

# HOMOLOGICAL LOCALIZATIONS OF EILENBERG-MAC LANE SPECTRA

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ABSTRACT. We discuss the Bousfield localization  $L_E X$  for any spectrum  $E$  and any  $HR$ -module  $X$ , where  $R$  is a ring with unit. Due to the splitting property of  $HR$ -modules, it is enough to study the localization of Eilenberg–Mac Lane spectra. Using general results about stable  $f$ -localizations, we give a method to compute the localization of an Eilenberg–Mac Lane spectrum  $L_E HG$  for any spectrum  $E$  and any abelian group  $G$ . We describe  $L_E HG$  explicitly when  $G$  is one of the following: finitely generated abelian groups,  $p$ -adic integers, Prüfer groups, and subrings of the rationals. The results depend basically on the  $E$ -acyclicity patterns of the spectrum  $H\mathbb{Q}$  and the spectrum  $H\mathbb{Z}/p$  for each prime  $p$ .

## 1. INTRODUCTION

Homological localizations were first defined by Adams [Ada73]. Bousfield developed the theory further by proving the existence of homological localizations in the category of spaces [Bou75] and in the category of spectra [Bou79b].

Given any spectrum  $E$ , a homological localization functor with respect to  $E$  is a homotopy idempotent transformation  $L_E: Ho^s \rightarrow Ho^s$ , where  $Ho^s$  is the stable homotopy category, that turns  $E$ -homology equivalences into homotopy equivalences in a universal way. Homological localizations are special cases of  $f$ -localizations in the sense of [Dro96] and commute with the suspension operator.

In [CG05], we presented a general study of  $f$ -localizations of  $HR$ -module spectra and discussed the preservation of several structures under the effect of these functors. In this paper, we restrict our attention to homological localizations in order to obtain more explicit results. In fact, we translate to spectra some of the results of [Bou82], by using ideas of [CG05] to simplify the arguments.

In [Bou79b], Bousfield determined the homological localizations of connective spectra with respect to connective homology theories. A spectrum  $X$  is connective if  $\pi_k(X) = 0$  for  $k < 0$ . If either  $E$  or  $X$  fail to be connective, then  $L_E X$  is somehow unpredictable. For example, the spectrum  $L_K S$ , where  $K$  denotes complex  $K$ -theory and  $S$  is the sphere spectrum, has infinitely many nonzero homotopy groups in both positive and negative dimensions (see [Rav84, Theorem 8.10] or [CG05, Corollary 5.15]).

We study  $L_E X$  where  $E$  is any homology theory (not necessarily connective) and  $X$  is any  $HR$ -module spectrum for a ring  $R$  with unit. Since any  $HR$ -module splits as a wedge of suspensions of Eilenberg–Mac Lane spectra, we focus on the study of  $L_E HG$  for any homology theory  $E$  and any abelian group  $G$ , where  $HG$  denotes the Eilenberg–Mac Lane spectrum associated to  $G$ . We describe all possible homological localizations in the case of finitely generated abelian groups and other groups, including the  $p$ -adic integers, the Prüfer groups  $\mathbb{Z}/p^\infty$ , and subrings of the rationals. For example, in the case of the spectrum  $H\mathbb{Z}$ , by the general approach

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of [CG05] we know any of its localization has at most one nonzero homotopy group and that this group has the structure of a *rigid ring* in the sense of [CRT00]. We prove that, for homological localizations of  $H\mathbb{Z}$ , the only rigid rings that appear are subrings of the rationals or products of  $p$ -adic integers for different primes.

The computations of these localizations depend on the  $E$ -acyclicity patterns of the spectra  $H\mathbb{Z}/p$  and  $H\mathbb{Q}$ , and on the set of primes  $p$  such that  $G$  is uniquely  $p$ -divisible, similarly as in [Bou82].

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## 2. HOMOLOGICAL LOCALIZATION OF SPECTRA

We will work in the stable homotopy category of spectra  $Ho^s$  (see [Ada74]). Any spectrum  $E$  in  $Ho^s$  gives rise to a homology theory defined as  $E_k(X) = \pi_k(E \wedge X)$  for any spectrum  $X$  and any  $k \in \mathbb{Z}$ . Homological localization with respect to the homology theory  $E$  is a functor that transforms homology equivalences with respect to this theory into homotopy equivalences in a universal way. It is unique up to homotopy and idempotent.

A map of spectra  $f: X \rightarrow Y$  is an  $E$ -equivalence if the map  $f_*: E_k(X) \rightarrow E_k(Y)$  is an isomorphism for all  $k \in \mathbb{Z}$ . A spectrum  $X \in Ho^s$  is called  $E$ -acyclic if  $E_k(X) = 0$  for all  $k \in \mathbb{Z}$ , that is, if  $E \wedge X$  is contractible. A spectrum  $Z$  is  $E$ -local if each  $E$ -equivalence  $X \rightarrow Y$  induces a homotopy equivalence  $F(Y, Z) \simeq F(X, Z)$ , or equivalently if  $F(W, Z) = 0$  for each  $E$ -acyclic spectrum  $W$ , where  $F(X, Y)$  denotes the function spectrum from  $X$  to  $Y$ . An  $E$ -localization of a spectrum  $X$  is a map  $\eta_X: X \rightarrow L_EX$ , where  $X$  is an  $E$ -local spectrum and  $\eta_X$  is an  $E$ -equivalence. Homological localization is universal in the following sense: the localization map  $\eta_X$  is initial among maps from  $X$  to  $E$ -local spectra and it is terminal among all  $E$ -equivalences with domain  $X$ .

The class of all the  $E$ -acyclic spectra for a given spectrum  $E$  is denoted by  $\langle E \rangle$  and called the *Bousfield class* or the *acyclicity class* of  $E$ . Given two spectra  $E$  and  $F$ , the  $E$ -localization functor and the  $F$ -localization functor are equivalent if and only if  $\langle E \rangle = \langle F \rangle$ . By Ohkawa's theorem, there is only a set of Bousfield classes [DP01], and therefore a set of non-equivalent homological localization functors.

## 3. ACYCLICITY PATTERNS OF $H\mathbb{Z}/p$ AND LOCALIZATION OF $HR$ -MODULES

In this section we study how the  $E$ -acyclicity patterns of  $H\mathbb{Z}/p$  determine the localization  $L_{E\mathbb{Z}/p}X$  for any spectrum  $E$  and any  $HR$ -module spectrum  $X$ . For any spectrum  $E$  and any abelian group  $G$ , let  $EG = E \wedge MG$ , where  $MG$  is the Moore spectrum associated to  $G$ .

A spectrum  $E$  is called a *stable  $R$ -GEM* if it is homotopy equivalent to a wedge of suspensions of Eilenberg–Mac Lane spectra, i.e.,  $E \simeq \bigvee_{k \in \mathbb{Z}} \Sigma^k HA_k$ , where each  $A_k$  is an  $R$ -module (hence, each  $HA_k$  is an  $HR$ -module spectrum). If  $R = \mathbb{Z}$ , then stable  $\mathbb{Z}$ -GEMs are called simply *stable GEMs*. The Eilenberg–Mac Lane spectrum  $HG$  is an  $R$ -GEM if  $G$  is an  $R$ -module. The stable  $R$ -GEMs are precisely the spectra that admit a module structure over the ring spectrum  $HR$  (see for example [CG05, Proposition 4.4]).

The splitting property of  $HR$ -modules allows us to describe their localization easily. Note that every  $HR$ -module is an  $H\mathbb{Z}$ -module trivially via the morphism  $\mathbb{Z} \rightarrow R$  that sends the unit of  $\mathbb{Z}$  to the unit of the ring  $R$ . And also that homological localizations commute with suspension, i.e.,  $L_E \Sigma^k HG \simeq \Sigma^k L_E HG$  for all  $k \in \mathbb{Z}$ .

$\mathbb{Z}$ , since the desuspension of an  $E$ -equivalence is again an  $E$ -equivalence [CG05, Proposition 2.4].

**Proposition 3.1.**  $L_E(\bigvee_{k \in \mathbb{Z}} \Sigma^k HA_k) \simeq \bigvee_{k \in \mathbb{Z}} (\Sigma^k L_E HA_k)$  for any spectrum  $E$ .

*Proof.* The spectrum  $\bigvee_{k \in \mathbb{Z}} (\Sigma^k L_E HA_k)$  is  $E$ -local, since the natural map

$$\bigvee_{k \in \mathbb{Z}} \Sigma^k L_E HA_k \longrightarrow \prod_{k \in \mathbb{Z}} \Sigma^k L_E HA_k$$

is a homotopy equivalence, because by [CG05, Theorem 5.6] for each value of  $k$ , at most two nonzero homotopy groups appear in  $L_E HA_k$ . Now, the map

$$\bigvee_{k \in \mathbb{Z}} \Sigma^k HA_k \longrightarrow \bigvee_{k \in \mathbb{Z}} \Sigma^k L_E HA_k$$

is an  $E$ -equivalence, because it is a wedge of  $E$ -equivalences.  $\square$

In [Bou79a], Bousfield showed that

$$(3.1) \quad \langle E \rangle = \langle E\mathbb{Q} \rangle \vee \bigvee_{p \in \mathcal{P}} \langle EZ/p \rangle,$$

for any spectrum  $E$ , where  $\mathcal{P}$  is the set of all primes. In fact, what this means essentially is that we can recover  $L_E X$  for any  $E$  and  $X$  from information on what happens rationally,  $L_{E\mathbb{Q}} X$ , and at each prime,  $L_{EZ/p} X$ . The following result of Bousfield [Bou79b, Proposition 2.9] illustrates this fact. Recall that a commutative diagram of spectra

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ h \downarrow & & \downarrow g \\ Z & \xrightarrow{i} & W \end{array}$$

is an *arithmetic square* if there is a map  $j: W \rightarrow \Sigma X$  such that

$$X \xrightarrow{(f,h)} Y \wedge Z \xrightarrow{(g,-i)} W \xrightarrow{j} \Sigma X$$

is a cofiber sequence of spectra.

**Proposition 3.2.** For all spectra  $E$  and  $X$ , there is an arithmetic square

$$(3.2) \quad \begin{array}{ccc} L_E X & \longrightarrow & \prod_{p \in \mathcal{P}} L_{EZ/p} X \\ \downarrow & & \downarrow \\ L_{E\mathbb{Q}} X & \longrightarrow & L_{E\mathbb{Q}}(\prod_{p \in \mathcal{P}} L_{EZ/p} X), \end{array}$$

where  $\mathcal{P}$  is the set of all primes.  $\square$

The  $E\mathbb{Q}$ -localizations were completely determined in [Bou79b]. For any spectrum  $E$ , all these localizations are equivalent to rationalization. In fact,  $L_{E\mathbb{Q}} X = L_{M\mathbb{Q}} X = X \wedge M\mathbb{Q}$  for all  $E$  and  $X$ .

The computation of  $L_{EZ/p} X$  for any  $E$  and any  $HR$ -module spectrum  $X$  depends on the  $E$ -acyclicity types of the spectrum  $H\mathbb{Z}/p$  for each prime  $p$ . Note that if  $\Sigma^i H\mathbb{Z}/p$  is  $E$ -acyclic for some  $i \in \mathbb{Z}$ , then  $\Sigma^k H\mathbb{Z}/p$  is  $E$ -acyclic for all  $k \in \mathbb{Z}$ , since homological localizations commute with suspension.

**Proposition 3.3.** If  $H\mathbb{Z}/p$  is  $E$ -acyclic, then  $L_{EZ/p} X = 0$  for any  $HR$ -module spectrum  $X$ .

*Proof.* It is enough to check that  $M\mathbb{Z}/p$  is  $E$ -acyclic, because in this case  $E\mathbb{Z}/p \wedge X \simeq E \wedge M\mathbb{Z}/p \wedge X = 0$ . The spectrum  $M\mathbb{Z}/p \wedge X$  is obviously  $E\mathbb{Q}$ -acyclic and  $E\mathbb{Z}/q$ -acyclic for  $q \neq p$ . In the case  $q = p$ , we have that

$$M\mathbb{Z}/p \wedge X \wedge E\mathbb{Z}/p \simeq X' \wedge H\mathbb{Z} \wedge M\mathbb{Z}/p \wedge E\mathbb{Z}/p = 0$$

since  $X$  is an  $H\mathbb{Z}$ -module and therefore splits as  $H\mathbb{Z} \wedge X'$  for some spectrum  $X'$ , and  $H\mathbb{Z} \wedge E\mathbb{Z}/p \simeq E \wedge H\mathbb{Z}/p = 0$ . Now using the decomposition (3.1), we have that  $X$  is  $E\mathbb{Z}/p$ -acyclic.  $\square$

**Lemma 3.4.** *If  $H\mathbb{Z}/p$  is not  $E$ -acyclic and  $f: X \rightarrow Y$  is an  $E\mathbb{Z}/p$ -equivalence, then it is an  $H\mathbb{Z}/p$ -equivalence.*

*Proof.* Since homological localizations commute with suspension, if we smash  $f$  with any spectrum the resulting map is an  $E\mathbb{Z}/p$ -equivalence. In particular, if we smash with the spectrum  $H\mathbb{Z}$ , the map  $f \wedge H\mathbb{Z}$  induces an equivalence

$$E\mathbb{Z}/p \wedge H\mathbb{Z} \wedge X \simeq E\mathbb{Z}/p \wedge H\mathbb{Z} \wedge Y.$$

The spectrum  $E\mathbb{Z}/p \wedge H\mathbb{Z} \simeq E \wedge H\mathbb{Z}/p$  is an  $H\mathbb{Z}/p$ -module, so its homotopy groups are  $\mathbb{Z}/p$ -vector spaces and it splits as a wedge  $\bigvee_{k \in I} \Sigma^k H\mathbb{Z}/p$  (there may be repetitions in the index set  $I$ ) and this wedge is non-trivial since by hypothesis  $E \wedge H\mathbb{Z}/p \neq 0$ . Hence,  $f$  induces an equivalence

$$\bigvee_{k \in I} \Sigma^k H\mathbb{Z}/p \wedge X \simeq \bigvee_{k \in I} \Sigma^k H\mathbb{Z}/p \wedge Y$$

which turns  $f$  into an  $H\mathbb{Z}/p$ -equivalence.  $\square$

The following theorem allows us to compute the localization  $L_{E\mathbb{Z}/p}$  of connective spectra or  $HR$ -modules when the spectrum  $H\mathbb{Z}/p$  is not  $E$ -acyclic.

**Theorem 3.5.** *If  $H\mathbb{Z}/p$  is not  $E$ -acyclic, then  $L_{E\mathbb{Z}/p}X \simeq L_{M\mathbb{Z}/p}X$  for every spectrum  $X$  that is connective or an  $HR$ -module.*

*Proof.* If  $X$  is a connective spectrum, then  $L_{M\mathbb{Z}/p}X \simeq L_{H\mathbb{Z}/p}X$  (see [Bou79b, Theorem 3.1]). The localization map  $X \rightarrow L_{M\mathbb{Z}/p}X \simeq L_{H\mathbb{Z}/p}X$  is an  $M\mathbb{Z}/p$ -equivalence and therefore an  $E\mathbb{Z}/p$ -equivalence. Moreover, the spectrum  $L_{H\mathbb{Z}/p}X$  is  $E\mathbb{Z}/p$ -local since by Lemma 3.4 every  $H\mathbb{Z}/p$ -local spectrum is  $E\mathbb{Z}/p$ -local.

If  $X$  is an  $HR$ -module, the result follows from the above and Proposition 3.1.  $\square$

The case  $L_{M\mathbb{Z}/p}X$  can be computed using [Bou79b, Proposition 2.5]:

**Proposition 3.6.** *For any spectrum  $X$ , we have that*

$$L_{M\mathbb{Z}/p}X \simeq F(\Sigma^{-1}M\mathbb{Z}/p^\infty, X),$$

and there is a splittable exact sequence

$$0 \rightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi_k(X)) \rightarrow \pi_k(L_{M\mathbb{Z}/p}X) \rightarrow \text{Hom}(\mathbb{Z}/p^\infty, \pi_{k-1}(X)) \rightarrow 0$$

for any  $k \in \mathbb{Z}$ .  $\square$

In the particular case when  $X$  is an Eilenberg–Mac Lane spectrum  $HG$ , we have that  $L_{M\mathbb{Z}/p}HG \simeq HA \vee \Sigma HB$  where  $A \cong \text{Ext}(\mathbb{Z}/p^\infty, G)$  and  $B \cong \text{Hom}(\mathbb{Z}/p^\infty, G)$ .

## 4. LOCALIZATIONS OF EILENBERG-MACLANE SPECTRA

In the study of homological localizations of  $HR$ -module spectra, we can focus our attention on the particular case of homological localizations of Eilenberg–Mac Lane spectra  $L_E HG$ , by Proposition 3.1. Homological localizations are a particular example of homotopical localizations or  $f$ -localizations. These localizations in the stable homotopy category have been studied in [CG05]. In that paper, we proved that the localization of any Eilenberg–Mac Lane spectrum has at most two nonzero homotopy groups in dimensions zero and one (see [CG05, Theorem 5.6]). Thanks to the Bousfield arithmetic square, to compute  $L_E HG$  it is enough to determine  $L_{E\mathbb{Z}/p} HG$  for every prime  $p$ , since in the rational case  $L_{E\mathbb{Q}} HG = H(\mathbb{Q} \otimes G)$  for any spectrum  $E$ .

An abelian group  $G$  is called *uniquely  $p$ -divisible* if for every  $g \in G$  there exists a unique  $h \in G$  such that  $g = ph$ . This condition is equivalent to saying that  $\mathbb{Z}/p \otimes G = 0$  and  $\text{Tor}(\mathbb{Z}/p, G) = 0$ .

**Lemma 4.1.** *For any spectrum  $E$ , the group  $\pi_k(E)$  is uniquely  $p$ -divisible for all  $k \in \mathbb{Z}$  if and only if  $E\mathbb{Z}/p = 0$ .*

*Proof.* The result follows using the exact sequence

$$\mathbb{Z}/p \otimes \pi_k(E) \longrightarrow \pi_k(E\mathbb{Z}/p) \longrightarrow \text{Tor}(\mathbb{Z}/p, \pi_{k-1}(E)),$$

which is valid for every  $k \in \mathbb{Z}$ . □

As a particular case, we have that the abelian group  $(H\mathbb{Z})_k(E)$  is uniquely  $p$ -divisible for all  $k \in \mathbb{Z}$  if and only if  $H\mathbb{Z}/p$  is  $E$ -acyclic. Note also that if  $\pi_k(E)$  is uniquely  $p$ -divisible, then  $(H\mathbb{Z})_k(E)$  is uniquely  $p$ -divisible.

**Proposition 4.2.** *If  $H\mathbb{Z}/p$  is not  $E$ -acyclic, then  $L_{E\mathbb{Z}/p} HG = 0$  if and only if  $G$  is uniquely  $p$ -divisible.*

*Proof.* If  $L_{E\mathbb{Z}/p} HG = 0$ , then  $E \wedge H\mathbb{Z}/p \wedge MG = 0$ . Since  $E \wedge H\mathbb{Z}/p$  is an  $H\mathbb{Z}/p$ -module spectrum, we have that

$$E \wedge H\mathbb{Z}/p \wedge MG = \bigvee_{k \in \mathbb{Z}} \Sigma^k H\mathbb{Z}/p \wedge MG = 0.$$

Therefore,  $H\mathbb{Z}/p \wedge MG = M\mathbb{Z}/p \wedge HG = 0$  and thus  $G$  is uniquely  $p$ -divisible by Lemma 4.1.

On the other hand, if  $G$  is uniquely  $p$ -divisible, then by Lemma 4.1 we have that  $HG \wedge M\mathbb{Z}/p = 0$  and hence  $L_{E\mathbb{Z}/p} HG = 0$ . □

By means of Theorem 3.5 and Proposition 4.2, one can now compute the localization  $L_{E\mathbb{Z}/p} HG$  depending on the  $E$ -acyclicity patterns of  $H\mathbb{Z}/p$ . If  $H\mathbb{Z}/p$  is  $E$ -acyclic, then  $L_{E\mathbb{Z}/p} HG = 0$ . If  $H\mathbb{Z}/p$  is not  $E$ -acyclic, then  $L_{E\mathbb{Z}/p} HG \simeq L_{M\mathbb{Z}/p} HG$  if  $G$  is not uniquely  $p$ -divisible and zero otherwise. The arithmetic square (3.2) in the case  $X = HG$  is the following:

$$(4.1) \quad \begin{array}{ccc} L_E HG & \longrightarrow & \prod_{p \in \mathcal{P}} L_{E\mathbb{Z}/p} HG \\ \downarrow & & \downarrow \\ H(G \otimes \mathbb{Q}) & \longrightarrow & M\mathbb{Q} \wedge (\prod_{p \in \mathcal{P}} L_{E\mathbb{Z}/p} HG), \end{array}$$

where  $\mathcal{P}$  is the set of all primes  $p$  such that  $H\mathbb{Z}/p$  is not  $E$ -acyclic and  $G$  is not uniquely  $p$ -divisible.

**Theorem 4.3.** *Let  $A_p = \text{Ext}(\mathbb{Z}/p^\infty, G)$ ,  $B_p = \text{Hom}(\mathbb{Z}/p^\infty, G)$ , and let  $\mathcal{P}$  be the set of primes such that  $H\mathbb{Z}/p$  is not  $E$ -acyclic and  $G$  is not uniquely  $p$ -divisible. For any spectrum  $E$  and any abelian group  $G$ , we have the following:*

(i) If  $H\mathbb{Q}$  is  $E$ -acyclic, then

$$L_E HG = \prod_{p \in \mathcal{P}} (HA_p \vee \Sigma HB_p).$$

(ii) If  $H\mathbb{Q}$  is not  $E$ -acyclic, then there is a cofiber sequence of spectra

$$L_E HG \longrightarrow H(\mathbb{Q} \otimes G) \vee \prod_{p \in \mathcal{P}} (HA_p \vee \Sigma HB_p) \longrightarrow M\mathbb{Q} \wedge \prod_{p \in \mathcal{P}} (HA_p \vee \Sigma HB_p).$$

*Proof.* The result follows from Proposition 3.3, Proposition 4.2, Theorem 3.5 and the arithmetic square (4.1).  $\square$

## 5. SOME EXAMPLES

In this section, we compute homological localizations of Eilenberg–MacLane spectra and  $HR$ -module spectra in some concrete examples. First, we compute  $L_E X$  for some non-connective homology theories  $E$  and any  $HR$ -module spectrum  $X$ .

**5.1. Localization with respect to  $n$ -th Morava  $K$ -theory  $K(n)$ .** Let  $n \geq 0$ ,  $p$  a fixed prime and  $K(n)$  the spectrum of the  $n$ -th Morava  $K$ -theory at  $p$ . Recall that  $\pi_* K(n) \cong \mathbb{Z}/p[v_n^{-1}, v_n]$  where  $|v_n| = 2(p^n - 1)$  for  $n \geq 1$ .

If  $n = 0$ , then  $K(0) = H\mathbb{Q} = M\mathbb{Q}$  and so  $L_{K(0)} HG = H(G \otimes \mathbb{Q})$ . In the case  $n \geq 1$  we know that  $K(n) \wedge M\mathbb{Q} = 0$  and  $H\mathbb{Z}/p$  is  $K(n)$ -acyclic for every prime  $p$ , because  $K(n) \wedge H\mathbb{Z}/p = 0$  for all primes  $p$  (see for example [Rav84, Theorem 2.1]). Thus  $L_{K(n)} HG = 0$  for  $n \geq 1$ . Hence,

**Proposition 5.1.** *For any  $HR$ -module  $X$ , its localization with respect to  $K(n)$  is either zero if  $n \geq 1$ , or rationalization if  $n = 0$ , i.e.,  $L_{K(0)} X = X \wedge M\mathbb{Q}$ .  $\square$*

**5.2. Localization with respect to Johnson–Wilson spectra  $E(n)$ .** The Bousfield class of  $E(n)$  splits as a wedge of Morava  $K$ -theories,  $\langle E(n) \rangle = \langle K(0) \vee \dots \vee K(n) \rangle$  (see [Rav84, Theorem 2.1]); therefore  $L_{E(n)} HG = L_{K(0)} HG = H(G \otimes \mathbb{Q})$  since  $L_{K(i)} HG = 0$  if  $i \geq 0$ .

**5.3. Localization with respect to complex  $K$ -theory.** The spectrum  $H\mathbb{Z}/p$  is  $K$ -acyclic for every prime  $p$  and  $K\mathbb{Q} \neq 0$ , so  $L_K HG = H(G \otimes \mathbb{Q})$ . Therefore, we infer the following:

**Proposition 5.2.** *For any  $HR$ -module spectrum  $X$ , its localization with respect to  $E(n)$  or  $K$ -theory is rationalization.  $\square$*

In the next examples, we use Theorem 4.3 to compute all the possible homological localizations of the spectrum  $HG$  with respect to any  $E$  for some families of abelian groups. Given any spectrum  $E$  and any abelian group  $G$ , we have the following acyclicity patterns that determine the localization  $L_E HG$  completely. These patterns are the stable analogues of Condition I and Condition II of [Bou82, Section 4]:

- Pattern I:  $E\mathbb{Q} = 0$  and  $E \wedge H\mathbb{Z}/p = 0$  for all primes  $p$ .
- Pattern II:  $E\mathbb{Q} \neq 0$  and  $E \wedge H\mathbb{Z}/p = 0$  for all primes  $p$ .
- Pattern III:  $E\mathbb{Q} = 0$  and  $E \wedge H\mathbb{Z}/p \neq 0$  for all primes  $p$  in a set of primes  $\mathcal{P}$ .
- Pattern IV:  $E\mathbb{Q} \neq 0$  and  $E \wedge H\mathbb{Z}/p \neq 0$  for all primes  $p$  in a set of primes  $\mathcal{P}$ .

Note that if Pattern I holds, we have that  $L_E HG = 0$  for any abelian group  $G$ .

**5.4. Localizations of  $H\mathbb{Z}$ .** The abelian group of the integers is not uniquely  $p$ -divisible for any prime  $p$ . If Pattern II holds, then  $L_E H\mathbb{Z} = H\mathbb{Q}$ . We have that

$$\mathrm{Hom}(\mathbb{Z}/p^\infty, \mathbb{Z}) = 0 \quad \text{and} \quad \mathrm{Ext}(\mathbb{Z}/p^\infty, \mathbb{Z}) = \widehat{\mathbb{Z}}_p,$$

where  $\widehat{\mathbb{Z}}_p$  is the ring of  $p$ -adic integers. If Pattern III holds, then  $L_E H\mathbb{Z} = H(\prod_{p \in \mathcal{P}} \widehat{\mathbb{Z}}_p)$ . And if Pattern IV holds, then taking  $\pi_0$  in the square (4.1) we have the following pullback diagram of abelian groups:

$$\begin{array}{ccc} \pi_0(L_E H\mathbb{Z}) & \longrightarrow & \prod_{p \in \mathcal{P}} \widehat{\mathbb{Z}}_p \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes \prod_{p \in \mathcal{P}} \widehat{\mathbb{Z}}_p, \end{array}$$

where  $\mathcal{P}$  is the set of all primes  $p$  such that  $E \wedge H\mathbb{Z}/p \neq 0$ . So  $L_E H\mathbb{Z} = H\mathbb{Z}_{\mathcal{P}}$ .

In [CG05, Theorem 5.12] we proved that every  $f$ -localization of the spectrum  $H\mathbb{Z}$  has at most one nonzero homotopy group, which acquires the structure of a rigid ring in the sense of [CRT00]. A ring  $A$  with unit is *rigid* if evaluation at 1 induces an isomorphism of abelian groups  $\mathrm{Hom}(A, A) \cong A$ . In the special case of homological localizations we get the following:

**Proposition 5.3.** *For any spectrum  $E$ , we have that  $L_E H\mathbb{Z}$  is either zero or  $HA$ , where the rigid ring  $A$  is a subring of  $\mathbb{Q}$  or a product of  $p$ -adic integers for different primes.  $\square$*

**5.5. Localizations of  $H\mathbb{Z}/p^k$  for a prime  $p$ .** The group  $\mathbb{Z}/p^k$  is uniquely  $q$ -divisible for every  $q \neq p$  and moreover  $L_{E\mathbb{Q}} H\mathbb{Z}/p^k \simeq H(\mathbb{Q} \otimes \mathbb{Z}/p^k) = 0$  for all  $p$ . We have that

$$\mathrm{Hom}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^k) = 0 \quad \text{and} \quad \mathrm{Ext}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^k) = \mathbb{Z}/p^k,$$

hence  $L_E H\mathbb{Z}/p^k = 0$  under Pattern II and  $L_E H\mathbb{Z}/p^k = H\mathbb{Z}/p^k$  under Pattern III or Pattern IV.

**5.6. Localizations of  $H\mathbb{Q}$ .** The group  $\mathbb{Q}$  is uniquely  $p$ -divisible for every prime  $p$ , so  $L_{E\mathbb{Z}/p} H\mathbb{Q} = 0$  for all  $p$ . If Pattern III holds, then  $L_E H\mathbb{Q} = 0$  and  $L_E H\mathbb{Q} = H\mathbb{Q}$  under Pattern II or Pattern IV.

**5.7. Localization of  $H\mathbb{Z}_{\mathcal{R}}$  for a set of primes  $\mathcal{R}$ .** For every prime  $p \in \mathcal{R}$ , we have that

$$\mathrm{Hom}(\mathbb{Z}/p^\infty, \mathbb{Z}_{\mathcal{R}}) = 0 \quad \text{and} \quad \mathrm{Ext}(\mathbb{Z}/p^\infty, \mathbb{Z}_{\mathcal{R}}) = \widehat{\mathbb{Z}}_p.$$

In fact,  $\mathrm{Hom}(\mathbb{Z}/p^\infty, G) = 0$  if  $G$  is a torsion-free abelian group and  $\mathrm{Ext}(\mathbb{Z}/p^\infty, G) = 0$  if and only if  $G$  is  $p$ -divisible. If Pattern II holds, then  $L_E H\mathbb{Z}_{\mathcal{R}} = H\mathbb{Q}$  because  $\mathbb{Q} \otimes \mathbb{Z}_{\mathcal{R}} \cong \mathbb{Q}$ . If Pattern III holds, then  $L_E H\mathbb{Z}_{\mathcal{R}} = H(\prod_{p \in \mathcal{R} \cap \mathcal{P}} \widehat{\mathbb{Z}}_p)$ . And if Pattern IV holds, then  $L_E H\mathbb{Z}_{\mathcal{R}} = H\mathbb{Z}_{\mathcal{R} \cap \mathcal{P}}$ , where  $\mathcal{P}$  is the set of all primes  $p$  such that  $H\mathbb{Z}/p$  is not  $E$ -acyclic. Note that this case generalizes the cases of the localization of  $H\mathbb{Z}$  (when  $\mathcal{R} = \emptyset$ ) and  $H\mathbb{Q}$  (when  $\mathcal{R}$  is the set of all primes).

**5.8. Localizations of  $H\mathbb{Z}/p^\infty$ .** The group  $\mathbb{Z}/p^\infty$  is uniquely  $p$ -divisible for every prime  $q \neq p$ . In this case,  $L_{E\mathbb{Q}} H\mathbb{Z}/p^\infty = 0$  since  $\mathbb{Q} \otimes \mathbb{Z}/p^\infty = 0$  for all  $p$ . We have that

$$\mathrm{Hom}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty) = \widehat{\mathbb{Z}}_p \quad \text{and} \quad \mathrm{Ext}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty) = 0.$$

Thus, under Pattern II,  $L_E H\mathbb{Z}/p^\infty = 0$ . If Pattern III holds, then  $L_E H\mathbb{Z}/p^\infty \simeq \Sigma H\widehat{\mathbb{Z}}_p$ . And if Pattern IV holds, then by Theorem 4.3 we have a cofiber sequence of spectra

$$L_E H\mathbb{Z}/p^\infty \longrightarrow \Sigma H\widehat{\mathbb{Z}}_p \longrightarrow \Sigma H\widehat{\mathbb{Q}} \longrightarrow \Sigma H(\widehat{\mathbb{Q}}/\widehat{\mathbb{Z}}_p),$$

where  $\widehat{\mathbb{Q}} \cong \widehat{\mathbb{Z}}_p \otimes \mathbb{Q}$  are the  $p$ -adic rationals. Hence,  $L_E H\mathbb{Z}/p^\infty = H(\widehat{\mathbb{Q}}/\widehat{\mathbb{Z}}_p) \simeq H\mathbb{Z}/p^\infty$ .

**5.9. Localization of  $H\widehat{\mathbb{Z}}_p$ .** We only have to focus on the prime  $p$ , because  $\widehat{\mathbb{Z}}_p$  is uniquely  $q$  divisible for all primes  $q \neq p$ . In this case, we have that

$$\mathrm{Hom}(\mathbb{Z}/p^\infty, \widehat{\mathbb{Z}}_p) = 0 \quad \text{and} \quad \mathrm{Ext}(\mathbb{Z}/p^\infty, \widehat{\mathbb{Z}}_p) = \widehat{\mathbb{Z}}_p.$$

If Pattern II holds, then  $L_E H\widehat{\mathbb{Z}}_p = H\widehat{\mathbb{Q}}_p$ . And if Pattern III or Pattern IV hold, then  $L_E H\widehat{\mathbb{Z}}_p = H\widehat{\mathbb{Z}}_p$ .

The following table summarizes the results obtained for the homological localizations of Eilenberg–Mac Lane spectra for different groups. The set  $\mathcal{P}$  is the set of primes  $p$  such that  $H\mathbb{Z}/p$  is not  $E$ -acyclic.

	Pattern A	Pattern B	Pattern C	Pattern D
$L_E H\mathbb{Z}$	0	$H\mathbb{Q}$	$\prod_{p \in \mathcal{P}} H\widehat{\mathbb{Z}}_p$	$H\mathbb{Z}_{\mathcal{P}}$
$L_E H\mathbb{Z}/p^k$	0	0	$H\mathbb{Z}/p^k$	$H\mathbb{Z}/p^k$
$L_E H\mathbb{Q}$	0	$H\mathbb{Q}$	0	$H\mathbb{Q}$
$L_E H\mathbb{Z}_{\mathcal{R}}$	0	$H\mathbb{Q}$	$\prod_{p \in \mathcal{P} \cap \mathcal{R}} H\widehat{\mathbb{Z}}_p$	$H\mathbb{Z}_{\mathcal{P} \cap \mathcal{R}}$
$L_E H\mathbb{Z}/p^\infty$	0	0	$\Sigma H\widehat{\mathbb{Z}}_p$	$H\mathbb{Z}/p^\infty$
$L_E H\widehat{\mathbb{Z}}_p$	0	$H\widehat{\mathbb{Q}}_p$	$H\widehat{\mathbb{Z}}_p$	$H\widehat{\mathbb{Z}}_p$

**5.10. Localization of  $HG$  where  $G$  is a finitely generated abelian group.**

Every finitely generated abelian group splits as a direct sum  $G = \bigoplus_{i=1}^n C_i$  where each  $C_i$  is either  $\mathbb{Z}$  or  $\mathbb{Z}/p^k$  for some prime  $p$  and  $k \geq 1$ . Since  $HG \simeq \bigvee_{i=1}^n HC_i$ , then  $L_E HG = \bigvee_{i=1}^n L_E HC_i$  and the localization of each  $HC_i$  is determined using the results of sections 5.4 and 5.5.

**5.11. Localization of  $HG$  where  $G$  is a divisible abelian group.**

If  $G$  is a divisible abelian group, then  $G \cong R \oplus T$ , where  $R = \bigoplus_i \mathbb{Q}$  and  $T = \bigoplus_p (\bigoplus_{j \geq 1} \mathbb{Z}/p^j)$ . In this case  $L_E HG \simeq L_E HR \vee L_E HT$ . Since  $R$  is a retract of  $\prod_i \mathbb{Q}$ , we have that  $L_E HR = HR$  or  $L_E HR = 0$  depending on whether  $H\mathbb{Q}$  is  $E$ -local or  $E$ -acyclic. The localization  $L_E HT$  can be determined using the exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}_{\mathcal{P}} \longrightarrow \mathbb{Q} \longrightarrow \bigoplus_{p \in \mathcal{P}} \mathbb{Z}/p^\infty \longrightarrow 0$$

together with the results of sections 5.6 and 5.8, and the fact that homological localizations preserve cofiber sequences.

## 6. LOCALIZATION OF REDUCED EILENBERG-MAC LANE SPECTRA

In all the examples we have studied, except in the case of  $H\mathbb{Z}/p^\infty$ , all the homological localizations of  $HG$  have at most one nonzero homotopy group in dimension zero. This property also holds when the group  $G$  is abelian reduced. An abelian group is *reduced* if it does not have nontrivial divisible subgroups. We say that the Eilenberg–Mac Lane spectrum  $HG$  is *reduced* if the group  $G$  is reduced.

**Theorem 6.1.** *If  $HG$  is reduced, then  $L_E HG$  is either zero or  $HA$  for some abelian group  $A$  and for any spectrum  $E$ .*

*Proof.* If  $G$  is reduced, then  $\mathrm{Hom}(\mathbb{Z}/p^\infty, G) = 0$ . The result follows now from Theorem 4.3.  $\square$

Any abelian group  $G$  splits as a direct sum  $G \cong G_1 \oplus G_2$ , where  $G_1$  is the maximal divisible subgroup of  $G$  and  $G_2$  is reduced. Moreover,  $G_1$  splits as a direct sum of  $\mathbb{Q}$ 's and  $\mathbb{Z}/p^\infty$  for several primes  $p$ . Hence, by Theorem 4.3 and the results in the previous section, the only possibility for the homological localization of an Eilenberg–Mac Lane spectrum  $HG$  to have a nonzero homotopy group in dimension one, is that some  $\mathbb{Z}/p^\infty$  appears as a factor of the decomposition of  $G$  and that  $L_E H\mathbb{Z}/p^\infty = \Sigma H\widehat{\mathbb{Z}}_p$ .

**Corollary 6.2.** *If  $\mathbb{Z}/p^\infty$  does not occur as a direct summand of  $G$  for any prime  $p$ , then  $L_E HG$  is either zero or  $HA$  for some abelian group  $A$  and any spectrum  $E$ .*

□

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