

# HOMOTOPY THEORY OF MODULES OVER OPERADS AND NON- $\Sigma$ OPERADS IN MONOIDAL MODEL CATEGORIES

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## 1. INTRODUCTION

There are many interesting situations in which algebraic structure can be described by operads [1, 12, 13, 14, 17, 20, 27, 32, 33, 34, 35]. Let  $(\mathbf{C}, \otimes, \mathbf{k})$  be a symmetric monoidal closed category (Section 2) with all small limits and colimits. It is possible to define two types of operads (Definition 6.1) in this setting, as well as algebras and modules over these operads. One type, called  $\Sigma$ -operad, is based on finite sets and incorporates symmetric group actions; the other type, called non- $\Sigma$  operad, is based on ordered sets and has no symmetric group contribution. (In this paper we use the term  $\Omega$ -operad for non- $\Sigma$  operad, where  $\Omega = \mathbf{O}$  for “Ordered.”) Given an operad  $\mathcal{O}$ , we are interested in the possibility of doing homotopy theory in the categories of  $\mathcal{O}$ -modules and  $\mathcal{O}$ -algebras, which in practice means putting a Quillen model structure on these categories of modules and algebras. In this setting,  $\mathcal{O}$ -algebras are left  $\mathcal{O}$ -modules concentrated at 0 (Section 6.3).

Of course, to get started we need some kind of homotopy theoretic structure on  $\mathbf{C}$  itself; this structure should mesh appropriately with the monoidal structure on  $\mathbf{C}$ . The basic assumption is this.

**Basic Assumption 1.1.** *From now on in this paper we assume that  $(\mathbf{C}, \otimes, \mathbf{k})$  is a symmetric monoidal closed category (Section 2) with all small limits and colimits, that  $\mathbf{C}$  is a cofibrantly generated model category (Definition 11.5) in which the generating cofibrations and acyclic cofibrations have small domains, and that with respect to this model structure  $(\mathbf{C}, \otimes, \mathbf{k})$  is a monoidal model category (Definition 11.7).*

The main theorem is this.

**Theorem 1.2.** *Assume that  $\mathbf{C}$  satisfies Basic Assumption 1.1 and in addition satisfies the monoid axiom (Definition 13.25). Let  $\mathcal{O}$  be an  $\Omega$ -operad in  $\mathbf{C}$ . Then the category of left  $\mathcal{O}$ -modules and the category of  $\mathcal{O}$ -algebras both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the weak equivalences and fibrations in  $\mathbf{C}$ .*

*Remark 1.3.* Given any  $\Omega$ -operad  $\mathcal{O}$ , there is an associated  $\Sigma$ -operad  $\mathcal{O} \cdot \Sigma$ , such that algebras over  $\mathcal{O} \cdot \Sigma$  are the same as algebras over  $\mathcal{O}$  (Section 7). It follows easily from the above theorem that if  $\mathcal{O}'$  is a  $\Sigma$ -operad which is a retract of  $\mathcal{O} \cdot \Sigma$ , then the category of algebras over  $\mathcal{O}'$  has a natural model category structure.

The above remark shows how to handle algebras over certain  $\Sigma$ -operads. We can do a lot better if  $\mathbf{C}$  satisfies a strong cofibrancy condition. In setting up the machinery for Theorems 1.2 and 1.4, we introduce model category structures on

the category of (*symmetric*) sequences in  $\mathcal{C}$  (Definition 3.1) and on the category of (*symmetric*) arrays in  $\mathcal{C}$  (Definition 13.1).

**Theorem 1.4.** *Assume that  $\mathcal{C}$  satisfies Basic Assumption 1.1 and in addition that every symmetric array (resp. symmetric sequence) in  $\mathcal{C}$  is cofibrant in the model category structure described below (Theorems 13.2 and 12.2). Then for any  $\Sigma$ -operad  $\mathcal{O}$  in  $\mathcal{C}$ , the category of left  $\mathcal{O}$ -modules (resp.  $\mathcal{O}$ -algebras) has a natural model category structure. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the weak equivalences and fibrations in  $\mathcal{C}$ .*

**1.1. Some examples of interest.** The hypotheses of these theorems may seem restrictive, but in fact they allow, especially in the case of Theorem 1.2, for many interesting examples including the case  $(\mathbf{sSet}, \times, *)$  of simplicial sets [5, 9, 15, 37], the case  $(\mathbf{Ch}_k, \otimes, k)$  of unbounded chain complexes over a commutative ring with unit [22, 29], and the case  $(\mathbf{Sp}^\Sigma, \wedge, S)$  of symmetric spectra [24]. In a related paper [16], we improve Theorem 1.2 to  $\Sigma$ -operads for the case  $(\mathbf{Sp}^\Sigma, \wedge, S)$  of symmetric spectra.

**1.2. Relationship to previous work.** One of the main theorems of Schwede and Shipley [39] is that the category of monoids in  $(\mathcal{C}, \otimes, k)$  has a natural model category structure, provided the *monoid axiom* (Definition 13.25) is satisfied. Theorem 1.2 improves this result to left modules and algebras over any  $\Omega$ -operad.

One of the main theorems of Hinich [18, 19] is that for unbounded chain complexes over a field of characteristic zero, the category of algebras over any  $\Sigma$ -operad has a natural model category structure. Theorem 1.4 improves this result to the category of left modules, and also provides (Section 14) a simplified conceptual proof of Hinich's original result. In this rational case our theorem is this.

**Theorem 1.5.** *Let  $k$  be a field of characteristic zero and let  $(\mathbf{Ch}_k, \otimes, k)$  be the symmetric monoidal closed category of unbounded chain complexes over  $k$ . Let  $\mathcal{O}$  be any  $\Sigma$ -operad or  $\Omega$ -operad. Then the category of left  $\mathcal{O}$ -modules (resp.  $\mathcal{O}$ -algebras) has a natural model category structure. The weak equivalences are the objectwise homology isomorphisms (resp. homology isomorphisms) and the fibrations are the objectwise dimensionwise surjections (resp. dimensionwise surjections).*

Another theorem of Hinich [18] is that for unbounded chain complexes over a commutative ring with unit, the category of algebras over any  $\Sigma$ -operad of the form  $\mathcal{O} \cdot \Sigma$  for some  $\Omega$ -operad  $\mathcal{O}$ , has a natural model category structure. Theorem 1.2 improves this result to the category of left modules. Our theorem is this.

**Theorem 1.6.** *Let  $k$  be a commutative ring with unit and let  $(\mathbf{Ch}_k, \otimes, k)$  be the symmetric monoidal closed category of unbounded chain complexes over  $k$ . Let  $\mathcal{O}$  be any  $\Omega$ -operad. Then the category of left  $\mathcal{O}$ -modules (resp.  $\mathcal{O}$ -algebras) has a natural model category structure. The weak equivalences are the objectwise homology isomorphisms (resp. homology isomorphisms) and the fibrations are the objectwise dimensionwise surjections (resp. dimensionwise surjections).*

One of the main theorems of Elmendorf and Mandell [6] is that the category of simplicial multifunctors from a small multicategory (enriched over simplicial sets) to the category of symmetric spectra has a natural simplicial model category structure. Their proof involves a filtration in the underlying category of certain pushouts

of algebras. We have benefitted from their paper and our proofs of Theorems 1.2 and 1.4 exploit similar filtrations (Section 13). In the special case of unbounded chain complexes, the analysis of certain pushouts reduces to an analysis of certain coproducts and filtration constructions are not required. We have included a shortened proof for this special case (Section 14).

The framework presented in this paper for doing homotopy theory in the categories of modules and algebras over an operad is largely influenced by Rezk [38].

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## 2. PRELIMINARIES ON GROUP ACTIONS

Here, for reference purposes, we collect certain basic properties of group actions and adjunctions involving group actions. Some of the statements we organize into propositions. Their proofs are left to the reader.

*Remark 2.1.* The model category assumptions on  $\mathbf{C}$  stated in Basic Assumption 1.1 will not be needed until Section 12.

**2.1. Symmetric monoidal closed categories.** By Basic Assumption 1.1,  $(\mathbf{C}, \otimes, \mathbf{k})$  is a symmetric monoidal closed category with all small limits and colimits.

In particular,  $\mathbf{C}$  has an initial object  $\emptyset$  and a terminal object  $*$ . See [30, VII] for monoidal categories and [30, VII.7] for symmetric monoidal categories. By *closed* we mean there exists a functor

$$\mathbf{C}^{\text{op}} \times \mathbf{C} \longrightarrow \mathbf{C}, \quad (Y, Z) \longmapsto \text{Map}(Y, Z),$$

which we call *mapping object* (or cotensor object), which fits into isomorphisms

$$(2.2) \quad \text{hom}_{\mathbf{C}}(X \otimes Y, Z) \cong \text{hom}_{\mathbf{C}}(X, \text{Map}(Y, Z)),$$

natural in  $X, Y, Z$ .

*Remark 2.3.* This condition is stronger than only requiring each functor  $- \otimes Y : \mathbf{C} \longrightarrow \mathbf{C}$  to have a specified right adjoint  $\text{Map}(Y, -) : \mathbf{C} \longrightarrow \mathbf{C}$ .

**2.2. Group actions and  $G$ -objects.** The symmetric monoidal closed structure on  $\mathbf{C}$  induces a corresponding structure on certain diagram categories.

**Definition 2.4.** Let  $G$  be a finite group.  $\mathbf{C}^{G^{\text{op}}}$  is the category with objects the functors  $X : G^{\text{op}} \longrightarrow \mathbf{C}$  and morphisms their natural transformations.  $\mathbf{C}^G$  is the category with objects the functors  $X : G \longrightarrow \mathbf{C}$  and morphisms their natural transformations.

The diagram category  $\mathbf{C}^{G^{\text{op}}}$  (resp.  $\mathbf{C}^G$ ) is isomorphic to the category of objects in  $\mathbf{C}$  with a specified right (resp. left) action of  $G$ .

**Proposition 2.5.** *Let  $G$  be a finite group. Then  $(\mathbf{C}^{G^{\text{op}}}, \otimes, \mathbf{k})$  has a symmetric monoidal closed structure induced from the symmetric monoidal closed structure on  $(\mathbf{C}, \otimes, \mathbf{k})$ . In particular, there are isomorphisms*

$$\text{hom}_{\mathbf{C}^{G^{\text{op}}}}(X \otimes Y, Z) \cong \text{hom}_{\mathbf{C}^{G^{\text{op}}}}(X, \text{Map}(Y, Z))$$

*natural in  $X, Y, Z$ .*

The proposition remains true when  $\mathbf{C}^{G^{\text{op}}}$  is replaced by  $\mathbf{C}^G$ . We usually leave such corresponding statements to the reader.

**2.3. Copower constructions.** If  $X$  is a finite set, let's define  $|X|$  to be the number of elements in  $X$ .

**Definition 2.6.** Let  $X$  be a finite set and  $A \in \mathbf{C}$ . The *copowers*  $A \cdot X \in \mathbf{C}$  and  $X \cdot A \in \mathbf{C}$  are defined by the same construction:

$$A \cdot X := \coprod_X A, \quad X \cdot A := \coprod_X A,$$

the coproduct in  $\mathbf{C}$  of  $|X|$ -copies of  $A$ .

Sometimes extra structure on the set  $X$  induces extra structure on the objects  $A \cdot X$  and  $X \cdot A$ . For example, if  $G$  is a finite group, then  $A \cdot G$  and  $G \cdot A$  have naturally occurring right and left actions of  $G$ , respectively, and hence  $A \cdot G \in \mathbf{C}^{G^{\text{op}}}$  and  $G \cdot A \in \mathbf{C}^G$ .

*Remark 2.7.* In the literature, copower is sometimes indicated by a tensor product symbol, but because several tensor products already appear in this paper, we are using the usual dot notation as in [30].

When  $\mathbf{C} = \mathbf{sSet}$  there are natural isomorphisms  $A \cdot G \cong A \times G$ , when  $\mathbf{C} = \mathbf{Ch}_k$  there are natural isomorphisms  $A \cdot G \cong A \otimes_k [G]$ , and when  $\mathbf{C} = \mathbf{Sp}^\Sigma$  there are natural isomorphisms  $A \cdot G \cong A \wedge G_+$ . Since left Kan extensions may be calculated objectwise in terms of copower, the copower construction appears in several adjunctions below.

**2.4.  $G$ -orbits and  $G$ -fixed points.**

**Definition 2.8.** Let  $G$  be a finite group. If  $Y : G^{\text{op}} \times G \rightarrow \mathbf{C}$  and  $Z : G \times G^{\text{op}} \rightarrow \mathbf{C}$  are functors, then  $Y_G \in \mathbf{C}$  and  $Z^G \in \mathbf{C}$  are defined by

$$Y_G := \text{coend } Y, \quad Z^G := \text{end } Z.$$

The universal properties satisfied by these coends and ends are convenient when working with  $Y_G$  and  $Z^G$ , but the reader may take the following calculations as definitions. There are natural isomorphisms,

$$Y_G \cong \text{colim}(G^{\text{op}} \xrightarrow{\text{diag}} G^{\text{op}} \times G^{\text{op}} \cong G^{\text{op}} \times G \xrightarrow{Y} \mathbf{C}),$$

$$Z^G \cong \text{lim}(G^{\text{op}} \xrightarrow{\text{diag}} G^{\text{op}} \times G^{\text{op}} \cong G \times G^{\text{op}} \xrightarrow{Z} \mathbf{C}).$$

### 2.5. Adjunctions.

**Proposition 2.9.** *Let  $G$  be a finite group,  $H \subseteq G$  a subgroup, and  $l : H \rightarrow G$  the inclusion of groups. Let  $G_1, G_2$  be finite groups and  $A_2 \in \mathbf{C}^{G_2^{\text{op}}}$ . There are adjunctions*

$$(2.10) \quad \mathbf{C} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\text{lim}} \end{array} \mathbf{C}^{H^{\text{op}}} \begin{array}{c} \xrightarrow{- \cdot_H G} \\ \xleftarrow{l^*} \end{array} \mathbf{C}^{G^{\text{op}}}, \quad \mathbf{C}^{G_1^{\text{op}}} \begin{array}{c} \xrightarrow{- \otimes A_2} \\ \xleftarrow{\text{Map}(A_2, -)^{G_2}} \end{array} \mathbf{C}^{(G_1 \times G_2)^{\text{op}}},$$

with left adjoints on top. In particular, there are isomorphisms

$$\begin{aligned} \text{hom}_{\mathbf{C}^{G^{\text{op}}}}(A \cdot_H G, B) &\cong \text{hom}_{\mathbf{C}^{H^{\text{op}}}}(A, B), \\ \text{hom}_{\mathbf{C}^{G^{\text{op}}}}(A \cdot (H \setminus G), B) &\cong \text{hom}_{\mathbf{C}}(A, B^H), \\ \text{hom}_{\mathbf{C}^{G^{\text{op}}}}(A \cdot G, B) &\cong \text{hom}_{\mathbf{C}}(A, B), \\ \text{hom}_{\mathbf{C}^{(G_1 \times G_2)^{\text{op}}}}(A_1 \otimes A_2, X) &\cong \text{hom}_{\mathbf{C}^{G_1^{\text{op}}}}(A_1, \text{Map}(A_2, X)^{G_2}), \end{aligned}$$

natural in  $A, B$  and  $A_1, X$ .

*Remark 2.11.* The restriction functor  $l^*$  is sometimes dropped from the notation, as in the natural isomorphisms in Proposition 2.9.

### 3. SEQUENCES AND SYMMETRIC SEQUENCES

In preparation for defining operads, we consider sequences and symmetric sequences of objects in  $\mathbf{C}$ . We introduce a symmetric monoidal structure  $\otimes$  on  $\text{SymSeq}$ , and a symmetric monoidal structure  $\hat{\otimes}$  on  $\text{Seq}$ . Both of these are relatively simple;  $\otimes$  is a form of the symmetric monoidal structure that is used in the construction of symmetric spectra [23, 24], while  $\hat{\otimes}$  is defined in a way that is very similar to the definition of the graded tensor product of chain complexes. These monoidal products possess appropriate adjoints, which can be interpreted as mapping objects. For instance, there are objects  $\text{Map}^{\otimes}(B, C)$  and  $\text{Map}^{\hat{\otimes}}(Y, Z)$  which fit into isomorphisms

$$\begin{aligned} \text{hom}(A \otimes B, C) &\cong \text{hom}(A, \text{Map}^{\otimes}(B, C)), \\ \text{hom}(X \hat{\otimes} Y, Z) &\cong \text{hom}(X, \text{Map}^{\hat{\otimes}}(Y, Z)), \end{aligned}$$

natural in the symmetric sequences  $A, B, C$  and the sequences  $X, Y, Z$ .

**3.1. Sequences and symmetric sequences.** Let's define the sets  $\mathbf{n} := \{1, \dots, n\}$  for each  $n \geq 0$ , where  $\mathbf{0} := \emptyset$  denotes the empty set. When regarded as a totally ordered set,  $\mathbf{n}$  is given its natural ordering.

**Definition 3.1.** Let  $n \geq 0$ .

- $\Sigma$  is the category of finite sets and their bijections.  $\Omega$  is the category of totally ordered finite sets and their order preserving bijections.
- A *symmetric sequence* in  $\mathbf{C}$  is a functor  $A : \Sigma^{\text{op}} \rightarrow \mathbf{C}$ . A *sequence* in  $\mathbf{C}$  is a functor  $X : \Omega^{\text{op}} \rightarrow \mathbf{C}$ .  $\text{SymSeq} := \mathbf{C}^{\Sigma^{\text{op}}}$  is the category of symmetric sequences in  $\mathbf{C}$  and their natural transformations.  $\text{Seq} := \mathbf{C}^{\Omega^{\text{op}}}$  is the category of sequences in  $\mathbf{C}$  and their natural transformations.
- A (symmetric) sequence  $A$  is *concentrated at  $n$*  if  $A[\mathbf{s}] = \emptyset$  for all  $s \neq n$ .

**3.2. Small skeletons and equivalent categories.** The indexing categories for symmetric sequences and sequences are not small, but they have small skeletons, which will be useful for calculations.

**Definition 3.2.**

- $\Sigma_n$  is the category with exactly one object  $\mathbf{n}$  and morphisms the bijections of sets.  $\Omega_n$  is the category with exactly one object  $\mathbf{n}$  and morphisms the identity map.
- $\Sigma'$  is the category with objects the sets  $\mathbf{n}$  for  $n \geq 0$  and morphisms the bijections of sets.  $\Omega'$  is the category with objects the totally ordered sets  $\mathbf{n}$  for  $n \geq 0$  and morphisms the identity maps.
- A *small symmetric sequence* in  $\mathbf{C}$  is a functor  $A : \Sigma'^{\text{op}} \rightarrow \mathbf{C}$ . A *small sequence* in  $\mathbf{C}$  is a functor  $X : \Omega'^{\text{op}} \rightarrow \mathbf{C}$ .  $\text{SymSeq}' := \mathbf{C}^{\Sigma'^{\text{op}}}$  is the category of small symmetric sequences in  $\mathbf{C}$  and their natural transformations.  $\text{Seq}' := \mathbf{C}^{\Omega'^{\text{op}}}$  is the category of small sequences in  $\mathbf{C}$  and their natural transformations.

The indexing categories  $\Sigma'^{\text{op}} \cong \coprod_{n \geq 0} \Sigma_n^{\text{op}}$  and  $\Omega'^{\text{op}} \cong \coprod_{n \geq 0} \Omega_n^{\text{op}}$  are coproducts of categories, hence giving functors  $A : \Sigma'^{\text{op}} \rightarrow \mathbf{C}$  and  $X : \Omega'^{\text{op}} \rightarrow \mathbf{C}$  is the same as giving collections  $\{A[\mathbf{n}]\}_{n \geq 0}$  and  $\{X[\mathbf{n}]\}_{n \geq 0}$  of objects in  $\mathbf{C}$  such that each  $A[\mathbf{n}]$  is equipped with a right action of the symmetric group  $\Sigma_n$ .

The inclusions  $i : \Sigma' \rightarrow \Sigma$  and  $i : \Omega' \rightarrow \Omega$  are equivalences of categories,

$$(3.3) \quad \Sigma \begin{array}{c} \xrightarrow{j} \\ \xleftarrow{i} \end{array} \Sigma', \quad \Omega \begin{array}{c} \xrightarrow{j} \\ \xleftarrow{i} \end{array} \Omega'.$$

In particular,  $i$  has a left adjoint  $j$  with corresponding unit  $\eta : \text{id} \rightarrow ij$  and counit  $\varepsilon : ji \rightarrow \text{id}$  isomorphisms.

*Remark 3.4.* Giving the equivalence of categories  $j : \Sigma \rightarrow \Sigma'$  is the same as giving a choice of bijection  $\eta_T : T \rightarrow |\mathbf{T}| = ij(T)$  (i.e., a choice of ordering) for each finite set  $T$ .

The equivalences of categories (3.3) induce equivalences of categories,

$$(3.5) \quad \text{SymSeq} \rightleftarrows \text{SymSeq}', \quad \text{Seq} \rightleftarrows \text{Seq}'.$$

**3.3. Ordered and unordered tensor products.** Given any non-empty totally ordered finite set  $N$  (i.e., a non-empty finite set  $N$  equipped with a bijection  $\eta : N \rightarrow |\mathbf{N}|$ ) and a collection of objects  $\{X_n\}_{n \in N}$  in  $\mathbf{C}$  indexed by  $N$ , there is a naturally occurring *ordered tensor product*  $\otimes_{n \in N} X_n \in \mathbf{C}$  defined by

$$\otimes_{n \in N} X_n := X_{\eta^{-1}(1)} \otimes \cdots \otimes X_{\eta^{-1}(|N|)}.$$

Similarly, given any non-empty finite set  $T$  and a collection of objects  $\{X_t\}_{t \in T}$  in  $\mathbf{C}$  indexed by  $T$ , we would like to define a naturally occurring *unordered tensor product*  $\otimes_{t \in T} X_t \in \mathbf{C}$ . One approach is to simply choose an ordering of  $T$  (i.e., choose a bijection  $\eta : T \rightarrow |\mathbf{T}|$ ) and declare that

$$\otimes_{t \in T} X_t := X_{\eta^{-1}(1)} \otimes \cdots \otimes X_{\eta^{-1}(|T|)}.$$

This approach is fine, but an isomorphic and more intrinsic definition would be to replace the choice of ordering with a choice of colimit. The idea is, instead of choosing a particular ordering of the unordered set  $T$ , take all possible orderings

of  $T$ , coproduct the corresponding ordered tensor products together, and identify them. The construction is this.

**Definition 3.6.** Let  $T$  be a non-empty finite set.

- $\Sigma_T$  is the category with exactly one object  $T$  and morphisms the bijections of sets. Let  $i_T : \Sigma_T \rightarrow \Sigma$  denote the inclusion of categories.
- The *category of orderings of  $T$*  is the over category  $\text{Ord}(T) := i_T \downarrow |\mathbf{T}|$ ; i.e., the objects are the bijections  $\eta : T \rightarrow |\mathbf{T}|$  and the morphisms  $\varphi : \eta \rightarrow \eta'$  are the commutative diagrams

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & T \\ \eta \downarrow \cong & \cong & \downarrow \eta' \\ |\mathbf{T}| & \xlongequal{\quad} & |\mathbf{T}|. \end{array}$$

Let  $T$  be a non-empty finite set and  $\{X_t\}_{t \in T}$  a collection of objects in  $\mathbf{C}$  indexed by  $T$ . There is a functor  $\otimes X : \text{Ord}(T) \rightarrow \mathbf{C}$  defined objectwise by

$$(\otimes X)(\eta : T \rightarrow |\mathbf{T}|) := X_{\eta^{-1}(1)} \otimes \cdots \otimes X_{\eta^{-1}(|\mathbf{T}|)}$$

which is useful for defining unordered tensor products.

**Definition 3.7.** Let  $T$  be a non-empty finite set and  $\{X_t\}_{t \in T}$  a collection of objects in  $\mathbf{C}$  indexed by  $T$ . The *unordered tensor product*  $\otimes_{t \in T} X_t \in \mathbf{C}$  is defined by

$$\otimes_{t \in T} X_t := \text{colim}(\otimes X).$$

*Remark 3.8.* Since every object in  $\text{Ord}(T)$  is terminal, for each bijection  $\eta : T \rightarrow |\mathbf{T}|$  there is a natural isomorphism,

$$\otimes_{t \in T} X_t \cong X_{\eta^{-1}(1)} \otimes \cdots \otimes X_{\eta^{-1}(|\mathbf{T}|)}.$$

#### 4. TENSOR PRODUCTS FOR (SYMMETRIC) SEQUENCES

Sequences and symmetric sequences have naturally occurring tensor products.

**Definition 4.1.** Let  $A_1, \dots, A_t$  be symmetric sequences and let  $X_1, \dots, X_t$  be sequences. The *tensor products*  $A_1 \otimes \cdots \otimes A_t \in \text{SymSeq}$  and  $X_1 \hat{\otimes} \cdots \hat{\otimes} X_t \in \text{Seq}$  are the left Kan extensions of objectwise tensor along coproduct of sets,

$$\begin{array}{ccc} (\Sigma^{\text{op}})^{\times t} \xrightarrow{A_1 \times \cdots \times A_t} \mathbf{C}^{\times t} \xrightarrow{\otimes} \mathbf{C} & & (\Omega^{\text{op}})^{\times t} \xrightarrow{X_1 \times \cdots \times X_t} \mathbf{C}^{\times t} \xrightarrow{\otimes} \mathbf{C} \\ \downarrow \Pi & & \downarrow \Pi \\ \Sigma^{\text{op}} \xrightarrow[A_1 \otimes \cdots \otimes A_t]{\text{left Kan extension}} \mathbf{C}, & & \Omega^{\text{op}} \xrightarrow[X_1 \hat{\otimes} \cdots \hat{\otimes} X_t]{\text{left Kan extension}} \mathbf{C}. \end{array}$$

*Remark 4.2.* The reader may wish to drop the hat in the tensor product notation  $\hat{\otimes}$  for sequences. We have included it only to avoid confusion later, since both the tensor product of sequences and of symmetric sequences appear in this paper, and sometimes in the same formula (Section 7).

**4.1. Calculations.** This gives a conceptual definition of the tensor products, but the reader may take either of the following calculations as a definition.

**Proposition 4.3.** *Let  $A_1, \dots, A_t$  be symmetric sequences and  $S \in \Sigma$ , with  $s := |S|$ . Let  $X_1, \dots, X_t$  be sequences and  $M \in \Omega$ , with  $m := |M|$ . There are natural isomorphisms,*

$$(4.4) \quad (A_1 \otimes \cdots \otimes A_t)[S] \cong \coprod_{\substack{\pi: S \rightarrow \mathbf{t} \\ \text{in Set}}} A_1[\pi^{-1}(1)] \otimes \cdots \otimes A_t[\pi^{-1}(t)],$$

$$(4.5) \quad \cong \coprod_{s_1 + \cdots + s_t = s} A_1[\mathbf{s}_1] \otimes \cdots \otimes A_t[\mathbf{s}_t]_{\Sigma_{s_1} \times \cdots \times \Sigma_{s_t}} \Sigma_s,$$

$$(4.6) \quad (X_1 \hat{\otimes} \cdots \hat{\otimes} X_t)[M] \cong \coprod_{\substack{\pi: M \rightarrow \mathbf{t} \\ \text{in OrdSet}}} X_1[\pi^{-1}(1)] \otimes \cdots \otimes X_t[\pi^{-1}(t)],$$

$$(4.7) \quad \cong \coprod_{m_1 + \cdots + m_t = m} X_1[\mathbf{m}_1] \otimes \cdots \otimes X_t[\mathbf{m}_t],$$

*Remark 4.8.* Giving a map of sets  $\pi: S \rightarrow \mathbf{t}$  is the same as giving an ordered partition  $(I_1, \dots, I_t)$  of  $S$ . Whenever  $\pi$  is not surjective, at least one  $I_j$  will be the empty set  $\mathbf{0}$ .

*Proof.* Left to the reader.  $\square$

**4.2. Tensor powers.** It will be useful to extend the definition of tensor powers  $A^{\otimes t}$  and  $X^{\hat{\otimes} n}$  to situations in which the integers  $t$  and  $n$  are replaced, respectively, by a finite set  $T$  or a finite ordered set  $N$ . The calculations in Proposition 4.3 suggest how to proceed. We introduce here the suggestive bracket notation used by Rezk [38].

**Definition 4.9.** Let  $A$  be a symmetric sequence and  $S, T \in \Sigma$ . Let  $X$  be a sequence and  $M, N \in \Omega$ . The *tensor powers*  $A^{\otimes T} \in \text{SymSeq}$  and  $X^{\hat{\otimes} N} \in \text{Seq}$  are defined objectwise by

$$(4.10) \quad \begin{aligned} (A^{\otimes T})[S] &:= A[S, T] := \coprod_{\substack{\pi: S \rightarrow T \\ \text{in Set}}} \otimes_{t \in T} A[\pi^{-1}(t)], \quad T \neq \emptyset, \\ (A^{\otimes \emptyset})[S] &:= A[S, \emptyset] := \coprod_{\substack{\pi: S \rightarrow \emptyset \\ \text{in Set}}} \mathbf{k}, \\ (X^{\hat{\otimes} N})[M] &:= X[M, N] := \coprod_{\substack{\pi: M \rightarrow N \\ \text{in OrdSet}}} \otimes_{n \in N} X[\pi^{-1}(n)], \quad N \neq \emptyset, \\ (X^{\hat{\otimes} \emptyset})[M] &:= X[M, \emptyset] := \coprod_{\substack{\pi: M \rightarrow \emptyset \\ \text{in OrdSet}}} \mathbf{k}. \end{aligned}$$

We will use the abbreviations  $A^{\otimes \emptyset} := A^{\otimes \emptyset}$  and  $X^{\hat{\otimes} \emptyset} := X^{\hat{\otimes} \emptyset}$ .

*Remark 4.11.* The reader will notice there is no hat appearing in the bracket notation  $X[M, N]$  and  $X[M, \emptyset]$  for sequences.

*Remark 4.12.* The tensor products indexed by  $T$  (resp. indexed by  $N$ ) are regarded as unordered (resp. ordered) (Section 3.3). **Set** is the category of sets and their maps. **OrdSet** is the category of totally ordered sets and their order preserving maps.

The above constructions give functors

$$\begin{aligned} \text{SymSeq} \times \Sigma^{\text{op}} \times \Sigma &\longrightarrow \mathbf{C}, & (A, S, T) &\longmapsto A[S, T], \\ \text{Seq} \times \Omega^{\text{op}} \times \Omega &\longrightarrow \mathbf{C}, & (X, M, N) &\longmapsto X[M, N], \\ \text{SymSeq} \times \text{SymSeq} &\longrightarrow \text{SymSeq}, & (A, B) &\longmapsto A \otimes B, \\ \text{Seq} \times \text{Seq} &\longrightarrow \text{Seq}, & (X, Y) &\longmapsto X \hat{\otimes} Y. \end{aligned}$$

Observe that the unit for the tensor product  $\otimes$  on  $\text{SymSeq}$  and the unit for the tensor product  $\hat{\otimes}$  on  $\text{Seq}$ , both denoted “1”, are given by the same formula

$$1[S] := \begin{cases} \mathbf{k}, & \text{for } |S| = 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is useful to make some simple calculations.

**Proposition 4.13.** *Let  $A, B$  be symmetric sequences. There are natural isomorphisms,*

$$\begin{aligned} A \otimes 1 &\cong A, & A^{\otimes 0} &\cong A[-, \mathbf{0}] \cong 1, & A \otimes B &\cong B \otimes A \\ A \otimes \emptyset &\cong \emptyset, & A^{\otimes 1} &\cong A[-, \mathbf{1}] \cong A, & (A^{\otimes t})[\mathbf{0}] &\cong A[\mathbf{0}, \mathbf{t}] \cong A[\mathbf{0}]^{\otimes t}, \quad t \geq 0. \end{aligned}$$

*Proof.* These are verified directly from Definition 4.9.  $\square$

Similar calculations are true for sequences.

**4.3. Mapping objects for (symmetric) sequences.** Let  $B, C$  be symmetric sequences and  $T \in \Sigma$ . Let  $Y, Z$  be sequences and  $N \in \Omega$ . There are functors

$$\begin{aligned} \Sigma \times \Sigma^{\text{op}} &\longrightarrow \mathbf{C}, & (S, S') &\longmapsto \text{Map}(B[S], C[T \amalg S']), \\ \Omega \times \Omega^{\text{op}} &\longrightarrow \mathbf{C}, & (M, M') &\longmapsto \text{Map}(Y[M], Z[N \amalg M']), \end{aligned}$$

which are useful for defining the mapping objects of  $(\text{SymSeq}, \otimes, 1)$  and  $(\text{Seq}, \hat{\otimes}, 1)$ .

**Definition 4.14.** Let  $B, C$  be symmetric sequences and  $T \in \Sigma$ . Let  $Y, Z$  be sequences and  $N \in \Omega$ . The *mapping objects*  $\text{Map}^{\otimes}(B, C) \in \text{SymSeq}$  and  $\text{Map}^{\hat{\otimes}}(Y, Z) \in \text{Seq}$  are defined objectwise by the ends

$$\begin{aligned} \text{Map}^{\otimes}(B, C)[T] &:= \text{Map}(B, C[T \amalg -])^{\Sigma}, \\ \text{Map}^{\hat{\otimes}}(Y, Z)[N] &:= \text{Map}(Y, Z[N \amalg -])^{\Omega}. \end{aligned}$$

**4.4. Universal properties.** Hence  $\text{Map}^{\otimes}(B, C)$  satisfies objectwise the universal property

$$(4.15) \quad \begin{array}{ccc} & & \text{Map}(B[S], C[T \amalg S]) \\ & \xrightarrow{f[S]} & \downarrow (\text{id}, (\text{id} \amalg \zeta)^*) \\ \text{Map}^{\otimes}(B, C)[T] & \xrightarrow{\tau[S]} & \text{Map}(B[S], C[T \amalg S']) \\ & \xrightarrow{\tau[S']} & \uparrow (\zeta^*, \text{id}) \\ & & \text{Map}(B[S'], C[T \amalg S']) \end{array} \quad \begin{array}{c} S \\ \uparrow \zeta \\ S' \end{array}$$

that each wedge  $f$  factors uniquely through the terminal wedge  $\tau$  of  $\text{Map}^\otimes(B, C)[T]$ . The mapping objects  $\text{Map}^{\hat{\otimes}}(Y, Z)$  satisfy objectwise a similar universal property. These constructions give functors

$$\begin{aligned} \text{SymSeq}^{\text{op}} \times \text{SymSeq} &\longrightarrow \text{SymSeq}, & (B, C) &\longmapsto \text{Map}^\otimes(B, C), \\ \text{Seq}^{\text{op}} \times \text{Seq} &\longrightarrow \text{Seq}, & (Y, Z) &\longmapsto \text{Map}^{\hat{\otimes}}(Y, Z). \end{aligned}$$

**Proposition 4.16.** *Let  $A, B, C$  be symmetric sequences and let  $X, Y, Z$  be sequences. There are isomorphisms*

$$(4.17) \quad \text{hom}(A \otimes B, C) \cong \text{hom}(A, \text{Map}^\otimes(B, C)),$$

$$(4.18) \quad \text{hom}(X \hat{\otimes} Y, Z) \cong \text{hom}(X, \text{Map}^{\hat{\otimes}}(Y, Z)),$$

natural in  $A, B, C$  and  $X, Y, Z$ .

*Proof.* Consider (4.17). Use the calculation (4.4) and the universal property (4.15) with the natural correspondence (2.2) to verify that giving a map  $A \otimes B \rightarrow C$  is the same as giving a map  $A \rightarrow \text{Map}^\otimes(B, C)$ , and that the resulting correspondence is natural. Use a similar argument for the case of sequences.  $\square$

#### 4.5. Monoidal structures.

**Proposition 4.19.** *( $\text{SymSeq}, \otimes, 1$ ) and ( $\text{Seq}, \hat{\otimes}, 1$ ) have the structure of symmetric monoidal closed categories with all small limits and colimits.*

*Proof.* To verify the symmetric monoidal structure, use (4.4) to describe the required natural isomorphisms and to verify the appropriate diagrams commute. Proposition 4.16 verifies the symmetric monoidal structure is closed. Limits and colimits are calculated objectwise. Argue similarly for sequences.  $\square$

**4.6. Calculations.** Definition 4.14 gives objectwise a conceptual interpretation of the mapping objects, but the reader may take the following calculations as definitions.

**Proposition 4.20.** *Let  $B, C$  be symmetric sequences and  $T \in \Sigma$ , with  $t := |T|$ . Let  $Y, Z$  be sequences and  $N \in \Omega$ , with  $n := |N|$ . There are natural isomorphisms,*

$$(4.21) \quad \text{Map}^\otimes(B, C)[T] \cong \prod_{s \geq 0} \text{Map}(B[\mathbf{s}], C[\mathbf{t} + \mathbf{s}])^{\Sigma_s},$$

$$(4.22) \quad \text{Map}^{\hat{\otimes}}(Y, Z)[N] \cong \prod_{m \geq 0} \text{Map}(Y[\mathbf{m}], Z[\mathbf{n} + \mathbf{m}]).$$

*Proof.* To verify (4.21), use the universal property (4.15) and restrict to a small skeleton to obtain natural isomorphisms

$$\text{Map}^\otimes(B, C)[T] \cong \lim \left( \prod_{\substack{S \\ \text{in } \Sigma'}} \text{Map}(B[S], C[T \amalg S]) \xrightarrow{\cong} \prod_{\substack{\zeta: S' \rightarrow S \\ \text{in } \Sigma'}} \text{Map}(B[S], C[T \amalg S']) \right).$$

This verifies (4.21). Argue similarly for the case of sequences.  $\square$

## 5. CIRCLE PRODUCTS FOR (SYMMETRIC) SEQUENCES

We describe a *circle product*  $\circ$  on  $\mathbf{SymSeq}$  and a related circle product  $\hat{\circ}$  on  $\mathbf{Seq}$ . These are monoidal products which are *not* symmetric monoidal, and they figure in the definitions of  $\Sigma$ -operad and  $\Omega$ -operad respectively (Definition 6.1). Perhaps surprisingly, these monoidal products possess appropriate adjoints, which can be interpreted as mapping objects. For instance, there are objects  $\mathrm{Map}^\circ(B, C)$  and  $\mathrm{Map}^{\hat{\circ}}(Y, Z)$  which fit into isomorphisms

$$\begin{aligned} \mathrm{hom}(A \circ B, C) &\cong \mathrm{hom}(A, \mathrm{Map}^\circ(B, C)), \\ \mathrm{hom}(X \hat{\circ} Y, Z) &\cong \mathrm{hom}(X, \mathrm{Map}^{\hat{\circ}}(Y, Z)), \end{aligned}$$

natural in the symmetric sequences  $A, B, C$  and the sequences  $X, Y, Z$ .

The material in this section is largely influenced by Rezk [38]. Earlier work exploiting circle product  $\circ$  for symmetric sequences includes [10, 11, 40]; more recent work includes [7, 8, 25, 26]. The circle product  $\hat{\circ}$  is used in [2] for working with  $\Omega$ -operads and their algebras.

**5.1. Circle products (or composition products).** Let  $A, B$  be symmetric sequences and  $S \in \Sigma$ . Let  $X, Y$  be sequences and  $M \in \Omega$ . There are functors

$$\begin{aligned} \Sigma^{\mathrm{op}} \times \Sigma &\longrightarrow \mathbf{C}, & (T', T) &\longmapsto A[T'] \otimes B[S, T], \\ \Omega^{\mathrm{op}} \times \Omega &\longrightarrow \mathbf{C}, & (N', N) &\longmapsto X[N'] \otimes Y[M, N], \end{aligned}$$

which are useful for defining circle products of sequences and of symmetric sequences.

**Definition 5.1.** Let  $A, B$  be symmetric sequences and  $S \in \Sigma$ . Let  $X, Y$  be sequences and  $M \in \Omega$ . The *circle products* (or composition products)  $A \circ B \in \mathbf{SymSeq}$  and  $X \hat{\circ} Y \in \mathbf{Seq}$  are defined objectwise by the coends

$$\begin{aligned} (A \circ B)[S] &:= A \otimes_{\Sigma} (B^{\hat{\circ}-})[S] = A \otimes_{\Sigma} B[S, -], \\ (X \hat{\circ} Y)[M] &:= X \otimes_{\Omega} (Y^{\hat{\circ}-})[M] = X \otimes_{\Omega} Y[M, -]. \end{aligned}$$

*Remark 5.2.* The reader may wish to drop the hat in the circle product notation  $\hat{\circ}$  for sequences. We have included it only to avoid confusion later, since both the circle product of sequences and of symmetric sequences appear in this paper, and sometimes in the same formula (Section 7).

**5.2. Universal properties.** Hence  $A \circ B$  satisfies objectwise the universal property

$$(5.3) \quad \begin{array}{ccccc} T & & A[T] \otimes B[S, T] & & \\ \downarrow \xi & & \uparrow \xi^* \otimes [\mathrm{id}, \mathrm{id}] & \searrow i[T] & \\ & & A[T'] \otimes B[S, T] & \xrightarrow{\quad \bar{f} \quad} & (A \circ B)[S] \\ & & \downarrow \mathrm{id} \otimes [\mathrm{id}, \xi] & \nearrow i[T'] & \exists! \\ T' & & A[T'] \otimes B[S, T'] & & \end{array}$$

$f[T]$  (top curved arrow),  $f[T']$  (bottom curved arrow),  $\bar{f}$  (dotted arrow),  $\exists!$  (arrow from  $\bar{f}$  to  $(A \circ B)[S]$ )

that each wedge  $f$  factors uniquely through the initial wedge  $i$  of  $(A \circ B)[S]$ . A similar universal property is satisfied objectwise by the circle products  $X \hat{\circ} Y$ . These

constructions give functors

$$\begin{aligned} \text{SymSeq} \times \text{SymSeq} &\longrightarrow \text{SymSeq}, & (A, B) &\longmapsto A \circ B, \\ \text{Seq} \times \text{Seq} &\longrightarrow \text{Seq}, & (X, Y) &\longmapsto X \hat{\circ} Y. \end{aligned}$$

**5.3. Calculations.** Definition 5.1 gives objectwise a conceptual interpretation of the circle products, but the reader may take the following calculations as definitions.

**Proposition 5.4.** *Let  $A, B$  be symmetric sequences and  $S \in \Sigma$ , with  $s := |S|$ . Let  $X, Y$  be sequences and  $M \in \Omega$ , with  $m := |M|$ . There are natural isomorphisms,*

$$(5.5) \quad \begin{aligned} (A \circ B)[S] &\cong \coprod_{t \geq 0} A[\mathbf{t}] \otimes_{\Sigma_t} (B^{\otimes t})[s] \cong \coprod_{t \geq 0} A[\mathbf{t}] \otimes_{\Sigma_t} B[\mathbf{s}, \mathbf{t}], \\ (X \hat{\circ} Y)[M] &\cong \coprod_{n \geq 0} X[\mathbf{n}] \otimes (Y^{\hat{\circ} n})[\mathbf{m}] \cong \coprod_{n \geq 0} X[\mathbf{n}] \otimes Y[\mathbf{m}, \mathbf{n}]. \end{aligned}$$

*Proof.* Use the universal property (5.3) and restrict to a small skeleton to obtain natural isomorphisms

$$(A \circ B)[S] \cong \text{colim} \left( \coprod_{\substack{\xi: T \rightarrow T' \\ \text{in } \Sigma'}} A[T'] \otimes B[S, T] \rightrightarrows \coprod_{\substack{T \\ \text{in } \Sigma'}} A[T] \otimes B[S, T] \right).$$

This verifies (5.5). The other case is similar.  $\square$

Observe that the unit for the circle product  $\circ$  on  $\text{SymSeq}$  and the unit for the circle product  $\hat{\circ}$  on  $\text{Seq}$ , both denoted “ $I$ ”, are given by the same formula

$$I[S] := \begin{cases} k, & \text{for } |S| = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

**Definition 5.6.** Let  $A$  be a symmetric sequence,  $X$  a sequence, and  $Z \in \mathbf{C}$ . The corresponding functors  $A \circ (-) : \mathbf{C} \rightarrow \mathbf{C}$  and  $X \hat{\circ} (-) : \mathbf{C} \rightarrow \mathbf{C}$  are defined objectwise by,

$$\begin{aligned} A \circ (Z) &:= \coprod_{t \geq 0} A[\mathbf{t}] \otimes_{\Sigma_t} Z^{\otimes t}, \\ X \hat{\circ} (Z) &:= \coprod_{t \geq 0} X[\mathbf{t}] \otimes Z^{\otimes t}. \end{aligned}$$

The category  $\mathbf{C}$  embeds in  $\text{SymSeq}$  (resp.  $\text{Seq}$ ) as the full subcategory of symmetric sequences (resp. sequences) concentrated at 0, via the functor  $\hat{\cdot} : \mathbf{C} \rightarrow \text{SymSeq}$  (resp.  $\hat{\cdot} : \mathbf{C} \rightarrow \text{Seq}$ ) defined objectwise by

$$(5.7) \quad \hat{Z}[S] := \begin{cases} Z, & \text{for } |S| = 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is useful to make some simple calculations.

**Proposition 5.8.** *Let  $A, B$  be symmetric sequences and  $Z \in \mathbf{C}$ . There are natural isomorphisms,*

$$\begin{aligned} \emptyset \circ A &\cong \emptyset, & I \circ A &\cong A, & A \circ I &\cong A, \\ (A \circ \emptyset)[S] &\cong \begin{cases} A[\mathbf{0}], & \text{for } |S| = 0, \\ \emptyset, & \text{otherwise,} \end{cases} \\ (A \circ \hat{Z})[S] &\cong \begin{cases} A \circ (Z), & \text{for } |S| = 0, \\ \emptyset, & \text{otherwise,} \end{cases} \\ (A \circ B)[\mathbf{0}] &\cong A \circ (B[\mathbf{0}]). \end{aligned}$$

*Proof.* These can be verified directly from Proposition 5.4.  $\square$

Similar calculations are true for sequences.

**5.4. Properties of tensor and circle products.** It is useful to understand how tensor products and circle products interact.

**Proposition 5.9.** *Let  $A, B, C$  be symmetric sequences,  $X, Y, Z$  be sequences, and  $t \geq 0$ . There are natural isomorphisms*

$$(5.10) \quad \begin{aligned} (A \otimes B) \circ C &\cong (A \circ C) \otimes (B \circ C), \\ (X \hat{\otimes} Y) \hat{\circ} Z &\cong (X \hat{\circ} Z) \hat{\otimes} (Y \hat{\circ} Z), \\ (B^{\otimes t}) \circ C &\cong (B \circ C)^{\otimes t}, \\ (Y^{\hat{\otimes} t}) \hat{\circ} Z &\cong (Y \hat{\circ} Z)^{\hat{\otimes} t}. \end{aligned}$$

*Proof.* Using (4.5) and (5.5), there are natural isomorphisms

$$\begin{aligned} (A \otimes B) \circ C &\cong \prod_{s \geq 0} (A \otimes B)[\mathbf{s}] \otimes_{\Sigma_s} C^{\otimes s} \\ &\cong \prod_{s \geq 0} \prod_{s_1 + s_2 = s} A[\mathbf{s}_1] \otimes B[\mathbf{s}_2] \otimes_{\Sigma_{s_1} \times \Sigma_{s_2}} C^{\otimes (s_1 + s_2)} \\ &\cong \left( \prod_{s_1 \geq 0} A[\mathbf{s}_1] \otimes_{\Sigma_{s_1}} C^{\otimes s_1} \right) \otimes \left( \prod_{s_2 \geq 0} B[\mathbf{s}_2] \otimes_{\Sigma_{s_2}} C^{\otimes s_2} \right) \\ &\cong (A \circ C) \otimes (B \circ C). \end{aligned}$$

The argument for sequences is similar.  $\square$

**Proposition 5.11.** *Let  $A, B$  be symmetric sequences and let  $X, Y$  be sequences. Suppose  $Z \in \mathbf{C}$  and  $t \geq 0$ . There are natural isomorphisms*

$$\begin{aligned} (A \otimes B) \circ (Z) &\cong (A \circ (Z)) \otimes (B \circ (Z)), \\ (X \hat{\otimes} Y) \hat{\circ} (Z) &\cong (X \hat{\circ} (Z)) \hat{\otimes} (Y \hat{\circ} (Z)), \\ (B^{\otimes t}) \circ (Z) &\cong (B \circ (Z))^{\otimes t}, \\ (Y^{\hat{\otimes} t}) \hat{\circ} (Z) &\cong (Y \hat{\circ} (Z))^{\hat{\otimes} t}. \end{aligned}$$

*Proof.* Argue as in the proof of Proposition 5.9, or use the embedding (5.7) to deduce it as a special case.  $\square$

**Proposition 5.12.** *Let  $A, B, C$  be symmetric sequences and let  $X, Y, Z$  be sequences. There are natural isomorphisms*

$$(A \circ B) \circ C \cong A \circ (B \circ C),$$

$$(X \hat{\circ} Y) \hat{\circ} Z \cong X \hat{\circ} (Y \hat{\circ} Z).$$

*Proof.* Using (5.5) and (5.10), there are natural isomorphisms

$$\begin{aligned} A \circ (B \circ C) &\cong \prod_{t \geq 0} A[\mathbf{t}] \otimes_{\Sigma_t} (B \circ C)^{\otimes t} \\ &\cong \prod_{t \geq 0} A[\mathbf{t}] \otimes_{\Sigma_t} (B^{\otimes t}) \circ C \\ &\cong \prod_{s \geq 0} \prod_{t \geq 0} A[\mathbf{t}] \otimes_{\Sigma_t} (B^{\otimes t}) [\mathbf{s}] \otimes_{\Sigma_s} C^{\otimes s} \\ &\cong (A \circ B) \circ C. \end{aligned}$$

Argue similarly for the case of sequences.  $\square$

**Proposition 5.13.** *Let  $A, B$  be symmetric sequences and let  $X, Y$  be sequences. Suppose  $Z \in \mathcal{C}$ . There are natural isomorphisms*

$$(A \circ B) \circ (Z) \cong A \circ (B \circ (Z)),$$

$$(X \hat{\circ} Y) \hat{\circ} (Z) \cong X \hat{\circ} (Y \hat{\circ} (Z)).$$

*Proof.* Argue as in the proof of Proposition 5.12, or use the embedding (5.7) to deduce it as a special case.  $\square$

**5.5. Mapping sequences.** Let  $B, C$  be symmetric sequences and  $T \in \Sigma$ . Let  $Y, Z$  be sequences and  $N \in \Omega$ . There are functors

$$\begin{aligned} \Sigma \times \Sigma^{\text{op}} &\longrightarrow \mathcal{C}, & (S, S') &\longmapsto \text{Map}(B[S, T], C[S']), \\ \Omega \times \Omega^{\text{op}} &\longrightarrow \mathcal{C}, & (M, M') &\longmapsto \text{Map}(Y[M, N], Z[M']), \end{aligned}$$

which are useful for defining mapping objects of  $(\text{SymSeq}, \circ, I)$  and  $(\text{Seq}, \hat{\circ}, I)$ .

**Definition 5.14.** Let  $B, C$  be symmetric sequences and  $T \in \Sigma$ . Let  $Y, Z$  be sequences and  $N \in \Omega$ . The *mapping sequences*  $\text{Map}^\circ(B, C) \in \text{SymSeq}$  and  $\text{Map}^\hat{\circ}(Y, Z) \in \text{Seq}$  are defined objectwise by the ends

$$\begin{aligned} \text{Map}^\circ(B, C)[T] &:= \text{Map}((B^{\otimes T})[-], C)^\Sigma = \text{Map}(B[-, T], C)^\Sigma, \\ \text{Map}^\hat{\circ}(Y, Z)[N] &:= \text{Map}((Y^{\hat{\circ} N})[-], Z)^\Omega = \text{Map}(Y[-, N], Z)^\Omega. \end{aligned}$$

**5.6. Universal properties.** Hence  $\text{Map}^\circ(B, C)$  satisfies objectwise the universal property

$$(5.15) \quad \begin{array}{ccc} & & \text{Map}(B[S, T], C[S]) \\ & \nearrow f[S] & \downarrow ([\text{id}, \text{id}], \zeta^*) \\ \text{Map}^\circ(B, C)[T] & & \text{Map}(B[S, T], C[S']) \\ & \searrow \tau[S] & \uparrow ([\zeta, \text{id}], \text{id}) \\ & & \text{Map}(B[S', T], C[S']) \end{array} \quad \begin{array}{c} S \\ \uparrow \zeta \\ S' \end{array}$$

that each wedge  $f$  factors uniquely through the terminal wedge  $\tau$  of  $\text{Map}^\circ(B, C)[T]$ . A similar universal property is satisfied objectwise by the mapping objects  $\text{Map}^\circ(Y, Z)$ . These constructions give functors

$$\begin{aligned} \text{SymSeq}^{\text{op}} \times \text{SymSeq} &\longrightarrow \text{SymSeq}, & (B, C) &\longmapsto \text{Map}^\circ(B, C), \\ \text{Seq}^{\text{op}} \times \text{Seq} &\longrightarrow \text{Seq}, & (Y, Z) &\longmapsto \text{Map}^\circ(Y, Z). \end{aligned}$$

**Proposition 5.16.** *Let  $A, B, C$  be symmetric sequences and let  $X, Y, Z$  be sequences. There are isomorphisms*

$$(5.17) \quad \text{hom}(A \circ B, C) \cong \text{hom}(A, \text{Map}^\circ(B, C)),$$

$$(5.18) \quad \text{hom}(X \hat{\circ} Y, Z) \cong \text{hom}(X, \text{Map}^\circ(Y, Z)),$$

natural in  $A, B, C$  and  $X, Y, Z$ .

*Proof.* Use the universal properties (5.3) and (5.15) with the natural correspondence (2.2) to verify that giving a map  $A \circ B \rightarrow C$  is the same as giving a map  $A \rightarrow \text{Map}^\circ(B, C)$ , and that the resulting correspondence is natural. Argue similarly for the case of sequences.  $\square$

### 5.7. Monoidal structures.

**Proposition 5.19.**  *$(\text{SymSeq}, \circ, I)$  and  $(\text{Seq}, \hat{\circ}, I)$  have the structure of monoidal closed categories with all small limits and colimits. Circle product is not symmetric.*

*Proof.* To verify the monoidal structure, use (5.5) along with properties of  $\otimes$  from Proposition 4.19 to describe the required natural isomorphisms and to verify the appropriate diagrams commute. Proposition 5.16 verifies the monoidal structure is closed. Limits and colimits are calculated objectwise. Argue similarly for the case of sequences.  $\square$

**5.8. Calculations.** Definition 5.14 gives objectwise a conceptual interpretation of the mapping sequences, but the reader may take the following calculations as definitions.

**Proposition 5.20.** *Let  $B, C$  be symmetric sequences and  $T \in \Sigma$ , with  $t := |T|$ . Let  $Y, Z$  be sequences and  $N \in \Omega$ , with  $n := |N|$ . There are natural isomorphisms,*

$$(5.21) \quad \begin{aligned} \text{Map}^\circ(B, C)[T] &\cong \prod_{s \geq 0} \text{Map}((B^{\otimes t})[s], C[s])^{\Sigma_s} \cong \prod_{s \geq 0} \text{Map}(B[s, t], C[s])^{\Sigma_s}, \\ \text{Map}^\circ(Y, Z)[N] &\cong \prod_{m \geq 0} \text{Map}((Y^{\hat{\circ} n})[m], Z[m]) \cong \prod_{m \geq 0} \text{Map}(Y[m, n], Z[m]). \end{aligned}$$

*Proof.* Use the universal property (5.15) for  $\text{Map}^\circ(B, C)[T]$  and restrict to a small skeleton to obtain natural isomorphisms

$$\text{Map}^\circ(B, C)[T] \cong \lim \left( \prod_{\substack{S \\ \text{in } \Sigma'}} \text{Map}(B[S, T], C[S]) \right) \xrightarrow{\cong} \left( \prod_{\substack{\zeta: S' \rightarrow S \\ \text{in } \Sigma'}} \text{Map}(B[S, T], C[S']) \right).$$

This verifies (5.21). The case for sequences is similar.  $\square$

It is useful to make some simple calculations.

**Proposition 5.22.** *Let  $B, C$  be symmetric sequences,  $T \in \Sigma$ , and  $Z \in \mathbf{C}$ . There are natural isomorphisms,*

$$\begin{aligned} \text{Map}^\circ(B, *) &\cong *, \\ \text{Map}^\circ(\emptyset, C)[T] &\cong \begin{cases} C[\mathbf{0}], & \text{for } |T| = 0, \\ *, & \text{otherwise,} \end{cases} \\ \text{Map}^\circ(B, C)[\mathbf{0}] &\cong C[\mathbf{0}], \\ \text{Map}^\circ(B, C)[\mathbf{1}] &\cong \prod_{s \geq 0} \text{Map}(B[\mathbf{s}], C[\mathbf{s}])^{\Sigma_s}, \\ \text{Map}^\circ(\hat{Z}, C)[T] &\cong \text{Map}(Z^{\otimes |T|}, C[\mathbf{0}]). \end{aligned}$$

*Proof.* These can be verified directly from Proposition 5.20.  $\square$

Similar calculations are true for sequences.

**5.9. Circle products as Kan extensions.** Circle products can also be understood as Kan extensions. Let  $\mathbf{F}$  be the category of finite sets and their maps and let  $\text{Ord}\mathbf{F}$  be the category of totally ordered finite sets and their order preserving maps. Then circle products are left Kan extensions of  $- * -$  along projection onto source,

$$\begin{array}{ccc} (\text{Iso}(\mathbf{F} \rightarrow \cdot))^{\text{op}} & \xrightarrow{A * B} & \mathbf{C} \\ \downarrow \text{proj} & & \\ \Sigma^{\text{op}} & \xrightarrow[\text{left Kan extension}]{A \circ B} & \mathbf{C}, \end{array} \quad \begin{array}{ccc} (\text{Iso}(\text{Ord}\mathbf{F} \rightarrow \cdot))^{\text{op}} & \xrightarrow{X * Y} & \mathbf{C} \\ \downarrow \text{proj} & & \\ \Omega^{\text{op}} & \xrightarrow[\text{left Kan extension}]{X \hat{\circ} Y} & \mathbf{C}. \end{array}$$

The functors  $- * -$  are defined objectwise by

$$\begin{aligned} (A * B)(\pi : S \rightarrow T) &:= A[T] \otimes_{\otimes_{t \in T} B[\pi^{-1}(t)]}, \\ (X * Y)(\pi : M \rightarrow N) &:= X[N] \otimes_{\otimes_{n \in N} Y[\pi^{-1}(n)]}. \end{aligned}$$

*Remark 5.23.* The tensor products indexed by  $T$  (resp. indexed by  $N$ ) are regarded as unordered (resp. ordered) (Section 3.3).

## 6. OPERADS, MODULES, AND ALGEBRAS

In this section we define operads and the objects they act on.

### 6.1. Operads.

**Definition 6.1.**

- A  $\Sigma$ -operad is a monoid object in  $(\text{SymSeq}, \circ, I)$  and a *morphism of  $\Sigma$ -operads* is a morphism of monoid objects in  $(\text{SymSeq}, \circ, I)$ .
- An  $\Omega$ -operad is a monoid object in  $(\text{Seq}, \hat{\circ}, I)$  and a *morphism of  $\Omega$ -operads* is a morphism of monoid objects in  $(\text{Seq}, \hat{\circ}, I)$ .

These two types of operads were originally defined by May [32]; the  $\Sigma$ -operad has symmetric groups and the  $\Omega$ -operad is based on ordered sets and is called a non- $\Sigma$  operad [27, 32].

*Example 6.2.* More explicitly, for instance, a  $\Sigma$ -operad is a symmetric sequence  $\mathcal{O}$  together with maps  $m : \mathcal{O} \circ \mathcal{O} \rightarrow \mathcal{O}$  and  $\eta : I \rightarrow \mathcal{O}$  in  $\text{SymSeq}$  which make the diagrams

$$\begin{array}{ccc} \mathcal{O} \circ \mathcal{O} \circ \mathcal{O} & \xrightarrow{m_{\text{oid}}} & \mathcal{O} \circ \mathcal{O} \\ \downarrow \text{id} \circ m & & \downarrow m \\ \mathcal{O} \circ \mathcal{O} & \xrightarrow{m} & \mathcal{O} \end{array} \quad \begin{array}{ccccc} I \circ \mathcal{O} & \xrightarrow{\eta_{\text{oid}}} & \mathcal{O} \circ \mathcal{O} & \xleftarrow{\text{id} \circ \eta} & \mathcal{O} \circ I \\ \downarrow \cong & & \downarrow m & & \downarrow \cong \\ \mathcal{O} & \xlongequal{\quad} & \mathcal{O} & \xlongequal{\quad} & \mathcal{O} \end{array}$$

commute. If  $\mathcal{O}$  and  $\mathcal{O}'$  are  $\Sigma$ -operads, then a morphism of  $\Sigma$ -operads is a map  $f : \mathcal{O} \rightarrow \mathcal{O}'$  in  $\text{SymSeq}$  which makes the diagrams

$$\begin{array}{ccc} \mathcal{O} \circ \mathcal{O} & \xrightarrow{m} & \mathcal{O} \\ \downarrow f \circ f & & \downarrow f \\ \mathcal{O}' \circ \mathcal{O}' & \xrightarrow{m} & \mathcal{O}' \end{array} \quad \begin{array}{ccc} \mathcal{O} & \xleftarrow{\eta} & I \\ \downarrow f & & \downarrow \eta \\ \mathcal{O}' & \xleftarrow{\eta} & I \end{array}$$

commute.

**6.2. Modules over operads.** Similar to the case of any monoid object, we introduce operads because we are interested in the objects they act on. Compare the following definition with [29, Chapter VII].

**Definition 6.3.** Let  $Q, R, S$  be  $\Sigma$ -operads (resp.  $\Omega$ -operads).

- A *left  $R$ -module* is an object in  $(\text{SymSeq}, \circ, I)$  (resp. an object in  $(\text{Seq}, \hat{\circ}, I)$ ) with a left action of  $R$  and a *morphism of left  $R$ -modules* is a map which respects the left  $R$ -module structure.
- A *right  $S$ -module* is an object in  $(\text{SymSeq}, \circ, I)$  (resp. an object in  $(\text{Seq}, \hat{\circ}, I)$ ) with a right action of  $S$  and a *morphism of right  $S$ -modules* is a map which respects the right  $S$ -module structure.
- An  *$(R, S)$ -bimodule* is an object in  $(\text{SymSeq}, \circ, I)$  (resp. an object in  $(\text{Seq}, \hat{\circ}, I)$ ) with compatible left  $R$ -module and right  $S$ -module structures and a *morphism of  $(R, S)$ -bimodules* is a map which respects the  $(R, S)$ -bimodule structure.

*Example 6.4.* More explicitly, for instance, if  $R$  is a  $\Sigma$ -operad, then a left  $R$ -module is a symmetric sequence  $B$  together with a map  $m : R \circ B \rightarrow B$  in  $\text{SymSeq}$  which makes the diagrams

$$(6.5) \quad \begin{array}{ccc} R \circ R \circ B & \xrightarrow{m_{\text{oid}}} & R \circ B \\ \downarrow \text{id} \circ m & & \downarrow m \\ R \circ B & \xrightarrow{m} & B \end{array} \quad \begin{array}{ccc} I \circ B & \xrightarrow{\eta_{\text{oid}}} & R \circ B \\ \downarrow \cong & & \downarrow m \\ B & \xlongequal{\quad} & B \end{array}$$

commute. If  $B$  and  $B'$  are left  $R$ -modules, then a morphism of left  $R$ -modules is a map  $f : B \rightarrow B'$  in  $\text{SymSeq}$  which makes the left-hand diagram

$$(6.6) \quad \begin{array}{ccc} R \circ B & \xrightarrow{m} & B \\ \downarrow \text{id} \circ f & & \downarrow f \\ R \circ B' & \xrightarrow{m} & B' \end{array} \quad \begin{array}{ccc} R \circ B \circ S & \xrightarrow{m_{\text{oid}}} & B \circ S \\ \downarrow \text{id} \circ m & & \downarrow m \\ R \circ B & \xrightarrow{m} & B \end{array}$$

commute. If  $R$  and  $S$  are  $\Sigma$ -operads, then an  $(R, S)$ -bimodule is an object in  $(\text{SymSeq}, \circ, I)$  with left  $R$ -module and right  $S$ -module structures which make the right-hand diagram in (6.6) commute.

**6.3. Algebras over operads.** An algebra over an operad  $\mathcal{O}$  is a left  $\mathcal{O}$ -module which is concentrated at 0.

**Definition 6.7.** Let  $\mathcal{O}$  be an operad.

- An  $\mathcal{O}$ -algebra is an object  $X \in \mathbf{C}$  with a left  $\mathcal{O}$ -module structure on  $\hat{X}$ .
- Let  $X$  and  $X'$  be  $\mathcal{O}$ -algebras. A *morphism of  $\mathcal{O}$ -algebras* is a map  $f : X \rightarrow X'$  in  $\mathbf{C}$  such that  $\hat{f} : \hat{X} \rightarrow \hat{X}'$  is a morphism of left  $\mathcal{O}$ -modules.

Giving a symmetric sequence  $Y$  a left  $\mathcal{O}$ -module structure is the same as giving a morphism of operads

$$m : \mathcal{O} \rightarrow \text{Map}^\circ(Y, Y).$$

Similarly, giving an object  $X \in \mathbf{C}$  an  $\mathcal{O}$ -algebra structure is the same as giving a morphism of operads

$$m : \mathcal{O} \rightarrow \text{Map}^\circ(\hat{X}, \hat{X}).$$

This is the original definition given by May [32] of an  $\mathcal{O}$ -algebra structure on  $X$ , where  $\text{Map}^\circ(\hat{X}, \hat{X})$  is called the *endomorphism operad of  $X$*  (see Proposition 5.22), and motivates the suggestion in [27, 32] that  $\mathcal{O}[\mathbf{t}]$  should be thought of as objects of parameters for  $t$ -ary operations.

Algebras over an operad can also be described as objects in  $\mathbf{C}$  with a left action of a particular monad (or triple). Each  $\Sigma$ -operad  $\mathcal{O}$  determines a functor  $\mathcal{O}(-) : \mathbf{C} \rightarrow \mathbf{C}$  defined objectwise by

$$\mathcal{O}(X) := \mathcal{O} \circ (X) = \coprod_{t \geq 0} \mathcal{O}[\mathbf{t}] \otimes_{\Sigma_t} X^{\otimes t},$$

along with natural transformations  $m : \mathcal{O}\mathcal{O} \rightarrow \mathcal{O}$  and  $\eta : \text{id} \rightarrow \mathcal{O}$  which make appropriate associativity and unit diagrams commute. Giving an  $\mathcal{O}$ -algebra structure on  $X$  is the same as giving a left action of  $\mathcal{O}(-)$  on  $X$ ; i.e., a map  $m : \mathcal{O}(X) \rightarrow X$  in  $\mathbf{C}$  which makes the appropriate associativity and unit diagrams commute. From this perspective, operads correspond to special functors in such a way that circle product corresponds to composition, but because these functors have such simple descriptions in terms of symmetric sequences, operads are easier to work with than arbitrary functors.

## 7. FREELY ADDING $\Sigma$ -ACTIONS

**Proposition 7.1.** *There is an adjunction*

$$\text{Seq} \begin{array}{c} \xleftarrow{-\cdot \Sigma} \\ \xrightarrow{\Phi^*} \end{array} \text{SymSeq}$$

with left adjoint on top and  $\Phi : \Omega \rightarrow \Sigma$  the forgetful functor.

*Proof.* There is a chain of adjunctions

$$\text{Seq} \rightleftarrows \text{Seq}' \begin{array}{c} \xleftarrow{-\cdot \Sigma} \\ \xrightarrow{\quad} \end{array} \text{SymSeq}' \rightleftarrows \text{SymSeq},$$

with left adjoints on top; here,  $(A \cdot \Sigma)[\mathbf{s}] := A[\mathbf{s}] \cdot \Sigma_{\mathbf{s}}$ . Composition gives the desired adjunction, with left adjoint also denoted  $-\cdot \Sigma$ .  $\square$

**7.1. Properties of the left adjoint.** It is useful to understand how tensor products and circle products behave with respect to  $- \cdot \Sigma$ .

**Proposition 7.2.** *Let  $A, B$  be sequences and  $t \geq 0$ . There are natural isomorphisms,*

$$(7.3) \quad (A \cdot \Sigma) \otimes (B \cdot \Sigma) \cong (A \hat{\otimes} B) \cdot \Sigma,$$

$$(7.4) \quad (B \cdot \Sigma)^{\otimes t} \cong (B^{\hat{\otimes} t}) \cdot \Sigma,$$

$$(7.5) \quad (A \cdot \Sigma) \circ (B \cdot \Sigma) \cong (A \hat{\circ} B) \cdot \Sigma,$$

in  $\text{SymSeq}$ .

*Proof.* To verify (7.3), let  $s \geq 0$  and consider the natural isomorphisms,

$$\begin{aligned} ((A \cdot \Sigma) \otimes (B \cdot \Sigma))[s] &\cong \coprod_{s_1 + s_2 = s} (A[s_1] \cdot \Sigma_{s_1}) \otimes (B[s_2] \cdot \Sigma_{s_2}) \cdot_{\Sigma_{s_1} \times \Sigma_{s_2}} \Sigma_s \\ &\cong ((A \hat{\otimes} B) \cdot \Sigma)[s]. \end{aligned}$$

To verify (7.5), consider the natural isomorphisms,

$$(A \cdot \Sigma) \circ (B \cdot \Sigma) \cong \coprod_{t \geq 0} (A \cdot \Sigma)[t] \otimes_{\Sigma_t} (B^{\hat{\otimes} t}) \cdot \Sigma \cong (A \hat{\circ} B) \cdot \Sigma.$$

□

## 7.2. Naturally occurring maps.

**Proposition 7.6.** *Let  $B \in \text{Seq}$  and  $C \in \text{SymSeq}$ . There is a map*

$$i : B \hat{\circ} C \longrightarrow (B \cdot \Sigma) \circ C$$

in  $\text{Seq}$ , natural in  $B, C$ .

*Proof.* Each  $M \in \Omega$  may be regarded as an unordered set; use this together with (4.4) and (4.6) to look for naturally occurring maps

$$i_M : (B \hat{\circ} C)[M] \longrightarrow ((B \cdot \Sigma) \circ C)[M],$$

for each  $M \in \Omega$ .

□

## 7.3. Adjunctions.

**Proposition 7.7.** *Let  $\mathcal{O}$  be an  $\Omega$ -operad. There are adjunctions*

$$\text{Lt}_{\mathcal{O}} \begin{array}{c} \xrightarrow{-\cdot \Sigma} \\ \xleftarrow{U} \end{array} \text{Lt}_{\mathcal{O} \cdot \Sigma} \quad \text{Alg}_{\mathcal{O}} \begin{array}{c} \xleftarrow{U} \\ \xrightarrow{-\cdot \Sigma} \end{array} \text{Alg}_{\mathcal{O} \cdot \Sigma}$$

with left adjoints on top and  $U$  the forgetful functor. The right-hand adjunction is an isomorphism of categories.

*Proof.* Use Propositions 7.2 and 7.6. □

## 8. LIMITS AND COLIMITS FOR MODULES AND ALGEBRAS

We have introduced modules and algebras over an operad, but to work with them, we need to understand how to build basic constructions such as limits and colimits. The material in this section is largely influenced by Rezk [38].

### 8.1. Reflexive coequalizers and filtered colimits.

**Definition 8.1.** A pair of maps of the form  $X_0 \begin{smallmatrix} \xleftarrow{d_0} \\ \xleftarrow{d_1} \end{smallmatrix} X_1$  in  $\mathcal{C}$  is called a *reflexive pair* if there exists  $s_0 : X_0 \rightarrow X_1$  in  $\mathcal{C}$  such that  $d_0 s_0 = \text{id}$  and  $d_1 s_0 = \text{id}$ . A *reflexive coequalizer* is the coequalizer of a reflexive pair.

Reflexive coequalizers will be useful for building colimits in the categories of modules and algebras over an operad.

**Proposition 8.2.**

- (a) Suppose  $X_{-1} \leftarrow X_0 \begin{smallmatrix} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{smallmatrix} X_1$  and  $Y_{-1} \leftarrow Y_0 \begin{smallmatrix} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{smallmatrix} Y_1$  are reflexive coequalizer diagrams in  $\mathcal{C}$ . Then their objectwise tensor product

$$X_{-1} \otimes Y_{-1} \leftarrow X_0 \otimes Y_0 \begin{smallmatrix} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{smallmatrix} X_1 \otimes Y_1$$

is a reflexive coequalizer diagram in  $\mathcal{C}$ .

- (b) Suppose  $X, Y : \mathcal{D} \rightarrow \mathcal{C}$  are filtered diagrams. Then objectwise tensor product of their colimiting cones is a colimiting cone. In particular, there are natural isomorphisms

$$\text{colim}_{d \in \mathcal{D}} (X_d \otimes Y_d) \cong (\text{colim}_{d \in \mathcal{D}} X_d) \otimes (\text{colim}_{d \in \mathcal{D}} Y_d).$$

in  $\mathcal{C}$ .

*Proof.* Consider (a) and the diagram

$$\begin{array}{ccc} X_0 \otimes Y_1 & \begin{smallmatrix} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{smallmatrix} & X_1 \otimes Y_1 \\ \Downarrow & & \Downarrow \\ X_0 \otimes Y_0 & \begin{smallmatrix} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{smallmatrix} & X_1 \otimes Y_0 \end{array}$$

Look for a naturally occurring cone into  $X_{-1} \otimes Y_{-1}$ , and verify this cone is initial with respect to all cones. Use relations satisfied by the reflexive pairs and note that tensoring with any  $X \in \mathcal{C}$  preserves colimiting cones. Verification of (b) is similar to (a), except instead of properties satisfied by reflexive pairs, use properties satisfied by filtered diagrams.  $\square$

Hence objectwise tensor product of diagrams in  $(\mathcal{C}, \otimes, k)$  respects certain colimiting cones. Objectwise circle product of diagrams in  $(\text{SymSeq}, \circ, I)$  and  $(\text{Seq}, \hat{\circ}, I)$  behave similarly.

**Proposition 8.3.**

- (a) Suppose  $A_{-1} \leftarrow A_0 \begin{smallmatrix} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{smallmatrix} A_1$  and  $B_{-1} \leftarrow B_0 \begin{smallmatrix} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{smallmatrix} B_1$  are reflexive coequalizer diagrams in  $\text{SymSeq}$ . Then their objectwise circle product

$$(8.4) \quad A_{-1} \circ B_{-1} \leftarrow A_0 \circ B_0 \begin{smallmatrix} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{smallmatrix} A_1 \circ B_1$$

is a reflexive coequalizer diagram in  $\text{SymSeq}$ .

- (b) Suppose  $A, B : \mathcal{D} \rightarrow \text{SymSeq}$  are filtered diagrams. Then objectwise circle product of their colimiting cones is a colimiting cone. In particular, there are natural isomorphisms

$$\text{colim}_{d \in \mathcal{D}} (A_d \circ B_d) \cong (\text{colim}_{d \in \mathcal{D}} A_d) \circ (\text{colim}_{d \in \mathcal{D}} B_d).$$

in  $\text{SymSeq}$ .

- (c) For sequences, the corresponding statements in (a) and (b) remain true; i.e., when  $(\text{SymSeq}, \circ, I)$  is replaced by  $(\text{Seq}, \hat{\circ}, I)$ .

*Proof.* Consider (a). We want to verify that (8.4) is a colimiting cone, hence sufficient to verify it is initial with respect to all cones. Use the universal property (5.3) together with (4.10) and Proposition 8.2(a). Verification of (b) is similar, except use Proposition 8.2(b).  $\square$

**Proposition 8.5.**

- (a) Suppose  $A_{-1} \longleftarrow A_0 \rightleftarrows A_1$  is a reflexive coequalizer diagram in  $\text{SymSeq}$  and  $Z_{-1} \longleftarrow Z_0 \rightleftarrows Z_1$  is a reflexive coequalizer diagram in  $\mathcal{C}$ . Then their objectwise evaluation

$$A_{-1} \circ (Z_{-1}) \longleftarrow A_0 \circ (Z_0) \rightleftarrows A_1 \circ (Z_1)$$

is a reflexive coequalizer diagram in  $\mathcal{C}$ .

- (b) Suppose  $A : \mathcal{D} \rightarrow \text{SymSeq}$  and  $Z : \mathcal{D} \rightarrow \mathcal{C}$  are filtered diagrams. Then objectwise evaluation of their colimiting cones is a colimiting cone. In particular, there are natural isomorphisms

$$\text{colim}_{d \in \mathcal{D}} (A_d \circ (Z_d)) \cong (\text{colim}_{d \in \mathcal{D}} A_d) \circ (\text{colim}_{d \in \mathcal{D}} Z_d).$$

in  $\mathcal{C}$ .

- (c) For sequences, the corresponding statements in (a) and (b) remain true; i.e., when  $(\text{SymSeq}, \circ, I)$  is replaced by  $(\text{Seq}, \hat{\circ}, I)$ .

*Proof.* Argue as a special case of Proposition 8.3 using the embedding in (5.7).  $\square$

**8.2. Free-forgetful adjunctions.**

**Definition 8.6.** Let  $\mathcal{O}, R, S$  be operads.

- $\text{Lt}_{\mathcal{O}}$  is the category of left  $\mathcal{O}$ -modules and their morphisms.
- $\text{Rt}_{\mathcal{O}}$  is the category of right  $\mathcal{O}$ -modules and their morphisms.
- $\text{Bi}_{(R,S)}$  is the category of  $(R, S)$ -bimodules and their morphisms.
- $\text{Alg}_{\mathcal{O}}$  is the category of  $\mathcal{O}$ -algebras and their morphisms.

It will be useful to establish the following free-forgetful adjunctions.

**Proposition 8.7.**

- (a) Let  $\mathcal{O}, R, S$  be  $\Sigma$ -operads. There are adjunctions

$$\text{SymSeq} \begin{array}{c} \xrightarrow{\mathcal{O} \circ -} \\ \xleftarrow{U} \end{array} \text{Lt}_{\mathcal{O}}, \quad \text{SymSeq} \begin{array}{c} \xrightarrow{- \circ \mathcal{O}} \\ \xleftarrow{U} \end{array} \text{Rt}_{\mathcal{O}}, \quad \text{SymSeq} \begin{array}{c} \xrightarrow{R \circ - \circ S} \\ \xleftarrow{U} \end{array} \text{Bi}_{(R,S)},$$

with left adjoints on top and  $U$  the forgetful functor.

- (b) Let  $\mathcal{O}, R, S$  be  $\Omega$ -operads. There are adjunctions

$$\text{Seq} \begin{array}{c} \xrightarrow{\mathcal{O} \hat{\circ} -} \\ \xleftarrow{U} \end{array} \text{Lt}_{\mathcal{O}}, \quad \text{Seq} \begin{array}{c} \xrightarrow{- \hat{\circ} \mathcal{O}} \\ \xleftarrow{U} \end{array} \text{Rt}_{\mathcal{O}}, \quad \text{Seq} \begin{array}{c} \xrightarrow{R \hat{\circ} - \hat{\circ} S} \\ \xleftarrow{U} \end{array} \text{Bi}_{(R,S)},$$

with left adjoints on top and  $U$  the forgetful functor.

*Proof.* To verify the first adjunction of (a), it is enough to show there are isomorphisms

$$\text{hom}_{\text{Lt}_{\mathcal{O}}}(\mathcal{O} \circ B, C) \cong \text{hom}_{\text{SymSeq}}(B, UC)$$

natural in  $B, C$ . For this, it is enough to verify the universal property that given any map  $f : B \rightarrow UC$  in  $\mathbf{SymSeq}$ , there exists a unique map  $\bar{f} : \mathcal{O} \circ B \rightarrow C$  in  $\mathbf{Lt}_{\mathcal{O}}$  such that  $f$  factors through the map

$$B \cong \text{id} \circ B \xrightarrow{\eta^{\text{oid}}} \mathcal{O} \circ B = U(\mathcal{O} \circ B)$$

via the map  $U\bar{f}$ . The other cases are similar.  $\square$

There are similar free-forgetful adjunctions for algebras over an operad.

**Proposition 8.8.** *Let  $\mathcal{O}$  be a  $\Sigma$ -operad and  $\mathcal{O}'$  be an  $\Omega$ -operad. There are adjunctions*

$$\mathbf{C} \begin{array}{c} \xrightarrow{\mathcal{O} \circ (-)} \\ \xleftarrow{U} \end{array} \mathbf{Alg}_{\mathcal{O}}, \quad \mathbf{C} \begin{array}{c} \xrightarrow{\mathcal{O}' \circ (-)} \\ \xleftarrow{U} \end{array} \mathbf{Alg}_{\mathcal{O}'},$$

with left adjoints on top and  $U$  the forgetful functor.

*Proof.* Argue as in the proof of Proposition 8.7.  $\square$

### 8.3. Construction of colimits.

**Proposition 8.9.** *Let  $\mathcal{O}, R, S$  be operads. Reflexive coequalizers and filtered colimits exist in  $\mathbf{Lt}_{\mathcal{O}}, \mathbf{Rt}_{\mathcal{O}}, \mathbf{Bi}_{(R,S)}$ , and  $\mathbf{Alg}_{\mathcal{O}}$ , and are preserved (and created) by the forgetful functors in Propositions 8.7 and 8.8.*

*Proof.* Let  $\mathcal{O}$  be a  $\Sigma$ -operad and consider the case of left  $\mathcal{O}$ -modules. Suppose  $A_0 \rightrightarrows A_1$  is a reflexive pair in  $\mathbf{Lt}_{\mathcal{O}}$  and consider the solid commutative diagram

$$\begin{array}{ccccc} \mathcal{O} \circ \mathcal{O} \circ A_{-1} & \longleftarrow & \mathcal{O} \circ \mathcal{O} \circ A_0 & \rightrightarrows & \mathcal{O} \circ \mathcal{O} \circ A_1 \\ \begin{array}{c} \downarrow d_0 \\ \downarrow d_1 \end{array} & & \begin{array}{c} \downarrow m \circ \text{id} \\ \downarrow \text{id} \circ m \end{array} & & \begin{array}{c} \downarrow m \circ \text{id} \\ \downarrow \text{id} \circ m \end{array} \\ \mathcal{O} \circ A_{-1} & \longleftarrow & \mathcal{O} \circ A_0 & \rightrightarrows & \mathcal{O} \circ A_1 \\ \begin{array}{c} \uparrow s_0 \\ \downarrow m \end{array} & & \begin{array}{c} \downarrow m \\ \downarrow m \end{array} & & \begin{array}{c} \downarrow m \\ \downarrow m \end{array} \\ A_{-1} & \longleftarrow & A_0 & \rightrightarrows & A_1 \end{array}$$

in  $\mathbf{SymSeq}$ , with bottom row the reflexive coequalizer diagram of the underlying reflexive pair in  $\mathbf{SymSeq}$ . By Proposition 8.3, the rows are reflexive coequalizer diagrams and hence there exist unique dotted arrows  $m, s_0, d_0, d_1$  in  $\mathbf{SymSeq}$  which make the diagram commute. By uniqueness,  $s_0 = \eta^{\text{oid}}$ ,  $d_0 = m \circ \text{id}$ , and  $d_1 = \text{id} \circ m$ . Verify that  $m$  gives  $A_{-1}$  the structure of a left  $\mathcal{O}$ -module and also verify that the bottom row is a reflexive coequalizer diagram in  $\mathbf{Lt}_{\mathcal{O}}$ . First check the diagram lives in  $\mathbf{Lt}_{\mathcal{O}}$ , then check the colimiting cone is initial with respect to all cones in  $\mathbf{Lt}_{\mathcal{O}}$ . The case for filtered colimits is similar. The cases for  $\mathbf{Rt}_{\mathcal{O}}, \mathbf{Bi}_{(R,S)}$ , and  $\mathbf{Alg}_{\mathcal{O}}$  can be argued similarly. Argue similarly for  $\Omega$ -operads.  $\square$

**Proposition 8.10.** *Let  $\mathcal{O}, R, S$  be operads. All small colimits exist in  $\mathbf{Lt}_{\mathcal{O}}, \mathbf{Rt}_{\mathcal{O}}, \mathbf{Bi}_{(R,S)}$ , and  $\mathbf{Alg}_{\mathcal{O}}$ .*

*Proof.* Let  $\mathcal{O}$  be a  $\Sigma$ -operad and consider the case of left  $\mathcal{O}$ -modules. Suppose  $A : D \rightarrow \mathbf{Lt}_{\mathcal{O}}$  is a small diagram. Want to show that  $\text{colim } A$  exists. This colimit

may be calculated by a reflexive coequalizer in  $\mathbf{Lt}_{\mathcal{O}}$  of the form,

$$\operatorname{colim} A \cong \operatorname{colim} \left( \operatorname{colim}_{d \in \mathbf{D}} (\mathcal{O} \circ A_d) \begin{array}{c} \xleftarrow{(\text{moid})_*} \\ \xleftarrow{(\text{idom})_*} \end{array} \operatorname{colim}_{d \in \mathbf{D}} (\mathcal{O} \circ \mathcal{O} \circ A_d) \right),$$

provided the indicated colimits appearing in this reflexive pair exist in  $\mathbf{Lt}_{\mathcal{O}}$ . The underlying category  $\mathbf{SymSeq}$  has all small colimits, and left adjoints preserve colimiting cones, hence there is a commutative diagram

$$\begin{array}{ccc} \operatorname{colim}_{d \in \mathbf{D}} (\mathcal{O} \circ A_d) & \begin{array}{c} \xleftarrow{(\text{moid})_*} \\ \xleftarrow{(\text{idom})_*} \end{array} & \operatorname{colim}_{d \in \mathbf{D}} (\mathcal{O} \circ \mathcal{O} \circ A_d) \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{O} \circ \left( \operatorname{colim}_{d \in \mathbf{D}} A_d \right) & \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{array} & \mathcal{O} \circ \left( \operatorname{colim}_{d \in \mathbf{D}} (\mathcal{O} \circ A_d) \right) \end{array}$$

in  $\mathbf{Lt}_{\mathcal{O}}$ . The colimits in the bottom row exist since they are in the underlying category  $\mathbf{SymSeq}$  (we have dropped the notation for the forgetful functor  $U$ ), hence the colimits in the top row exist in  $\mathbf{Lt}_{\mathcal{O}}$ . Therefore  $\operatorname{colim} A$  exists. The cases for  $\mathbf{Rt}_{\mathcal{O}}$ ,  $\mathbf{Bi}_{(R,S)}$ , and  $\mathbf{Alg}_{\mathcal{O}}$  can be argued similarly. Argue similarly for  $\Omega$ -operads.  $\square$

*Example 8.11.* For instance, if  $\mathcal{O}$  is a  $\Sigma$ -operad and  $A, B \in \mathbf{Lt}_{\mathcal{O}}$ , then the coproduct  $A \amalg B$  in  $\mathbf{Lt}_{\mathcal{O}}$  may be calculated by a reflexive coequalizer of the form

$$A \amalg B \cong \operatorname{colim} \left( \mathcal{O} \circ (A \amalg B) \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathcal{O} \circ ((\mathcal{O} \circ A) \amalg (\mathcal{O} \circ B)) \right)$$

in the underlying category  $\mathbf{SymSeq}$ . The coproducts appearing inside the parentheses are in the underlying category  $\mathbf{SymSeq}$ .

**8.4. Colimits for right modules.** Colimits in right modules over an operad are particularly simple.

**Proposition 8.12.** *Let  $\mathcal{O}$  be an operad. The forgetful functors from right  $\mathcal{O}$ -modules in Proposition 8.7 preserve (and create) all small colimits.*

*Proof.* Let  $\mathcal{O}$  be a  $\Sigma$ -operad and suppose  $A : \mathbf{D} \rightarrow \mathbf{Rt}_{\mathcal{O}}$  is a small diagram. By the proof of Proposition 8.10,  $\operatorname{colim} A$  may be calculated in the underlying category by the colimit in  $\mathbf{SymSeq}$  of the reflexive pair in the top row of the commutative diagram

$$\begin{array}{ccc} \left( \operatorname{colim}_{d \in \mathbf{D}} A_d \right) \circ \mathcal{O} & \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{array} & \left( \operatorname{colim}_{d \in \mathbf{D}} (A_d \circ \mathcal{O}) \right) \circ \mathcal{O} \\ \downarrow \cong & & \downarrow \cong \\ \operatorname{colim}_{d \in \mathbf{D}} (A_d \circ \mathcal{O}) & \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \end{array} & \operatorname{colim}_{d \in \mathbf{D}} (A_d \circ \mathcal{O} \circ \mathcal{O}), \end{array}$$

and since the functor  $- \circ \mathcal{O} : \mathbf{SymSeq} \rightarrow \mathbf{SymSeq}$  preserves colimiting cones, this is the same as calculating the colimit of the bottom row in  $\mathbf{SymSeq}$ , which is the colimit of the underlying diagram of  $A$  in  $\mathbf{SymSeq}$ . We have dropped the notation for the forgetful functor  $U$ . Argue similarly for  $\Omega$ -operads.  $\square$

### 8.5. Construction of limits.

**Proposition 8.13.** *Let  $\mathcal{O}, R, S$  be operads. All small limits exist in  $\text{Lt}_{\mathcal{O}}$ ,  $\text{Rt}_{\mathcal{O}}$ ,  $\text{Bi}_{(R,S)}$ , and  $\text{Alg}_{\mathcal{O}}$ , and are preserved (and created) by the forgetful functors in Propositions 8.7 and 8.8.*

*Proof.* This can be argued similar to the proof of Proposition 8.9.  $\square$

## 9. BASIC CONSTRUCTIONS FOR MODULES

In this section we present some basic constructions for modules. The material in this section is largely influenced by Rezk [38]. Compare the following definition with [29, Chapter VI.5].

### 9.1. Circle products (mapping sequences) over an operad.

**Definition 9.1.** Let  $R$  be a  $\Sigma$ -operad (resp.  $\Omega$ -operad),  $A$  a right  $R$ -module, and  $B$  a left  $R$ -module. Define  $A \circ_R B \in \text{SymSeq}$  (resp.  $A \hat{\circ}_R B \in \text{Seq}$ ) by the reflexive coequalizer

$$A \circ_R B := \text{colim} \left( A \circ B \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{d_1} \end{array} A \circ R \circ B \right),$$

$$\left( \text{resp. } A \hat{\circ}_R B := \text{colim} \left( A \hat{\circ} B \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{d_1} \end{array} A \hat{\circ} R \hat{\circ} B \right) \right),$$

with  $d_0$  induced by  $m : A \circ R \rightarrow A$  and  $d_1$  induced by  $m : R \circ B \rightarrow B$  (resp.  $d_0$  induced by  $m : A \hat{\circ} R \rightarrow A$  and  $d_1$  induced by  $m : R \hat{\circ} B \rightarrow B$ ).

**Definition 9.2.** Let  $S$  be a  $\Sigma$ -operad (resp.  $\Omega$ -operad) and let  $B$  and  $C$  be right  $S$ -modules. Define  $\text{Map}_S^\circ(B, C) \in \text{SymSeq}$  (resp.  $\text{Map}_S^\hat{\circ}(B, C) \in \text{Seq}$ ) by the equalizer

$$\text{Map}_S^\circ(B, C) := \lim \left( \text{Map}^\circ(B, C) \begin{array}{c} \xrightarrow{d^0} \\ \xleftarrow{d^1} \end{array} \text{Map}^\circ(B \circ S, C) \right),$$

$$\left( \text{resp. } \text{Map}_S^\hat{\circ}(B, C) := \lim \left( \text{Map}^\hat{\circ}(B, C) \begin{array}{c} \xrightarrow{d^0} \\ \xleftarrow{d^1} \end{array} \text{Map}^\hat{\circ}(B \hat{\circ} S, C) \right) \right),$$

with  $d^0$  induced by  $m : B \circ S \rightarrow B$  and  $d^1$  induced by  $m : C \circ S \rightarrow C$  (resp.  $d^0$  induced by  $m : B \hat{\circ} S \rightarrow B$  and  $d^1$  induced by  $m : C \hat{\circ} S \rightarrow C$ ).

**9.2. Adjunctions.** Compare the following adjunctions with [29, Chapter VI.8].

**Proposition 9.3.** *Let  $Q, R, S$  be  $\Sigma$ -operads. There are isomorphisms,*

$$(9.4) \quad \text{hom}_{\text{Rt}_S}(A \circ B, C) \cong \text{hom}(A, \text{Map}_S^\circ(B, C)),$$

$$(9.5) \quad \text{hom}(A \circ_R B, C) \cong \text{hom}_{\text{Rt}_R}(A, \text{Map}^\circ(B, C)),$$

$$(9.6) \quad \text{hom}_{(Q,S)}(A \circ_R B, C) \cong \text{hom}_{(Q,R)}(A, \text{Map}_S^\circ(B, C)),$$

*natural in  $A, B, C$ .*

*Remark 9.7.* In (9.4),  $A$  is a symmetric sequence, and both  $B$  and  $C$  have right  $S$ -module structures. In (9.5),  $A$  has a right  $R$ -module structure,  $B$  has a left  $R$ -module structure, and  $C$  is a symmetric sequence. In (9.6),  $A$  has a  $(Q, R)$ -bimodule structure,  $B$  has a  $(R, S)$ -bimodule structure, and  $C$  has a  $(Q, S)$ -bimodule structure.

*Proof.* The natural correspondence (9.6) implies both (9.4) and (9.5), and its proof can be built up from those for (9.4) and (9.5). Use the natural correspondences (5.17) together with the commutative diagrams satisfied by each object as defined in Section 6.  $\square$

There is a corresponding statement for  $\Omega$ -operads.

**Proposition 9.8.** *Let  $Q, R, S$  be  $\Omega$ -operads. There are isomorphisms,*

$$\begin{aligned} \mathrm{hom}_{\mathrm{Rt}_S}(A \hat{\circ} B, C) &\cong \mathrm{hom}(A, \mathrm{Map}_S^{\hat{\circ}}(B, C)), \\ \mathrm{hom}(A \hat{\circ}_R B, C) &\cong \mathrm{hom}_{\mathrm{Rt}_R}(A, \mathrm{Map}^{\hat{\circ}}(B, C)), \\ \mathrm{hom}_{(Q,S)}(A \hat{\circ}_R B, C) &\cong \mathrm{hom}_{(Q,R)}(A, \mathrm{Map}_S^{\hat{\circ}}(B, C)), \end{aligned}$$

*natural in  $A, B, C$ .*

*Proof.* Argue as in the proof of Proposition 9.3.  $\square$

### 9.3. Cancellation and associativity properties.

**Proposition 9.9.** *Let  $R$  be a  $\Sigma$ -operad (resp.  $\Omega$ -operad),  $A$  a right  $R$ -module, and  $B$  a left  $R$ -module. There are natural isomorphisms*

$$\begin{aligned} A \circ_R R &\cong A & R \circ_R B &\cong B, \\ (\text{resp. } A \hat{\circ}_R R &\cong A & R \hat{\circ}_R B &\cong B). \end{aligned}$$

*Proof.* Suppose  $R$  is a  $\Sigma$ -operad. We want to verify  $A$  is naturally isomorphic to a particular coequalizer. Look for a naturally occurring cone into  $A$  and verify it is initial with respect to all cones. The other cases are similar.  $\square$

**Proposition 9.10.** *Let  $R, S$  be  $\Sigma$ -operads (resp.  $\Omega$ -operads),  $A$  a right  $R$ -module,  $B$  an  $(R, S)$ -bimodule, and  $C$  a left  $S$ -module. There are natural isomorphisms*

$$\begin{aligned} (A \circ_R B) \circ_S C &\cong A \circ_R (B \circ_S C), \\ (\text{resp. } (A \hat{\circ}_R B) \hat{\circ}_S C &\cong A \hat{\circ}_R (B \hat{\circ}_S C)). \end{aligned}$$

*Proof.* Use Proposition 8.3.  $\square$

### 9.4. Change of operads adjunction.

**Proposition 9.11.** *Let  $f : R \rightarrow S$  be a morphism of  $\Sigma$ -operads (resp.  $\Omega$ -operads). There is an adjunction*

$$\mathrm{Lt}_R \begin{array}{c} \xrightarrow{f_*} \\ \xleftarrow{f^*} \end{array} \mathrm{Lt}_S,$$

*with left adjoint  $f_* := S \circ_R -$  (resp.  $f_* := S \hat{\circ}_R -$ ) and  $f^*$  the forgetful functor. In particular, there are isomorphisms*

$$\begin{aligned} \mathrm{hom}_{\mathrm{Lt}_S}(S \circ_R A, B) &\cong \mathrm{hom}_{\mathrm{Lt}_R}(A, f^*(B)), \\ (\text{resp. } \mathrm{hom}_{\mathrm{Lt}_S}(S \hat{\circ}_R A, B) &\cong \mathrm{hom}_{\mathrm{Lt}_R}(A, f^*(B))), \end{aligned}$$

*natural in  $A, B$ .*

*Proof.* Look for natural transformations  $\eta : \mathrm{id} \rightarrow f^* f_*$  and  $\varepsilon : f_* f^* \rightarrow \mathrm{id}$  and verify that  $f_* \xrightarrow{f_* \eta} f_* f^* f_* \xrightarrow{\varepsilon f_*} f_*$  and  $f^* \xrightarrow{\eta f^*} f^* f_* f^* \xrightarrow{f^* \varepsilon} f^*$  each factor the identity.  $\square$

## 10. BASIC CONSTRUCTIONS FOR ALGEBRAS

Here we present some corresponding constructions for algebras.

## 10.1. Circle products over an operad.

**Definition 10.1.** Let  $R$  be a  $\Sigma$ -operad (resp.  $\Omega$ -operad),  $A$  a right  $R$ -module, and  $Z$  an  $R$ -algebra. Define  $A \circ_R (Z) \in \mathbf{C}$  (resp.  $A \hat{\circ}_R (Z) \in \mathbf{C}$ ) by the reflexive coequalizer

$$A \circ_R (Z) := \operatorname{colim} \left( A \circ (Z) \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{d_1} \end{array} (A \circ R) \circ (Z) \right),$$

$$\left( \text{resp. } A \hat{\circ}_R (Z) := \operatorname{colim} \left( A \hat{\circ} (Z) \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{d_1} \end{array} (A \hat{\circ} R) \hat{\circ} (Z) \right) \right),$$

with  $d_0$  induced by  $m : A \circ R \rightarrow A$  and  $d_1$  induced by  $m : R \circ (Z) \rightarrow Z$  (resp.  $d_0$  induced by  $m : A \hat{\circ} R \rightarrow A$  and  $d_1$  induced by  $m : R \hat{\circ} (Z) \rightarrow Z$ ).

## 10.2. Cancellation and associativity properties.

**Proposition 10.2.** Let  $R$  be a  $\Sigma$ -operad (resp.  $\Omega$ -operad) and  $Z$  an  $R$ -algebra. There are natural isomorphisms

$$R \circ_R (Z) \cong Z, \quad (\text{resp. } R \hat{\circ}_R (Z) \cong Z).$$

*Proof.* Argue as a special case of Proposition 9.9 by taking  $B := \hat{Z}$ .  $\square$

**Proposition 10.3.** Let  $R, S$  be  $\Sigma$ -operads (resp.  $\Omega$ -operads),  $A$  a right  $R$ -module,  $B$  an  $(R, S)$ -bimodule, and  $Z$  an  $S$ -algebra. There are natural isomorphisms

$$(A \circ_R B) \circ_S (Z) \cong A \circ_R (B \circ_S (Z)),$$

$$\left( \text{resp. } (A \hat{\circ}_R B) \hat{\circ}_S (Z) \cong A \hat{\circ}_R (B \hat{\circ}_S (Z)) \right).$$

*Proof.* Argue as a special case of Proposition 9.10 by taking  $C := \hat{Z}$ .  $\square$

## 10.3. Change of operads adjunction.

**Proposition 10.4.** Let  $f : R \rightarrow S$  be a morphism of  $\Sigma$ -operads (resp.  $\Omega$ -operads). There is an adjunction

$$\operatorname{Alg}_R \begin{array}{c} \xrightarrow{f_*} \\ \xleftarrow{f^*} \end{array} \operatorname{Alg}_S,$$

with left adjoint  $f_* := S \circ_R (-)$  (resp.  $f_* := S \hat{\circ}_R (-)$ ) and  $f^*$  the forgetful functor. In particular, there are isomorphisms

$$\operatorname{hom}_{\operatorname{Alg}_S}(S \circ_R (A), B) \cong \operatorname{hom}_{\operatorname{Alg}_R}(A, f^*(B)),$$

$$\left( \text{resp. } \operatorname{hom}_{\operatorname{Alg}_S}(S \hat{\circ}_R (A), B) \cong \operatorname{hom}_{\operatorname{Alg}_R}(A, f^*(B)) \right),$$

natural in  $A, B$ .

*Proof.* Argue as in the proof of Proposition 9.11.  $\square$

11. MODEL CATEGORIES AND DEFINITIONS

In this section we establish some notation; the definitions appearing below are only intended to make precise Basic Assumption 1.1. We assume the reader is familiar with model categories. A useful introduction is given in [4], from which we have taken the model category axioms listed below. See also the original articles by Quillen [36, 37], and the more recent [3, 15, 21, 22].

In this paper, our primary method of constructing model categories from existing ones involves the additional structure of a cofibrantly generated model category together with (possibly transfinite) small object arguments. Schwede and Shipley provide an account of these techniques in [39, Section 2] which will be sufficient for our purposes. The reader unfamiliar with the small object argument may consult [4, Section 7.12] for a useful introduction; after which the (possibly transfinite) versions in [21, 22, 39] appear quite natural.

11.1. Model categories.

**Definition 11.1.** In  $\mathcal{C}$ , a map  $i : A \rightarrow B$  has the *left lifting property* (LLP) with respect to a map  $p : X \rightarrow Y$  if every solid commutative diagram of the form

$$(11.2) \quad \begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow \xi & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

has a lift  $\xi$ . In  $\mathcal{C}$ , a map  $p : X \rightarrow Y$  has the *right lifting property* (RLP) with respect to a map  $i : A \rightarrow B$  if every solid commutative diagram of the form (11.2) has a lift  $\xi$ .

**Definition 11.3.** A *model category* is a category  $\mathcal{C}$  with three distinguished subcategories of  $\mathcal{C}$ :

- $\mathcal{W}$  the subcategory of *weak equivalences*
- $\mathcal{Fib}$  the subcategory of *fibrations*
- $\mathcal{Cof}$  the subcategory of *cofibrations*

each of which contains all objects of  $\mathcal{C}$ . A map which is both a fibration (resp. cofibration) and a weak equivalence is called an *acyclic fibration* (resp. *acyclic cofibration*). An object  $X \in \mathcal{C}$  is called *cofibrant* (resp. *fibrant*) if  $\emptyset \rightarrow X$  is a cofibration (resp.  $X \rightarrow *$  is a fibration). We require the following axioms:

- (MC1) Finite limits and colimits exist in  $\mathcal{C}$ .
- (MC2) If  $f$  and  $g$  are maps in  $\mathcal{C}$  such that  $gf$  is defined and if two of the three maps  $f, g, gf$  are weak equivalences, then so is the third.
- (MC3) If  $f$  is a retract of  $g$  and  $g$  is a fibration, cofibration, or weak equivalence, then so is  $f$ .
- (MC4) Cofibrations have the LLP with respect to acyclic fibrations. Acyclic cofibrations have the LLP with respect to fibrations.
- (MC5) Any map  $f$  can be factored in two ways: (i)  $f = pi$ , where  $i$  is a cofibration and  $p$  is an acyclic fibration, and (ii)  $f = pi$ , where  $i$  is an acyclic cofibration and  $p$  is a fibration.

*Remark 11.4.* The definition above describes what was originally called a “closed” model category [37]; following [4] we have dropped the term “closed” in this paper.

**11.2. Cofibrantly generated model categories.** To construct model category structures on (symmetric) sequences and on modules and algebras over an operad, we will require some extra conditions on the model category structure of  $\mathcal{C}$ .

**Definition 11.5.** A model category  $\mathcal{C}$  is *cofibrantly generated* if it has all small limits and colimits and there exists a set  $\mathcal{I}$  of cofibrations and a set  $\mathcal{J}$  of acyclic cofibrations satisfying certain properties [39, Definition 2.2]; these properties will imply that  $\mathcal{I}$  and  $\mathcal{J}$  completely determine the model category structure.

*Remark 11.6.* The maps in  $\mathcal{I}$  (resp.  $\mathcal{J}$ ) are called *generating cofibrations* (resp. *generating acyclic cofibrations*). The properties referred to in Definition 11.5 are useful for creating a model category structure on the target  $\mathcal{D}$  of a left adjoint, from an existing cofibrantly generated model structure on the source  $\mathcal{C}$ . Sometimes the single left adjoint is replaced by a set of left adjoints with the same target.

**11.3. Monoidal model categories.**

**Definition 11.7.** A *monoidal model category* is a model category  $\mathcal{C}$  with a symmetric monoidal closed structure  $(\mathcal{C}, \otimes, \mathbf{k})$  such that the following axiom is satisfied:

(ENR) If  $j : A \rightarrow B$  is a cofibration and  $p : X \rightarrow Y$  is a fibration, then the pullback corner map

$$\mathrm{Map}(B, X) \longrightarrow \mathrm{Map}(A, X) \times_{\mathrm{Map}(A, Y)} \mathrm{Map}(B, Y)$$

is a fibration that is an acyclic fibration if either  $j$  or  $p$  is a weak equivalence.

*Remark 11.8.* A model category with a symmetric monoidal closed structure  $(\mathcal{C}, \otimes, \mathbf{k})$  satisfies the axiom (ENR) if and only if it satisfies the axiom:

(ENR') If  $i : K \rightarrow L$  and  $j : A \rightarrow B$  are cofibrations, then the pushout corner map

$$L \otimes A \coprod_{K \otimes A} K \otimes B \longrightarrow L \otimes B$$

is a cofibration that is an acyclic cofibration if either  $i$  or  $j$  is a weak equivalence.

In particular, Definition 11.7 is equivalent to the definition by Schwede and Shipley [39, Section 3] of a monoidal model category. Lewis and Mandell [28] refer to (ENR) as the *enrichment axiom*, and also include an additional condition involving the unit of  $\otimes$  which we will not require in this paper.

## 12. MODEL CATEGORIES FOR $G$ -OBJECTS AND (SYMMETRIC) SEQUENCES

So far in this paper, except in Section 1, we have only used the property that  $(\mathcal{C}, \otimes, \mathbf{k})$  is a symmetric monoidal closed category with all small limits and colimits. In this section, we begin to make use of the model category assumptions on  $(\mathcal{C}, \otimes, \mathbf{k})$  described in Basic Assumption 1.1.

In this section, we establish natural model category structures on  $G$ -objects and (symmetric) sequences, and investigate how well tensor product and circle product mesh with this model category structure.

**12.1. Model category structure for  $G$ -objects.**

**Theorem 12.1.** *Let  $G$  be a finite group. Then the category  $\mathcal{C}^{G^{\mathrm{op}}}$  of right  $G$  objects has a natural model category structure. The weak equivalences are the objectwise weak equivalences and the fibrations are the objectwise fibrations. The model structure is cofibrantly generated.*

*Proof.* The model category structure on  $\mathbb{C}^{G^{\text{op}}}$  is created by the adjunction

$$\mathbb{C} \begin{array}{c} \xrightarrow{-\cdot G} \\ \xleftarrow{U} \end{array} \mathbb{C}^{G^{\text{op}}}$$

with left adjoint on top and  $U$  the forgetful functor. Define a map  $f$  in  $\mathbb{C}^{G^{\text{op}}}$  to be a weak equivalence (resp. fibration) if  $Uf$  is a weak equivalence (resp. fibration) in  $\mathbb{C}$ . Define a map  $f$  in  $\mathbb{C}^{G^{\text{op}}}$  to be a cofibration if it has the LLP with respect to all acyclic fibrations in  $\mathbb{C}^{G^{\text{op}}}$ . To verify this gives a model category structure, argue as in the proof of Theorem 12.2. By construction, the model category structure is cofibrantly generated.  $\square$

## 12.2. Model category structure for (symmetric) sequences.

**Theorem 12.2.** *The symmetric monoidal closed categories  $(\text{SymSeq}, \otimes, 1)$  of symmetric sequences and  $(\text{Seq}, \hat{\otimes}, 1)$  of sequences have natural model category structures. The weak equivalences are the objectwise weak equivalences and the fibrations are the objectwise fibrations. The model structures are cofibrantly generated and give  $(\text{SymSeq}, \otimes, 1)$  and  $(\text{Seq}, \hat{\otimes}, 1)$  the structure of monoidal model categories.*

*Proof.* The model category structure on  $\text{SymSeq}$  is created by the set of adjunctions

$$\mathbb{C} \begin{array}{c} \xrightarrow{-\cdot \Sigma_s} \\ \xleftarrow{U} \end{array} \mathbb{C}^{\Sigma_s^{\text{op}}} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\text{in}_s^*} \end{array} \text{SymSeq}' \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \text{SymSeq}, \quad s \geq 0,$$

with left adjoints on top,  $U$  the forgetful functor, and inclusion  $\text{in}_s : \Sigma_s^{\text{op}} \rightarrow \Sigma'^{\text{op}}$ . The right-hand adjunction is the equivalence of categories (3.5). Define a map  $f$  in  $\text{SymSeq}$  to be a weak equivalence (resp. fibration) if  $U(\text{in}_s^*(f))$  is a weak equivalence (resp. fibration) in  $\mathbb{C}$  for every  $s \geq 0$ . Define a map  $f$  in  $\text{SymSeq}$  to be a cofibration if it has the LLP with respect to all acyclic fibrations in  $\text{SymSeq}$ .

We want to verify the model category axioms (MC1)-(MC5). We already know (MC1) is satisfied, and verifying (MC2) and (MC3) is clear. The arguments in the proof of [39, Lemma 2.3] use (possibly transfinite) small object arguments to verify (MC5). The first part of (MC4) is satisfied by definition, and the second part of (MC4) follows from the usual lifting and retract argument, as described in the proof of [39, Lemma 2.3]. This verifies the model category axioms. By construction, the model category is cofibrantly generated. By Theorem 12.4, this gives  $(\text{SymSeq}, \otimes, 1)$  the structure of a monoidal model category.

The model category structure on  $\text{Seq}$  is created by the set of adjunctions

$$\mathbb{C} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{U} \end{array} \mathbb{C}^{\Omega_s^{\text{op}}} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\text{in}_s^*} \end{array} \text{Seq}' \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \text{Seq}, \quad s \geq 0,$$

with left adjoints on top,  $U$  the forgetful functor, and inclusion  $\text{in}_s : \Omega_s^{\text{op}} \rightarrow \Omega'^{\text{op}}$ . Argue as in the  $\text{SymSeq}$  case.  $\square$

*Remark 12.3.* Since the right adjoints in this proof all commute with filtered colimits, the smallness conditions needed for the (possibly transfinite) small object arguments in [39, Lemma 2.3] are satisfied. Also, condition (1) of [39, Lemma 2.3] is easily verified since colimits in  $\text{SymSeq}$  and  $\text{Seq}$  are computed objectwise in the underlying category, and since acyclic cofibrations in  $\mathbb{C}$  are preserved under coproducts, pushouts, and transfinite compositions.

**12.3. Pushout corner map for tensor products.** Here we verify that tensor products of symmetric sequences mesh nicely with the model structure.

**Theorem 12.4.**

- (a) *In symmetric sequences (resp. sequences), if  $i : K \rightarrow L$  and  $j : A \rightarrow B$  are cofibrations, then the pushout corner map*

$$L \otimes A \coprod_{K \otimes A} K \otimes B \longrightarrow L \otimes B ,$$

(resp.  $L \hat{\otimes} A \coprod_{K \hat{\otimes} A} K \hat{\otimes} B \longrightarrow L \hat{\otimes} B$ ),

*is a cofibration that is an acyclic cofibration if either  $i$  or  $j$  is a weak equivalence.*

- (b) *In symmetric sequences (resp. sequences), if  $j : A \rightarrow B$  is a cofibration and  $p : X \rightarrow Y$  is a fibration, then the pullback corner map*

$$\mathrm{Map}^\otimes(B, X) \longrightarrow \mathrm{Map}^\otimes(A, X) \times_{\mathrm{Map}^\otimes(A, Y)} \mathrm{Map}^\otimes(B, Y) ,$$

(resp.  $\mathrm{Map}^{\hat{\otimes}}(B, X) \longrightarrow \mathrm{Map}^{\hat{\otimes}}(A, X) \times_{\mathrm{Map}^{\hat{\otimes}}(A, Y)} \mathrm{Map}^{\hat{\otimes}}(B, Y)$ ),

*is a fibration that is an acyclic fibration if either  $j$  or  $p$  is a weak equivalence.*

We prove this theorem in section 12.8.

**12.4. Pushout corner map for circle products.** These model structures also mesh nicely with circle product, provided an additional cofibrancy condition is satisfied. A version of the following theorem is given by Rezk [38] for symmetric sequences of simplicial sets, using a model category structure with fewer weak equivalences.

**Theorem 12.5.** *Let  $A$  be a cofibrant symmetric sequence (resp. cofibrant sequence).*

- (a) *In symmetric sequences (resp. sequences), if  $i : K \rightarrow L$  and  $j : A \rightarrow B$  are cofibrations, then the pushout corner map*

$$L \circ A \coprod_{K \circ A} K \circ B \longrightarrow L \circ B ,$$

(resp.  $L \hat{\circ} A \coprod_{K \hat{\circ} A} K \hat{\circ} B \longrightarrow L \hat{\circ} B$ ),

*is a cofibration that is an acyclic cofibration if either  $i$  or  $j$  is a weak equivalence.*

- (b) *In symmetric sequences (resp. sequences), if  $j : A \rightarrow B$  is a cofibration and  $p : X \rightarrow Y$  is a fibration, then the pullback corner map*

$$\mathrm{Map}^\circ(B, X) \longrightarrow \mathrm{Map}^\circ(A, X) \times_{\mathrm{Map}^\circ(A, Y)} \mathrm{Map}^\circ(B, Y) ,$$

(resp.  $\mathrm{Map}^{\hat{\circ}}(B, X) \longrightarrow \mathrm{Map}^{\hat{\circ}}(A, X) \times_{\mathrm{Map}^{\hat{\circ}}(A, Y)} \mathrm{Map}^{\hat{\circ}}(B, Y)$ ),

*is a fibration that is an acyclic fibration if either  $j$  or  $p$  is a weak equivalence.*

We prove this theorem in section 12.8.

### 12.5. Fixed points and the pullback corner map.

**Proposition 12.6.** *Let  $G$  be a finite group and  $H \subseteq G$  a subgroup. In  $\mathcal{C}^{G^{\text{op}}}$ , suppose  $j : A \rightarrow B$  is a cofibration and  $p : X \rightarrow Y$  is a fibration. Then in  $\mathcal{C}$  the pullback corner map*

$$\text{Map}(B, X)^H \longrightarrow \text{Map}(A, X)^H \times_{\text{Map}(A, Y)^H} \text{Map}(B, Y)^H$$

*is a fibration that is an acyclic fibration if either  $j$  or  $p$  is a weak equivalence.*

*Proof.* Suppose  $j : A \rightarrow B$  is a cofibration and  $p : X \rightarrow Y$  is an acyclic fibration. Let  $i : C \rightarrow D$  be a cofibration in  $\mathcal{C}$ . We want to verify the pullback corner map satisfies the right lifting property with respect to  $i$ .

$$(12.7) \quad \begin{array}{ccc} C & \longrightarrow & \text{Map}(B, X)^H \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ D & \longrightarrow & \text{Map}(A, X)^H \times_{\text{Map}(A, Y)^H} \text{Map}(B, Y)^H. \end{array}$$

The solid commutative diagram (12.7) in  $\mathcal{C}$  has a lift if and only if the solid diagram (12.8) in  $\mathcal{C}^{G^{\text{op}}}$  has a lift,

$$(12.8) \quad \begin{array}{ccc} C \cdot (H \backslash G) & \longrightarrow & \text{Map}(B, X) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ D \cdot (H \backslash G) & \longrightarrow & \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y). \end{array}$$

if and only if the solid diagram (12.9) in  $\mathcal{C}^{G^{\text{op}}}$  has a lift.

$$(12.9) \quad \begin{array}{ccc} A & \longrightarrow & \text{Map}(D \cdot (H \backslash G), X) \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ B & \longrightarrow & \text{Map}(C \cdot (H \backslash G), X) \times_{\text{Map}(C \cdot (H \backslash G), Y)} \text{Map}(D \cdot (H \backslash G), Y). \end{array}$$

Hence it is sufficient to verify that the right-hand vertical map in (12.9) is an acyclic fibration in  $\mathcal{C}$ , and hence in  $\mathcal{C}^{G^{\text{op}}}$ . The map  $i \cdot \text{id} : C \cdot (H \backslash G) \rightarrow D \cdot (H \backslash G)$  is isomorphic in  $\mathcal{C}$  to a coproduct of cofibrations in  $\mathcal{C}$ , hence is itself a cofibration in  $\mathcal{C}$ , and the (ENR) axiom finishes the argument for this case. The other cases are similar.  $\square$

### 12.6. Calculations for mapping sequences.

**Proposition 12.10.** *Let  $B$  and  $X$  be symmetric sequences and  $t \geq 1$ . Then for each  $s \geq 0$  there is a natural isomorphism in  $\mathcal{C}$ ,*

$$\text{Map}(B[\mathbf{s}, \mathbf{t}], X[\mathbf{s}])^{\Sigma_s} \cong \coprod_{s_1 + \dots + s_t = s} \text{Map}(B[\mathbf{s}_1] \otimes \dots \otimes B[\mathbf{s}_t], X[\mathbf{s}])^{\Sigma_{s_1} \times \dots \times \Sigma_{s_t}}.$$

*Proof.* Use the calculation in Proposition 4.3.  $\square$

### 12.7. Tensor products and (acyclic) cofibrations for $G$ -objects.

**Proposition 12.11.** *Let  $G_1, \dots, G_n$  be finite groups.*

- (a) *Suppose for  $k = 1, \dots, n$  that  $j_k : A_k \rightarrow B_k$  is a cofibration between cofibrant objects in  $\mathcal{C}^{G_k^{\text{op}}}$ . Then the induced map*

$$j_1 \otimes \cdots \otimes j_n : A_1 \otimes \cdots \otimes A_n \rightarrow B_1 \otimes \cdots \otimes B_n$$

*is a cofibration in  $\mathcal{C}^{(G_1 \times \cdots \times G_n)^{\text{op}}}$  that is an acyclic cofibration if each  $j_k$  is a weak equivalence.*

- (b) *Suppose for  $k = 1, \dots, n$  that  $A_k$  is a cofibrant object in  $\mathcal{C}^{G_k^{\text{op}}}$ . Then  $A_1 \otimes \cdots \otimes A_n$  is a cofibrant object in  $\mathcal{C}^{(G_1 \times \cdots \times G_n)^{\text{op}}}$ .*

*Remark 12.12.* By the right-hand adjunction in (2.10),  $-\otimes A_k$  preserves initial objects. In particular, if  $A_1, \dots, A_n$  in the statement of (a) are all initial objects, then  $A_1 \otimes \cdots \otimes A_n$  is an initial object in  $\mathcal{C}^{(G_1 \times \cdots \times G_n)^{\text{op}}}$ .

*Proof.* For each  $n$ , statement (b) is a special case of statement (a), hence it is sufficient to verify (a). By induction on  $n$ , it is enough to verify the case  $n = 2$ . Suppose for  $k = 1, 2$  that  $j_k : A_k \rightarrow B_k$  is a cofibration between cofibrant objects in  $\mathcal{C}^{G_k^{\text{op}}}$ . The induced map  $j_1 \otimes j_2 : A_1 \otimes A_2 \rightarrow B_1 \otimes B_2$  factors as

$$A_1 \otimes A_2 \xrightarrow{j_1 \otimes \text{id}} B_1 \otimes A_2 \xrightarrow{\text{id} \otimes j_2} B_1 \otimes B_2,$$

hence it is sufficient to verify each of these is a cofibration in  $\mathcal{C}^{(G_1 \times G_2)^{\text{op}}}$ . Consider any acyclic fibration  $p : X \rightarrow Y$  in  $\mathcal{C}^{(G_1 \times G_2)^{\text{op}}}$ . We want to show that  $j_1 \otimes \text{id}$  has the LLP with respect to  $p$ .

$$\begin{array}{ccc} A_1 \otimes A_2 & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ B_1 \otimes A_2 & \longrightarrow & Y \end{array} \quad \begin{array}{ccc} A_1 & \longrightarrow & \text{Map}(A_2, X)^{G_2} \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ B_1 & \longrightarrow & \text{Map}(A_2, Y)^{G_2} \end{array}$$

The left-hand solid commutative diagram in  $\mathcal{C}^{(G_1 \times G_2)^{\text{op}}}$  has a lift if and only if the right-hand solid diagram in  $\mathcal{C}^{G_1^{\text{op}}}$  has a lift. We know  $A_2$  is cofibrant in  $\mathcal{C}^{G_2^{\text{op}}}$ , hence by Proposition 12.6 the right-hand solid diagram has a lift, finishing the argument that  $j_1 \otimes \text{id}$  is a cofibration in  $\mathcal{C}^{(G_1 \times G_2)^{\text{op}}}$ . Similarly,  $\text{id} \otimes j_2$  is a cofibration in  $\mathcal{C}^{(G_1 \times G_2)^{\text{op}}}$ . The case for acyclic cofibrations is similar.  $\square$

The following proposition is also useful.

**Proposition 12.13.** *Let  $G_1$  and  $G_2$  be finite groups. Suppose for  $k = 1, 2$  that  $j_k : A_k \rightarrow B_k$  is a cofibration in  $\mathcal{C}^{G_k^{\text{op}}}$ . Then the pushout corner map*

$$(12.14) \quad B_1 \otimes A_2 \amalg_{A_1 \otimes A_2} A_1 \otimes B_2 \longrightarrow B_1 \otimes B_2$$

*is a cofibration in  $\mathcal{C}^{(G_1 \times G_2)^{\text{op}}}$  that is an acyclic cofibration if either  $j_1$  or  $j_2$  is a weak equivalence.*

*Proof.* Suppose for  $k = 1, 2$  that  $j_k : A_k \rightarrow B_k$  is a cofibration in  $\mathcal{C}^{G_k^{\text{op}}}$ . Consider any acyclic fibration  $p : X \rightarrow Y$  in  $\mathcal{C}^{(G_1 \times G_2)^{\text{op}}}$ . We want to show that the pushout

corner map (12.14) has the LLP with respect to  $p$ . The solid commutative diagram

$$\begin{array}{ccc} B_1 \otimes A_2 \amalg_{A_1 \otimes A_2} A_1 \otimes B_2 & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ B_1 \otimes B_2 & \longrightarrow & Y \end{array}$$

in  $\mathbf{C}^{(G_1 \times G_2)^{\text{op}}}$  has a lift if and only if the solid diagram

$$\begin{array}{ccc} A_1 & \longrightarrow & \text{Map}(B_2, X)^{G_2} \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ B_1 & \longrightarrow & \text{Map}(A_2, X)^{G_2} \times_{\text{Map}(A_2, Y)^{G_2}} \text{Map}(B_2, Y)^{G_2} \end{array}$$

in  $\mathbf{C}^{G_1^{\text{op}}}$  has a lift. We know  $A_2 \rightarrow B_2$  is a cofibration in  $\mathbf{C}^{G_2^{\text{op}}}$ , hence Proposition 12.6 finishes the argument that (12.14) is a cofibration in  $\mathbf{C}^{(G_1 \times G_2)^{\text{op}}}$ . The other cases are similar.  $\square$

## 12.8. Proofs for the pushout corner map theorems.

*Proof of Theorem 12.4.* Statements (a) and (b) are equivalent. This can be verified using the natural correspondence (4.17) together with the various lifting characterizations [4, Proposition 3.13] satisfied by any closed model category. Hence it is sufficient to verify statement (b).

Suppose  $j : A \rightarrow B$  is an acyclic cofibration and  $p : X \rightarrow Y$  is a fibration. We want to verify each pullback corner map

$$\text{Map}^{\otimes}(B, X)[\mathbf{t}] \longrightarrow \text{Map}^{\otimes}(A, X)[\mathbf{t}] \times_{\text{Map}^{\otimes}(A, Y)[\mathbf{t}]} \text{Map}^{\otimes}(B, Y)[\mathbf{t}],$$

is an acyclic fibration in  $\mathbf{C}$ . By Proposition 4.20 it is sufficient to show each map

$$\begin{array}{c} \text{Map}(B[\mathbf{s}], X[\mathbf{t} + \mathbf{s}])^{\Sigma_{\mathbf{s}}} \\ \downarrow \\ \text{Map}(A[\mathbf{s}], X[\mathbf{t} + \mathbf{s}])^{\Sigma_{\mathbf{s}}} \times_{\text{Map}(A[\mathbf{s}], Y[\mathbf{t} + \mathbf{s}])^{\Sigma_{\mathbf{s}}}} \text{Map}(B[\mathbf{s}], Y[\mathbf{t} + \mathbf{s}])^{\Sigma_{\mathbf{s}}} \end{array}$$

is an acyclic fibration in  $\mathbf{C}$ . Proposition 12.6 completes the proof for this case. The other cases are similar.  $\square$

*Proof of Theorem 12.5.* Statements (a) and (b) are equivalent. This can be verified using the natural correspondence (5.17) together with the various lifting characterizations [4, Proposition 3.13] satisfied by any closed model category. Hence it is sufficient to verify statement (b).

Suppose  $j : A \rightarrow B$  is an acyclic cofibration between cofibrant objects and  $p : X \rightarrow Y$  is a fibration. We want to verify each pullback corner map

$$\text{Map}^{\circ}(B, X)[\mathbf{t}] \longrightarrow \text{Map}^{\circ}(A, X)[\mathbf{t}] \times_{\text{Map}^{\circ}(A, Y)[\mathbf{t}]} \text{Map}^{\circ}(B, Y)[\mathbf{t}],$$

is an acyclic fibration in  $\mathbf{C}$ . If  $t = 0$ , this map is an isomorphism by a calculation in Proposition 5.22. If  $t \geq 1$ , by Proposition 5.20 it is sufficient to show each map

$$\begin{array}{c} \text{Map}(B[\mathbf{s}, \mathbf{t}], X[\mathbf{s}])^{\Sigma_s} \\ \downarrow \\ \text{Map}(A[\mathbf{s}, \mathbf{t}], X[\mathbf{s}])^{\Sigma_s} \times_{\text{Map}(A[\mathbf{s}, \mathbf{t}], Y[\mathbf{s}])^{\Sigma_s}} \text{Map}(B[\mathbf{s}, \mathbf{t}], Y[\mathbf{s}])^{\Sigma_s} \end{array}$$

is an acyclic fibration in  $\mathbf{C}$ . By Propositions 12.10 and 12.6, it is enough to verify each map

$$A[\mathbf{s}_1] \otimes \cdots \otimes A[\mathbf{s}_t] \longrightarrow B[\mathbf{s}_1] \otimes \cdots \otimes B[\mathbf{s}_t],$$

is an acyclic cofibration in  $\mathbf{C}^{(\Sigma_{s_1} \times \cdots \times \Sigma_{s_t})^{\text{op}}}$ . Proposition 12.11 completes the proof for this case. The other cases are similar.  $\square$

### 12.9. A special case.

*Remark 12.15.* Consider Theorem 12.5. It is useful to note that  $L \circ \emptyset$  and  $K \circ \emptyset$  may not be isomorphic, and similarly  $\text{Map}^\circ(\emptyset, X)$  and  $\text{Map}^\circ(\emptyset, Y)$  may not be isomorphic. On the other hand, Theorem 12.5 reduces the proof of the following proposition to a trivial inspection at the emptyset  $\mathbf{0}$ .

**Proposition 12.16.** *Let  $B$  be a cofibrant symmetric sequence (resp. cofibrant sequence).*

- (a) *In symmetric sequences (resp. sequences), if  $i : K \rightarrow L$  is a cofibration, then the induced map*

$$K \circ B \rightarrow L \circ B, \quad (\text{resp. } K \hat{\circ} B \rightarrow L \hat{\circ} B),$$

*is a cofibration that is an acyclic cofibration if  $i$  is a weak equivalence.*

- (b) *In symmetric sequences (resp. sequences), if  $p : X \rightarrow Y$  is a fibration, then the induced map*

$$\text{Map}^\circ(B, X) \rightarrow \text{Map}^\circ(B, Y), \quad (\text{resp. } \text{Map}^\hat{\circ}(B, X) \rightarrow \text{Map}^\hat{\circ}(B, Y)),$$

*is a fibration that is an acyclic fibration if  $p$  is a weak equivalence.*

*Proof.* Statements (a) and (b) are equivalent. This can be verified using the natural correspondence (5.17) together with the various lifting characterizations satisfied by any closed model category. Hence it is sufficient to verify (b). Suppose  $B$  is cofibrant and  $p : X \rightarrow Y$  is an acyclic fibration. We want to verify each induced map

$$\text{Map}^\circ(B, X)[\mathbf{t}] \rightarrow \text{Map}^\circ(B, Y)[\mathbf{t}]$$

is an acyclic fibration in  $\mathbf{C}$ . Theorem 12.5(b) implies this for  $t \geq 1$ . For  $t = 0$ , it is enough to note that  $X[\mathbf{0}] \rightarrow Y[\mathbf{0}]$  is an acyclic fibration. The other case is similar.  $\square$

## 13. PROOFS

In this section we give proofs of Theorems 1.2 and 1.4. When working with left modules over an operad, we are led naturally to replace  $(\mathbf{C}, \otimes, \mathbf{k})$  with  $(\text{SymSeq}, \otimes, 1)$  as the underlying monoidal model category, and hence to working with symmetric arrays.

### 13.1. Arrays and symmetric arrays.

#### Definition 13.1.

- A *symmetric array* in  $\mathbf{C}$  is a symmetric sequence in  $\mathbf{SymSeq}$ ; i.e. a functor  $A : \Sigma^{\text{op}} \rightarrow \mathbf{SymSeq}$ . An *array* in  $\mathbf{C}$  is a sequence in  $\mathbf{Seq}$ ; i.e. a functor  $A : \Omega^{\text{op}} \rightarrow \mathbf{Seq}$ .
- $\mathbf{SymArray} := \mathbf{SymSeq}^{\Sigma^{\text{op}}} \cong \mathbf{C}^{\Sigma^{\text{op}} \times \Sigma^{\text{op}}}$  is the category of symmetric arrays in  $\mathbf{C}$  and their natural transformations.  $\mathbf{Array} := \mathbf{Seq}^{\Omega^{\text{op}}} \cong \mathbf{C}^{\Omega^{\text{op}} \times \Omega^{\text{op}}}$  is the category of arrays in  $\mathbf{C}$  and their natural transformations.

All of the statements and constructions which were previously described in terms of  $(\mathbf{C}, \otimes, \mathbf{k})$  are equally true for  $(\mathbf{SymSeq}, \otimes, 1)$  and  $(\mathbf{Seq}, \hat{\otimes}, 1)$ , and we usually cite and use the appropriate statements and constructions without further comment.

**Theorem 13.2.** *The categories  $\mathbf{SymArray}$  of symmetric arrays and  $\mathbf{Array}$  of arrays have natural model category structures. The weak equivalences are the objectwise weak equivalences and the fibrations are the objectwise fibrations. The model structures are cofibrantly generated.*

*Proof.* This is a special case of Theorem 12.2 with  $(\mathbf{C}, \otimes, \mathbf{k})$  replaced by  $(\mathbf{SymSeq}, \otimes, 1)$  and  $(\mathbf{Seq}, \hat{\otimes}, 1)$ .  $\square$

### 13.2. Model category structures in $\Sigma$ -operad case.

*Proof of Theorem 1.4.* The model category structure on  $\mathbf{Lt}_{\mathcal{O}}$  (resp.  $\mathbf{Alg}_{\mathcal{O}}$ ) is created by the adjunction

$$\mathbf{SymSeq} \begin{array}{c} \xrightarrow{\mathcal{O} \circ -} \\ \xleftarrow{U} \end{array} \mathbf{Lt}_{\mathcal{O}} \quad \left( \text{resp. } \mathbf{C} \begin{array}{c} \xrightarrow{\mathcal{O} \circ (-)} \\ \xleftarrow{U} \end{array} \mathbf{Alg}_{\mathcal{O}} \right)$$

with left adjoint on top and  $U$  the forgetful functor. Define a map  $f$  in  $\mathbf{Lt}_{\mathcal{O}}$  to be a weak equivalence (resp. fibration) if  $U(f)$  is a weak equivalence (resp. fibration) in  $\mathbf{SymSeq}$ . Similarly, define a map  $f$  in  $\mathbf{Alg}_{\mathcal{O}}$  to be a weak equivalence (resp. fibration) if  $U(f)$  is a weak equivalence (resp. fibration) in  $\mathbf{C}$ . Define a map  $f$  in  $\mathbf{Lt}_{\mathcal{O}}$  (resp.  $\mathbf{Alg}_{\mathcal{O}}$ ) to be a cofibration if it has the LLP with respect to all acyclic fibrations in  $\mathbf{Lt}_{\mathcal{O}}$  (resp.  $\mathbf{Alg}_{\mathcal{O}}$ ).

Consider the case of  $\mathbf{Lt}_{\mathcal{O}}$ . We want to verify the model category axioms (MC1)-(MC5). We already know (MC1) is satisfied, and verifying (MC2) and (MC3) is clear. The arguments in the proof of [39, Lemma 2.3] use (possibly transfinite) small object arguments to reduce (MC5) to verifying Proposition 13.4 below. The first part of (MC4) is satisfied by definition, and the second part of (MC4) follows from the usual lifting and retract argument, as described in the proof of [39, Lemma 2.3]. This verifies the model category axioms. By construction, the model category is cofibrantly generated.

Consider the case of  $\mathbf{Alg}_{\mathcal{O}}$ . Argue similar to the case of  $\mathbf{Lt}_{\mathcal{O}}$ , except use Proposition 13.4 together with Remark 13.7. By construction, the model category is cofibrantly generated.  $\square$

*Remark 13.3.* Since the forgetful functors in this proof commute with filtered colimits, the smallness conditions needed for the (possibly transfinite) small object arguments in [39, Lemma 2.3] are satisfied.

### 13.3. Analysis of pushouts for $\Sigma$ -operad case.

**Proposition 13.4.** *Let  $\mathcal{O}$  be a  $\Sigma$ -operad and  $A \in \text{Lt}_{\mathcal{O}}$ . Assume every object in  $\text{SymArray}$  is cofibrant. Then every (possibly transfinite) composition of pushouts in  $\text{Lt}_{\mathcal{O}}$  of the form*

$$(13.5) \quad \begin{array}{ccc} \mathcal{O} \circ X & \longrightarrow & A \\ \downarrow \text{id} \circ i & & \downarrow j \\ \mathcal{O} \circ Y & \longrightarrow & A \amalg_{\mathcal{O} \circ X} \mathcal{O} \circ Y, \end{array}$$

such that  $i : X \rightarrow Y$  an acyclic cofibration in  $\text{SymSeq}$ , is a weak equivalence in the underlying category  $\text{SymSeq}$ .

*Remark 13.6.* The proof of this proposition verifies the stronger statement that the (possibly transfinite) composition of pushouts in  $\text{Lt}_{\mathcal{O}}$  is an *acyclic cofibration* in the underlying category  $\text{SymSeq}$ .

*Remark 13.7.* If  $X, Y, A$  are concentrated at 0, then the pushout diagram (13.5) is concentrated at 0. To verify this, use Proposition 5.8 and the construction of colimits described in the proof of Proposition 8.10.

This subsection is devoted to proving Proposition 13.4, which we cited above in the proof of Theorem 1.4. A first step in analyzing the pushouts in (13.5) is an analysis of certain coproducts. The following proposition is motivated by a similar construction given in [14, Section 2.3] and [31, Section 13] in the context of algebras over an operad.

**Proposition 13.8.** *Let  $\mathcal{O}$  be a  $\Sigma$ -operad,  $A \in \text{Lt}_{\mathcal{O}}$ , and  $Y \in \text{SymSeq}$ . Consider any coproduct in  $\text{Lt}_{\mathcal{O}}$  of the form*

$$(13.9) \quad A \amalg (\mathcal{O} \circ Y).$$

There exists a symmetric array  $\mathcal{O}_A$  and natural isomorphisms

$$A \amalg (\mathcal{O} \circ Y) \cong \mathcal{O}_A \circ (Y) = \coprod_{q \geq 0} \mathcal{O}_A[\mathbf{q}]_{\otimes_{\Sigma_q}} Y^{\otimes q}$$

in the underlying category  $\text{SymSeq}$ . If  $Q \in \Sigma$  and  $q := |Q|$ , then  $\mathcal{O}_A[Q]$  is naturally isomorphic to a colimit of the form

$$\mathcal{O}_A[Q] \cong \text{colim} \left( \coprod_{p \geq 0} \mathcal{O}[\mathbf{p} + \mathbf{q}]_{\otimes_{\Sigma_p}} A^{\otimes p} \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{d_1} \end{array} \coprod_{p \geq 0} \mathcal{O}[\mathbf{p} + \mathbf{q}]_{\otimes_{\Sigma_p}} (\mathcal{O} \circ A)^{\otimes p} \right),$$

in  $\text{SymSeq}$ , with  $d_0$  induced by operad multiplication and  $d_1$  induced by  $m : \mathcal{O} \circ A \rightarrow A$ .

*Proof.* The coproduct in (13.9) may be calculated by a reflexive coequalizer in  $\text{Lt}_{\mathcal{O}}$  of the form,

$$A \amalg (\mathcal{O} \circ Y) \cong \text{colim} \left( (\mathcal{O} \circ A) \amalg (\mathcal{O} \circ Y) \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{d_1} \end{array} (\mathcal{O} \circ \mathcal{O} \circ A) \amalg (\mathcal{O} \circ Y) \right),$$

The maps  $d_0$  and  $d_1$  are induced by maps  $m : \mathcal{O} \circ \mathcal{O} \rightarrow \mathcal{O}$  and  $m : \mathcal{O} \circ A \rightarrow A$ , respectively. There are natural isomorphisms in the underlying category  $\text{SymSeq}$ ,

$$\begin{aligned} (\mathcal{O} \circ A) \amalg (\mathcal{O} \circ Y) &\cong \mathcal{O} \circ (A \amalg Y) \\ &\cong \coprod_{t \geq 0} \mathcal{O}[t] \otimes_{\Sigma_t} (A \amalg Y)^{\otimes t} \\ &\cong \coprod_{q \geq 0} \left( \coprod_{p \geq 0} \mathcal{O}[\mathbf{p} + \mathbf{q}] \otimes_{\Sigma_p} A^{\otimes p} \right) \otimes_{\Sigma_q} Y^{\otimes q}, \end{aligned}$$

and similarly,

$$(\mathcal{O} \circ \mathcal{O} \circ A) \amalg (\mathcal{O} \circ Y) \cong \coprod_{q \geq 0} \left( \coprod_{p \geq 0} \mathcal{O}[\mathbf{p} + \mathbf{q}] \otimes_{\Sigma_p} (\mathcal{O} \circ A)^{\otimes p} \right) \otimes_{\Sigma_q} Y^{\otimes q}.$$

The maps  $d_0$  and  $d_1$  similarly factor in the underlying category  $\text{SymSeq}$ .  $\square$

*Remark 13.10.* We have used the natural isomorphisms

$$(A \amalg Y)^{\otimes t} \cong \coprod_{p+q=t} \Sigma_{p+q} \cdot_{\Sigma_p \times \Sigma_q} A^{\otimes p} \otimes Y^{\otimes q},$$

in the proof of Proposition 13.8.

**Definition 13.11.** Let  $i : X \rightarrow Y$  be a morphism in  $\text{SymSeq}$  and  $t \geq 1$ . Define  $Q_0^t := X^{\otimes t}$  and  $Q_t^t := Y^{\otimes t}$ . For  $0 < q < t$  define  $Q_q^t$  inductively by the pushout diagrams

$$\begin{array}{ccc} \Sigma_t \cdot_{\Sigma_{t-q} \times \Sigma_q} X^{\otimes(t-q)} \otimes Q_{q-1}^q & \xrightarrow{\text{Pr}_*} & Q_{q-1}^t \\ \downarrow i_* & & \downarrow i_* \\ \Sigma_t \cdot_{\Sigma_{t-q} \times \Sigma_q} X^{\otimes(t-q)} \otimes Y^{\otimes q} & \longrightarrow & Q_q^t \end{array}$$

in  $\text{SymSeq}^{\Sigma_t}$ .

*Remark 13.12.* The construction  $Q_t^{t-1}$  can be thought of as a  $\Sigma_t$ -equivariant version of the colimit of a punctured  $t$ -cube (Proposition 13.23). If the category  $\mathbf{C}$  is pointed, there is a natural isomorphism  $Y^{\otimes t} / Q_{t-1}^t \cong (Y/X)^{\otimes t}$ .

The following proposition provides a useful description of certain pushouts of left modules, and is motivated by a similar construction given in [6, section 12] in the context of simplicial multifunctors of symmetric spectra.

**Proposition 13.13.** *Let  $\mathcal{O}$  be a  $\Sigma$ -operad,  $A \in \text{Lt}_{\mathcal{O}}$ , and  $i : X \rightarrow Y$  in  $\text{SymSeq}$ . Consider any pushout diagram in  $\text{Lt}_{\mathcal{O}}$  of the form,*

$$(13.14) \quad \begin{array}{ccc} \mathcal{O} \circ X & \xrightarrow{f} & A \\ \downarrow \text{id} \circ i & & \downarrow \\ \mathcal{O} \circ Y & \longrightarrow & A \amalg_{(\mathcal{O} \circ X)} (\mathcal{O} \circ Y). \end{array}$$

The pushout in (13.14) is naturally isomorphic to a filtered colimit of the form

$$(13.15) \quad A \amalg_{(\mathcal{O} \circ X)} (\mathcal{O} \circ Y) \cong \text{colim} \left( A_0 \xrightarrow{j_1} A_1 \xrightarrow{j_2} A_2 \xrightarrow{j_3} \dots \right)$$

in the underlying category  $\mathbf{SymSeq}$ , with  $A_0 := \mathcal{O}_A[\mathbf{0}] \cong A$  and  $A_t$  defined inductively by pushout diagrams in  $\mathbf{SymSeq}$  of the form

$$(13.16) \quad \begin{array}{ccc} \mathcal{O}_A[\mathbf{t}] \otimes_{\Sigma_t} Q_{t-1}^t & \xrightarrow{f_*} & A_{t-1} \\ \downarrow \text{id} \otimes_{\Sigma_t} i_* & & \downarrow j_t \\ \mathcal{O}_A[\mathbf{t}] \otimes_{\Sigma_t} Y^{\otimes t} & \xrightarrow{\xi_t} & A_t \end{array}$$

*Proof.* The pushout in (13.14) may be calculated by a reflexive coequalizer in  $\mathbf{Lt}_{\mathcal{O}}$  of the form

$$A \amalg_{(\mathcal{O} \circ X)} (\mathcal{O} \circ Y) \cong \text{colim} \left( A \amalg (\mathcal{O} \circ Y) \begin{array}{c} \xleftarrow{\bar{i}} \\ \xleftarrow{\bar{f}} \end{array} A \amalg (\mathcal{O} \circ X) \amalg (\mathcal{O} \circ Y) \right)$$

The maps  $\bar{i}$  and  $\bar{f}$  are induced by maps  $\text{id} \circ i_*$  and  $\text{id} \circ f_*$  which fit into the commutative diagram

$$(13.17) \quad \begin{array}{ccccc} \mathcal{O}_A \circ (X \amalg Y) & \longleftarrow & \mathcal{O} \circ (A \amalg X \amalg Y) & \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{d_1} \end{array} & \mathcal{O} \circ ((\mathcal{O} \circ A) \amalg X \amalg Y) \\ \bar{i} \downarrow \parallel \bar{f} & & \text{id} \circ i_* \downarrow \parallel \text{id} \circ f_* & & \text{id} \circ i_* \downarrow \parallel \text{id} \circ f_* \\ \mathcal{O}_A \circ (Y) & \longleftarrow & \mathcal{O} \circ (A \amalg Y) & \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{d_1} \end{array} & \mathcal{O} \circ ((\mathcal{O} \circ A) \amalg Y) \end{array}$$

in  $\mathbf{Lt}_{\mathcal{O}}$ , with rows reflexive coequalizer diagrams, and maps  $i_*$  and  $f_*$  in  $\mathbf{SymSeq}$  induced by  $i : X \rightarrow Y$  and  $f : X \rightarrow A$ . By Proposition 13.8, the coproduct in (13.14) may be calculated by the colimit of the left hand column of (13.17) in the underlying category  $\mathbf{SymSeq}$ . We want to reconstruct this colimit via a suitable filtered colimit.

Using (13.17), there exist maps  $\bar{f}_{q,p}$  which make the diagrams

$$\begin{array}{ccc} \mathcal{O}_A \circ (X \amalg Y) \cong \coprod_{q \geq 0} \coprod_{p \geq 0} ( ) & \xleftarrow{\text{in}_{q,p}} & (\mathcal{O}_A[\mathbf{p} + \mathbf{q}] \otimes_{\Sigma_p \times \Sigma_q} X^{\otimes p} \otimes Y^{\otimes q}) \\ \downarrow \bar{f} & & \downarrow \exists \bar{f}_{q,p} \\ \mathcal{O}_A \circ (Y) \cong \coprod_{t \geq 0} ( ) & \xleftarrow{\text{in}_q} & (\mathcal{O}_A[\mathbf{q}] \otimes_{\Sigma_q} Y^{\otimes q}) \end{array}$$

in  $\mathbf{SymSeq}$  commute. Similarly, there exist maps  $\bar{i}_{q,p}$  which make the diagrams

$$\begin{array}{ccc} \mathcal{O}_A \circ (X \amalg Y) \cong \coprod_{q \geq 0} \coprod_{p \geq 0} ( ) & \xleftarrow{\text{in}_{q,p}} & (\mathcal{O}_A[\mathbf{p} + \mathbf{q}] \otimes_{\Sigma_p \times \Sigma_q} X^{\otimes p} \otimes Y^{\otimes q}) \\ \downarrow \bar{i} & & \downarrow \exists \bar{i}_{q,p} \\ \mathcal{O}_A \circ (Y) \cong \coprod_{t \geq 0} ( ) & \xleftarrow{\text{in}_{p+q}} & (\mathcal{O}_A[\mathbf{p} + \mathbf{q}] \otimes_{\Sigma_{p+q}} Y^{\otimes (p+q)}) \end{array}$$

in  $\mathbf{SymSeq}$  commute. Let  $\varphi : \mathcal{O}_A \circ (Y) \rightarrow \cdot$  be a morphism in  $\mathbf{SymSeq}$  and define  $\varphi_q := \varphi \text{in}_q$ . Then  $\varphi \bar{i} = \varphi \bar{f}$  if and only if the diagrams

$$\begin{array}{ccc} \mathcal{O}_A[\mathbf{p} + \mathbf{q}] \otimes_{\Sigma_p \times \Sigma_q} X^{\otimes p} \otimes Y^{\otimes q} & \xrightarrow{\bar{f}_{q,p}} & \mathcal{O}_A[\mathbf{q}] \otimes_{\Sigma_q} Y^{\otimes q} \\ \downarrow \bar{i}_{q,p} & & \downarrow \varphi_q \\ \mathcal{O}_A[\mathbf{p} + \mathbf{q}] \otimes_{\Sigma_{p+q}} Y^{\otimes(p+q)} & \xrightarrow{\varphi_{p+q}} & \cdot \end{array}$$

commute for every  $p, q \geq 0$ . Since  $\bar{i}_{q,0} = \text{id}$  and  $\bar{f}_{q,0} = \text{id}$ , it is sufficient to consider  $q \geq 0$  and  $p > 0$ .

To motivate the construction (13.16), it is useful to describe a preliminary construction which also calculates the pushout in (13.14). Define  $A_0 := \mathcal{O}_A[\mathbf{0}] \cong A$  and for each  $t \geq 1$  define  $A_t$  by the pushout diagram

$$(13.18) \quad \begin{array}{ccc} \coprod_{\substack{p+q=t \\ q \geq 0, p > 0}} \mathcal{O}_A[\mathbf{p} + \mathbf{q}] \otimes_{\Sigma_p \times \Sigma_q} X^{\otimes p} \otimes Y^{\otimes q} & \xrightarrow{f_*} & A_{t-1} \\ \downarrow i_* & & \downarrow j_t \\ \mathcal{O}_A[\mathbf{t}] \otimes_{\Sigma_t} Y^{\otimes t} & \xrightarrow{\xi_t} & A_t \end{array}$$

in  $\mathbf{SymSeq}$ . The maps  $f_*$  and  $i_*$  are induced by the appropriate maps  $\bar{f}_{q,p}$  and  $\bar{i}_{q,p}$ . Verify that (13.15) is satisfied.

The collection of maps  $\bar{f}_{q,p}$  and  $\bar{i}_{q,p}$  satisfy many compatibility relations. This suggests we replace the coproduct in (13.18), which is isomorphic to

$$\mathcal{O}_A[\mathbf{t}] \otimes_{\Sigma_t} \left[ (X \amalg Y)^{\otimes t} - Y^{\otimes t} \right],$$

with an appropriate (possibly smaller) pushout construction. Here,  $(X \amalg Y)^{\otimes t} - Y^{\otimes t}$  means the coproduct of all factors in  $(X \amalg Y)^{\otimes t}$  except  $Y^{\otimes t}$ .

Define  $A_0 := \mathcal{O}_A[\mathbf{0}] \cong A$  and for each  $t \geq 1$  define  $A_t$  by the pushout diagram (13.16) in  $\mathbf{SymSeq}$ . The maps  $f_*$  and  $i_*$  are induced by the appropriate maps  $\bar{f}_{q,p}$  and  $\bar{i}_{q,p}$ . Verify that (13.15) is satisfied.  $\square$

**Proposition 13.19.** *Let  $\mathcal{O}$  be a  $\Sigma$ -operad,  $A \in \mathbf{Lt}_{\mathcal{O}}$ , and  $i : X \rightarrow Y$  a cofibration (resp. acyclic cofibration) in  $\mathbf{SymSeq}$ . Assume every object in  $\mathbf{SymArray}$  is cofibrant. Consider any pushout diagram in  $\mathbf{Lt}_{\mathcal{O}}$  of the form,*

$$(13.20) \quad \begin{array}{ccc} \mathcal{O} \circ X & \longrightarrow & A \\ \downarrow \text{id} \circ i & & \downarrow j \\ \mathcal{O} \circ Y & \longrightarrow & A \amalg_{\mathcal{O} \circ X} \mathcal{O} \circ Y. \end{array}$$

*Then each map  $j_t : A_{t-1} \rightarrow A_t$  in the filtration (13.15) is a cofibration (resp. acyclic cofibration) in  $\mathbf{SymSeq}$ . In particular,  $j$  is a cofibration (resp. acyclic cofibration) in the underlying category  $\mathbf{SymSeq}$ .*

*Proof.* Suppose  $i : X \rightarrow Y$  is an acyclic cofibration in  $\mathbf{SymSeq}$ . We want to show each  $j_t : A_{t-1} \rightarrow A_t$  is an acyclic cofibration in  $\mathbf{SymSeq}$ . By the construction of  $j_t$  in Proposition 13.13, it is sufficient to verify each  $\text{id}_{\otimes_{\Sigma_t}} i_*$  in (13.16) is an acyclic

cofibration. Suppose  $p : C \rightarrow D$  is a fibration in  $\mathbf{SymSeq}$ . We want to verify  $\mathrm{id} \otimes_{\Sigma_t} i_*$  has the LLP with respect to  $p$ .

$$\begin{array}{ccc} \mathcal{O}_A[\mathbf{t}] \otimes_{\Sigma_t} Q_{t-1}^t & \longrightarrow & C \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \mathcal{O}_A[\mathbf{t}] \otimes_{\Sigma_t} Y^{\otimes t} & \longrightarrow & D \end{array} \quad \begin{array}{ccc} \mathcal{O}_A[\mathbf{t}] \otimes Q_{t-1}^t & \longrightarrow & C \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \mathcal{O}_A[\mathbf{t}] \otimes Y^{\otimes t} & \longrightarrow & D, \end{array}$$

The left-hand solid commutative diagram in  $\mathbf{SymSeq}$  has a lift if and only if the right-hand solid diagram in  $\mathbf{SymSeq}^{\Sigma_t^{\mathrm{op}}}$  has a lift. Hence it is sufficient to verify that the solid diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathrm{Map}^{\otimes}(Y^{\otimes t}, C) \\ \downarrow & \nearrow \text{dotted} & \downarrow (*) \\ \mathcal{O}_A[\mathbf{t}] & \longrightarrow & \mathrm{Map}^{\otimes}(Q_{t-1}^t, C) \times_{\mathrm{Map}^{\otimes}(Q_{t-1}^t, D)} \mathrm{Map}^{\otimes}(Y^{\otimes t}, D) \end{array}$$

in  $\mathbf{SymSeq}^{\Sigma_t^{\mathrm{op}}}$  has a lift. We know  $\mathcal{O}_A[\mathbf{t}]$  is cofibrant in  $\mathbf{SymSeq}^{\Sigma_t^{\mathrm{op}}}$ , hence sufficient to verify  $(*)$  is an acyclic fibration. By Theorem 12.4 it is enough to verify  $i_* : Q_{t-1}^t \rightarrow Y^{\otimes t}$  is an acyclic cofibration in the underlying category  $\mathbf{SymSeq}$ , and Proposition 13.21 finishes the proof. The other case is similar.  $\square$

#### 13.4. Punctured cube.

**Proposition 13.21.** *Let  $i : X \rightarrow Y$  be a cofibration (resp. acyclic cofibration) in  $\mathbf{SymSeq}$ . Then the induced map  $i_* : Q_{t-1}^t \rightarrow Y^{\otimes t}$  is a cofibration (resp. acyclic cofibration) in the underlying category  $\mathbf{SymSeq}$ .*

Before proving this proposition, we establish some notation.

**Definition 13.22.** Let  $t \geq 2$ .

- $\mathbf{Cube}_t$  is the category with objects the vertices  $(v_1, \dots, v_t) \in \{0, 1\}^t$  of the unit  $t$ -cube; there is at most one morphism between any two objects, and there is a morphism

$$(v_1, \dots, v_t) \longrightarrow (v'_1, \dots, v'_t)$$

if and only if  $v_i \leq v'_i$  for every  $1 \leq i \leq t$ . In particular,  $\mathbf{Cube}_t$  is the category associated to a partial order on the set  $\{0, 1\}^t$ .

- The *punctured cube*  $\mathbf{pCube}_t$  is the full subcategory of  $\mathbf{Cube}_t$  with all objects except the terminal object  $(1, \dots, 1)$  of  $\mathbf{Cube}_t$ .

Let  $i : X \rightarrow Y$  be a morphism in  $\mathbf{SymSeq}$ . It will be useful to introduce an associated functor  $w : \mathbf{pCube}_t \rightarrow \mathbf{SymSeq}$  defined objectwise by

$$w(v_1, \dots, v_t) := c_1 \otimes \cdots \otimes c_t \quad \text{with} \quad c_i := \begin{cases} X, & \text{for } v_i = 0, \\ Y, & \text{for } v_i = 1, \end{cases}$$

and with morphisms induced by  $i : X \rightarrow Y$ . In particular, for  $t = 3$  the diagram  $w$  looks like,

$$\begin{array}{ccccc}
 & & X \otimes X \otimes Y & \longrightarrow & X \otimes Y \otimes Y \\
 & \swarrow & \uparrow & & \uparrow \\
 & Y \otimes X \otimes Y & X \otimes X \otimes X & \longrightarrow & X \otimes Y \otimes X \\
 & \uparrow & \swarrow & \searrow & \\
 Y \otimes X \otimes X & \longrightarrow & Y \otimes Y \otimes X & & 
 \end{array}$$

**Proposition 13.23.** *Let  $i : X \rightarrow Y$  be a morphism in  $\mathbf{SymSeq}$  and  $t \geq 2$ . There are natural isomorphisms*

$$Q_{t-1}^t \cong \operatorname{colim}(w : \mathbf{pCube}_t \rightarrow \mathbf{SymSeq})$$

in the underlying category  $\mathbf{SymSeq}$ .

*Proof.* Use the  $t$  maps in  $\mathbf{SymSeq}$  obtained from the map

$$\Sigma_t \cdot \Sigma_1 \times \Sigma_{t-1} X \otimes Y^{\otimes(t-1)} \rightarrow Q_{t-1}^t$$

in Definition 13.11 to define a cone into  $Q_{t-1}^t$  and verify this cone is initial with respect to all cones.  $\square$

*Proof of Proposition 13.21.* Suppose  $i : X \rightarrow Y$  is an acyclic cofibration in  $\mathbf{SymSeq}$ . The colimit of the diagram  $w : \mathbf{pCube}_t \rightarrow \mathbf{SymSeq}$  may be computed inductively using pushout corner maps, and hence by Proposition 13.23 there are natural isomorphisms

$$\begin{aligned}
 Q_1^2 &\cong Y \otimes X \amalg_{X \otimes X} X \otimes Y, \\
 Q_2^3 &\cong Y \otimes Y \otimes X \amalg_{(Y \otimes X \amalg_{X \otimes X} X \otimes Y) \otimes X} (Y \otimes X \amalg_{X \otimes X} X \otimes Y) \otimes Y, \dots
 \end{aligned}$$

in the underlying category  $\mathbf{SymSeq}$ . The same argument provides an inductive construction of the induced map  $i_* : Q_{t-1}^t \rightarrow Y^{\otimes t}$  in the underlying category  $\mathbf{SymSeq}$ ; using the natural isomorphisms in Proposition 13.23, for each  $t \geq 2$  the  $Q_{t-1}^t$  fit into pushout squares

$$\begin{array}{ccc}
 Q_{t-2}^{t-1} \otimes X & \xrightarrow{\operatorname{id} \otimes i} & Q_{t-2}^{t-1} \otimes Y \\
 \downarrow i_* \otimes \operatorname{id} & & \downarrow i_* \otimes \operatorname{id} \\
 Y^{\otimes(t-1)} \otimes X & \longrightarrow & Q_{t-1}^t \\
 \downarrow \operatorname{id} \otimes i & \searrow i_* & \downarrow i_* \\
 & & Y^{\otimes t}
 \end{array}$$

in the underlying category  $\mathbf{SymSeq}$  with induced map  $i_* : Q_{t-1}^t \rightarrow Y^{\otimes t}$  the indicated pushout corner map. By iterated applications of Theorem 12.4,  $i_*$  is an acyclic cofibration in  $\mathbf{SymSeq}$ . The case for cofibrations is similar.  $\square$

*Remark 13.24.* This construction of  $i_*$  by iterated pushout corner maps is used in the proof of the main theorem in [39].

*Proof of Proposition 13.4.* By Proposition 13.19, each map  $j$  is an acyclic cofibration in the underlying category  $\mathbf{SymSeq}$ . Noting that (possibly transfinite) compositions of acyclic cofibrations are acyclic cofibrations, completes the proof.  $\square$

**13.5. Model category structures in  $\Omega$ -operad case.** In the following subsections we prove Theorem 1.2. The argument and required constructions are related to those of the previous subsections, but different enough to require some exposition. The strong cofibrancy condition exploited in Theorem 1.4 is replaced here by the weaker *monoid axiom* [39], but at the cost of dropping all  $\Sigma$ -actions; i.e., working with  $\Omega$ -operads instead of  $\Sigma$ -operads.

**Definition 13.25.** A monoidal model category satisfies the *monoid axiom* if every map which is a (possibly transfinite) composition of pushouts of maps in

$$(13.26) \quad \{\text{acyclic cofibrations}\} \otimes \mathbf{C}$$

is a weak equivalence.

*Remark 13.27.* In this definition, (13.26) is notation for the collection of maps of the form

$$f \otimes \text{id} : K \otimes B \longrightarrow L \otimes B$$

such that  $f : K \longrightarrow L$  is an acyclic cofibration and  $B \in \mathbf{C}$ .

*Proof of Theorem 1.2.* The model category structure on  $\mathbf{Lt}_{\mathcal{O}}$  (resp.  $\mathbf{Alg}_{\mathcal{O}}$ ) is created by the adjunction

$$\mathbf{Seq} \begin{array}{c} \xrightarrow{\mathcal{O} \hat{\delta} -} \\ \xleftarrow{U} \end{array} \mathbf{Lt}_{\mathcal{O}} \quad \left( \text{resp.} \quad \mathbf{C} \begin{array}{c} \xrightarrow{\mathcal{O} \hat{\delta} (-)} \\ \xleftarrow{U} \end{array} \mathbf{Alg}_{\mathcal{O}} \right)$$

with left adjoint on top and  $U$  the forgetful functor. Define a map  $f$  in  $\mathbf{Lt}_{\mathcal{O}}$  to be a weak equivalence (resp. fibration) if  $U(f)$  is a weak equivalence (resp. fibration) in  $\mathbf{Seq}$ . Similarly, define a map  $f$  in  $\mathbf{Alg}_{\mathcal{O}}$  to be a weak equivalence (resp. fibration) if  $U(f)$  is a weak equivalence (resp. fibration) in  $\mathbf{C}$ . Define a map  $f$  in  $\mathbf{Lt}_{\mathcal{O}}$  (resp.  $\mathbf{Alg}_{\mathcal{O}}$ ) to be a cofibration if it has the LLP with respect to all acyclic fibrations in  $\mathbf{Lt}_{\mathcal{O}}$  (resp.  $\mathbf{Alg}_{\mathcal{O}}$ ).

Consider the case of  $\mathbf{Lt}_{\mathcal{O}}$ . We want to verify the model category axioms (MC1)-(MC5). We already know (MC1) is satisfied, and verifying (MC2) and (MC3) is clear. The arguments in the proof of [39, Lemma 2.3] use (possibly transfinite) small object arguments to reduce (MC5) to verifying Proposition 13.29 below. The first part of (MC4) is satisfied by definition, and the second part of (MC4) follows from the usual lifting and retract argument, as described in the proof of [39, Lemma 2.3]. This verifies the model category axioms. By construction, the model category is cofibrantly generated.

Consider the case of  $\mathbf{Alg}_{\mathcal{O}}$ . Argue similar to the case of  $\mathbf{Lt}_{\mathcal{O}}$ , except use Proposition 13.29 together with Remark 13.31. By construction, the model category is cofibrantly generated.  $\square$

*Remark 13.28.* Since the forgetful functors in this proof commute with filtered colimits, the smallness conditions needed for the (possibly transfinite) small object arguments in [39, Lemma 2.3] are satisfied.

13.6. Analysis of pushouts for  $\Omega$ -operad case.

**Proposition 13.29.** *Let  $\mathcal{O}$  be an  $\Omega$ -operad and  $A \in \text{Lt}_{\mathcal{O}}$ . Assume  $(\mathbf{C}, \otimes, \mathbf{k})$  satisfies the monoid axiom. Then every (possibly transfinite) composition of pushouts in  $\text{Lt}_{\mathcal{O}}$  of the form*

$$(13.30) \quad \begin{array}{ccc} \mathcal{O} \hat{\circ} X & \longrightarrow & A \\ \downarrow \text{id} \hat{\circ} i & & \downarrow j \\ \mathcal{O} \hat{\circ} Y & \longrightarrow & A \amalg_{\mathcal{O} \hat{\circ} X} \mathcal{O} \hat{\circ} Y, \end{array}$$

such that  $i : X \rightarrow Y$  is an acyclic cofibration in  $\text{Seq}$ , is a weak equivalence in the underlying category  $\text{Seq}$ .

*Remark 13.31.* If  $X, Y, A$  are concentrated at 0, then the pushout diagram (13.30) is concentrated at 0. To verify this, argue as in Remark 13.7.

This subsection is devoted to proving Proposition 13.29, which we cited above in the proof of Theorem 1.2. Similar to the previous subsections, a first step in analyzing the pushouts in (13.30) is an analysis of certain coproducts. The following is an  $\Omega$ -operad version of Proposition 13.8.

**Proposition 13.32.** *Let  $\mathcal{O}$  be an  $\Omega$ -operad,  $A \in \text{Lt}_{\mathcal{O}}$ , and  $Y \in \text{Seq}$ . Consider any coproduct in  $\text{Lt}_{\mathcal{O}}$  of the form*

$$(13.33) \quad A \amalg (\mathcal{O} \hat{\circ} Y).$$

There exists an array  $\mathcal{O}_A$  and natural isomorphisms

$$A \amalg (\mathcal{O} \hat{\circ} Y) \cong \mathcal{O}_A \hat{\circ} (Y) = \coprod_{q \geq 0} \mathcal{O}_A[\mathbf{q}] \hat{\otimes} Y^{\hat{\otimes} q}$$

in the underlying category  $\text{Seq}$ . If  $Q \in \Omega$  and  $q := |Q|$ , then  $\mathcal{O}_A[Q]$  is naturally isomorphic to a colimit of the form

$$\text{colim} \left( \coprod_{p \geq 0} \mathcal{O}[\mathbf{p} + \mathbf{q}] \otimes \left[ \frac{\Sigma_{p+q}}{\Sigma_p \times \Sigma_q} \cdot A^{\hat{\otimes} p} \right] \xrightleftharpoons[d_1]{d_0} \coprod_{p \geq 0} \mathcal{O}[\mathbf{p} + \mathbf{q}] \otimes \left[ \frac{\Sigma_{p+q}}{\Sigma_p \times \Sigma_q} \cdot (\mathcal{O} \hat{\circ} A)^{\hat{\otimes} p} \right] \right),$$

in  $\text{Seq}$ , with  $d_0$  induced by operad multiplication and  $d_1$  induced by  $m : \mathcal{O} \hat{\circ} A \rightarrow A$ .

*Proof.* Verify the coproduct in (13.33) may be calculated by a reflexive coequalizer in  $\text{Lt}_{\mathcal{O}}$  of the form,

$$A \amalg (\mathcal{O} \hat{\circ} Y) \cong \text{colim} \left( (\mathcal{O} \hat{\circ} A) \amalg (\mathcal{O} \hat{\circ} Y) \xrightleftharpoons[d_1]{d_0} (\mathcal{O} \hat{\circ} \mathcal{O} \hat{\circ} A) \amalg (\mathcal{O} \hat{\circ} Y) \right).$$

The maps  $d_0$  and  $d_1$  are induced by maps  $m : \mathcal{O} \hat{\circ} \mathcal{O} \rightarrow \mathcal{O}$  and  $m : \mathcal{O} \hat{\circ} A \rightarrow A$ , respectively. There are natural isomorphisms in the underlying category  $\text{Seq}$ ,

$$\begin{aligned} (\mathcal{O} \hat{\circ} A) \amalg (\mathcal{O} \hat{\circ} Y) &\cong \mathcal{O} \hat{\circ} (A \amalg Y) \\ &\cong \coprod_{t \geq 0} \mathcal{O}[\mathbf{t}] \otimes (A \amalg Y)^{\hat{\otimes} t} \\ &\cong \coprod_{q \geq 0} \left( \coprod_{p \geq 0} \mathcal{O}[\mathbf{p} + \mathbf{q}] \otimes \left[ \frac{\Sigma_{p+q}}{\Sigma_p \times \Sigma_q} \cdot A^{\hat{\otimes} p} \right] \right) \hat{\otimes} Y^{\hat{\otimes} q}, \end{aligned}$$

and similarly,

$$(\mathcal{O} \hat{\circ} \mathcal{O} \hat{\circ} A) \amalg (\mathcal{O} \hat{\circ} Y) \cong \coprod_{q \geq 0} \left( \coprod_{p \geq 0} \mathcal{O}[\mathbf{p} + \mathbf{q}] \otimes \left[ \frac{\Sigma_{p+q}}{\Sigma_p \times \Sigma_q} \cdot (\mathcal{O} \hat{\circ} A)^{\hat{\circ} p} \right] \right) \hat{\circ} Y^{\hat{\circ} q}.$$

The maps  $d_0$  and  $d_1$  similarly factor in the underlying category  $\mathbf{Seq}$ . It is important to note that the ordering of all tensor power factors is respected, and that we are simply using the symmetric groups in the isomorphisms

$$(A \amalg Y)^{\hat{\circ} t} \cong \coprod_{p+q=t} \Sigma_{p+q} \cdot_{\Sigma_p \times \Sigma_q} A^{\hat{\circ} p} \hat{\circ} Y^{\hat{\circ} q}$$

to build convenient indexing sets for the tensor powers.  $\square$

**Definition 13.34.** Let  $i : X \rightarrow Y$  be a morphism in  $\mathbf{Seq}$  and  $t \geq 1$ . Define  $Q_0^t := X^{\hat{\circ} t}$  and  $Q_t^t := Y^{\hat{\circ} t}$ . For  $0 < q < t$  define  $Q_q^t$  inductively by the pushout diagrams

$$\begin{array}{ccc} \Sigma_t \cdot_{\Sigma_{t-q} \times \Sigma_q} X^{\hat{\circ}(t-q)} \hat{\circ} Q_{q-1}^q & \xrightarrow{\text{pr}_*} & Q_{q-1}^t \\ \downarrow i_* & & \downarrow i_* \\ \Sigma_t \cdot_{\Sigma_{t-q} \times \Sigma_q} X^{\hat{\circ}(t-q)} \hat{\circ} Y^{\hat{\circ} q} & \longrightarrow & Q_q^t \end{array}$$

in  $\mathbf{Seq}^{\Sigma_t}$ .

The following is an  $\Omega$ -operad version of Proposition 13.13, and provides a useful description of certain pushouts of left modules.

**Proposition 13.35.** Let  $\mathcal{O}$  be an  $\Omega$ -operad,  $A \in \mathbf{Lt}_{\mathcal{O}}$ , and  $i : X \rightarrow Y$  in  $\mathbf{Seq}$ . Consider any pushout diagram in  $\mathbf{Lt}_{\mathcal{O}}$  of the form,

$$(13.36) \quad \begin{array}{ccc} \mathcal{O} \hat{\circ} X & \xrightarrow{f} & A \\ \downarrow \text{id} \hat{\circ} i & & \downarrow \\ \mathcal{O} \hat{\circ} Y & \longrightarrow & A \amalg_{(\mathcal{O} \hat{\circ} X)} (\mathcal{O} \hat{\circ} Y). \end{array}$$

The pushout in (13.36) is naturally isomorphic to a filtered colimit of the form

$$(13.37) \quad A \amalg_{(\mathcal{O} \hat{\circ} X)} (\mathcal{O} \hat{\circ} Y) \cong \text{colim} \left( A_0 \xrightarrow{j_1} A_1 \xrightarrow{j_2} A_2 \xrightarrow{j_3} \dots \right)$$

in the underlying category  $\mathbf{Seq}$ , with  $A_0 := \mathcal{O}_A[\mathbf{0}] \cong A$  and  $A_t$  defined inductively by pushout diagrams in  $\mathbf{Seq}$  of the form

$$(13.38) \quad \begin{array}{ccc} \mathcal{O}_A[\mathbf{t}] \hat{\circ} Q_{t-1}^t & \xrightarrow{f_*} & A_{t-1} \\ \downarrow \text{id} \hat{\circ} i_* & & \downarrow j_t \\ \mathcal{O}_A[\mathbf{t}] \hat{\circ} Y^{\hat{\circ} t} & \xrightarrow{\xi_t} & A_t. \end{array}$$

*Proof.* Verify the pushout in (13.36) may be calculated by a reflexive coequalizer in  $\mathbf{Lt}_{\mathcal{O}}$  of the form

$$A \amalg_{(\mathcal{O} \hat{\circ} X)} (\mathcal{O} \hat{\circ} Y) \cong \text{colim} \left( A \amalg (\mathcal{O} \hat{\circ} Y) \begin{array}{c} \xleftarrow{\bar{i}} \\ \xleftarrow{\bar{f}} \end{array} A \amalg (\mathcal{O} \hat{\circ} X) \amalg (\mathcal{O} \hat{\circ} Y) \right)$$

The maps  $\bar{i}$  and  $\bar{f}$  are induced by maps  $\text{id} \hat{\circ} i_*$  and  $\text{id} \hat{\circ} f_*$  which fit into the commutative diagram

$$(13.39) \quad \begin{array}{ccccc} \mathcal{O}_A \hat{\circ} (X \amalg Y) & \longleftarrow & \mathcal{O} \hat{\circ} (A \amalg X \amalg Y) & \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{d_1} \end{array} & \mathcal{O} \hat{\circ} ((\mathcal{O} \hat{\circ} A) \amalg X \amalg Y) \\ \bar{i} \downarrow \parallel \bar{f} & & \text{id} \hat{\circ} i_* \downarrow \parallel \text{id} \hat{\circ} f_* & & \text{id} \hat{\circ} i_* \downarrow \parallel \text{id} \hat{\circ} f_* \\ \mathcal{O}_A \hat{\circ} (Y) & \longleftarrow & \mathcal{O} \hat{\circ} (A \amalg Y) & \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{d_1} \end{array} & \mathcal{O} \hat{\circ} ((\mathcal{O} \hat{\circ} A) \amalg Y) \end{array}$$

in  $\text{Lt}_{\mathcal{O}}$ , with rows reflexive coequalizer diagrams, and maps  $i_*$  and  $f_*$  in  $\text{Seq}$  induced by  $i : X \rightarrow Y$  and  $f : X \rightarrow A$ . By Proposition 13.32, the coproduct in (13.36) may be calculated by the colimit of the left hand column of (13.39) in the underlying category  $\text{Seq}$ . We want to reconstruct this colimit via a suitable filtered colimit.

Using (13.39), there exist maps  $\bar{f}_{q,p}$  which make the diagrams

$$\begin{array}{ccc} \mathcal{O}_A \hat{\circ} (X \amalg Y) \cong \coprod_{q \geq 0} \coprod_{p \geq 0} ( ) & \xleftarrow{\text{in}_{q,p}} & (\mathcal{O}_A[\mathbf{p} + \mathbf{q}] \hat{\otimes} [\Sigma_{p+q} \cdot \Sigma_p \times \Sigma_q X^{\hat{\otimes} p} \hat{\otimes} Y^{\hat{\otimes} q}]) \\ \downarrow \bar{f} & & \downarrow \exists \bar{f}_{q,p} \\ \mathcal{O}_A \hat{\circ} (Y) \cong \coprod_{t \geq 0} ( ) & \xleftarrow{\text{in}_q} & (\mathcal{O}_A[\mathbf{q}] \hat{\otimes} Y^{\hat{\otimes} q}) \end{array}$$

in  $\text{Seq}$  commute. Similarly, there exist maps  $\bar{i}_{q,p}$  which make the diagrams

$$\begin{array}{ccc} \mathcal{O}_A \hat{\circ} (X \amalg Y) \cong \coprod_{q \geq 0} \coprod_{p \geq 0} ( ) & \xleftarrow{\text{in}_{q,p}} & (\mathcal{O}_A[\mathbf{p} + \mathbf{q}] \hat{\otimes} [\Sigma_{p+q} \cdot \Sigma_p \times \Sigma_q X^{\hat{\otimes} p} \hat{\otimes} Y^{\hat{\otimes} q}]) \\ \downarrow \bar{i} & & \downarrow \exists \bar{i}_{q,p} \\ \mathcal{O}_A \hat{\circ} (Y) \cong \coprod_{t \geq 0} ( ) & \xleftarrow{\text{in}_{p+q}} & (\mathcal{O}_A[\mathbf{p} + \mathbf{q}] \hat{\otimes} Y^{\hat{\otimes} (p+q)}) \end{array}$$

in  $\text{Seq}$  commute. Let  $\varphi : \mathcal{O}_A \hat{\circ} (Y) \rightarrow \cdot$  be a morphism in  $\text{Seq}$  and define  $\varphi_q := \varphi \text{in}_q$ . Then  $\varphi \bar{i} = \varphi \bar{f}$  if and only if the diagrams

$$\begin{array}{ccc} \mathcal{O}_A[\mathbf{p} + \mathbf{q}] \hat{\otimes} [\Sigma_{p+q} \cdot \Sigma_p \times \Sigma_q X^{\hat{\otimes} p} \hat{\otimes} Y^{\hat{\otimes} q}] & \xrightarrow{\bar{f}_{q,p}} & \mathcal{O}_A[\mathbf{q}] \hat{\otimes} Y^{\hat{\otimes} q} \\ \downarrow \bar{i}_{q,p} & & \downarrow \varphi_q \\ \mathcal{O}_A[\mathbf{p} + \mathbf{q}] \hat{\otimes} Y^{\hat{\otimes} (p+q)} & \xrightarrow{\varphi_{p+q}} & \cdot \end{array}$$

commute for every  $p, q \geq 0$ . Since  $\bar{i}_{q,0} = \text{id}$  and  $\bar{f}_{q,0} = \text{id}$ , it is sufficient to consider  $q \geq 0$  and  $p > 0$ .

To motivate the construction (13.38), it is useful to describe a preliminary construction which also calculates the pushout in (13.36). Define  $A_0 := \mathcal{O}_A[\mathbf{0}] \cong A$

and for each  $t \geq 1$  define  $A_t$  by the pushout diagram

$$(13.40) \quad \begin{array}{ccc} \coprod_{\substack{p+q=t \\ q \geq 0, p > 0}} \mathcal{O}_A[\mathbf{p} + \mathbf{q}] \hat{\otimes} [\Sigma_{p+q} \cdot \Sigma_p \times \Sigma_q X^{\hat{\otimes} p} \hat{\otimes} Y^{\hat{\otimes} q}] & \xrightarrow{f_*} & A_{t-1} \\ \downarrow i_* & & \downarrow j_t \\ \mathcal{O}_A[\mathbf{t}] \hat{\otimes} Y^{\hat{\otimes} t} & \xrightarrow{\xi_t} & A_t \end{array}$$

in  $\mathbf{Seq}$ . The maps  $f_*$  and  $i_*$  are induced by the appropriate maps  $\bar{f}_{q,p}$  and  $\bar{i}_{q,p}$ . Verify that (13.37) is satisfied.

The collection of maps  $\bar{f}_{q,p}$  and  $\bar{i}_{q,p}$  satisfy many compatibility relations. This suggests we replace the coproduct in (13.40), which is isomorphic to

$$\mathcal{O}_A[\mathbf{t}] \hat{\otimes} \left[ (X \amalg Y)^{\hat{\otimes} t} - Y^{\hat{\otimes} t} \right],$$

with an appropriate pushout construction. Here,  $(X \amalg Y)^{\hat{\otimes} t} - Y^{\hat{\otimes} t}$  means the coproduct of all factors in  $(X \amalg Y)^{\hat{\otimes} t}$  except  $Y^{\hat{\otimes} t}$ .

Define  $A_0 := \mathcal{O}_A[\mathbf{0}] \cong A$  and for each  $t \geq 1$  define  $A_t$  by the pushout diagram (13.38) in  $\mathbf{Seq}$ . The maps  $f_*$  and  $i_*$  are induced by the appropriate maps  $\bar{f}_{q,p}$  and  $\bar{i}_{q,p}$ . Verify that (13.37) is satisfied.  $\square$

**Proposition 13.41.** *Let  $i : X \rightarrow Y$  be a cofibration (resp. acyclic cofibration) in  $\mathbf{Seq}$ . Then the induced map  $i_* : Q_{t-1}^t \rightarrow Y^{\hat{\otimes} t}$  is a cofibration (resp. acyclic cofibration) in the underlying category  $\mathbf{Seq}$ .*

*Proof.* Argue as in the proof of Proposition 13.21, replacing  $(\mathbf{SymSeq}, \otimes, 1)$  with  $(\mathbf{Seq}, \hat{\otimes}, 1)$ .  $\square$

**Proposition 13.42.** *Assume that  $\mathbf{C}$  satisfies Basic Assumption 1.1 and in addition satisfies the monoid axiom. Then  $(\mathbf{Seq}, \hat{\otimes}, 1)$  satisfies the monoid axiom.*

*Proof.* Since colimits in  $\mathbf{Seq}$  are calculated objectwise, use (4.7) together with an argument that the pushout of a coproduct  $\amalg_{\alpha} f_{\alpha}$  of a finite set of maps can be written as a finite composition of pushouts of the maps  $f_{\alpha}$ .  $\square$

*Proof of Proposition 13.29.* By Proposition 13.35,  $j$  is a (possibly transfinite) composition of pushouts of maps of the form  $\text{id} \hat{\otimes} i_*$ , and Propositions 13.42 and 13.41 finish the argument.  $\square$

#### 14. SHORTENED PROOF FOR CHAIN COMPLEXES

In this section, we include a shortened proof of Theorem 1.4, for the special case  $(\mathbf{Ch}_k, \otimes, k)$  of unbounded chain complexes over a field of characteristic zero.

Consider the proof of Theorem 1.4 given in Section 13.2, for the case of  $\mathbf{Lt}_{\mathcal{O}}$ ; for the special case of  $\mathbf{Alg}_{\mathcal{O}}$  see Remark 13.7. The (possibly transfinite) small object arguments only require the pushouts in Proposition 13.4 to be constructed from a set of *generating* acyclic cofibrations. In the special case of chain complexes [22, Section 2.3], a set of generating acyclic cofibrations for  $\mathbf{SymSeq}$  may be chosen such that each has the form  $i : \emptyset \rightarrow D$ . Since acyclic cofibrations are preserved under (possibly transfinite) compositions, in this special case, the proof of Theorem 1.4 reduces to the following proposition.

**Proposition 14.1.** *Let  $\mathcal{O}$  be a  $\Sigma$ -operad and  $D \in \mathbf{SymSeq}$  such that  $i : \emptyset \rightarrow D$  is an acyclic cofibration. Assume every object in  $\mathbf{SymArray}$  is cofibrant. Consider any pushout diagram in  $\mathbf{Lt}_{\mathcal{O}}$  of the form,*

$$\begin{array}{ccc} \mathcal{O} \circ \emptyset & \longrightarrow & A \\ \downarrow \text{id} \circ i & & \downarrow j \\ \mathcal{O} \circ D & \longrightarrow & A \amalg (\mathcal{O} \circ D). \end{array}$$

*Then  $j$  is an acyclic cofibration in the underlying category  $\mathbf{SymSeq}$ .*

*Remark 14.2.* Suppose  $G$  is any finite group. Since  $k$  is a field of characteristic zero, every  $k[G]$ -module is projective. It follows that every symmetric array in  $\mathbf{Ch}_k$  is cofibrant.

*Proof.* By Proposition 13.8, it is enough to verify

$$\mathcal{O}_A \circ (i) : \mathcal{O}_A \circ (\emptyset) \rightarrow \mathcal{O}_A \circ (D)$$

is an acyclic cofibration in  $\mathbf{SymSeq}$ . Consider any fibration  $p : X \rightarrow Y$  in  $\mathbf{SymSeq}$ . We want to show that  $\mathcal{O}_A \circ (i)$  has the LLP with respect to  $p$ .

$$\begin{array}{ccc} \mathcal{O}_A \circ (\emptyset) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \mathcal{O}_A \circ (D) & \longrightarrow & Y \end{array} \quad \begin{array}{ccc} \mathcal{O}_A \circ \hat{\emptyset} & \longrightarrow & \hat{X} \\ \downarrow & \nearrow & \downarrow \hat{p} \\ \mathcal{O}_A \circ \hat{D} & \longrightarrow & \hat{Y} \end{array}$$

The left-hand solid commutative diagram in  $\mathbf{SymSeq}$  has a lift if and only if the right-hand solid diagram in  $\mathbf{SymArray}$  has a lift. Hence it is sufficient to verify that the solid diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & \text{Map}^\circ(\hat{D}, \hat{X}) \\ \downarrow & \nearrow & \downarrow (*) \\ \mathcal{O}_A & \longrightarrow & \text{Map}^\circ(\hat{\emptyset}, \hat{X}) \times_{\text{Map}^\circ(\hat{\emptyset}, \hat{Y})} \text{Map}^\circ(\hat{D}, \hat{Y}) \end{array}$$

in  $\mathbf{SymArray}$  has a lift. We know  $\mathcal{O}_A$  is cofibrant in  $\mathbf{SymArray}$ , hence it is sufficient to verify  $(*)$  is an acyclic fibration. By Proposition 5.22, it is enough to show each map

$$\text{Map}^\circ(D^{\otimes t}, X) \longrightarrow \text{Map}^\circ(\emptyset^{\otimes t}, X) \times_{\text{Map}^\circ(\emptyset^{\otimes t}, Y)} \text{Map}^\circ(D^{\otimes t}, Y)$$

is an acyclic fibration in  $\mathbf{SymSeq}$ . Theorem 12.4 together with Propositions 12.6 and 12.11 finish the proof.  $\square$

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