leq \infty$. The cohomology of $Q(\infty D)$ is the cohomology of A with support on D.

In fact we may reduce constructions to the case when the divisor D is a single point P. Evidently, $Q(\infty P)$ is a skyscraper sheaf concentrated at P, so we may localize at P to obtain

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}(\infty P)_P \longrightarrow Q(\infty P) \longrightarrow 0.$$

Notice that $\mathcal{O}(\infty P)_P = \mathcal{O}(\infty tors)$.

Since A is a smooth curve, the local ring $\mathcal{O}_P$ is a discrete valuation ring, and if we choose a local uniformizer t_P any element of $\Gamma(Q(\infty P))$ may be represented by an element of the form

$$a_{-1}t_P^{-1} + a_{-3}t_P^{-3} + \cdots + a_{-n}t_P^{-n}$$

for suitable scalars a_{-i}. Thus the sequence becomes

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_P[1/t_P] \longrightarrow \mathcal{O}_P/t_P^{\infty} \longrightarrow 0.$$

This gives the basis of the Thom isomorphism.

Lemma 7.2. *A choice of local uniformizer at P gives isomorphisms*

$$\mathcal{O}((a+r)P)/\mathcal{O}(rP) = Q((a+r)P)/Q(rP) \cong Q(aP),$$

and hence

$$Q(\infty P) \otimes \mathcal{O}(rP) \cong Q(\infty P). \quad \square$$

Note that it is immediate from the Riemann-Roch formula that for $0 \leq a \leq \infty$ the cohomology group $H^0(A; Q(aP))$ is a dimensional, and $H^1(A; Q(aP)) = 0$.

Now we may assemble these sheaves for each point. Indeed, we have a diagram

$$\begin{array}{ccccc} \mathcal{O} & \longrightarrow & \mathcal{O}(\infty D) & \longrightarrow & Q(\infty D) \\ \downarrow & & \downarrow & & \\ \mathcal{O} & \longrightarrow & \mathcal{O}(\infty(D+D')) & \longrightarrow & Q(\infty(D+D')) \end{array}$$

of sheaves, and hence a map $Q(\infty D) \longrightarrow Q(\infty(D+D'))$.

Proposition 7.3. *If P, P' are distinct points of A then the natural map*

$$Q(\infty P) \oplus Q(\infty P') \xrightarrow{\cong} Q(\infty(P+P'))$$

is an isomorphism.

Proof: We apply the Snake Lemma to the diagram

$$\begin{array}{ccccc} \mathcal{O}\oplus\mathcal{O} & \longrightarrow & \mathcal{O}(\infty P)\oplus\mathcal{O}(\infty Q) & \longrightarrow & Q(\infty P)\oplus Q(\infty P') \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathcal{O}(\infty(P+P')) & \longrightarrow & Q(\infty(P+P')) \end{array}$$

in the abelian category of sheaves on A. The first vertical is obviously surjective with kernel $\mathcal{O}$. The kernel of the second vertical is also $\mathcal{O}$, since if f and f' are local sections of $\mathcal{O}(\infty P)$ and $\mathcal{O}(\infty P')$ (ie f only has poles at P and f' only at P') then $f+f'=0$ implies that f and f' are regular. Finally we must show that $\mathcal{O}(\infty(P+P'))$ is the sheaf quotient of $\mathcal{O}\longrightarrow\mathcal{O}(\infty P)\oplus\mathcal{O}(\infty P')$. However, this may be verified stalkwise, where it is clear. $\square$

Let us now collect what we need for the construction. To give a Thom isomorphism for $Q(\infty A\langle d\rangle)$ we need to choose local uniformizers t_P at each point P of exact order d. For example we explained in Section 6 how coordinate data determines a function t_d vanishing on $A\langle d\rangle$ to the first order at all points of $A\langle d\rangle$, and we could take $t_P=t_d$ for all points P of exact order d.

Corollary 7.4. *The natural map gives an isomorphism*

$$\bigoplus_d Q(\infty A\langle d\rangle)\xrightarrow{\cong}Q(\infty tors),$$

and a choice of coordinate t_P at each $P\in A\langle d\rangle$ gives a Thom isomorphism

$$T_d: Q(\infty A\langle d\rangle)\otimes\mathcal{O}(A\langle d\rangle)\xrightarrow{\cong}Q(\infty A\langle d\rangle).$$

The sheaf $Q(\infty A\langle d\rangle)$ has no higher cohomology and its global sections are

$$\Gamma Q(\infty A\langle d\rangle)=VA/\{f\mid f \text{ is regular on } A\langle d\rangle\}.\quad\square$$

Remark 7.5. This corresponds to the fact that there is a rational splitting

$$\Sigma E\mathcal{F}_+\simeq\bigvee_H\Sigma E\langle H\rangle$$

where $E\langle H\rangle=cofibre(E[\subset H]_+\longrightarrow E[\subseteq H]_+)$ [6, 2.2.3].

8. A cohomology theory associated to an elliptic curve.

We are now ready to state and prove the main theorem. Indeed, the paper so far has consisted entirely of motivation and repackaging of known results by way of preparation.

Theorem 8.1. *Given an elliptic curve A over a field k of characteristic zero, and coordinate data (x_e,y_e), there is an associated 2-periodic rational $\mathbb{T}$-equivariant cohomology theory of type A, so that for any representation W with $W^{\mathbb{T}}=0$ we have*

$$\widetilde{EA}^i_{\mathbb{T}}(S^W)=H^i(A;\mathcal{O}(-D(W)))$$

and

$$\widetilde{EA}^{\mathbb{T}}_i(S^W)=H^i(A;\mathcal{O}(D(W)))$$

where the divisor $D(W)$ is defined by taking

$$D(W)=\sum_n a_nA[n] \text{ when } W=\sum_n a_nz^n.$$

This association is functorial for isomorphisms of elliptic curves with coordinate data.

Remark 8.2. (i) The elliptic curve can be recovered from the cohomology theory. Indeed, we may form the graded ring

$$\widetilde{EA}^0_{\mathbb{T}}(S^{-*z}) := \{\widetilde{EA}^0_{\mathbb{T}}(S^{-az})\}_{a\geq 0}$$

from the products $S^{-az}\wedge S^{-bz} \longrightarrow S^{-(a+b)z}$, and the elliptic curve can be recovered from the cohomology theory via

$$A = \mathrm{Proj}(\widetilde{EA}^0_{\mathbb{T}}(S^{-*z})),$$

as commented in Section 5.
(ii) The coordinate data on A can therefore be recovered from suitable elements of homology:

$$x_e \in \widetilde{EA}_0^{\mathbb{T}}(S^{2z}) \text{ and } y_e \in \widetilde{EA}_0^{\mathbb{T}}(S^{3z}).$$

Remark 8.3. We have not required the field to be algebraically closed. To see the advantage of this, note that even for the multiplicative group, the individual points of order n are only defined over k if k contains appropriate roots of unity. However $\mathbb{G}_m[n]$ (defined by $1-z^n$) and hence also $\mathbb{G}_m\langle n\rangle$ (defined by the cyclotomic polynomial $\phi_n(z)$) are defined over $\mathbb{Q}$. Hence equivariant K theory itself is defined over $\mathbb{Q}$. For an elliptic curve A we require that there is a basis for $\Gamma\mathcal{O}(a\mathbb{G}\langle d\rangle))$ consisting of functions defined over k.

Proof: We must describe a vector space $V = VA$, an R-module TA and an R-map

$$qA : t^{\mathcal{F}}_* \otimes VA \longrightarrow TA.$$

It is easy to describe VA and TA; indeed, we take

$$VA = \Gamma\mathcal{O}(\infty tors)$$

consisting of rational functions whose poles are all at torsion points, and torsion module

$$TA = \Gamma(Q(\infty tors)).$$

The splitting

$$Q(\infty tors) \cong \bigoplus_d Q(\infty A\langle d\rangle)$$

of 7.4 gives

$$TA = \bigoplus_d TA\langle d\rangle$$

where

$$TA\langle d\rangle = VA/\{f \mid f \text{ is regular on } A\langle d\rangle\}.$$

It is not hard to describe the R-module structure on TA. The direct sum splitting of TA corresponds to the splitting

$$R \cong \prod_d \mathbb{Q}[c],$$

and $TA\langle d\rangle$ is a $\mathbb{Q}[c]$-module where c acts as multiplication by the function t_d defining $A\langle d\rangle$. Since the order of any pole is finite, $TA\langle d\rangle$ is a torsion $\mathbb{Q}[c]$-module. Notice that the definition of the Thom isomorphism is arranged so that the composite

$$\phi_D : Q(\infty D) = Q(\infty D)\otimes\mathcal{O} \longrightarrow Q(\infty D)\otimes\mathcal{O}(D) \cong Q(\infty D)$$

is multiplication by t_d.

Definition 8.4. If $u : \mathcal{F} \longrightarrow \mathbb{Z}$ is a function positive almost everywhere, we define

$$qA : c^u \otimes V \longrightarrow TA = \bigoplus_d TA\langle d \rangle$$

by specifying its dth component

$$qA(c^u \otimes f)_d = \overline{t_d^{u(d)} f}.$$

Lemma 8.5. *The definition does determine an R-map $qA : t_*^{\mathcal{F}} \otimes V \longrightarrow TA$.*

Proof: Since any function is regular at all but finitely many points, the map qA maps into the sum.

Now, R-maps $q : t_*^{\mathcal{F}} \otimes V \longrightarrow \bigoplus_d T_d$ are determined by the idempotent pieces $q_d : \mathbb{Q}[c, c^{-1}] \otimes V \longrightarrow T_d$, and conversely, any set of $\mathbb{Q}[c]$-maps q_d so that $q_d(c^0 \otimes f)$ is non-zero for only finitely many d determines an R-map q. It is easy to see that the components of qA (ie $q_d(c^s \otimes f) = qA(c^{s\delta(d)} \otimes f)_d$) have these properties, and that the function they determine agrees with $qA(c^u \otimes f)$ wherever it is defined. □

Now we can check that the resulting homology and cohomology of spheres agrees with the cohomology of the corresponding divisors on the elliptic curve.

Consider the complex representation W with $W^{\mathbb{T}} = 0$ and the corresponding function $w(H) = \dim_{\mathbb{C}}(W^H)$. We see from 3.3 and 3.5 (as in the proof of 4.2 that

$$\widetilde{EA}_0^{\mathbb{T}}(S^W) = \ker(qA : c^w \otimes VA \longrightarrow TA)$$

and

$$\widetilde{EA}_1^{\mathbb{T}}(S^W) = \mathrm{cok}(qA : c^w \otimes V \longrightarrow TA)$$

and similarly with W replaced by $-W$. Since the kernel and cokernel are vector spaces over k, it is no loss of generality to extend scalars to assume it is algebraically closed. This is convenient because it is simpler to treat separate points of order n one at a time.

The following two lemmas complete the proof. □

Lemma 8.6. *If W is a representation with $W^{\mathbb{T}} = 0$ then*

$$\widetilde{EA}_0^{\mathbb{T}}(S^W) = H^0(A; \mathcal{O}(D(W))),$$

and if $W \neq 0$,

$$\widetilde{EA}_0^{\mathbb{T}}(S^{-W}) = 0.$$

Proof: By definition

$$qA(c^w \otimes f)_d = \overline{t_d^{w(d)} f}.$$

Since the function t_d vanishes to exactly the first order on $A\langle d \rangle$, the condition that f lies in the kernel is that $\mathrm{ord}_P(f) \geq -w(d)$ for each point P of exact order d. Since $D(W) = \Sigma_P w(d_P)(P)$ we have

$$\ker(qA : c^w \otimes W \longrightarrow TA) = \{f \in VA \mid \mathrm{div}(f) + D(W) \geq 0\}$$

as required.

Replacing W by $-W$, the second statement is immediate. □

Remark 8.7. The proof is local and therefore shows the kernel is actually the subsheaf $\mathcal{O}(D(W))$ of the constant sheaf VA.

The calculation of the odd cohomology is less elementary.

Proposition 8.8. *If W is a representation with $W^{\mathbb{T}} = 0$ then*

$$\widetilde{EA}_1^{\mathbb{T}}(S^{-W}) = H^1(A; \mathcal{O}(-D(W))),$$

and if $W \neq 0$,

$$\widetilde{EA}_1^{\mathbb{T}}(S^{W}) = 0.$$

Proof: We have to calculate $\mathrm{cok}(qA : c^{-w} \otimes VA \longrightarrow TA)$. The following proof that this is $H^1(A; \mathcal{O}(-D(W)))$ is that given in [14, Proposition II.3].

We have already considered the kernel, and we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(-D(W)) \longrightarrow VA \longrightarrow Q(-D(W)) \longrightarrow 0.$$

The exact sequence in cohomology ends

$$VA \xrightarrow{\phi} H^0(A; Q(-D(W))) \longrightarrow H^1(A; \mathcal{O}(-D(W))) \longrightarrow 0,$$

so it remains to observe that $\mathrm{cok}(\phi)$ may be identified with $\mathrm{cok}(qA^{-w})$.

However $Q(-D(W))$ is a skyscraper sheaf concentrated its space of sections is $W/W(D)$, where

$$W = \{(x_P)_P \mid x_P \in VA, \text{ and almost all } x_P \in k\}$$

is the space of adèles (for torsion points P) and

$$W(D) = \{(x_P) \in W \mid \mathrm{ord}_P(x_P) + \mathrm{ord}_P(D) \geq 0\}.$$

Thus $\mathrm{cok}(\phi) = W/(W(-D(W)) + VA) = \mathrm{cok}(qA^{-w})$ as required. □

Remark 8.9. It is possible to give a more explicit proof as follows. First, one checks any element $(g_1, g_2, \ldots) \in \bigoplus_d TA\langle d \rangle$ is congruent to one with $g_2 = g_3 = \cdots = 0$. Now, identify a subspace of the correct codimension in the image. Using divisors one sees the cokernel must be at least this big. Finally, the cokernel is naturally dual to $H^0(A; \mathcal{O}(D(W))$, and hence naturally isomorphic to $H^1(A; \mathcal{O}(-D(W)))$ by Serre duality.

9. Multiplicative properties.

Theorem 9.1. *If E is constructed from a 1-dimensional group scheme (ie if $E = E\mathbb{G}$ or EA) then E is a commutative ring spectrum.*

For the rest of this section we suppose $E = (N \longrightarrow t_*^{\mathcal{F}} \otimes V)$, and that there is a short exact sequence

$$0 \longrightarrow N \xrightarrow{\beta} t_*^{\mathcal{F}} \otimes V \xrightarrow{q} Q \longrightarrow 0.$$

It is natural to use the geometric terminology, and talk of V as a space of sections (of an imagined bundle), and $N(c^0)$ as the space of regular sections

First we note that E is flat.

Lemma 9.2. *A spectrum E with monomorphic structure β map is flat.*

Proof: Tensor product on $\mathcal{A}_s$ is defined termwise. First, note that $t_*^{\mathcal{F}} \otimes V$ is exact for tensor product with objects P with $\mathcal{E}^{-1}P \cong t_*^{\mathcal{F}} \otimes W$ for some W, so the tensor product is exact on the vertex part.

For the nub, we use the fact that the category $\mathcal{A}_s$ is of flat dimension 1 by [6, 23.3.5], together with the fact that N is a submodule of $t_*^{\mathcal{F}} \otimes V$. □

It follows that tensor product with E models the smash product.

Lemma 9.3. *Suppose that V has a commutative and associative product (so we may refer to it as an* algebra *of sections).*

If the product of two regular sections is regular then the associated object E admits a commutative and associative product.

Proof: By hypothesis the product on $t_*^{\mathcal{F}} \otimes V$ takes $N \otimes_R N$ to N, and therefore gives a map

$$E \otimes E \longrightarrow E$$

in $\mathcal{A}_s$. Associativity and commutativity are inherited from V. □

Corollary 9.4. *(i) If V is an affine algebra of functions the product of two regular sections is regular.*
(i) If V is an elliptic algebra of functions then the product of two regular sections is regular.

Proof: Suppose s and t are sections. We must show that if $q(s) = 0$ and $q(t) = 0$ then $q(st) = 0$. This is clear since regularity is detected one point at at time and $\mathrm{ord}_x(fg) = \mathrm{ord}_x(f) + \mathrm{ord}_x(g)$. □

Now that we have a product structure we can tie up topological and geometric duality in a satisfactory way.

Lemma 9.5. *Spanier-Whitehead duality for spheres corresponds to Serre duality in the sense that the Serre duality pairing*

$$\begin{array}{ccc} H^1(A;\mathcal{O}(-D(W))) \otimes H^0(A;\mathcal{O}(D(W))) & \longrightarrow & H^1(A;\mathcal{O}) \\ \| & & \| \\ [S^0, S^{-W} \wedge \Sigma EA]^{\mathbb{T}} \otimes [S^0, S^W \wedge EA]^{\mathbb{T}} & & [S^0, \Sigma EA]^{\mathbb{T}} \end{array}$$

is induced by the algebraically obvious Spanier-Whithead pairing

$$S^{-W} \wedge EA \wedge S^W \wedge EA \simeq S^{-W} \wedge S^W \wedge EA \wedge EA \longrightarrow S^0 \wedge EA \wedge EA \longrightarrow EA.$$

Proof: Both maps are induced by multiplication of functions and a residue map (see [14, Chapter II]). □

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Department of Pure Mathematics, Hicks Building, Sheffield S3 7RH. UK.
E-mail address: j.greenlees@sheffield.ac.uk

Department of Mathematics, MIT, Cambridge, MA 02139-4307, USA.
E-mail address: mjh@math.mit.edu

Department of Mathematics, MIT, Cambridge, MA 02139-4307, USA.
E-mail address: ioanid@math.mit.edu