A HOMOTOPY THEORY FOR STACKS

SHARON HOLLANDER

Abstract. We give a homotopy theoretic characterization of stacks on a site $\mathcal{C}$ as the *homotopy sheaves* of groupoids on $\mathcal{C}$. We use this characterization to construct a model category in which stacks are the fibrant objects. We compare different definitions of stacks and show that they lead to Quillen equivalent model categories. In addition, we show that these model structures are Quillen equivalent to the S^2-nullification of Jardine's model structure on sheaves of simplicial sets on $\mathcal{C}$.

1. Introduction

Stacks arise as classifying objects for moduli problems in algebraic geometry. This means that, in some sense, maps from a scheme X into a stack correspond to isomorphism classes of families of certain objects over X. A standard example is the stack of all curves: a map from a scheme X into this stack corresponds to an isomorphism class of families of curves over X. Other examples include the stack representing vector bundles and the stack representing curves of genus g with n marked points. In algebraic geometry, stacks are regarded as a generalization of schemes, and many of the usual constructions for schemes are extended so as to make sense for stacks as well. For example, one can define cohomology groups for a stack. These groups yield important information about general properties of the objects which the stack classifies.

Recently, stacks have also come up in algebraic topology. Complex oriented cohomology theories give rise to Hopf algebroids which corepresent stacks on the category of affine schemes, and these stacks map to the moduli stack of formal groups. In recent work of Hopkins and Miller, it has been shown conversely that, in good situations, stacks over the moduli stack of formal groups give rise to spectra which are approximations (often localizations) of the sphere spectrum. These spectra play a key role in modern attempts at understanding and calculating the stable homotopy groups of spheres.

There are many different definitions of stacks. The main purpose of this paper is to show that all of these definitions can be interpreted in terms of homotopy theory, and to show that from this point of view they are natural and easy to compare.

One definition of stacks is based on the concept of category fibered in groupoids [DM, Gi] and another based on the concept of lax presheaf of groupoids [Brn, Bry]. In each definition, part of the information encoded in a stack $\mathcal{M}$ is an assignment to each scheme X of a groupoid $\mathcal{M}(X)$. These assignments are required to satisfy 'descent conditions', which are often somewhat cumbersome. We will show that, for the definitions of stack commonly in use, the descent conditions can be given a simple homotopy theoretic interpretation.

Date: May 25, 2001.

The descent conditions describe the circumstances under which we require that local data glue together to yield global data. Naively, one might require "isomorphism classes of -" to satisfy the sheaf condition. However, for very fundamental reasons, this almost never happens in examples. Taking isomorphism classes is a localization process, and such processes rarely preserve limits such as those which arise in the statement of the sheaf condition. Instead, one can ask that an assignment of groupoids satisfy a sheaf condition with respect to the *best approximation to the limit* which is invariant under taking isomorphism classes. This is called the homotopy limit, and denoted holim. Stacks are assignments which satisfy this modified sheaf condition, so in this sense, stacks are the *homotopy sheaves.*

We propose the following definition of stack as a reference point, as it is conceptually the simplest:

Definition 1.1. *Let $\mathcal{C}$ be a Grothendieck topology. A presheaf of groupoids, F on $\mathcal{C}$ is a* **stack** *if for every cover $\{U_i \to X\}$ in $\mathcal{C}$, we have an equivalence of categories*

$$F(X) \xrightarrow{\quad\sim\quad} \operatorname{holim}\Big(\prod F(U_i) \Longrightarrow \prod F(U_{ij}) \Rrightarrow \prod F(U_{ijk}) \dots \Big).$$

Here $U_{i_0 \dots i_n}$ denotes the iterated fiber product $U_{i_0} \times_X \cdots \times_X U_{i_n}$, and the homotopy limit is taken in the category of groupoids (see sections 3-4).

We will show that all other definitions of stack commonly in use can be given similar homotopy theoretic interpretations. Not only the definition but many properties of stacks which are of interest are homotopy theoretic in nature, and this homotopy theoretic perspective both simplifies the task of comparing the different definitions as well as illuminates the sense in which stacks are the "right" classifying spaces for algebraic problems. In particular, the previous definition gives an alternate category of stacks which is equivalent from the point of view of homotopy theory but much easier to work with, and which is related in a simple way to familiar homotopy theoretic categories.

In more detail, for each of the definitions of stack, we will construct a model category in which the stacks are the fibrant objects. In these model categories, constructions that are commonly performed on stacks (such as 2-category pullbacks, stackification, sheaves over a stack and others) have easy homotopy-theoretic interpretations [Holl]. Moreover, homotopy classes of maps from an object $X \in \mathcal{C}$ to a stack $\mathcal{M}$ correspond to the isomorphism classes of $\mathcal{M}(X)$, and the homotopies themselves correspond to isomorphisms in $\mathcal{M}(X)$. We will see that all of these different model categories are Quillen equivalent. This is the formal way of saying they are all models for the same underlying homotopy theory. This equivalence makes precise the sense in which, when dealing with stacks, it is enough to consider presheaves of groupoids satisfying descent conditions.

More precisely, we will analyze three different categories in which stacks can be defined (see section 5 for the definitions) and prove the following results. Let $\mathcal{C}$ be a Grothendieck topology.

Theorem 1.2. *There are adjoint pairs of functors between: categories fibered in groupoids over $\mathcal{C}$, presheaves of groupoids on $\mathcal{C}$, sheaves of groupoids on $\mathcal{C}$, and lax presheaves of groupoids on $\mathcal{C}$,*

$$lax - P(\mathcal{C}, \mathcal{G}rpd) \underset{\sim}{\overset{\sim}{\rightleftarrows}} \mathcal{G}rpd/\mathcal{C} \underset{p}{\overset{\Gamma}{\rightleftarrows}} P(\mathcal{C}, \mathcal{G}rpd) \underset{sh}{\overset{i}{\leftrightarrows}} Sh(\mathcal{C}, \mathcal{G}rpd),$$

where the right adjoints point to presheaves. All of these functors take stacks as defined in the domain category to stacks as defined in the range category and thus restrict to give adjoint pairs between the stacks in each of these categories.

Theorem 1.3. *There are simplicial model category structures on each of the above listed categories in which:*

1. *the stacks are the fibrant objects,*
2. *in $P(\mathcal{C}, \mathcal{G}rpd)$ or $Sh(\mathcal{C}, \mathcal{G}rpd)$, a weak equivalence is a map satisfying the local lifting conditions (see 8.2),*
3. *if the topology on $\mathcal{C}$ has enough points, the weak equivalences in $P(\mathcal{C}, \mathcal{G}rpd)$ are the stalkwise equivalences of groupoids,*
4. *all of the adjoint pairs listed above are Quillen equivalences,*
5. *the fundamental groupoid of the simplicial* Hom *set between $X \in \mathcal{C}$ and a stack $\mathcal{M}$, the homotopy function complex $h\operatorname{Hom}(X, \mathcal{M})$ is equivalent to the groupoid $\mathcal{M}(X)$. In particular, $[X, \mathcal{M}]$ is the set of isomorphism classes of $\mathcal{M}(X)$.*

Presheaves of groupoids, which will be our preferred setting for talking about stacks, is closely related to presheaves of simplicial sets. The homotopy theory of the latter has been developed by Jardine [Ja], and is the basis on which Morel and Voevodsky build the $\mathbb{A}^1$-homotopy theory of schemes, see [MV].

Theorem 1.4. *The local model structure on $P(\mathcal{C}, \mathcal{G}rpd)$ is Quillen equivalent to Jardine's model category structure on $P(\mathcal{C}, s\mathcal{S}et)$ localized with respect to the maps $\partial\Delta^n \otimes X \to \Delta^n \otimes X$, for each $X \in \mathcal{C}$ and $n > 2$.*

This theorem says that the homotopy theory of stacks is recovered from Jardine's model category by eliminating all higher homotopies.

1.1. **Notation and Assumptions.** So as not run into set theoretic problems, we assume that the Grothendieck topology $\mathcal{C}$ is a small category. We also assume that the topology on $\mathcal{C}$ is subcanonical in order to construct the desired model structure on $Sh(\mathcal{C}, \mathcal{G}rpd)$. For $\{U_i \to X\}$ a cover in $\mathcal{C}$, and F a presheaf on $\mathcal{C}$,

- $U_{i_0 \dots i_n}$ denotes the iterated fiber product $U_{i_0} \times_X \cdots \times_X U_{i_n}$.
- $U_\bullet$ denotes the simplicial diagram in $Pre(\mathcal{C})$ with
$$(U_\bullet)_n = \coprod_I U_{i_0} \times_X \cdots \times_X U_{i_n},$$
where the coproduct is taken over all multi-indices of length n, and the face and boundary maps are defined by the various projection and diagonal maps. This is referred to as the *nerve* of the cover $\{U_i \to X\}$.
- To simplify notation $\coprod U_i$ will sometimes be denoted by U,
the coproduct $\coprod U_{ij}$ will be denoted by $U \times_X U$,
and $\coprod U_{ijk}$ by $U \times_X U \times_X U$.
- $F(U_\bullet) = \operatorname{Hom}(U_\bullet, F)$ denotes the cosimplicial diagram $F(U_\bullet)_n = \prod_I F(U_{i_0} \times_X \cdots \times_X U_{i_n})$ with coface and codegeneracy maps dual to those for $U_\bullet$.
- We will sometimes write $F(U)$ for $\prod F(U_i)$,
$F(U \times_X U)$ for $\prod F(U_{ij})$,
and $F(U \times_X U \times_X U)$ for $\prod F(U_{ijk})$.
- Similarly, a cover $\{V_i \to Y\}$ may be denoted by $V \to Y$, and the nerve of this cover by $\{V \to Y\}_\bullet$.

1.2. **Contents.** The following is an outline of the contents of this paper:

In section 2 we give necessary background information about groupoids, monoidal categories, enriched categories, model categories and localization.

In section 3 we construct a model structure on groupoids, and prove that it is Quillen equivalent to a localization of simplicial sets with respect to the map $S^2 \to *$, called the S^2 nullification of $s\mathcal{S}et$.

In section 4 we give some background on homotopy limits and colimits, and prove that the descent category is a model for the homotopy inverse limit of a cosimplicial diagram of groupoids.

In section 5 we review the definition of categories fibered in groupoids over a fixed base category $\mathcal{C}$. Then we construct an adjoint pair of functors between this category and the category of presheaves of groupoids on $\mathcal{C}$. We define stacks in each of these categories as the objects which satisfy a *homotopy sheaf condition*.

Section 6 contains a discussion of the classical definition of stacks [DM], and a proof that it is equivalent to our definition in terms of the homotopy sheaf condition.

In section 7 we put model structures on the categories described in section 5 and on the category of sheaves of groupoids. The weak equivalences are defined to be object, respectively. fiberwise. We note that the pairs of adjoint functors between the different categories that were defined previously are Quillen pairs. We also observe that these model structures can be localized with respect to the *local* equivalences $\operatorname{holim} U_\bullet \to X$, and in these local model structures the fibrant objects are the stacks.

In section 8 we give a characterization of pointwise weak equivalences for presheaves of groupoids in terms of Dan Dugger's *local lifting conditions*. We use this to prove that the local model categories are all Quillen equivalent. We also obtain that the local model category structure on presheaves of groupoids is Quillen equivalent to the S^2 nullification of Jardine's model category structure on presheaves of simplicial sets, and conclude that when the Grothendieck topology on $\mathcal{C}$ has enough points, the weak equivalences in the local model category structure are precisely the pointwise equivalences of groupoids.

Appendix A contains a discussion of limits and colimits in the category $\mathcal{G}rpd/\mathcal{C}$ of categories fibered in groupoids.

In appendix B we define the category of lax presheaves of groupoids and describe the adjoint pair between lax presheaves and categories fibered in groupoids. This is an equivalence of categories and hence allows us to translate all the results from categories fibered in groupoids to lax presheaves.

1.3. **Acknowledgments.** A great many thanks are owed to Dan Dugger, Gustavo Granja, and Mike Hopkins, for many helpful conversations and ideas, without which this paper would not exist.

The author has recently learned about the paper of Jardine [Ja2], which appears to treat some of the questions dealt with here. Although his approach is quite different, it is possible that there is some overlap in the results.

2. Preliminaries

A groupoid is a small category in which all morphisms are invertible. $\mathcal{G}rpd$ denotes the full subcategory of $\mathcal{C}at$ whose objects are groupoids. In this section we will define the notion of a category with a groupoid action. Many of the categories we will discuss in the future have groupoid actions, and many of their properties follow from similar properties of groupoids. We show that such categories have a natural simplicial structure, determined by the action of the fundamental groupoid of the simplicial sets. We also review the concept of a model category, and quote the results about localization which we will need later.

2.1. Groupoids.

Recall, the nerve embedding, $N : \mathcal{C}at \to s\mathcal{S}et$. For $\mathcal{C} \in \mathcal{C}at$, $N(\mathcal{C})_n$ is the set of n-tuples of composable morphisms

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{n-2}} X_{n-1} \xleftarrow{f_{n-1}} X_n$$

with the convention that a 0-tuple is just an object. For $i \neq 0, n$, the boundary maps d_i send $(f_0, \dots f_i, f_{i+1}, \dots, f_{n-1}) \mapsto (f_1, \dots f_{i-1} \circ f_i, \dots, f_{n-1})$; d_0 leaves out f_0, and d_n leaves out f_n. In particular $N(G)_1 \xrightarrow{d_0} N(G)_0$ is the domain function, and d_1 is the range function. The degeneracy maps s_i insert an identity morphism in the ith position.

We begin by noting that $\mathcal{G}rpd$ is complete and cocomplete since the (co)limit of a diagram of groupoids in $\mathcal{C}at$ is still a groupoid.

Note 2.1. Recall that the limit of a diagram in $\mathcal{C}at$ is the category whose objects (morphisms) are the limit of the sets of objects (morphisms) in the diagram. To construct a colimit in $\mathcal{C}at$ one takes as objects the colimit of the sets of objects and for morphisms the formal compositions of elements in the colimit of the morphisms, modulo the obvious relations.

We will need the following characterization of the functors in $\mathcal{G}rpd$ that are equivalences of categories.

Lemma 2.2. *The functor $G \xrightarrow{F} H \in \mathcal{G}rpd$ is an equivalence of categories, if and only if the following two conditions hold:*

- *F induces a bijection on isomorphism classes of objects.*
- *For every $a \in G$, the induced map $\mathrm{Aut}_G(a) \to \mathrm{Aut}_H(F(a))$ is an isomorphism.*

Definition 2.3. [GZ] *Let $s\mathcal{S}et \xrightarrow{\pi_{oid}} \mathcal{G}rpd$ be the functor which assigns to a simplicial set X the groupoid with objects X_0 and morphisms freely generated under composition by the members of X_1 and their formal inverses subject to the relations $d_2x \circ d_0x = d_1x$ for each $x \in X_2$.*

The proof of the following lemma is easy.

Lemma 2.4. *The functors $\pi_{oid} : s\mathcal{S}et \leftrightarrow \mathcal{G}rpd : N$ are an adjoint pair, in which N is the right adjoint, and π_{oid} is the left adjoint. The composition of functors $\pi_{oid} \circ N$ is naturally isomorphic to the identity functor of $\mathcal{G}rpd$.*

Note 2.5. The definition of π_{oid} used here is the one given in [GZ]. This groupoid is naturally equivalent to the one defined via homotopy classes of paths [GJ, p. 39]. The [GZ] definition is needed to form the adjoint pair (π_{oid}, N), and thus to define the simplicial structure on $\mathcal{G}rpd$ which is essential for many of our results.

Note 2.6. It follows from the previous note, as $\pi_{oid} \circ N \cong id$, and NG is a Kan complex, that for any $G \in \mathcal{G}rpd$, the isomorphism classes of G are in bijective correspondence with $\pi_0 NG$, and the automorphism group of an object $a \in G$, is isomorphic to $\pi_1(NG, a)$.

The category $\mathcal{G}rpd$ has an internal Hom, written $\mathcal{G}rpd(G, H)$, where the objects of $\mathcal{G}rpd(G, H)$ are the functors $G \to H$, and the isomorphisms are the natural isomorphisms between these functors.

Lemma 2.7. *Let G be a groupoid, then $\mathcal{G}rpd(G, -)$ is the right adjoint to the functor $G \times (-)$.*

Recall [EK] that a *closed category* (short for closed symmetric monoidal) is a category $\mathcal{M}$ with an internal Hom and an associative and commutative product $\otimes$ with a unit S, such that for all $X \in \mathcal{M}$, the functor $X \otimes (-)$ is the left adjoint of $\mathcal{M}(X, -)$.

By lemma 2.7, $\mathcal{G}rpd$ is a closed category with the categorical product and the internal Hom defined above.

Another example of a closed category is $s\mathcal{S}et$. The tensor product is just given by the categorical product, and the internal Hom is given by the formula

$$s\mathcal{S}et(X, Y)_n := \mathrm{Hom}_{s\mathcal{S}et}(\Delta^n \times X, Y)$$

where $\Delta^\bullet$ denotes the cosimplicial standard simplex [BK, p. 268].

Recall [Db], that a category $\mathcal{C}$ is *enriched* over a monoidal category $\mathcal{M}$, if there is a bifunctor from $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{M}$ assigning to each $X, Y \in \mathcal{C}$ an object $\mathcal{M}_{\mathcal{C}}(X, Y) \in \mathcal{M}$ (which we also denote by $\mathcal{M}(X, Y)$) for each object X an "identity" $S \to \mathcal{M}(X, X)$, and for each triple of objects $X, Y, Z \in \mathcal{C}$ a "composition" $\mathcal{M}(X, Y) \otimes \mathcal{M}(Y, Z) \to \mathcal{M}(X, Z)$ which is associative and unital. Moreover it is required that $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{hom}_{\mathcal{M}}(S, \mathcal{M}(X, Y))$. $\mathcal{C}$ is said to be *enriched with tensor and cotensor* if for all $G \in \mathcal{M}$ and $X, Y \in \mathcal{C}$ there are objects $X \bar{\otimes} G$ and $Y^G \in \mathcal{C}$ such that

$$\mathcal{M}(X \bar{\otimes} G, Y) = \mathcal{M}(X, Y^G) = \mathcal{M}(G, \mathcal{M}(X, Y)).$$

It then follows that this tensor and cotensor operations satisfy all the usual properties.

Note 2.8. In practice we will abuse notation and denote the tensor product of objects of $\mathcal{C}$ with objects $\mathcal{M}$ by $\otimes$.

Any closed monoidal category $\mathcal{M}$ is enriched with tensor and cotensor over itself. A category enriched with tensor and cotensor over simplicial sets is called a *simplicial category*. We will say that a category enriched with tensor and cotensor over groupoids has a *groupoid action*.

Proposition 2.9. *Let $\mathcal{C}$ be a category with a groupoid action. Then the assignment*

$$s\mathcal{S}et(X, Y) := N(\mathcal{G}rpd(X, Y))$$

gives $\mathcal{C}$ the structure of a simplicial category. Moreover, the tensor and cotensor are given by the formulas

$$Y \otimes S := Y \otimes \pi_{oid} S,$$
$$Y^S := Y^{\pi_{oid} S},$$

for any $X, Y \in \mathcal{C}, S \in s\mathcal{S}et$.

This proposition follows immediately from the following lemma.

Lemma 2.10. *Let $X \in s\mathcal{S}et, G \in \mathcal{G}rpd$, then the adjoint pair of functors π_{oid} and N satisfies*

$$N(\mathcal{G}rpd(\pi_{oid}X, G)) = s\mathcal{S}et(X, N(G)).$$

In particular, given $G, H \in \mathcal{G}rpd$,

$$N(\mathcal{G}rpd(H, G)) = s\mathcal{S}et(N(H), N(G)).$$

Proof. The nerve of $\mathcal{G}rpd(\pi_{oid}X, G)$ has 0-simplices the functors $\pi_{oid}X \to G$. By lemma 2.4 these are the elements of $\mathrm{Hom}_{s\mathcal{S}et}(X, N(G)) = s\mathcal{S}et(X, N(G))_0$. The n-simplices of $N(\mathcal{G}rpd(\pi_{oid}X, G))$ are n-tuples of composable natural isomorphisms between such functors. They can be naturally identified with functors $\pi_{oid}X \times \pi_{oid}\Delta^n = \pi_{oid}(X \times \Delta^n) \to G$ By another application of lemma 2.4 these are identified with the elements of $\mathrm{Hom}_{s\mathcal{S}et}(X \times \Delta^n, N(G))$. □

The following examples of categories with a groupoid action will be used throughout the rest of the paper.

Example 2.11 (Diagrams of Groupoids)**.** Let X and Y be diagrams of groupoids indexed by a category D, and G a groupoid. Then we define $\mathcal{G}rpd(X, Y)$, to be the groupoid with objects the natural transformations $X \to Y$, and with morphisms the natural isomorphisms $X \times \pi_{oid}\Delta^1 \to Y$, where $\pi_{oid}\Delta^1$ denotes the constant diagram (which assigns to each object the groupoid with two objects and a unique isomorphism between them). Then we have

$$(X \otimes G)(d) = X(d) \times G,$$
$$X^G(d) = \mathrm{Hom}(G, X(d)).$$

When $\mathcal{C}$ is a Grothendieck topology, diagrams indexed by $\mathcal{C}^{op}$ are called *presheaves of groupoids* on $\mathcal{C}$. The category of presheaves of groupoids on $\mathcal{C}$ is denoted $P(\mathcal{C}, \mathcal{G}rpd)$.

Example 2.12 (Sheaves of Groupoids on a Grothendieck Topology $\mathcal{C}$)**.** A *sheaf of groupoids* on $\mathcal{C}$ is a presheaf which satisfies the "sheaf condition": For every covering $\{U_i \to U\}$,

$$F(U) \to \prod F(U_i) \rightrightarrows \prod F(U_i \times_U U_j)$$

is an equalizer sequence. Equivalently, we could require $F(U)$ to be the limit of the cosimplicial diagram determined by the nerve of the covering as, in any category, the limit of a cosimplicial diagram $X^\bullet$ is the equalizer of $d^0, d^1 : X^0 \rightrightarrows X^1$. The category $Sh(\mathcal{C}, \mathcal{G}rpd)$ is the full subcategory of $P(\mathcal{C}, \mathcal{G}rpd)$ whose objects are the sheaves of groupoids on $\mathcal{C}$.

We list some of the important properties of sheaves and presheaves.

(1) There is a "sheafification functor" $P(\mathcal{C}, \mathcal{G}rpd) \xrightarrow{sh} Sh(\mathcal{C}, \mathcal{G}rpd)$ which is the left adjoint to the inclusion functor of sheaves in presheaves.
(2) The category $Sh(\mathcal{C}, \mathcal{G}rpd)$ inherits a $\mathcal{G}rpd$ action via the inclusion into $P(\mathcal{C}, \mathcal{G}rpd)$ as it is easy to check that for a sheaf $\mathcal{F}$, the presheaves $\mathcal{F} \otimes G$ and $\mathcal{F}^G$ are still sheaves.

For further elaboration of these points see [MM].

2.2. **Review of Model Categories.** We recall the definition of a model category structure on a category $\mathcal{C}$. Model categories are an abstract setting in which to do homotopy theory.

A model category [Ho, Q, DS], is a category $\mathcal{C}$, together with three distinguished classes of morphisms in $\mathcal{C}$, called cofibrations, fibrations, and weak equivalences, which are closed under composition and contain all identity morphisms, and satisfy the following properties:

- (MC1) Small limits and colimits exist in $\mathcal{C}$.
- (MC2) If f, g are morphisms with $g \circ f$ defined, and two of the three morphisms $f, g, g \circ f$ are weak equivalences then so is the third.
- (MC3) If f is a retract of g and g is a fibration, cofibration, or weak equivalence, then so is f.
- (MC4) Given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{l}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y. \end{array}$$

  where either (a) i is a cofibration and p a trivial fibration (a fibration which is also a weak equivalence), or (b) i is a trivial cofibration (a cofibration which is also a weak equivalence), and p a fibration, then there exists a lifting $l : B \to X$ making the above diagram commute.
- (MC5) Any morphism f can be factored functorially in two ways: (a) $f = p \circ i$ where i is a cofibration and p is a trivial fibration; and (b) $f = p \circ i$ where i is a trivial cofibration and p is a fibration.

An object X is called *cofibrant* if the map from the initial object $\emptyset$, to X is a cofibration. An object X is called *fibrant* if the map from X to the final object $*$, is a fibration. The category obtained from $\mathcal{C}$ by formally inverting the weak equivalences is called the *homotopy category* of $\mathcal{C}$, and denoted $Ho(\mathcal{C})$.

A set of (trivial) cofibrations are said to *generate* if the trivial fibrations (fibrations) are characterized by having the right lifting property (as in MC4) with respect to these morphisms.

A *simplicial model category* is a model category $\mathcal{C}$ which has a simplicial structure compatible with the model structure in the sense that the following axiom holds:

> (SM7) Given a cofibration $A \xrightarrow{i} B$ and a fibration $X \xrightarrow{p} Y$, the induced map
>
> $$s\mathcal{S}et(B, X) \to s\mathcal{S}et(A, X) \times_{s\mathcal{S}et(A,Y)} s\mathcal{S}et(B, Y)$$
>
> is a fibration. In addition, if either i or p is a weak equivalence then the above map is a trivial fibration.

A *Quillen pair* between model categories is an adjoint pair $L : \mathcal{C} \leftrightarrow \mathcal{D} : R$ where the left adjoint L preserves cofibrations and trivial cofibrations, or equivalently the right adjoint R preserves fibrations and trivial fibrations. Under these conditions, one can define the *derived functors* $\underline{L} : Ho(\mathcal{C}) \to Ho(\mathcal{D})$ and $\underline{R} : Ho(\mathcal{D}) \to Ho(\mathcal{C})$, and they form an adjoint pair. A Quillen pair is called a *Quillen equivalence* if, for $A \in \mathcal{C}$ cofibrant and $B \in \mathcal{D}$ fibrant, a morphism $LA \to B$ is a weak equivalence in $\mathcal{D}$ if and only if its adjoint $A \to RB$ is a weak equivalence in $\mathcal{C}$. A Quillen pair is a Quillen equivalence if and only if it induces an equivalence of categories between $Ho(\mathcal{C})$ and $Ho(\mathcal{D})$, see [Ho, p. 19].

Under mild conditions there is a procedure called localization which formally adds weak equivalences to a model category (a good reference is [Dg]). Let $\mathcal{C}$ be a simplicial model category and S a set of morphisms between cofibrant objects in $\mathcal{C}$. A fibrant object $X \in \mathcal{C}$ is called *S-local* if for all $f \in S$ the induced map $s\mathcal{S}et(f, X)$ is a weak equivalence. A morphism f in $\mathcal{C}$ is called an *S-equivalence* if for all S-local X, we have that $h\operatorname{Hom}(f, X)$ is a weak equivalence, (where $h\operatorname{Hom}$ is the homotopy function complex, see [Hi]). A model category is *left proper* if pushouts of weak equivalences along cofibrations are weak equivalences.

Note 2.13. A model category $\mathcal{C}$ is combinatorial if it is cofibrantly generated and the underlying category is locally presentable [Sm]. All the categories we will be working with here are locally presentable, (as they have underlying sets) and we will give explicit sets of generating cofibrations.

Theorem 2.14. *(J.Smith)* [Sm] *Let $\mathcal{C}$ be a left proper, combinatorial, simplicial model category and S a set of morphisms between cofibrant objects in $\mathcal{C}$. Then there exists a new model category structure on $\mathcal{C}$ in which*

- *the weak equivalences are S-equivalences,*
- *the cofibrations are the old cofibrations,*
- *the fibrations are maps with the right lifting property with respect to the maps which are cofibrations and also S-equivalences.*

In addition the fibrant objects of $\mathcal{C}$ are precisely the S-local objects, and this new model structure is again left proper, combinatorial, and simplicial.

This new model category is called the *S-localization* of $\mathcal{C}$ and denoted $S^{-1}\mathcal{C}$. Notice that all of the original weak equivalences in $\mathcal{C}$ are, by construction, S-equivalences. The following properties of localization will be used often.

Note 2.15. (P. Hirschhorn) [Hi] Let S and $\mathcal{C}$ be as in the above theorem, $\mathcal{D}$ be a model category, and $L : \mathcal{C} \leftrightarrow \mathcal{D} : R$ a Quillen pair such that L takes morphisms in S to weak equivalences in $\mathcal{D}$. Then

(a) The pair (L, R) is also a Quillen pair $L : S^{-1}\mathcal{C} \leftrightarrow \mathcal{D} : R$.

(b) In particular, if S' is a set of morphisms between cofibrant objects in $\mathcal{D}$, and L takes morphisms in S to S'-equivalences, there is a Quillen pair $L : S^{-1}\mathcal{C} \leftrightarrow (S')^{-1}\mathcal{D} : R$.

(c) [Hi, Theorem 3.4.20] If $L : \mathcal{C} \leftrightarrow \mathcal{D} : R$ is a Quillen equivalence and S is a set of morphisms between cofibrant objects in $\mathcal{C}$, then $L : S^{-1}\mathcal{C} \leftrightarrow (LS)^{-1}\mathcal{D} : R$ is also a Quillen equivalence.

(d) If S', S are sets of morphisms in $\mathcal{C}$ then the two model structures $S^{-1}(S')^{-1}\mathcal{C} = (S')^{-1}S^{-1}\mathcal{C}$ agree.

3. Model Category Structure on Groupoids

In this section we will describe a model category structure on $\mathcal{G}rpd$ which appears in [An, Bo], a proof can be found in [St]. This model category structure will enable us to prove that the descent category, which appears prominently in the definition of stacks, is a model for the homotopy inverse limit of a cosimplicial groupoid. With this in mind the various definitions of stacks can be interpreted as different incarnations of presheaves of groupoids satisfying a 'homotopy sheaf condition'.

Under the nerve embedding, functors between categories become maps between simplicial sets, and natural transformations between functors give rise to homotopies between the corresponding maps. If $F \xrightarrow{\phi} G$, and $F \xrightarrow{\xi} H$ are natural transformations, we obtain homotopies between $N(F)$ and $N(G)$, and from $N(F)$ to $N(H)$. Though there is not necessarily a natural transformation from G to H corresponding to the composite homotopy. Thus, our intuitive notion of homotopy in $\mathcal{C}at$, as a natural transformation between functors, does not correspond to the one defined via the nerve embedding in $s\mathcal{S}et$. However, if our categories are groupoids, this problem does not arise since all natural transformations are natural isomorphisms. This close relationship between our intuition for what homotopy should be in $\mathcal{G}rpd$ and the notion of homotopy defined via the nerve, motivates the model category structure on $\mathcal{G}rpd$ we define here, where a map f in $\mathcal{G}rpd$ is a weak equivalence or fibration if and only if $N(f)$ is one.

We will sometimes abuse notation and denote the groupoid $\pi_{oid}(\Delta^i)$ by Δ^i. This is the groupoid with $i+1$ objects with unique isomorphisms between them. Similarly, we will sometimes denote $\pi_{oid}(\partial\Delta^i)$ by $\partial\Delta^i$. BG denotes the groupoid with one object whose automorphism group is the group G.

Theorem 3.1. *There is a left proper, simplicial, cofibrantly generated model category structure on $\mathcal{G}rpd$ in which:*

- *weak equivalences are functors which induce an equivalence of categories,*
- *fibrations are the functors with the right lifting property with respect to the map $\Delta^0 \to \Delta^1$,*
- *cofibrations are functors which are injections on objects.*

The generating trivial cofibration is the morphism $\Delta^0 \to \Delta^1$, and the generating cofibrations are the morphisms $\partial\Delta^i \to \Delta^i, i = 0, 1, 2$.

Note 3.2. In this model category structure all objects are both fibrant and cofibrant, so all weak equivalences are homotopy equivalences.

Lemma 3.3. *Let $G \xrightarrow{f} H$ be a map of groupoids. The following are equivalent:*

- *f is a weak equivalence in $\mathcal{G}rpd$*
- *Nf is weak equivalence in $s\mathcal{S}et$*

Similarly, the following are equivalent:

- *f is a (trivial) fibration in $\mathcal{G}rpd$.*
- *Nf is a (trivial) Kan fibration in $s\mathcal{S}et$.*
- *f has the right lifting property with respect to $\Delta^0 \to \Delta^1$ (with respect to $\partial\Delta^n \to \Delta^n$ for $n = 0, 1, 2$).*

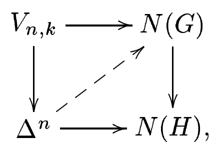
Note that the morphisms $\partial\Delta^i \to \Delta^i, i = 0, 1, 2$, are

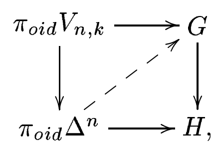
- $\emptyset \to \star$,
- $\{\star, \star\} \to I$

- $\Delta^2 \times (B\mathbb{Z} \to \star)$.

Proof. If f is a weak equivalence in $\mathcal{G}rpd$, it is an equivalence of categories and so Nf is a homotopy equivalence in $s\mathcal{S}et$. Since the nerve of a groupoid is a Kan complex, if Nf is a weak equivalence, it must be a homotopy equivalence, and so $\pi_{oid}Nf = f$ is an equivalence of categories.

Kan fibrations of simplicial sets are characterized by having the right lifting property with respect to the maps $V_{n,k} \to \Delta^n, n \geq 1$ and trivial Kan fibrations are characterized by having the right lifting property with respect to the maps $\partial\Delta^n \to \Delta^n$. Given a morphism $G \to H$ of groupoids, it is equivalent to construct a lifting in either of the diagrams





so we can characterize the maps in $\mathcal{G}rpd$ whose nerves are fibrations as the maps which have the right lifting property with respect to $\pi_{oid}\, V_{n,k} \to \pi_{oid}\, \Delta^n$. Similarly, the maps whose nerves are trivial fibrations are characterized as the morphisms with the right lifting property with respect to the maps $\pi_{oid}\, \partial\, \Delta^n \to \pi_{oid}\, \Delta^n$. Now notice that the inclusions $\pi_{oid}V_{i,k} \to \pi_{oid}\Delta^i$ are isomorphisms for $i > 1$, and that the inclusions $\pi_{oid}\partial\Delta^i \to \pi_{oid}\Delta^i$ are isomorphisms for $i > 2$. ☐

Note 3.4. The previous lemma gives sets of generating cofibrations and trivial cofibrations for the model structure in Theorem 3.1.

Proof of Theorem 3.1. For MC1-MC5 see [St].

For SM7, we need to show that given $A \xrightarrow{i} B$ a cofibration and $X \xrightarrow{p} Y$ a fibration, the induced map

$$s\mathcal{S}et(B, X) \to s\mathcal{S}et(A, X) \times_{s\mathcal{S}et(A,Y)} s\mathcal{S}et(B, Y)$$

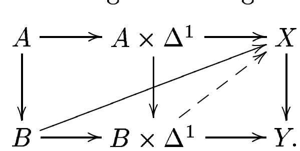
is a fibration of simplicial sets. In addition, we need to show that if either i or p is a weak equivalence, then the above map is a trivial fibration. The simplicial structure on $\mathcal{G}rpd$ is defined by taking the nerve of the internal Hom, and N commutes with limits, so we can rewrite the above map as

$$N\left(\mathcal{G}rpd(B, X) \to \mathcal{G}rpd(A, X) \times_{\mathcal{G}rpd(A,Y)} \mathcal{G}rpd(B, Y)\right);$$

which is a (trivial) fibration if and only if the map

$$\mathcal{G}rpd(B, X) \to \mathcal{G}rpd(A, X) \times_{\mathcal{G}rpd(A,Y)} \mathcal{G}rpd(B, Y)$$

is one. By lemma 3.3, this is the case if and only if this map has the right lifting property with respect to $\Delta^0 \to \Delta^1$ $(\partial\Delta^i \to \Delta^i, i = 0, 1, 2)$. In the first case, the desired lifting is equivalent to a lifting in the diagram



This follows since $(A\times\Delta^1)\coprod_A B \to B\times\Delta^1$ is a trivial cofibration. Similarly, in the second case, the desired lifting exists because the map $(A\times\Delta^i)\coprod_{A\times\partial\Delta^i}(B\times\partial\Delta^i) \to B\times\Delta^i$ is a cofibration.

To show that the model category structure is left proper we must show that the pushout of a weak equivalence along a cofibration is again a weak equivalence. We have already observed above that this is true when the weak equivalence is a cofibration so, by MC5, it suffices to show that the pushout P of a trivial fibration $A \xrightarrow{j} C$ along a cofibration $A \xrightarrow{i} B$ is a weak equivalence. This follows from the following more general proposition. □

Proposition 3.5. *Let A, B, C be small categories, and $A \xrightarrow{i} B$ be a functor which is a monomorphism on objects, and $A \xrightarrow{j} C$ a surjective equivalence of categories. Then the induced functor to the pushout in $\mathfrak{Cat}$, $B \to P := C\coprod_A B$ is also a surjective equivalence of categories.*

Proof. First note that the universal map $B \xrightarrow{p} P$ is surjective on objects. If $b, b' \in \operatorname{ob} B$, then $p(b) = p(b')$ if and only if $b = b'$ or there exist $a, a' \in \operatorname{ob} A$ with $i(a) = b, i(a') = b'$ and $j(a) = j(a')$. In the latter case there is a unique map $a \longrightarrow a' \in A$ which maps to the identity of $j(a)$ and we will call the image of this map in B the canonical map $b \longrightarrow b'$. For b not in the image of A the canonical map $b \longrightarrow b$ is defined to be the identity.

It is clear that p induces an isomorphism on components so it remains to show that p induces an isomorphism

$$\operatorname{Hom}_B(b, b') \longrightarrow \operatorname{Hom}_P(p(b), p(b')).$$

For β, β' objects of P, let $W(\beta, \beta')$ denote the set of words formed by formal compositions of morphisms in B and C such that the first map in the word has domain β, the last map has range β' and consecutive maps have domains and ranges whose images in P agree. Recall that $\operatorname{Hom}_P(\beta, \beta')$ is the quotient of $W(\beta, \beta')$ by the equivalence relation generated by the composition in B, composition in C and $i(f) \sim j(f)$ for f a morphism in A.

Let b, b' be objects of B and write $\beta = p(b), \beta' = p(b')$. We will define functions $\phi_{b,b'} : W(\beta, \beta') \longrightarrow \operatorname{Hom}_B(b, b')$ which are constant on the equivalence classes of $W(\beta, \beta')$ and so determine functions $\operatorname{Hom}_P(\beta, \beta') \longrightarrow \operatorname{Hom}_B(b, b')$. It will be immediate from the construction that these are inverse to p and this will complete the proof.

The functions $\phi_{b,b'}$ are defined by induction on the length of words as follows. Let w be a word of length 1. If w is a morphism $c \xrightarrow{f} c' \in C$ then let a, a' be the unique objects in A such that $i(a) = b, i(a') = b', j(a) = c, j(a') = c'$ and let $a \xrightarrow{g} a'$ denote the unique morphism in A such that $j(g) = f$. Define $\phi_{b,b'}(w) = i(g)$. If w is a morphism $b_1 \xrightarrow{f} b_2 \in B$ define $\phi_{b,b'}(w)$ to be the composite $b \longrightarrow b_1 \xrightarrow{f} b_2 \longrightarrow b'$ where the unlabeled arrows are canonical morphisms.

Now suppose $\phi_{b,b'}$ has been defined on words of length $\leq n$ and let $w = w'f$ where w' is a word of length n and f is a morphism in B or in C. Let b'' be an arbitrary object of B mapping to the range of w' and define $\phi_{b,b'}(w)$ as the composite $b \xrightarrow{\phi_{b,b''}(w')} b'' \xrightarrow{\phi_{b'',b'}(f)} b'$. It follows from the construction that the value of $\phi_{b,b'}$ is independent of the choice of b'' and that $\phi_{b,b'}$ is constant on the equivalence classes of $W(\beta, \beta')$. □

Corollary 3.6. *With this model category structure on* $\mathcal{G}rpd$*, the adjoint pair* π_{oid} : $s\mathcal{S}et \leftrightarrow \mathcal{G}rpd : N$ *is a Quillen pair.*

Remark 3.7. The previous corollary implies that π_{oid} preserves trivial cofibrations, and hence is equivalent to the usual fundamental groupoid functor.

To end this section we give an alternative description of the homotopy theory of groupoids. Consider the model category on $s\mathcal{S}et$ which is the localization of the usual model structure with respect to the map

$$\partial\Delta^3 \to \Delta^3.$$

We will call this the S^2 *nullification* of $s\mathcal{S}et$, following [DF]. Notice that the maps

$$\partial\Delta^n \to \Delta^n, n > 2.$$

are all weak equivalences in this localized model structure, so we could equivalently localize $s\mathcal{S}et$ with respect to this set of maps.

Lemma 3.8. *In the* S^2 *nullification of* $s\mathcal{S}et$*, weak equivalences are the maps which induce an isomorphism on* π_0 *and* π_1 *at all base points.*

Theorem 3.9. *The adjoint pair*

$$s\mathcal{S}et \underset{N}{\overset{\pi_{oid}}{\rightleftarrows}} \mathcal{G}rpd,$$

is a Quillen equivalence between $\mathcal{G}rpd$ *and the* S^2 *nullification of* $s\mathcal{S}et$*.*

4. Homotopy Limits and Colimits

It is well known how to define homotopy limits and colimits in simplicial model categories. One can write down explicit formulas going back to [BK]. In this section, we will give simplified formulas for homotopy (co)limits in case the simplicial structure comes from a groupoid action (2.9). Our main concern will be the homotopy limit of a cosimplicial diagram, and dually the homotopy colimit of a simplicial diagram. Our simplified formula for the former will allow us in section 7 to interpret the descent conditions for stacks in a homotopy-theoretic manner.

Let $\mathcal{C}$ be a simplicial model category. The homotopy limit of an I-diagram X in $\mathcal{C}$ with each $X(i)$ fibrant is the equalizer of the two natural maps

$$\prod_i X(i)^{N(I/i)} \rightrightarrows \prod_{j \xrightarrow{\alpha} i} X(i)^{N(I/j)},$$

where I/i denotes the category of objects over i. Similarly, the homotopy colimit of an objectwise cofibrant I-diagram X is the coequalizer of the two maps

$$\coprod_{i \xrightarrow{\alpha} j} X(i) \otimes N(j/I) \rightrightarrows \coprod_i X(i) \otimes N(i/I),$$

where j/I denotes the category of objects under j. An exposition of these constructions for simplicial sets appears in [BK, GJ], and for a general simplicial model category in [Hi]. For Y a fibrant object and $\mathbf{X} \in \mathcal{C}^I$ objectwise cofibrant, these functors satisfy the equation

$$s\mathcal{S}et(\operatorname{hocolim} \mathbf{X}, Y) = \operatorname{holim} s\mathcal{S}et(\mathbf{X}, Y). \tag{4.1}$$

Note 4.2. When the simplicial structure on $\mathcal{C}$ is derived from a groupoid structure, the above formula is obtained by applying N to the equality

$$\mathcal{G}rpd(\operatorname{hocolim} \mathbf{X}, Y) = \operatorname{holim} \mathcal{G}rpd(\mathbf{X}, Y).$$

Theorem 4.3. *Let $\mathcal{C}$ be a simplicial model category whose simplicial structure derives from a groupoid action, and let $X^\bullet$ be a cosimplicial object in $\mathcal{C}$, with each X^i fibrant. Then a model for the homotopy inverse limit of $X^\bullet$ is given by the equalizer of the natural maps*

$$\prod_{i=0}^{2} (X^i)^{\Delta^i} \rightrightarrows \prod_{[j]\to[i]}^{i\le 2, j\le 1} (X^i)^{\Delta^j}.$$

Proof. First, notice that writing $\operatorname{sk}_2 \Delta^\bullet$ for the 2-skeleton of $\Delta^\bullet$, the inclusion $\pi_{oid} \operatorname{sk}_2 \Delta^\bullet \to \pi_{oid}\Delta^\bullet$ is an isomorphism. It follows that $TotX^\bullet$, the space of maps from $\Delta^\bullet$ to $X^\bullet$, is isomorphic to $Tot_2 X^\bullet$, the space of maps from the restriction $\Delta^\bullet|_{\Delta[2]}$ to $X^\bullet|_{\Delta[2]}$ where $\Delta[2]$ denotes the full subcategory of Δ with objects $[0], [1]$ and $[2]$. Since the map $\pi_{oid} \operatorname{sk}_1 \Delta^2 \longrightarrow \pi_{oid}\Delta^2$ is surjective, $Tot_2 X^\bullet$ is given by the equalizer in the statement of the theorem.

It now suffices to show that the homotopy limit of $X^\bullet$ is naturally homotopy equivalent to $TotX^\bullet$. Using the definition of the homotopy limit in a simplicial model category given above, this is an easy consequence of the following proposition. □

Proposition 4.4. *There is a homotopy equivalence of cosimplicial groupoids*

$$F : \pi_{oid} N(\Delta/[\bullet]) \leftrightarrow \pi_{oid}\Delta^{\bullet} : G.$$

Proof. Morphisms in $\pi_{oid} N(\Delta/[n])$ are generated by the commutative triangles

$$\begin{array}{ccc} [k] & \longrightarrow & [m] \\ & \searrow \quad \swarrow & \\ & [n] & \end{array}$$

and their formal inverses. Let $\pi_{oid} N(\Delta/[n]) \xrightarrow{F_n} \pi_{oid}\Delta^n$ be the functor which sends

- the object $[m] \to [n]$ to the vertex $[0] \xrightarrow{e_m} [m] \to [n]$, where $e_k : [0] \to [k]$ denotes the map which sends 0 to k.
- a generating morphism as above to the 1-simplex in Δ^n which is the unique map filling in the following diagram

$$\begin{array}{ccccc} [0] & \xrightarrow{d^1} & [1] & \xleftarrow{d^0} & [0] \\ \downarrow{\scriptstyle e_k} & & \vdots{\scriptstyle !} & & \downarrow{\scriptstyle e_m} \\ [k] & \longrightarrow & \vdots & \longrightarrow & [m] \\ & \searrow & \downarrow & \swarrow & \\ & & [n]. & & \end{array}$$

One can check easily that F is well defined and natural in n, and so defines a morphism

$$\pi_{oid} N(\Delta/[\bullet]) \xrightarrow{F} \pi_{oid}\Delta^{\bullet} \in c\mathcal{G}rpd.$$

Let G_n be the functor which is defined

- on objects by including $[0] \to [n]$ in $\Delta/[n]$.
- on generating morphisms $[1] \to [n]$ as the composition

$$\begin{array}{ccccc} [0] & \xrightarrow{d^1} & [1] & \xrightarrow{(d^0)^{-1}} & [0] \\ & \searrow & \downarrow & \swarrow & \\ & & [n]. & & \end{array}$$

Again it is easy to check that G_n is well defined and natural in $[n]$, and so defines a morphism $\pi_{oid}\Delta^{\bullet} \xrightarrow{G} \pi_{oid} N(\Delta/[\bullet])$ in $c\mathcal{G}rpd$.

The composition $F \circ G$ is the identity. There is a natural transformation $G \circ F \xrightarrow{\alpha} id$ defined on objects $[m] \to [n] \in \pi_{oid} N(\Delta/[n])$ by the triangle

$$\begin{array}{ccc} [0] & \xrightarrow{e_m} & [m] \\ & \searrow \quad \swarrow & \\ & [n]. & \end{array}$$

∎

The groupoid $\mathrm{Tot}_2(X^\bullet)$ will also be called the *descent category* of $X^\bullet$. From now on, when we refer to the homotopy limit of a cosimplicial groupoid $X^\bullet$ we will mean the simpler model $\mathrm{Tot}_2(X^\bullet)$. The following corollary gives an explicit description of this groupoid.

Corollary 4.5. *The homotopy inverse limit of a cosimplicial groupoid $X^\bullet$ is the groupoid whose*

- *objects are pairs $(a, d^1(a) \xrightarrow{\alpha} d^0(a))$, with $a \in obX^0, \alpha \in morX^1$, such that $s^0(\alpha) = id_x$, and $d^0(\alpha) \circ d^2(\alpha) = d^1(\alpha)$,*
- *morphisms $(a, \alpha) \to (a', \alpha')$ are maps $a \xrightarrow{\beta} a'$, such that the following diagram commutes*

$$\begin{array}{ccc} d^1(a) & \xrightarrow{d^1(\beta)} & d^1(a') \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha'} \\ d^0(a) & \xrightarrow{d^0(\beta)} & d^0(a') \end{array}$$

Dually we have the following theorem giving a formula for homotopy colimits of simplicial diagrams.

Theorem 4.6. *Let $\mathcal{C}$ be a simplicial model category whose simplicial structure derives from a groupoid action and let $X_\bullet \in s\mathcal{C}$, be such that each X_i is cofibrant. Then the homotopy colimit of $X_\bullet$ is naturally homotopy equivalent to the coequalizer of the maps*

$$\coprod_{[n]\longrightarrow[m]}^{n\leq 1, m\leq 2} X_m \otimes \Delta^n \rightrightarrows \coprod_{n=0}^{2} X_n \otimes \Delta^n.$$

5. Categories Fibered in Groupoids

There are different categories in which the descent condition can be formulated, and in which stacks can be defined. In this section we will discuss the category of *categories fibered in groupoids* over $\mathcal{C}$, [DM, Gi]. This category is denoted $\mathcal{G}rpd/\mathcal{C}$.

After discussing some important properties of $\mathcal{G}rpd/\mathcal{C}$, we will be able to define an adjoint pair of functors

$$\mathcal{G}rpd/\mathcal{C} \underset{p}{\overset{\Gamma}{\rightleftarrows}} P(\mathcal{C}, \mathcal{G}rpd)$$

satisfying the following properties:

- For F in $P(\mathcal{C}, \mathcal{G}rpd)$, the map $F(X) \to \Gamma p F(X)$ is an equivalence of groupoids, for all $X \in \mathcal{C}$.
- For $\mathcal{E} \in \mathcal{G}rpd/\mathcal{C}$, the map $p\Gamma\mathcal{E} \to \mathcal{E}$ is an equivalence of categories over $\mathcal{C}$.

When $\mathcal{C}$ has a Grothendieck topology, we will define stacks in both categories so that the pair (p, Γ) restricts to an adjoint pair between the subcategories of stacks. In section 7, we will define model structures on these categories such that the adjunction above induces a Quillen equivalence.

5.1. **Categories Fibered in Groupoids over $\mathcal{C}$.** One should think of a category fibered in groupoids over $\mathcal{C}$ as the analogue in $\mathcal{C}at$ of a fibration over $\mathcal{C}$ with fibers which are groupoids. Recall that if $X \xrightarrow{f} Y$ is a fibration of topological spaces, given a path I in Y, and $x \in X$ such that $f(x) = I(1)$, we can lift I to a path I' in X, with $I'(1) = x$. One can use these liftings to define a map from the fiber over $I(1)$ to the fiber over $I(0)$. This map is only determined up to homotopy but a homotopy between two liftings is again determined up to homotopy and so on. Similarly, a category fibered in groupoids over $\mathcal{C}$, $\mathcal{E} \xrightarrow{F} \mathcal{C}$, satisfies a path lifting condition, where the lift is unique only up to isomorphism. However, since in groupoids there are no nontrivial homotopies between homotopies, this isomorphism is unique. More precisely, a morphism $X \to Y \in \mathcal{C}$, determines a pullback functor from the fiber over Y to the fiber over X, which is unique up to a unique natural isomorphism.

Here is a standard example to motivate the definition.

Example 5.1 (Vector Bundles on $\mathcal{T}op$)**.** Let $Vec(\mathcal{T}op)$ be the category whose objects are vector bundles $E_Y \twoheadrightarrow Y$, and whose morphisms are pullback squares

$$\begin{array}{ccc} E_Y & \longrightarrow & E_X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

The projection functor $Vec(\mathcal{T}op) \to \mathcal{T}op$ is an example of a category fibered in groupoids over $\mathcal{T}op$. Here are some ways in which it resembles a fibration:

- The fact that we can pull back vector bundles tells us that there is 'path lifting'

$$\begin{array}{ccc} * & \xrightarrow{(E \twoheadrightarrow Z)} & Vec(\mathcal{T}op) \\ {\scriptstyle 1}\downarrow & \nearrow & \downarrow \\ I & \xrightarrow{Y \xrightarrow{f} Z} & \mathcal{T}op. \end{array}$$

  A lifting in this diagram is a choice of a bundle $E' \twoheadrightarrow Y$ and an isomorphism $E' \xrightarrow{\sim} f^*E$. Two different choices will necessarily be canonically isomorphic.
- All the fibers of this functor are groupoids.

Now we give the definition of a category fibered in groupoids, which formalizes the 'path lifting' condition described above.

Definition 5.2. [DM] *The category $\mathcal{G}rpd/\mathcal{C}$ is the full subcategory of $\mathcal{C}at/\mathcal{C}$ whose objects are functors $\mathcal{E} \xrightarrow{F} \mathcal{C}$ satisfying the following properties:*

(1) Given $Y \xrightarrow{f} X \in \mathcal{C}$, and $X' \in \mathcal{E}$ such that $F(X') = X$, there exists $Y' \xrightarrow{f'} X' \in \mathcal{E}$ such that $F(f') = f$.

(2) Given a diagram in $\mathcal{E}$, over the commutative diagram in $\mathcal{C}$,

$$\begin{array}{ccccccc} & & Y' & \xRightarrow{F} & & & Y \\ & & \downarrow{\scriptstyle f'} & & & \swarrow{\scriptstyle h} & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{g'} & X' & \xRightarrow{F} & Z & \xrightarrow{g} & X, \end{array}$$

with $F(f') = f, F(g') = g$, there exists a unique h' such that $g' \circ h' = f'$ and $F(h') = h$.

This definition may seem involved but it becomes very simple when we look at the functors $F_{X'}$ induced by F on the over categories

$$\mathcal{E}/X' \xrightarrow{F_{X'}} \mathcal{C}/X,$$

where $X' \in \mathcal{E}$, and $F(X') = X$. The conditions for $\mathcal{E} \xrightarrow{F} \mathcal{C}$ to be a category fibered in groupoids over $\mathcal{C}$ are equivalent to the following simple requirements of the functors $F_{X'}$:

(1) $F_{X'}$ is surjective.

(2) For every pair of objects $Y', Z' \in \mathcal{E}/X'$ with $F_{X'}(Y') = Y, F_{X'}(Z') = Z$ the induced map

$$\mathrm{Hom}_{\mathcal{E}/X'}(Y', Z') \to \mathrm{Hom}_{\mathcal{C}/X}(Y, Z)$$

is a bijection.

Together these conditions are equivalent to saying that the functors $F_{X'}$ are surjective equivalences of categories.

Let $\mathcal{E}_X$ denote the fiber category over X in $\mathcal{E}$. This has objects those of $\mathcal{E}$ lying over X and morphisms those of $\mathcal{E}$ lying over id_X. It is easy to see that if $\mathcal{E} \to \mathcal{C} \in \mathcal{G}rpd/\mathcal{C}$, the fiber categories $\mathcal{E}_X$ are groupoids.

Example 5.3. The simplest examples of categories fibered in groupoids over $\mathcal{C}$ are the projection functors $\mathcal{C}/X \to \mathcal{C}$ for each $X \in \mathcal{C}$. If $Y \xrightarrow{f} X$ is an object of $\mathcal{C}/X$, then $(\mathcal{C}/X)/f \cong \mathcal{C}/Y$, and so conditions 1. and 2. above are trivially satisfied. Notice that $(\mathcal{C}/X)_Y$ is the discrete groupoid whose set of objects is $\mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Another class of simple examples are $\mathcal{C} \times G \xrightarrow{pr} \mathcal{C}$, for $G \in \mathcal{G}rpd$. Here the fibers over each $X \in \mathcal{C}$ are canonically isomorphic to G.

Categories fibered in groupoids are enriched over $\mathcal{G}rpd$ in a natural way.

Lemma 5.4. *$\mathcal{G}rpd/\mathcal{C}$ is enriched with tensor and cotensor over $\mathcal{G}rpd$. The objects of $\mathcal{G}rpd(\mathcal{E}, \mathcal{E}')$ are the functors $\mathcal{E} \to \mathcal{E}'$ over $\mathcal{C}$, and the morphisms are the natural isomorphisms between such functors covering the identity natural automorphism of $id_{\mathcal{C}}$. Moreover, the tensor is given by the formula*

$$\mathcal{E} \otimes G := \mathcal{E} \times_{\mathcal{C}} (\mathcal{C} \times G),$$

and the cotensor $\mathcal{E}^G$ is the category of functors from $(G \to *)$ to $(\mathcal{E} \to \mathcal{C})$.

Proposition 5.5. *Let $\mathcal{E} \xrightarrow{F} \mathcal{C} \in \mathcal{G}rpd/\mathcal{C}$, $X' \in \mathcal{E}$, and let $X = F(X')$. Then*

(1) there is a section

$$\begin{array}{ccc} & & \mathcal{E} \\ & \overset{G}{\nearrow} & \downarrow F \\ \mathcal{C}/X & \longrightarrow & \mathcal{C} \end{array}$$

such that $G(id_X) = X'$.

(2) If $G, G' : \mathcal{C}/X \to \mathcal{E}$ are two such sections and $G(id_X) \xrightarrow{f} G'(id_X)$ a morphism in $\mathcal{E}_X$, then there is a unique natural isomorphism $G \xrightarrow{\phi} G'$ over $id_{\mathcal{C}}$, with $\phi(id_X) = f$.

Proof. First notice that giving a section $\mathcal{C}/X \xrightarrow{G} \mathcal{E}$ over $\mathcal{C}$ with $G(id_X) = X'$ is the same as giving a section

$$\begin{array}{ccc} \mathcal{E}/X' & \longrightarrow & \mathcal{E} \\ \uparrow\downarrow & & \downarrow F \\ \mathcal{C}/X & \longrightarrow & \mathcal{C}. \end{array}$$

sending id_X to $id_{X'}$.

1) Define G on objects $Y \in \mathcal{C}/X$, to be an arbitrary choice of $Y' \in \mathcal{E}/X'$ with $F_{X'}(Y') = Y$, (this is possible since $\mathcal{E}/X' \to \mathcal{C}/X$ is a surjection). For a pair of objects $Y, Z \in \mathcal{C}/X$, define

$$\mathrm{Hom}_{\mathcal{C}/X}(Y, Z) \xrightarrow{G} \mathrm{Hom}_{\mathcal{E}/X'}(Y', Z')$$

to be the inverse of the bijection

$$\mathrm{Hom}_{\mathcal{E}/X'}(Y', Z') \xrightarrow{F_{X'}} \mathrm{Hom}_{\mathcal{C}/X}(Y, Z).$$

To show this construction gives a functor, consider a pair of composable morphisms $f, g \in \mathcal{C}/X$. The morphisms $G(f) \circ G(g)$ and $G(f \circ g)$ have the same domain and range and the same image, $f \circ g$, in $\mathcal{C}/X$, therefore they must be equal.

2) Suppose G' is another such functor. Then for each object $(Y \to X) \in \mathcal{C}/X$ there is a unique isomorphism

$$\begin{array}{ccc} G(Y \to X) & \longrightarrow & G(id_X) \\ {\scriptstyle \exists!}\downarrow & & \downarrow{\scriptstyle f} \\ G'(Y \to X) & \longrightarrow & G'(id_X). \end{array}$$

lying over the identity of Y. By uniqueness, this collection of isomorphism forms a natural isomorphism $G \xrightarrow{\phi} G'$, and ϕ is the unique natural isomorphism $G \to G'$ over $id_{\mathcal{C}}$ which evaluated at id_X is f. □

Corollary 5.6. *For each $X \in \mathcal{C}$, the natural map*

$$\mathcal{G}rpd(\mathcal{C}/X, \mathcal{E}) \to \mathcal{E}_X$$

given by evaluation at id_X is a surjective equivalence of groupoids. There is a left inverse which is unique up to unique natural isomorphism.

This corollary says that given $\mathcal{E} \to \mathcal{C}$ there is a functorial "rigidification" of the fibers. Later we will use this method of rigidification to construct a functor from $\mathcal{G}rpd/\mathcal{C}$ to $P(\mathcal{C}, \mathcal{G}rpd)$.

In a similar fashion we can prove:

Proposition 5.7. *Let $\mathcal{E} \to \mathcal{C}$ be a category fibered in groupoids, and $Y \xrightarrow{f} X$ morphism in $\mathcal{C}$. There are "pullback" functors $\mathcal{E}_X \xrightarrow{f^*} \mathcal{E}_Y$ which are unique up to a unique natural isomorphism covering id_Y.*

Proof. To construct the functor on objects $X' \in \mathcal{E}_X$, we arbitrarily lift $Y \to X$ using condition 1 of Definition 5.2. Once the functor has been defined on objects, condition 2 of Definition 5.2 yields a map $Y' \to Y''$ for each morphism $X' \to X'' \in \mathcal{E}_X$. Finally, the uniqueness in condition 2 implies that this assignment is a functor and that any two assignments are naturally isomorphic over id_Y. □

Now we can give a definition of stack in $\mathcal{G}rpd/\mathcal{C}$.

Definition 5.8. *Let $\mathcal{C}$ be a category with a Grothendieck topology. A category fibered in groupoids $\mathcal{E} \xrightarrow{F} \mathcal{C}$ is a stack if for all covers $\{U_i \to X\}$ the map*

$$\mathcal{G}rpd(\mathcal{C}/X, \mathcal{E}) \to \operatorname{holim} \mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$$

is an equivalence of groupoids.

We will compare this definition with the usual definition [DM] in the next section.

5.2. **Adjoint Pair Between $\mathcal{G}rpd/\mathcal{C}$ and $P(\mathcal{C}, \mathcal{G}rpd)$.** Let $\mathcal{E} \to \mathcal{C}$ be a category fibered in groupoids. By Corollary 5.6, the assignment to each $X \in \mathcal{C}$ of the sections $\mathcal{G}rpd(\mathcal{C}/X, \mathcal{E})$ is a functor such that $\mathcal{G}rpd(\mathcal{C}/X, \mathcal{E}) \xrightarrow{\sim} \mathcal{E}_X$.

Definition 5.9. *Let $\Gamma : \mathcal{G}rpd/\mathcal{C} \to P(\mathcal{C}, \mathcal{G}rpd)$ be the functor which sends $\mathcal{E} \to \mathcal{C}$ to the presheaf $\Gamma\mathcal{E}(X) := \mathcal{G}rpd_{\mathcal{G}rpd/\mathcal{C}}(\mathcal{C}/X, \mathcal{E})$.*

Let $p : P(\mathcal{C}, \mathcal{G}rpd) \to \mathcal{G}rpd/\mathcal{C}$ be the functor defined by setting pF to be the category whose

- *objects are pairs (X, a) with $a \in F(X)$,*
- *morphisms $(X, a) \to (Y, b)$ are pairs (f, α) where $X \xrightarrow{f} Y \in \mathcal{C}$ and $a \xrightarrow{\alpha} F(f)b$ is an isomorphism in $\mathcal{F}(X)$.*

The composition of two morphisms $(X,a) \xrightarrow{(f,\alpha)} (Y,b) \xrightarrow{(g,\beta)} (Z,c)$ *is the pair* $(g \circ f, F(f)(\beta)\circ\alpha)$.

It is easy to check that both p and Γ preserve the groupoid action on their domain categories. Under p presheaves of groupoids sit inside $\mathcal{G}rpd/\mathcal{C}$ as the "trivializable bundles" (see example 5.1).

Theorem 5.10. *The functors*

$$P(\mathcal{C},\mathcal{G}rpd) \underset{\Gamma}{\overset{p}{\rightleftarrows}} \mathcal{G}rpd/\mathcal{C},$$

form an adjoint pair with p the left adjoint. The unit of the adjunction is an objectwise equivalence, and the counit is a fiberwise equivalence of groupoids.

Proof. We will define natural transformations $\eta : id \to \Gamma p$, and $\epsilon : p\Gamma \to id$. It will be clear from their definition that they satisfy the equations required to form the the unit and counit of an adjunction.

Define $\epsilon : p\Gamma\mathcal{E} \to \mathcal{E}$ on objects by sending $(X, \phi : \mathcal{C}/X \to \mathcal{E})$ to $\phi(id_X) \in \mathcal{E}$, and on morphisms by sending $(f : X \to X', \xi : \phi \to f^*(\phi'))$ to the composite $\phi'(f)\circ\xi(id_X)$. It follows from Corollary 5.6, that ϵ is a fiberwise equivalence.

Define $\eta : F \to \Gamma pF$ to be the map of presheaves which sends an object $a \in F(X)$ to the section $\phi_a : \mathcal{C}/X \to pF$ defined by $\phi_a(Y \xrightarrow{f} X) = (Y, F(f)a); \phi_a(Y \xrightarrow{g} Z) = (g, id)$, and a morphism $a \xrightarrow{\alpha} b \in F(X)$ to the natural transformation $\xi : \phi_a \to \phi_b$ defined by $\xi(Y \xrightarrow{f} X) = F(f)(\alpha)$. By construction $F(X)$ is the fiber over X in pF. Another application of Corollary 5.6 shows that $\mathcal{G}rpd(\mathcal{C}/X, pF) \xrightarrow{\sim} pF_X = F(X)$, and so η is an objectwise equivalence. □

The existence of this adjoint pair now motivates the following definition of stack in $P(\mathcal{C},\mathcal{G}rpd)$.

Definition 5.11. *A presheaf F of groupoids on* $\mathcal{C}$ *is a* **stack** *if for all covers* $\{U_i \to X\}$ *the map* $F(X) \to \operatorname{holim} F(U_\bullet)$ *is an equivalence of groupoids.*

With this definition, a category fibered in groupoids $\mathcal{E} \xrightarrow{F} \mathcal{C}$ is a stack if and only if $\Gamma\mathcal{E}$ is a stack in $P(\mathcal{C},\mathcal{G}rpd)$, so our adjoint pair restricts to one between the stacks in $\mathcal{G}rpd/\mathcal{C}$ and the stacks in $P(\mathcal{C},\mathcal{G}rpd)$.

6. Stacks

In this section we will discuss the usual definition of stacks in $\mathcal{G}rpd/\mathcal{C}$ [DM] used in algebraic geometry, and show that it is equivalent to the definition we have given using homotopy limits (Definition 5.8).

We start with an example that will hopefully provide intuition for the descent/homotopy sheaf condition.

Example 6.1 (Principal G-bundles on X). Consider the functor $\pi_0 BG$ which assigns to a space the set of isomorphism classes of principal G bundles over it. Locally all bundles are trivial, so gluing together isomorphism classes via the sheaf condition yields only the isomorphism class of the trivial bundles. The sheafification of $\pi_0 BG$ is just the constant assignment of the isomorphism class of the trivial bundle. In particular, $\pi_0 BG$ is not generally a sheaf.

Yet there is a sense in which isomorphism classes of principal G-bundles are *determined locally.* A cover, principal G-bundles on each member of the cover, and *coherent isomorphisms* between their restrictions to the intersections determine a G-bundle on the total space. More precisely, given an open cover $\{U_i \subset X\}$ and

- G-bundles $E_i \to U_i$,
- isomorphisms we call *gluing data* $\alpha_{ij} : E_i|_{U_i \cap U_j} \to E_j|_{U_i \cap U_j}$,
- satisfying $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ when restricted to $U_i \cap U_j \cap U_k$,

there is a principal G-bundle $E \to X$, and isomorphisms $f_i : E|_{U_i} \to E_i$, compatible with the gluing data:

$$\begin{array}{ccc} E_i|_{U_i\cap U_j} & \xleftarrow{f_i} & E|_{U_i \cap U_j} \\ & \searrow^{\alpha_{ij}} & \downarrow f_j \\ & & E_j|_{U_i \cap U_j}. \end{array}$$

Let $BG(X)$ denote the groupoid of principal G-bundles on X and isomorphisms between them, and $U_\bullet$ the nerve of the cover $\{U_i \subset X\}$. We can translate the above property as saying:

Given an object $a \in \prod BG(U_i)$, and an isomorphism $d^1 a \xrightarrow{\alpha} d^0 a$, which is coherent in the sense that $d^0(\alpha) \circ d^2(\alpha) = d^1(\alpha)$, then up to isomorphism a is in the image of $BG(X)$.

This is essentially what it means for $BG(X)$ to be the homotopy inverse limit of the cosimplicial diagram of groupoids $BG(U_\bullet)$.

Let $\mathcal{E} \to \mathcal{C}$ be a category fibered in groupoids, and assume that for each $X \xrightarrow{f} Y$ we have chosen pullback functors $\mathcal{E}_Y \xrightarrow{f^*} \mathcal{E}_X$. Given a morphism $U_i \to U \in \mathcal{C}$, we will sometimes abuse notation and denote the pullback of an element $a \in \mathcal{E}_U$ to $\mathcal{E}_{U_i}$ by $a|_{U_i}$. In defining some of the maps below, we will also make implicit use of the natural isomorphisms $(a|_{U_i})|_{U_{ij}} \cong a|_{U_{ij}}$.

Definition 6.2. [Gi] [DM] *A stack in $\mathcal{G}rpd/\mathcal{C}$ is an object $\mathcal{E} \to \mathcal{C}$ which satisfies the following properties for any cover $\{U_i \to X\}$:*

(1) given $a, b \in \mathcal{E}_X$, the following is equalizer sequence

$$\mathrm{Hom}_{\mathcal{E}_X}(a,b) \to \prod \mathrm{Hom}_{\mathcal{E}_{U_i}}(a|_{U_i}, b|_{U_i}) \rightrightarrows \prod \mathrm{Hom}_{\mathcal{E}_{U_{ij}}}(a|_{U_{ij}}, b|_{U_{ij}}),$$

(2) given $a_i \in \mathcal{E}_{U_i}$ and isomorphisms

$$a_i|_{U_{ij}} \xrightarrow{\alpha_{ij}} a_j|_{U_{ij}},$$

satisfying the cocycle condition

$$\alpha_{jk}|_{U_{ijk}} \circ \alpha_{ij}|_{U_{ijk}} = \alpha_{ik}|_{U_{ijk}},$$

then there exist $a \in \mathcal{E}_X$, and isomorphisms $a|_{U_i} \xrightarrow{\beta_i} a_i$, such that the following square commutes

$$\begin{array}{ccc} a|_{U_{ij}} & \xrightarrow{\beta_i|_{U_{ij}}} & a_i|_{U_{ij}} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \alpha_{ij}} \\ a|_{U_{ij}} & \xrightarrow{\beta_j|_{U_{ij}}} & a_j|_{U_{ij}}. \end{array} \tag{6.3}$$

In this case, we say that $\mathcal{E} \to \mathcal{C}$ satisfies descent*.*

Note 6.4. Note that pulling back the square 6.3 along the diagonal map $\Delta : U_i \to U_{ii}$ shows that the family of isomorphisms α_{ij} must satisfy the added condition $\Delta^*(\alpha_{ii}) = id_{U_i}$ and so we might as well have added this requirement to the cocycle condition.

This definition seems very complicated, but it can be considerably simplified if we recall the description of the homotopy inverse limit of a cosimplicial groupoid given in Corollary 4.5.

Proposition 6.5. *A category fibered in groupoids $\mathcal{E} \to \mathcal{C}$ is a stack in the sense of Definition 6.2, if and only if for all covers $\{U_i \to X\}$*

$$\mathcal{G}rpd(\mathcal{C}/X, \mathcal{E}) \to \operatorname{holim} \mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E}) \tag{6.6}$$

is an equivalence, i.e. if $\mathcal{E} \to \mathcal{C}$ is a stack in the sense of Definition 5.8.

Proof. We begin by showing that condition 1. in Definition 6.2 is equivalent to the requirement that for objects $F_a, F_b \in \mathcal{G}rpd(\mathcal{C}/X, \mathcal{E})$, the set of morphisms $F_a \to F_b$ is in bijective correspondence with the set of morphisms between their images in $\operatorname{holim} \mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$.

Consider objects $F_a, F_b \in \mathcal{G}rpd(\mathcal{C}/X, \mathcal{E})$, and let $a = F_a(id_X)$ and $b = F_b(id_X)$ in $\mathcal{E}_X$. Evaluation at $id_{(-)}$ induces bijections

$$\begin{array}{ccccc} \operatorname{Hom}(F_a, F_b) & \longrightarrow & \prod \operatorname{Hom}(F_a|_{U_i}, F_b|_{U_i}) & \Longrightarrow & \prod \operatorname{Hom}(F_a|_{U_{ij}}, F_b|_{U_{ij}}) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \operatorname{Hom}_{\mathcal{E}_X}(a,b) & \longrightarrow & \prod \operatorname{Hom}_{\mathcal{E}_{U_i}}(a|_{U_i}, b|_{U_i}) & \Longrightarrow & \prod \operatorname{Hom}_{\mathcal{E}_{U_{ij}}}(a|_{U_{ij}}, b|_{U_{ij}}). \end{array}$$

It follows that the top line is an equalizer if and only if the bottom one is. By corollary 4.5, the top line is an equalizer if and only if $\operatorname{Hom}(F_a, F_b)$ is in bijective correspondence with the set of maps from the image of F_a to the image of F_b in $\operatorname{holim} \mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$. The requirement that the bottom line be an equalizer is condition 1. in Definition 6.2.

To finish the proof we have to show that condition 2. is equivalent to the requirement that every object in $\operatorname{holim} \mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$ be isomorphic to one in the image of $\mathcal{G}rpd(\mathcal{C}/X, \mathcal{E})$. This follows from the description of morphisms in Corollary 4.5 once we show that specifying an object in holim $\mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$ is equivalent to specifying descent datum as in condition 2. of Definition 6.2.

By corollary 4.5, an object of holim $\mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$, consists of an object $F_c \in \prod \mathcal{G}rpd(\mathcal{C}/U_i, \mathcal{E})$, and an isomorphism $d^1 F_c \xrightarrow{\alpha} d^0 F_c$, satisfying $d^0(\alpha) \circ d^2(\alpha) = d^1(\alpha)$ and $s^0(\alpha) = id_{F_c}$. For any $U \xrightarrow{f} V$, and $F_a \in \mathcal{G}rpd(\mathcal{C}/V, \mathcal{E})$ with $F_a(id_V) = a$, the evaluation $F_a|_U(id_U)$ is a choice of pullback of a along f, and so $F_a|_U(id_U)$ is canonically isomorphic to the pullback f^*a, which we chose in advance. Evaluating at id_{U_i} determines $c \in \prod \mathcal{E}_{U_i}$, and isomorphisms $\alpha_{ij} = \alpha(id_{U_{ij}})$ satisfying the cocycle condition. Composing with the canonical isomorphisms $c|_{U_{ij}} \cong F_c|_{U_{ij}}(id_{U_{ij}})$, we obtain isomorphisms $c|_{U_i} \xrightarrow{\bar{\alpha}_{ij}} c|_{U_j}$, satisfying the cocycle condition.

Conversely, given $c \in \prod \mathcal{E}_{U_i}$ and α_{ij}, as in condition 2. satisfying $\Delta^*(\alpha_{ii}) = id_{U_i}$ (see Note 6.4), we can lift them to an object $F_c \in \prod \mathcal{G}rpd(\mathcal{C}/U_i, \mathcal{E})$, and an isomorphism $d^1 F_c \xrightarrow{\alpha} d^0 F_c$. Since these lifts are essentially unique they must also satisfy the cocycle condition and $s^0(\alpha) = id_{F_c}$ and hence determine an object of holim $\mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$. □

7. Model Structures

In this section we put model structures on $P(\mathcal{C}, \mathcal{G}rpd)$, $Sh(\mathcal{C}, \mathcal{G}rpd)$, and $\mathcal{G}rpd/\mathcal{C}$. In the first two subsections, we describe model structures on (pre)sheaves and categories fibered in groupoids. A morphism in $(sh)P(\mathcal{C}, \mathcal{G}rpd)$ will be a fibration or weak equivalence if it is one when evaluated at each object. In $\mathcal{G}rpd/\mathcal{C}$, the weak equivalences are the maps which induce an equivalence of groupoids on the fibers or, equivalently, maps which become weak equivalences in $P(\mathcal{C}, \mathcal{G}rpd)$ after applying Γ.

The above model category structure on $P(\mathcal{C}, \mathcal{G}rpd)$ is not very interesting because it does not see the topology on $\mathcal{C}$. In a Grothendieck topology there is a notion of locality. Just as sheaves are isomorphic if they are locally isomorphic, so too stacks should be equivalent if they are locally equivalent. Thus, there should be a model structure for which weak equivalences are those maps which locally are weak equivalences of groupoids. The most basic local equivalences are the maps $\operatorname{hocolim} U_\bullet \to X$, as stacks can be defined to be those presheaves which see this as an equivalence. This suggests that we should declare these to be new weak equivalences.

In the third subsection, we use Theorem 2.14 to localize the model structures on $P(\mathcal{C}, \mathcal{G}rpd)$, $Sh(\mathcal{C}, \mathcal{G}rpd)$, and $\mathcal{G}rpd/\mathcal{C}$, with respect to the set of maps

$$\operatorname{hocolim} U_\bullet \to X, \text{ where } \{U_i \to X\} \text{ a cover in } \mathcal{C},$$

We then observe that in these *local* model structures, the fibrant objects are the stacks.

In the next section we will prove that all these *local* model category structures on $P(\mathcal{C}, \mathcal{G}rpd)$, $Sh(\mathcal{C}, \mathcal{G}rpd)$, and $\mathcal{G}rpd/\mathcal{C}$ are Quillen equivalent. We will also prove that the weak equivalences in the local model structure on $P(\mathcal{C}, \mathcal{G}rpd)$ are the maps which, locally, are weak equivalences.

7.1. **Model Category Structure on (Pre)Sheaves of Groupoids.** In this subsection we construct a model category on both sheaves and presheaves of groupoids on a Grothendieck topology $\mathcal{C}$, using a set of "generators". More precisely, we will give a collection of objects X and define a map f to be a weak equivalence or a fibration if and only if the map of groupoids $\mathcal{G}rpd(X, f)$ is one for all X. This definition of weak equivalences and fibrations together with the smallness of the generators X implies that the sets of maps $\{X \otimes G \to X \otimes H\}$, where X is a generator and $G \to H$ is a generating (trivial) cofibration of groupoids, form sets of generating (trivial) cofibrations. In our case the "generators" X will be the representable functors.

Henceforth we will abuse notation and denote by X the sheaf $\operatorname{Hom}_{\mathcal{C}}(-, X)$ of discrete groupoids represented by the object $X \in \mathcal{C}$.

Theorem 7.1. *There are left proper, cofibrantly generated, model category structures on $P(\mathcal{C}, \mathcal{G}rpd)$, and $Sh(\mathcal{C}, \mathcal{G}rpd)$, where*

- *f is a weak equivalence or a fibration if $\mathcal{G}rpd(X, f)$ is one for all $X \in \mathcal{C}$,*
- *cofibrations are the maps with the left lifting property with respect to trivial fibrations.*

The maps of the form $X \to X \otimes \Delta^1$, for $X \in \mathcal{C}$, form a set of generating trivial cofibrations. The maps of the form $X \otimes \partial\Delta^i \to X \otimes \Delta^i$ for $X \in \mathcal{C}$ and $i = 0, 1, 2$ form a set of generating cofibrations.

Corollary 7.2. *The adjoint pair*

$$Sh(\mathcal{C}, \mathcal{G}rpd) \underset{sh}{\overset{i}{\rightleftarrows}} P(\mathcal{C}, \mathcal{G}rpd)$$

is a Quillen pair.

Proof. **Presheaves:** For MC1, note that limits and colimits are defined objectwise in $P(\mathcal{C}, \mathcal{G}rpd)$. MC2-MC4a are obvious. For $X \in \mathcal{C}$, the functor $\mathcal{G}rpd_{P(\mathcal{C},\mathcal{G}rpd)}(X, -)$ is evaluation at X, which commutes with all limits and colimits in $P(\mathcal{C}, \mathcal{G}rpd)$. It follows that X is small in $P(\mathcal{C}, \mathcal{G}rpd)$, hence the domains of the generating (trivial) cofibrations are small. This implies MC5a. Now note that cofibrations are, in particular, objectwise cofibrations. Since colimits are computed objectwise, it follows that pushouts and directed colimits of trivial cofibrations are again trivial cofibrations, which proves MC5b. Similarly, left properness follows from the left properness of $\mathcal{G}rpd$ and the fact that cofibrations are objectwise cofibrations. MC4b now follows by the same argument used in the proof of Theorem 3.1. SM7 follows immediately from SM7 for $\mathcal{G}rpd$.

Sheaves: MC1-MC4a, are obvious. The inclusion of sheaves in presheaves preserves filtered colimits so the domains of the generating (trivial) cofibrations are also small in sheaves, and MC5a follows. For MC5b, it suffices to show that the pushout in presheaves, of a sheaf along a generating trivial cofibration is still a sheaf. Consider the diagram

$$\begin{array}{ccc} X & \longrightarrow & F \\ \downarrow & & \downarrow \\ X \otimes \Delta^1 & \longrightarrow & (X \otimes \Delta^1) \coprod_X F, \end{array}$$

where F is a sheaf and $X \in \mathcal{C}$. The presheaf of groupoids $X \otimes \Delta^1 \coprod_X F$ has:

- object presheaf, the presheaf of objects in $F \coprod X$ and
- morphism presheaf, the presheaf of objects in $F^{\Delta^1} \coprod F^{\Delta^1} \times_F X \coprod X \times_F F^{\Delta^1} \coprod X \times_F F^{\Delta^1} \times_F X$.

The presheaves of objects and morphisms of $(X \otimes \Delta^1) \coprod_X F$ are sheaves, so $(X \otimes \Delta^1) \coprod_X F$ is a sheaf. MC4b follows by the same argument given in the proof of Theorem 3.1. SM7 follows immediately from SM7 for $\mathcal{G}rpd$.

Since $P(\mathcal{C}, \mathcal{G}rpd)$ is left proper, to show left properness for sheaves it suffices to show that the pushout in $P(\mathcal{C}, \mathcal{G}rpd)$ of a weak equivalence along a cofibration of sheaves is again a sheaf. Since we have already proven that the pushout of a sheaf along a trivial cofibration is a trivial cofibration whose range is a sheaf, we can assume that our weak equivalence is a trivial fibration.

We begin by noting that cofibrations of sheaves are, in particular, objectwise cofibrations, as sheafification preserves monomorphisms (and N and π_{oid} preserve cofibrations).

Trivial fibrations in $\mathcal{G}rpd$ are the surjective equivalences of categories, and so pushouts of trivial fibrations along objectwise cofibrations in $P(\mathcal{C}, \mathcal{G}rpd)$ are again trivial fibrations in $P(\mathcal{C}, \mathcal{G}rpd)$. Consider the diagram in $P(\mathcal{C}, \mathcal{G}rpd)$

$$\begin{array}{ccc} A & \xrightarrow{\sim} & F \\ \downarrow & & \downarrow \\ B & \xrightarrow{\sim} & B \coprod_A F. \end{array}$$

Let P denote the pushout $B \coprod_A F$. The argument given above to show that cofibrations are objectwise cofibrations shows also that the pushout in presheaves of a sheaf along a cofibration of sheaves is a sheaf on objects. Hence P is a sheaf on objects.

To see that the morphisms of P are a sheaf, recall that for each $X \in \mathcal{C}$, the map $B(X) \twoheadrightarrow P(X)$.is a surjective equivalence of categories.

Given a presheaf G, let $I(G)$ be the presheaf with $I(G)(X)$ the category with objects, the objects of $G(X)$ and a unique morphism between each pair of objects $G(X)$. There is a canonical map $G \to I(G)$ and if G is a sheaf on objects, then $I(G)$ is a sheaf. Since $B \to P$ is a trivial fibration, it is easy to check that $B \cong I(B) \times_{I(P)} P$. Using the following facts:

- the set of morphisms of a fiber product is the fiber product of the morphisms,
- the map $I(B) \to I(P)$ is a surjection on objects and morphisms,

it is not hard to check that P satisfies the sheaf condition. □

7.2. **Categories Fibered in Groupoids over $\mathcal{C}$.** In this subsection we construct a model category on $\mathcal{G}rpd/\mathcal{C}$ relative using the set of "generators" $\mathcal{C}/X \to \mathcal{C}$.

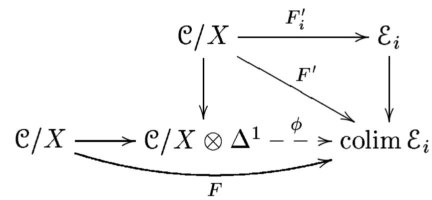
Theorem 7.3. *There is a left proper, cofibrantly generated, simplicial model category structure on $\mathcal{G}rpd/\mathcal{C}$ in which*

- *f is a weak equivalence or a fibration if $\mathcal{G}rpd_{\mathcal{G}rpd/\mathcal{C}}(\mathcal{C}/X, f)$ is one for all $X \in \mathcal{C}$,*
- *cofibrations are the maps with the left lifting property with respect to trivial fibrations.*

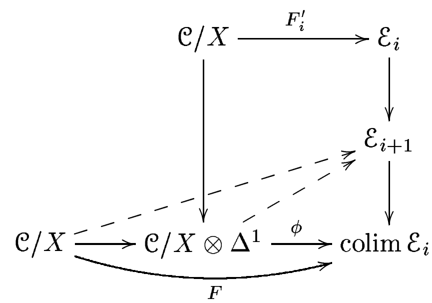
The maps of the form $\mathcal{C}/X \to (\mathcal{C}/X \otimes \Delta^1)$, for $X \in \mathcal{C}$, form a set of generating trivial cofibrations. The maps of the form $(\mathcal{C}/X \otimes \partial\Delta^i) \to (\mathcal{C}/X \otimes \Delta^i)$, for $X \in \mathcal{C}$ and $i = 0, 1, 2$ form a set of generating cofibrations.

Proof. For MC1, see Appendix A. MC2-MC4a are obvious. In order to apply the small object argument to prove MC5, we need to check that the objects $\mathcal{C}/X \otimes G \to \mathcal{C}$ with $G = (\partial)\Delta^i, i = 0, 1, 2$, are small with respect to the colimits which appear in the small object argument. First notice that sequential colimits in $\mathcal{G}rpd/\mathcal{C}$ agree with sequential colimits in $\mathrm{Cat}/\mathcal{C}$. For convenience, in the construction of the factorization for MC5a we will take pushouts along both the generating cofibrations and the generating trivial cofibrations.

Let $\mathcal{E}_i \to \mathcal{E}_{i+1}$ be constructed as usual, using the small object argument, and let consider a map $F : \mathcal{C}/X \longrightarrow \operatorname{colim} \mathcal{E}_i$. $F(id_X)$ lifts to some element X' in some $\mathcal{E}_i$, and we can extend this to a map $F'_i : \mathcal{C}/X \longrightarrow \mathcal{E}_i$. Let F' be the composition $\mathcal{C}/X \to \mathcal{E}_i \to \operatorname{colim} \mathcal{E}_i$. Then $F'(id_X) = F(id_X)$, and so there is a unique natural isomorphism $\phi : F \longrightarrow F'$ making the following diagram commute



The map $\mathcal{C}/X \to \mathcal{C}/X \otimes \Delta^1$ is one of the generating trivial cofibrations, so by construction we obtain a lift



Thus $\mathcal{C}/X$ is small with respect $\operatorname{colim} \mathcal{E}_i$. Since natural transformations between sections are determined uniquely by their evaluation on id_X, a similar argument shows that $\mathcal{C}/X \otimes (\partial)\Delta^i$ is small with respect to $\operatorname{colim} \mathcal{E}_i$. This completes the proof of MC5a.

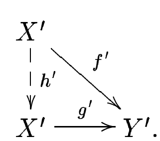
For MC5b, note that if $\mathcal{E} \to \mathcal{E}'$ has the left lifting property with respect to all fibrations, then in particular it has the left lifting property with respect to $\mathcal{E} \to \mathcal{C}$ and $(\mathcal{E}')^{\Delta^1} \to (\mathcal{E}')^{\partial\Delta^1}$, and therefore it is an equivalence of categories over $\mathcal{C}$. An equivalence of categories over $\mathcal{C}$ is clearly a weak equivalence. It follows that the cofibration constructed using the small object argument for MC5b is also a weak equivalence.

MC4b now follows by the same argument given in the proof of Theorem 3.1. SM7 follows immediately from the definition of (trivial) fibration in $\mathcal{G}rpd/\mathcal{C}$ and the adjunction formulas given by the simplicial structure.

To show left properness, it suffices to show that the pushout of a trivial fibration along a cofibration is a weak equivalence. We begin by noting that trivial fibrations are surjective equivalences of categories. Let $F : \mathcal{E}' \longrightarrow \mathcal{E}''$ be a trivial fibration and let $X', Y' \in \mathcal{E}, X'' = F(X'), Y'' = F(Y')$. Clearly F is surjective on objects and morphisms. We will show that the map

$$\operatorname{Hom}_{\mathcal{E}'}(X', Y') \to \operatorname{Hom}_{\mathcal{E}''}(X'', Y'')$$

is a bijection. If $F(f') = F(g')$ then f' and g' have the same image in $\mathcal{C}$ and so there is a unique isomorphism h' filling in the following triangle in $\mathcal{E}'$:



By the uniqueness of the lifting h', $F(h') = id_{X''} \in \mathcal{E}''$. Since F is a trivial fibration it follows that $h' = id_{X'}$.

Now note that cofibrations in $\mathcal{G}rpd/\mathcal{C}$ are inclusions on objects as this is the case for the generating cofibrations. Proposition 3.5 implies that the pushout in $\mathrm{Cat}/\mathcal{C}$ of a surjective equivalence of categories along an inclusion on objects is still an equivalence of categories over $\mathcal{C}$. This simultaneously implies that the pushout in $\mathrm{Cat}/\mathcal{C}$ coincides in this case with the pushout in $\mathcal{G}rpd/\mathcal{C}$ (see the proof of Proposition A.1) and completes the proof. □

Corollary 7.4. *The adjoint pair $p : P(\mathcal{C}, \mathcal{G}rpd) \leftrightarrow \mathcal{G}rpd/\mathcal{C} : \Gamma$ is a Quillen equivalence.*

Proof. This follows immediately from the definition of the model structures and Theorem 5.10. □

7.3. **Local Model Category Structures.** Recall that given $X \in \mathcal{C}$ we also denote by X the (pre)sheaf represented by X. For convenience, we will sometimes also denote by X the category fibered in groupoids $\mathcal{C}/X \to \mathcal{C}$. In any of the categories $P(\mathcal{C}, \mathcal{G}rpd), Sh(\mathcal{C}, \mathcal{G}rpd)$ or $\mathcal{G}rpd/\mathcal{C}$, we denote by S the set of maps

$$S = \{\operatorname{hocolim} U_\bullet \to X : \{U_i \to X\} \text{ is a cover in } \mathcal{C}\}$$

where $U_\bullet$ denotes (as usual) the nerve of the covering $\{U_i \to X\}$.

Proposition 7.5. *Let $\mathcal{M}$ be one of the categories $P(\mathcal{C}, \mathcal{G}rpd), Sh(\mathcal{C}, \mathcal{G}rpd)$ or $\mathcal{G}rpd/\mathcal{C}$. There is a model category structure on $\mathcal{M}$ which is the localization of the model structure of Theorems 7.1 or 7.3 with respect to the set of maps S.*

Proof. Since homotopy colimits of cofibrant objects are cofibrant, the domains and ranges of the morphisms in the localizing set are cofibrant. By Theorems 7.1 and 7.3, the model category structures on $P(\mathcal{C}, \mathcal{G}rpd)$, $Sh(\mathcal{C}, \mathcal{G}rpd)$ and $\mathcal{G}rpd/\mathcal{C}$ satisfy the hypothesis of Theorem 2.14, so the proposition follows. □

Let $\mathcal{M}$ be one of the categories $P(\mathcal{C}, \mathcal{G}rpd), Sh(\mathcal{C}, \mathcal{G}rpd)$ or $\mathcal{G}rpd/\mathcal{C}$. We will write $\mathcal{M}_L$ for the category $\mathcal{M}$ with the model structure given by the previous proposition.

Since in the old model structure on $\mathcal{M}$ every object is fibrant, and $X \in \mathcal{C}$ is cofibrant, an object $F \in \mathcal{M}_L$ is fibrant if and only if

$$\mathcal{G}rpd(X, F) \to \mathcal{G}rpd(\operatorname{hocolim} U_\bullet, F) = \operatorname{holim} \mathcal{G}rpd(U_\bullet, F)$$

is a weak equivalence for all covers. By definition of stack, this happens if and only if F is a stack. It follows that a fibrant replacement functor for $\mathcal{M}_L$ is a stackification functor.

Remark 7.6. Since stacks are the fibrant objects, and representables are cofibrant, it follows that when $\mathcal{M}$ is a stack, $h\operatorname{Hom}(X, \mathcal{M})$ is equivalent to the groupoid $\mathcal{M}(X)$. In particular, $[X, \mathcal{M}]$ is the set of isomorphism classes of $\mathcal{M}(X)$.

Remark 7.7. It is not hard to check that a small presentation (in the sense of [Dg, Definition 6.1]) of $P(\mathcal{C}, \mathcal{G}rpd)_L$ is given by the Yoneda embedding of $\mathcal{C}$ in $P(\mathcal{C}, \mathcal{G}rpd)$ and the set of maps

$$X \otimes \partial\Delta^n \to X \otimes \Delta^n, \text{ for all } X \in \mathcal{C}, n > 2$$

$$\operatorname{hocolim} U_\bullet \to X \text{ for all covers } \{U_i \to X\} \text{ in } \mathcal{C}.$$

This means that the local model category structure is the "quotient" of the universal model category generated by $\mathcal{C}$ by the relations given by the maps above.

8. Characterization of Local Equivalences

In this section we prove that a morphism f is a local weak equivalence if and only if it satisfies one of the following equivalent properties:

- f is an *isomorphism on sheaves of homotopy groups*,
- f satisfies the *local lifting conditions*,
- f is a *stalkwise weak equivalence* (when $\mathcal{C}$ has enough points).

Furthermore we prove that our local model structure $P(\mathcal{C}, \mathcal{G}rpd)_L$ is Quillen equivalent to the S^2 nullification of Jardine's model structure on presheaves of simplicial sets [Ja].

In subsection 8.1 we describe Jardine's model structure on presheaves of simplicial sets and show that it is the localization of the Heller model structure with respect to a set of maps S_{π_*}. There is an analogue of the Heller model structure for presheaves of groupoids which we denote by $P(\mathcal{C}, \mathcal{G}rpd)_H$. We prove that its localization with respect to $\pi_{oid}S_{\pi_*}$ has weak equivalences the isomorphisms on sheaves of homotopy groups, and is Quillen equivalent to the S^2 nullification of Jardine's model structure. The main theorem in this subsection is that the identity adjoint pair induces a Quillen equivalence

$$P(\mathcal{C}, \mathcal{G}rpd)_L \leftrightarrow (\pi_{oid}S_{\pi_*})^{-1}P(\mathcal{C}, \mathcal{G}rpd)_H. \tag{8.1}$$

It follows that $P(\mathcal{C}, \mathcal{G}rpd)_L$ is Quillen equivalent to the S^2 nullification of Jardine's model structure. We prove that (8.1) is a Quillen pair, and leave the proof that the weak equivalences are the same till 8.2.

In subsection 8.2 we introduce Dan Dugger's local lifting conditions, and prove that they are satisfied by a map $\phi \in P(\mathcal{C}, \mathcal{G}rpd)$ if and only if ϕ induces an isomorphism on sheaves of homotopy groups, and if and only if ϕ is a local weak equivalence. This completes the proof that (8.1) is a Quillen equivalence.

In subsection 8.3 we apply the characterization of local weak equivalences to show that the adjoint pairs

$$sh : P(\mathcal{C}, \mathcal{G}rpd) \leftrightarrow Sh(\mathcal{C}, \mathcal{G}rpd) : i$$

and

$$p : P(\mathcal{C}, \mathcal{G}rpd) \leftrightarrow \mathcal{G}rpd/\mathcal{C} : \Gamma$$

are Quillen equivalences between the local model structures on each of these categories.

8.1. Jardine's Model Structure.

In this subsection we compare the local model structure on presheaves of groupoids to Jardine's model structure on simplicial presheaves [Ja]. In order to define this model structure we will need the notion of sheaves of homotopy groups. Note that for a simplicial set X, and basepoint $a \in X_0$, $\pi_n(X, a)$ denotes the n-th homotopy group of the fibrant replacement of X with basepoint the image of a.

Definition 8.2. [Ja] *Let F be a presheaf of simplicial sets or groupoids. Then*

- *$\pi_0 F$ is the presheaf of sets defined by $(\pi_0 F)(X) := \pi_0(F(X))$.*
- *For $F \in P(\mathcal{C}, s\mathcal{S}et)$ and $a \in F(X)_0$, $\pi_n(F, a)$ is the presheaf of groups on $\mathcal{C}/X$ defined by*

$$\pi_n(F, a)(Y \xrightarrow{f} X) = \pi_n(F(Y), f^*a).$$

For $F \in P(\mathcal{C}, \mathcal{G}rpd)$ and $a \in \operatorname{ob} F(X)$, $\pi_n(F, a) := \pi_n(NF, a)$.

We say that a map $F \xrightarrow{\phi} G$ *of presheaves of simplicial sets or groupoids is an* isomorphism on sheaves of homotopy groups *if the induced maps* $sh\pi_0(\phi)$ *and* $sh\pi_n(\phi, a)$ *are isomorphisms for all* $a \in F(X)$*, and all* $X \in \mathcal{C}$.

Note that if F is a presheaf of groupoids then $\pi_i(F, a) = 0$ for $i > 1$, and $\pi_1(F, a)$ is the presheaf of groups $\mathrm{Aut}_F(a)$ on $\mathcal{C}/X$, where

$$\mathrm{Aut}_F(a)(Y \xrightarrow{f} X) := \mathrm{Aut}_{F(Y)}(f^* a).$$

Note also that if $F \to G$ is an objectwise weak equivalence, then the induced map of presheaves of homotopy groups is an isomorphism.

Reference 8.3 (Jardine's Model Structure [Ja])**.** There is a left proper, cofibrantly generated, simplicial model structure on $P(\mathcal{C}, s\mathcal{S}et)$ where

- cofibrations are the maps which are objectwise cofibrations,
- weak equivalences are the maps which are isomorphisms on sheaves of homotopy groups,
- fibrations are the maps with the right lifting property with respect to the trivial cofibrations.

The Jardine model category will be denoted by $P(\mathcal{C}, s\mathcal{S}et)_J$.

Proposition 8.4. *(a) There is a model structure on* $P(\mathcal{C}, \mathcal{G}rpd)$*, denoted* $(\pi_{oid} S_{\pi_*})^{-1} P(\mathcal{C}, \mathcal{G}rpd)_H$*, in which the cofibrations are objectwise and the weak equivalences are the isomorphisms on sheaves of homotopy groups.*

(b) The adjoint pair (π_{oid}, N) *induces a Quillen equivalence between* $(\pi_{oid} S_{\pi_*})^{-1} P(\mathcal{C}, \mathcal{G}rpd)_H$ *and the* S^2 *nullification of* $P(\mathcal{C}, s\mathcal{S}et)_J$.

To prove the proposition we will make use of the following model structure:

Reference 8.5 (Heller Model Structure [He, Sm])**.** There are left proper, cofibrantly generated, simplicial model structures on $P(\mathcal{C}, s\mathcal{S}et)$ and $P(\mathcal{C}, \mathcal{G}rpd)$ where

- cofibrations are the maps which are objectwise cofibrations,
- weak equivalences are the objectwise weak equivalences, and
- fibrations are the maps with the right lifting property with respect to the trivial cofibrations.

Proof. A proof for presheaves of simplicial sets is contained in [He], while the general case of a left proper combinatorial model category is contained in [Sm]. □

The categories of presheaves of simplicial sets and groupoids with the Heller model structure will be denoted $P(\mathcal{C}, s\mathcal{S}et)_H$ and $P(\mathcal{C}, \mathcal{G}rpd)_H$ respectively.

The following lemma will also be needed in the proof of Proposition 8.4.

Lemma 8.6. *(1) Let* S_{π_*} *be a set of generating trivial cofibrations in* $P(\mathcal{C}, s\mathcal{S}et)_J$*. Then the identity adjoint pair is an isomorphism*

$$(S_{\pi_*})^{-1} P(\mathcal{C}, s\mathcal{S}et)_H = P(\mathcal{C}, s\mathcal{S}et)_J.$$

(2) Consider the set of morphisms in $P(\mathcal{C}, s\mathcal{S}et)$:

$$\partial\Delta^n \otimes X \to \Delta^n \otimes X, \text{ for } n > 2, X \in \mathcal{C}$$

and let $(S^2)^{-1} P(\mathcal{C}, s\mathcal{S}et)_H$ *denote the localization of the Heller model structure with respect to these morphisms. The Quillen pair* (π_{oid}, N) *induces a Quillen equivalence:*

$$\pi_{oid} : (S^2)^{-1} P(\mathcal{C}, s\mathcal{S}et)_H \leftrightarrow P(\mathcal{C}, \mathcal{G}rpd)_H : N.$$

Proof of Proposition 8.4. Applying Theorem 2.15(c) and (d) we see that after localizing the above Quillen equivalences we still have Quillen equivalences

$$(S^2)^{-1}(S_{\pi_*})^{-1}P(\mathcal{C}, s\mathcal{S}et)_H = (S^2)^{-1}P(\mathcal{C}, s\mathcal{S}et)_J$$
$$(S^2)^{-1}(S_{\pi_*})^{-1}P(\mathcal{C}, s\mathcal{S}et)_H \leftrightarrow (\pi_{oid}S_{\pi_*})^{-1}P(\mathcal{C}, \mathcal{G}rpd)_H$$

It follows that $(\pi_{oid}S_{\pi_*})^{-1}P(\mathcal{C}, \mathcal{G}rpd)_H$ is Quillen equivalent to $(S^2)^{-1}P(\mathcal{C}, s\mathcal{S}et)_J$.

Now we will show that the weak equivalences in $(S^2)^{-1}P(\mathcal{C}, s\mathcal{S}et)_J$ are the isomorphisms on sheaves of homotopy groups in dimensions 0 and 1. As $F \to N\pi_{oid}F$ is a weak equivalence (because it is one in $(S^2)^{-1}P(\mathcal{C}, s\mathcal{S}et)_H$), morphisms which are isomorphisms on sheaves of homotopy groups in dimensions 0 and 1 are weak equivalences.

We claim that the fibrant replacement functor in $(S^2)^{-1}(S_{\pi_*})^{-1}P(\mathcal{C}, s\mathcal{S}et)_H$ can be constructed as a transfinite composition of fibrant replacement functors of $(S^2)^{-1}P(\mathcal{C}, s\mathcal{S}et)_H$ and $(S_{\pi_*})^{-1}P(\mathcal{C}, s\mathcal{S}et)_H$ [Dg2]. The desired number of compositions is a cardinal c such that all the generating trivial cofibrations in $(S^2)^{-1}P(\mathcal{C}, s\mathcal{S}et)_H$ and $(S_{\pi_*})^{-1}P(\mathcal{C}, s\mathcal{S}et)_H$ are small with respect to c. As fibrant replacement in $(S^2)^{-1}P(\mathcal{C}, s\mathcal{S}et)_H$ and in $(S_{\pi_*})^{-1}P(\mathcal{C}, s\mathcal{S}et)_H$ are isomorphisms on sheaves of homotopy groups in dimensions $0, 1$, the same is true for fibrant replacement in $(S^2)^{-1}(S_{\pi_*})^{-1}P(\mathcal{C}, s\mathcal{S}et)_H$. Now let $A \xrightarrow{f} B$ be a weak equivalence and let P denote a fibrant replacement functor in $(S^2)^{-1}(S_{\pi_*})^{-1}P(\mathcal{C}, s\mathcal{S}et)_H$. As Pf, $A \to PA$, and $B \to PB$ are isomorphisms on sheaves of homotopy groups in dimensions $0, 1$, so is f.

We now show that weak equivalences in $(\pi_{oid}S_{\pi_*})^{-1}P(\mathcal{C}, \mathcal{G}rpd)_H$ are the isomorphisms on sheaves of homotopy groups. As π_{oid} preserves weak equivalences between cofibrant objects it preserves all weak equivalences. It follows that all isomorphisms on sheaves of homotopy groups are weak equivalences. Since π_{oid} induces a surjective equivalence of categories

$$Ho((S^2)^{-1}(S_{\pi_*})^{-1}P(\mathcal{C}, s\mathcal{S}et)_H) \to Ho((\pi_{oid}S_{\pi_*})^{-1}P(\mathcal{C}, \mathcal{G}rpd)_H),$$

all the weak equivalences in $(\pi_{oid}S_{\pi_*})^{-1}P(\mathcal{C}, \mathcal{G}rpd)_H$ are the image under π_{oid} of weak equivalences in $(S^2)^{-1}(S_{\pi_*})^{-1}P(\mathcal{C}, s\mathcal{S}et)_H$ and therefore are isomorphisms on sheaves of homotopy groups.

□

Theorem 8.7. *The identity adjoint pair induces a Quillen pair*

$$P(\mathcal{C}, \mathcal{G}rpd)_L \leftrightarrow (\pi_{oid}S_{\pi_*})^{-1}P(\mathcal{C}, \mathcal{G}rpd)_H.$$

Proof. The cofibrations in the model structure on $P(\mathcal{C}, \mathcal{G}rpd)$ of Theorem 7.1 are in particular objectwise cofibrations, and the weak equivalences agree with those in $P(\mathcal{C}, \mathcal{G}rpd)_H$. So there is an induced Quillen pair

$$P(\mathcal{C}, \mathcal{G}rpd) \leftrightarrow P(\mathcal{C}, \mathcal{G}rpd)_H \leftrightarrow (\pi_{oid}S_{\pi_*})^{-1}P(\mathcal{C}, \mathcal{G}rpd)_H.$$

To complete the proof, by Theorem 2.15, it suffices to show that the maps $\operatorname{hocolim} U_\bullet \to X$ are isomorphisms on sheaves of homotopy groups. Note that in the model structure of Theorem 7.1 the homotopy colimit of the simplicial objects $U_\bullet$ agrees with the geometric realization $|U_\bullet|$, as the homotopy colimit of objectwise cofibrant diagrams can be constructed objectwise.

Let $Y \in \mathcal{C}$, and consider the map

$$\mathcal{G}rpd(Y, |U_\bullet|) = |\mathcal{G}rpd(Y, U_\bullet)| \to \mathcal{G}rpd(Y, X),$$

where the equality above holds because both the simplicial action and colimits are defined objectwise and Y is a discrete presheaf of groupoids. Using the fact that the Yoneda embedding preserves limits we see that $\mathcal{G}rpd(Y, U_\bullet)$ is the nerve of the map $\mathcal{G}rpd(Y, U) \to \mathcal{G}rpd(Y, X)$, that is, the simplicial groupoid:

$$\cdots \Rrightarrow \mathcal{G}rpd(Y, U) \times_{\mathcal{G}rpd(Y,X)} \mathcal{G}rpd(Y, U) \Rrightarrow \mathcal{G}rpd(Y, U).$$

As $\mathcal{G}rpd(Y, U)$ and $\mathcal{G}rpd(Y, X)$ are discrete groupoids, it follows that the simplicial set $\mathcal{G}rpd(Y, U_\bullet)$ has contractible components indexed by the image of $\mathcal{G}rpd(Y, U)$ in $\mathcal{G}rpd(Y, X)$. In other words $\mathcal{G}rpd(Y, |U_\bullet|)$ is homotopy equivalent to the discrete set of maps $Y \to X$ which factor through $U \to X$. It follows that $\pi_0|U_\bullet|$ is the presheaf of sets defined by the image of U in X, and the presheaves $\pi_1(|U_\bullet|, a)$ are trivial for all base points. Therefore the induced maps on π_1 are isomorphisms.

One checks easily that $|U_\bullet| \to X$ induces an isomorphism on $sh\pi_0$. □

Theorem 8.8. *The identity adjoint pair induces a Quillen equivalence*

$$P(\mathcal{C}, \mathcal{G}rpd)_L \leftrightarrow (\pi_{oid} S_{\pi_*})^{-1} P(\mathcal{C}, \mathcal{G}rpd)_H.$$

Furthermore, the weak equivalences in these two model structures agree.

Proof. To see that the left adjoint preserves weak equivalences, i.e. that the local weak equivalences are isomorphisms on sheaves of homotopy groups, factor a weak equivalence $f \in P(\mathcal{C}, \mathcal{G}rpd)_L$ as a cofibration i followed by a trivial fibration p. The cofibration i is a weak equivalence and so, by Theorem 8.7, its image is a trivial cofibration in $(\pi_{oid} S_{\pi_*})^{-1} P(\mathcal{C}, \mathcal{G}rpd)_H$. As p is an objectwise weak equivalence, it is also a weak equivalence in $(\pi_{oid} S_{\pi_*})^{-1} P(\mathcal{C}, \mathcal{G}rpd)_H$.

To complete the proof, it suffices to show that the weak equivalences in $(\pi_{oid} S_{\pi_*})^{-1} P(\mathcal{C}, \mathcal{G}rpd)_H$ are also weak equivalences in $P(\mathcal{C}, \mathcal{G}rpd)_L$. We use the characterization of weak equivalences in the next subsection to prove this in Theorem 8.13. □

Corollary 8.9. *If the Grothendieck topology on $\mathcal{C}$ has enough points, a morphism $f \in P(\mathcal{C}, \mathcal{G}rpd)$ is a local weak equivalence if and only if it is a stalkwise weak equivalence of groupoids.*

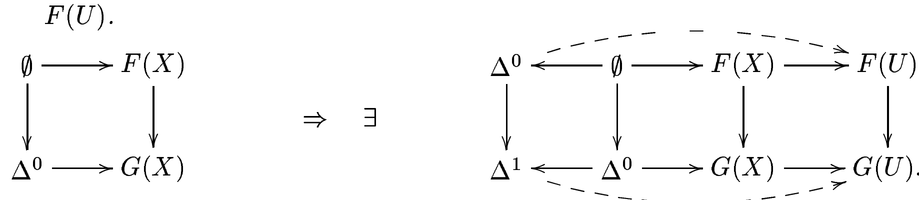
Proof. We have characterized the weak equivalences as those maps which induce isomorphisms on sheaves of homotopy groups, so the proof is exactly the same as the proof in [Ja] of the analogous result for $P(\mathcal{C}, s\mathcal{S}et)$. □

Corollary 8.10. *The local model structure on presheaves of groupoids $P(\mathcal{C}, \mathcal{G}rpd)_L$ is Quillen equivalent to the S^2-nullification of Jardine's model structure on presheaves of simplicial sets $(S^2)^{-1} P(\mathcal{C}, s\mathcal{S}et)_J$.*

8.2. **Characterization of Local Weak Equivalences.** In this subsection we give a characterization of the weak equivalences in $P(\mathcal{C}, \mathcal{G}rpd)_L$ in terms of Dan Dugger's local lifting conditions. This characterization allows us to complete the proof of Theorem 8.8, and prove in subsections 8.3 that the local model structures $P(\mathcal{C}, \mathcal{G}rpd)_L$, $Sh(\mathcal{C}, \mathcal{G}rpd)_L$ and $\mathcal{G}rpd/\mathcal{C}_L$ of section 7.3 are Quillen equivalent.

Definition 8.11. [Dg2] *A map $F \xrightarrow{\phi} G \in P(\mathcal{C}, \mathcal{G}rpd)$ is said to satisfy the* local lifting conditions *if:*

(1) *(Surjectivity on π_0). Given an isomorphism class in $G(X)$, not necessarily represented in $F(X)$, there is a cover $U \to X$ such that it is represented in $F(U)$.*



(2) *(Injectivity on π_0). If two isomorphism classes in $F(X)$ become identified in $G(X)$, there is a cover $U \to X$ such that they become identified in $F(U)$.*

(3) *(Surjectivity on π_1). If an element of the automorphism group of an object in $G(X)$ is not in the image of the automorphism group of an object lying over it in $F(X)$, then there is a cover U for which it is.*

(Recall that $B\mathbb{Z} \simeq S^1$.)

(4) *(Injectivity on π_1). If two elements in the automorphism group of some object in $F(X)$ become identified in $G(X)$, there is a cover U such that they become identified in $F(U)$.*

Theorem 8.12. [Dg2] *A map $F \xrightarrow{\phi} G \in P(\mathcal{C}, \mathcal{G}rpd)$ is an equivalence on sheaves of homotopy groups if and only of it satisfies the local lifting conditions.*

Proof. Recall that for F a presheaf, its sheafification shF, can be constructed by setting

$$shF(X) = \operatorname{colim}(\lim F(U) \rightrightarrows F(V))$$

where the colimit is taken over all covers $U \to X$ and $V \to U \times_X U$. It follows that if $a \in shF(X)$ then there exists a cover $U \to X$ such that a lifts to an element of $F(U)$. Similarly if $a, b \in F(X)$ have the same images in $shF(X)$ there exists a cover $U \to X$ so that they have the same image in $F(U)$. Conversely these two properties are enough to characterize the sheafification. It follows that conditions 1. and 2. are equivalent to $sh\pi_0\phi$ being an isomorphism, and conditions 3. and 4. are equivalent to $sh\,\mathrm{Aut}_\phi(a)$ being an isomorphism for all $a \in F(X), X \in \mathcal{C}$. $\square$

We use this theorem to prove the following result which completes the proof of Theorem 8.8.

Theorem 8.13. *A map $F \to G \in P(\mathcal{C}, \mathcal{G}rpd)$ satisfies the local lifting conditions if and only if it is a local equivalence.*

Proof. We may assume F and G are fibrant, as fibrant replacement is a local weak equivalence, and we have already seen that the local weak equivalences are isomorphisms on sheaves of homotopy groups. In this case, we need to show that $F \to G$ is an objectwise weak equivalence.

Consider a map $F \to G$ between stacks in $P(\mathcal{C}, \mathcal{G}rpd)$ which satisfies the local lifting conditions. First we show $F(X) \to G(X)$ is injective on automorphism groups. We are in the situation of 8.11(4), so we are guaranteed that there is a cover $U \to X$ and a lift in the diagram of 8.11(4). The descent condition for the cover $U \to X$ gives a commutative diagram

$$\begin{array}{ccccccc} B\mathbb{Z} & \longrightarrow & F(X) & & & & \\ \downarrow & & \downarrow & & & & \\ \Delta^0 & \longrightarrow & F(U) & \rightrightarrows & F(U \times_X U) & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & F(U \times_X U \times_X U). \end{array}$$

The image of $B\mathbb{Z}$ in each $F(U^i)$ is an identity morphism. Since $F(X) \xrightarrow{\sim} \operatorname{holim} F(U_\bullet)$, the image of $B\mathbb{Z}$ in $F(X)$ must be trivial also.

To show that $F(X) \to G(X)$ is surjective on automorphism groups, suppose we have a diagram as in 8.11(3). Consider again the descent condition for the cover $U \to X$, and the commutative diagram

$$\begin{array}{ccccccc} \Delta^0 & \longrightarrow & F(X) & & & & \\ \downarrow & & \downarrow & & & & \\ B\mathbb{Z} & \longrightarrow & F(U) & \rightrightarrows & F(U \times_X U) & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & F(U \times_X U \times_X U). \end{array}$$

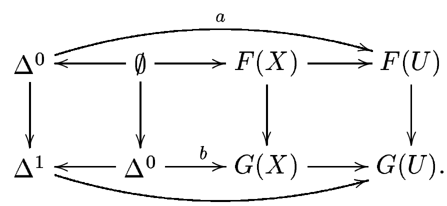
Let ϕ denote the image of $B\mathbb{Z}$ in $F(U)$. Then $d^0(\phi)$ and $d^1(\phi)$ are automorphisms of the same object in $F(U \times_X U)$, and they have the same image in $G(U \times_X U)$. Since $F \to G$ is an injection on automorphism groups, $d^0(\phi) = d^1(\phi)$, which gives us a lift of ϕ to $\operatorname{holim} F(U_\bullet)$. Since $F(X) \xrightarrow{\sim} \operatorname{holim} F(U_\bullet)$, there is a unique lift $B\mathbb{Z} \to F(X)$.

Next we show that $F(X) \to G(X)$ is an injection on connected components. Let $a, b \in F(X)$, be objects with isomorphic images in $G(X)$. By 8.11(2), we have a commutative diagram

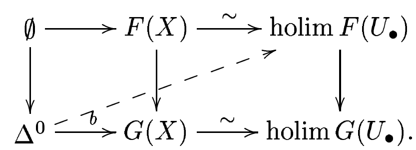
$$\begin{array}{ccccc} \partial\Delta^1 & \longrightarrow & F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow & \overset{\alpha}{\nearrow} & \downarrow \\ \Delta^1 & \longrightarrow & G(X) & \longrightarrow & G(U). \end{array}$$

We also have two maps $\Delta^1 \xrightarrow{d^i(\alpha)} F(U \times_X U)$, whose composition to $G(U \times_X U)$ is the same. Since $F \to G$ is injective on automorphism groups, it follows that $d^1(\alpha) = d_0(\alpha)$. This data gives a lifting of α to $\operatorname{holim} F(U_\bullet)$. Since $F(X) \xrightarrow{\sim} \operatorname{holim} F(U_\bullet)$, and the domain and range of α lie in $F(X)$, this in turn lifts uniquely to a morphism in $F(X)$.

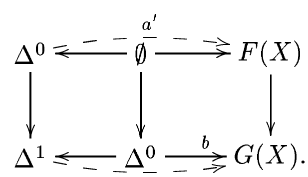
Lastly, we show that $F(X) \to G(X)$ is surjective on isomorphism classes. Consider the diagram from 8.11(1)



Let $a \in F(U)$ be the image of Δ^0, $b \in G(X)$, be the image of Δ^0 in $G(X)$, and $\beta : im(a) \longrightarrow im(b)$ be the image of Δ^1 in $G(U)$. Since $F \to G$ is an surjection on automorphism groups, we can lift $(d^1\beta)^{-1} \circ (d^0\beta) : im(d^0(a)) \to im(d^1(a))$, to some $\alpha : d^0(a) \longrightarrow d_1(a) \in F(U^2)$. Since $F \to G$ is an injection on automorphism groups, this lifting is unique. The image of $d^1(\alpha^{-1}) \circ d^0(\alpha) \circ d^2(\alpha)$ is trivial in $G(U^3)$, so it is also trivial in $F(U^3)$. Hence (a, α) is an element of $\operatorname{holim} F(U_\bullet)$, which determines a lifting in the diagram



Pick $a' \in F(X)$ whose image in $\operatorname{holim} F(U_\bullet)$ is isomorphic to (a, α). Then the image of a' in $G(X)$ is isomorphic to b, so we can fill in the following diagram



which completes the proof. □

Corollary 8.14. *Let $F \to G$ be an objectwise fibration, then the first of the local lifting conditions of 8.11 can be simplified to $1'$. (Surjectivity on π_0).*

The local lifting conditions $1', 2, 3, 4$ are preserved under pullbacks, so the pullback of an objectwise fibration which is a local weak equivalence is again an objectwise fibration which is a local weak equivalence.

8.3. Comparison of the Local Model Category Structures.

Proposition 8.15. *The adjoint pairs*

$$\mathcal{G}rpd/\mathcal{C} \underset{p}{\overset{\Gamma}{\rightleftarrows}} P(\mathcal{C}, \mathcal{G}rpd) \underset{sh}{\overset{i}{\leftrightarrows}} Sh(\mathcal{C}, \mathcal{G}rpd)$$

induce Quillen equivalences between the local model structures.

Proof. Let S denote the sets of morphisms

$$\operatorname{hocolim} U_\bullet \to X, \text{ for } \{U_i \to X\} \text{ a cover}$$

in $P(\mathcal{C}, \mathcal{G}rpd)$, $Sh(\mathcal{C}, \mathcal{G}rpd)$ and $\mathcal{G}rpd/\mathcal{C}$. Since homotopy colimits commute with the left adjoint in a Quillen pair, the set $S \in P(\mathcal{C}, \mathcal{G}rpd)$ is mapped by sh and p to the sets S in $Sh(\mathcal{C}, \mathcal{G}rpd)$ and $\mathcal{G}rpd/\mathcal{C}$ respectively. By Theorem 2.15, the adjoint pairs (sh, i), and (p, Γ) are still Quillen pairs between the local model category structures, and (p, Γ) is still a Quillen equivalence.

It remains to show that (sh, i) is a Quillen equivalence. By construction of the sheafification functor, the map $F \to shF$ satisfies the local lifting conditions, and so is a weak equivalence in $P(\mathcal{C}, \mathcal{G}rpd)_L$. Similarly, it is easy to check that if a map $f \in P(\mathcal{C}, \mathcal{G}rpd)$ satisfies the local lifting conditions then so does $sh(f)$.

We will now prove that sh preserves weak equivalences. Let $A \to B$ be a weak equivalence in $P(\mathcal{C}, \mathcal{G}rpd)_L$, and P denote a fibrant replacement functor on $P(\mathcal{C}, \mathcal{G}rpd)_L$. One can check directly that the sheafification of a stack F is a stack and so sheafification preserves fibrant replacement. We have the following commuting diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow\sim & & \big\downarrow\sim \\ PA & \longrightarrow & PB \end{array} \quad \overset{sh}{\Rightarrow} \quad \begin{array}{ccc} shA & \longrightarrow & shB \\ \big\downarrow\sim & & \big\downarrow\sim \\ sh(PA) & \longrightarrow & sh(PB). \end{array}$$

In $P(\mathcal{C}, \mathcal{G}rpd)_L$, the morphism $sh(PA) \to sh(PB)$ is a weak equivalence between fibrant objects (as $PA \xrightarrow{\sim} PB$ is a weak equivalence in $P(\mathcal{C}, \mathcal{G}rpd)_L$) and so is an objectwise weak equivalence. It follows that $sh(PA) \to sh(PB)$ is a weak equivalence in $Sh(\mathcal{C}, \mathcal{G}rpd)_L$, and therefore, so is $shA \to shB$.

Now we show that the forgetful functor i also preserves weak equivalences. Let f be any weak equivalence in $Sh(\mathcal{C}, \mathcal{G}rpd)_L$, and Pf its fibrant replacement in $P(\mathcal{C}, \mathcal{G}rpd)_L$. As $sh(Pf)$ is the fibrant replacement of f in sheaves it is also a weak equivalence, and so also an objectwise weak equivalence. It follows that $sh(Pf)$ is a weak equivalence in $P(\mathcal{C}, \mathcal{G}rpd)_L$, and therefore f is a weak equivalence also.

As both i and sh preserve weak equivalences, and the unit and counit are weak equivalences, the Quillen pair (sh, i) is a Quillen equivalence. □

Corollary 8.16. *A morphism $X \xrightarrow{f} Y \in Sh(\mathcal{C}, \mathcal{G}rpd)_L$ is a weak equivalence if and only if $i(f)$ is a weak equivalence in $P(\mathcal{C}, \mathcal{G}rpd)_L$. It follows that the weak equivalence in $Sh(\mathcal{C}, \mathcal{G}rpd)_L$ are the maps which satisfy the local lifting conditions. In particular, the weak equivalences in $Sh(\mathcal{C}, \mathcal{G}rpd)_L$ are the maps which are objectwise full and faithful, and satisfy 8.11(1).*

Proof. We show that if a morphism $X \xrightarrow{f} Y \in Sh(\mathcal{C}, \mathcal{G}rpd)_L$, is such that $i(f)$ is a weak equivalence in $P(\mathcal{C}, \mathcal{G}rpd)_L$ then f was already a weak equivalence in $Sh(\mathcal{C}, \mathcal{G}rpd)_L$. Let C denote a cofibrant replacement functor in $P(\mathcal{C}, \mathcal{G}rpd)_L$, and let F be a fibrant sheaf. Then the map $h\operatorname{Hom}(f, F) = s\mathcal{S}et(Cf, F)$ is a weak equivalence. As sheafification preserves fibrant replacement

$$s\mathcal{S}et(Cf, F) = s\mathcal{S}et(sh(Cf), F),$$

and so the map $s\mathcal{S}et(sh(Cf), F)$ is also a weak equivalence. As $sh(CX)$ and $sh(CY)$ are cofibrant as sheaves it follows that $sh(Cf)$ is a weak equivalence in

$Sh(\mathcal{C},\mathcal{G}rpd)_L$. We have the following commutative diagram in $Sh(\mathcal{C},\mathcal{G}rpd)$

$$\begin{array}{ccc} sh(CX) & \xrightarrow{sh(Cf)} & sh(CY) \\ \downarrow{\sim} & & \downarrow{\sim} \\ shX \cong X & \xrightarrow{f} & shY \cong Y. \end{array}$$

where the vertical arrows are weak equivalences because they are the sheafification of weak equivalences in $P(\mathcal{C},\mathcal{G}rpd)_L$. By a 2 out of 3 argument, it follows that $X \xrightarrow{f} Y$ is also a weak equivalence in $Sh(\mathcal{C},\mathcal{G}rpd)_L$.

To complete the proof, notice that for a morphism $X \xrightarrow{f} Y$ of sheaves, the local lifting conditions 2. - 4. are equivalent to f being objectwise full and faithful. $\square$

Appendix A. Limits and Colimits in $\mathcal{G}rpd/\mathcal{C}$

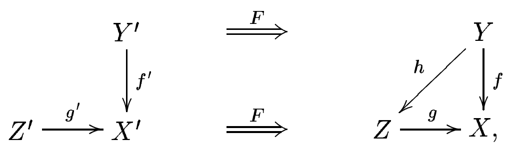
Theorem A.1. *Categories fibered in groupoids over* $\mathcal{C}$ *are closed under small limits and colimits.*

In order to prove this, we will need a few preliminaries.

Definition A.2. $F : \mathcal{E} \to \mathcal{C} \in \mathrm{Cat}/\mathcal{C}$ *is pre-fibered in groupoids if*

(1) Given $f : Y \to X \in \mathcal{C}$ *and* $X' \in \mathcal{E}$ *such that* $F(X') = X$*, there exists* $f' \in \mathcal{E}$ *such that* $F(f') = f$*.*

(2) Given a diagram in $\mathcal{E}$*, over the commutative diagram in* $\mathcal{C}$*,*

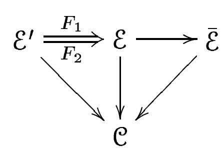


with $F(f') = f, F(g') = g$*, there exists* h' *such that* $g' \circ h' = f'$ *and* $F(h') = h$*. Moreover, given two such maps* h'_1, h'_2*, there exists an automorphism* $\phi \in Aut_{\mathcal{E}}(Y')$ *such that* $F(\phi) = id_Y$ *and* $h'_1 \circ \phi = h'_2$*.*

Thus, the difference between fibered and pre-fibered is that categories which are pre-fibered in groupoids only satisfy the uniqueness in condition 2) of Definition 5.2 in a weak form.

Proposition A.3. *Let* I *be a small category, and* $F : I \longrightarrow \mathcal{G}rpd/\mathcal{C}$*, a diagram. Then the colimit of* F *in* $\mathcal{C}at/\mathcal{C}$ *is pre-fibered in groupoids.*

Proof. The coproduct in $\mathcal{C}at/\mathcal{C}$ of a set of objects in $\mathcal{G}rpd/\mathcal{C}$ is again in $\mathcal{G}rpd/\mathcal{C}$ so it suffices to consider the case of a coequalizer diagram. Consider the diagram



where $F_1, F_2 \in \mathcal{G}rpd/\mathcal{C}$ and $\bar{\mathcal{E}}$ is the coequalizer of the two arrows in $\mathcal{C}at$. Recall that the coequalizer in $\mathcal{C}at$ has objects the coequalizer of the sets of objects, and morphisms the formal compositions of the coequalizer of the morphisms, modulo the relations given by composition in $\mathcal{E}$. Thus the map $\bar{\mathcal{E}} \longrightarrow \mathcal{C}$ clearly satisfies condition 1. of definition A.2.

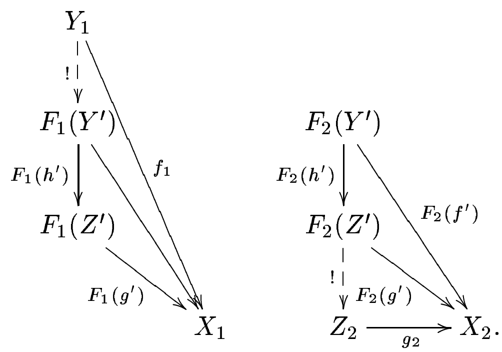
We now prove that it also satisfies condition 2. with an induction argument. Consider the diagrams

where the bared objects and morphisms represent objects and morphisms in $\bar{\mathcal{E}}$ projecting to the corresponding objects and morphisms in $\mathcal{C}$. Using the construction of $\bar{\mathcal{E}}$, we can factor $\bar{f}$ and $\bar{g}$ as formal compositions of maps in the image of $\mathcal{E}$ in $\bar{\mathcal{E}}$. Let $\bar{f} = (\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_n)$ and $\bar{g} = (\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_m)$, with $\mathrm{domain}(\bar{f}_i) = \mathrm{range}(\bar{f}_{i-1})$, $\mathrm{domain}(\bar{g}_i) = \mathrm{range}(\bar{g}_{i-1})$, and $\mathrm{range}(\bar{f}_n) = \mathrm{range}(\bar{g}_m) = \bar{X}$ in $\bar{\mathcal{E}}$.

Firstly, consider the case when $n = m = 0$. Let

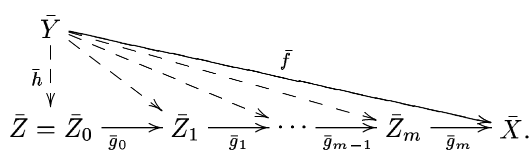
$$Y_1 \xrightarrow{f_1} X_1 \quad Z_2 \xrightarrow{g_2} X_2 \quad \in \mathcal{E}$$

be representatives of the maps $\bar{f}$ and $\bar{g}$ respectively. If there is $X' \in \mathcal{E}'$ such that $F_1(X') = X_1$ and $F_2(X') = X_2$, lift f, g to morphisms f', g' in $\mathcal{E}'$ whose range is X'. Since $\mathcal{E}' \in \mathcal{G}rpd/\mathcal{C}$, there is a unique $h' \in \mathcal{E}'$, projecting to $h \in \mathcal{C}$, such that $g' \circ h' = f'$. Since $\mathcal{E} \in \mathcal{G}rpd/\mathcal{C}$, there are unique isomorphisms in $\mathcal{E}$, projecting to identity morphisms in $\mathcal{C}$, filling in the diagrams



Then the map $\bar{h}$, defined as the formal composition $Y_1 \xrightarrow{\sim} F_1(Y') \sim F_2(Y') \to F_2(Z') \xrightarrow{\sim} Z_2$, is such that $\bar{g} \circ \bar{h} = \bar{f} \in \bar{\mathcal{E}}$. In general, there will not be an object X' such that $F_1(X') = X_1$ and $F_2(X') = X_2$, but a finite sequence of objects in $\mathcal{E}'$ such that their images under F_1 and F_2 form a chain connecting X_1 and X_2. The above argument is easily generalized to deal with this case. This completes the proof in the case when $n = m = 0$.

If $n = 0$ then we can use the previous case and induction on m to lift as indicated in the following diagram



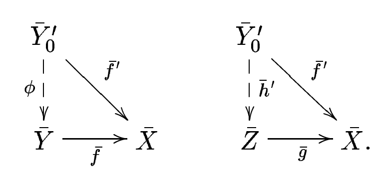
so the result is true in the case when $n = 0$ and m is arbitrary.

It is not hard to check that one can choose the lift $\bar{h}$ so that it is the image in $\bar{\mathcal{E}}$ of a formal composition of isomorphisms in $\mathcal{E}_Y$ followed by a lift of h to $\mathcal{E}$.

To complete the proof, notice that there is a lift of $f \in \mathcal{C}$ to a map

$$\bar{Y}'_0 \xrightarrow{\bar{f}'} \bar{X} \in \bar{\mathcal{E}}$$

which is in the image of $\mathcal{E}$. Then by the previous case, there is an isomorphism $\phi \in \bar{\mathcal{E}}$, projecting to $id_Y \in \mathcal{C}$, as well as a map $\bar{h}' \in \bar{\mathcal{E}}$ such that the following diagrams commute in $\bar{\mathcal{E}}$



We can now take $\bar{h} = \bar{h}' \circ \phi^{-1}$. Notice that if h is the identity, we can choose $\bar{h}$ to be an isomorphism. $\square$

Proposition A.4. *Let $\mathcal{E} \to \mathcal{C}$ be pre-fibered in groupoids. Let $\sim$ be the equivalence relation on $\mathcal{E}$ generated by setting $\alpha \sim id$ for the automorphisms $\alpha \in \mathcal{E}$ which satisfy:*
(1) α maps to an identity morphism in $\mathcal{C}$,
(2) there exists $f \in \mathcal{E}$ such that $f \circ \alpha = f$.
Then $\mathcal{E}/\sim \to \mathcal{C}$ is also pre-fibered in groupoids.

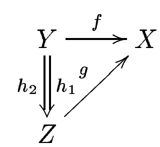
Proof. The map $\mathcal{E} \longrightarrow \mathcal{E}/\sim$ is surjective on morphisms and bijective on objects so this is obvious. $\square$

Proof of Theorem A.1. **Colimits:** Let I be a small category and $F : I \longrightarrow \mathcal{G}rpd/\mathcal{C}$ be a diagram. We denote by F' the composite $I \xrightarrow{F} \mathcal{G}rpd/\mathcal{C} \longrightarrow \mathcal{C}at/\mathcal{C}$. Let $\mathcal{E}_{\text{colim}}$ denote the colimit of F' in $\mathcal{C}at$. We will show that the colimit of F is the directed colimit of categories in $\mathcal{C}at/\mathcal{C}$,

$$\mathcal{E}_{\text{colim}} \to \mathcal{E}_{\text{colim}}/\sim \longrightarrow (\mathcal{E}_{\text{colim}}/\sim)/\sim \longrightarrow \cdots \tag{A.5}$$

Denote the i-th category in this diagram $\mathcal{E}^i_{\text{colim}}$ and the colimit $\mathcal{E} := \text{colim}_i(\mathcal{E}^i_{\text{colim}})$. Propositions A.3 and A.4 imply that Condition 1) and the existence part in Condition 2) of Definition 5.2 are still satisfied by $\mathcal{E}$.

To show the uniqueness part in Condition 2), suppose given a commutative diagram in $\mathcal{E}$:



such that h_1 and h_2 project to the same map in $\mathcal{C}$. Pick lifts h_1' and h_2' of h_1 and h_2 in some $\mathcal{E}^i_{\text{colim}}$. Then they also project to the same map in $\mathcal{C}$ so by Proposition A.3, there is an automorphism α of Y in $\mathcal{E}^i_{\text{colim}}$ mapping to an identity in $\mathcal{C}$ such that $h_2' \circ \alpha = h_1'$. It follows that $h_1' = h_2' \in \mathcal{E}^{i+1}_{\text{colim}}$ and so h_1 and h_2 agree in $\mathcal{E}$.

We still need to show that $\mathcal{E}$ is the colimit in $\mathcal{G}rpd/\mathcal{C}$, but this follows because any map $\mathcal{F} \to \mathcal{E}' \in \mathcal{C}at/\mathcal{C}$, with $\mathcal{E}' \in \mathcal{G}rpd/\mathcal{C}$ factors uniquely through $\mathcal{F}/\sim$.

Limits: Let $F : I \to \mathcal{G}rpd/\mathcal{C}$ be a diagram, and let $\lim F'$ denote its inverse limit in $\mathcal{C}at/\mathcal{C}$. If $\lim F' \in \mathcal{G}rpd/\mathcal{C}$ then it is the limit in $\mathcal{G}rpd/\mathcal{C}$ as this is a full subcategory of $\mathcal{C}at/\mathcal{C}$.

The objects and morphisms of $\lim F'$ are the inverse limits of the sets of objects and morphisms, so for each object $X' \in \lim F'$, the category $(\lim F')/X'$, is the limit of categories $F(i)/X_i'$, $i \in I$. It is easy to see that the map $(\lim F')/X' \to \mathcal{C}/X$

- is a bijection on Hom-sets, since this is the case for each of the constituent functors $F(i)/X_i' \to \mathcal{C}/X$,
- but it is not necessarily a surjection on objects even though each of the functors $F(i)/X_i' \to \mathcal{C}/X$ is.

It follows that if $\lim F'$ is not fibered in groupoids over $\mathcal{C}$, this is due to the failure of Condition 1) in Definition 5.2. However, in this case, the full subcategory of $\lim F'$ with objects all those X' such that $(\lim F')/X' \to \mathcal{C}/X$ is surjective on objects, clearly is fibered in groupoids and satisfies the universal property of the limit.

$\square$

Appendix B. Lax Presheaves of Groupoids

In this section we will define the category of lax presheaves of groupoids, denoted $lax - P(\mathcal{C}, \mathcal{G}rpd)$, and we will give an equivalence between this category and the category $\mathcal{G}rpd/\mathcal{C}$. When $\mathcal{C}$ has a Grothendieck topology, $lax - P(\mathcal{C}, \mathcal{G}rpd)$ is also used as an ambient category in which to define stacks and we observe that the two different definitions of stack agree under this equivalence.

Using this equivalence of categories, all the results proved in this paper for $\mathcal{G}rpd/\mathcal{C}$ can be transfered to $lax - P(\mathcal{C}, \mathcal{G}rpd)$.

One should think of a lax presheaf on $\mathcal{C}$ as a category fibered in groupoids *together with* a choice of pullback functors. The morphisms between lax presheaves are sufficiently flexible so that different choices of pullback functors for the same category fibered in groupoids correspond to canonically isomorphic lax presheaves.

The relation between the categories $\mathcal{G}rpd/\mathcal{C}$ and $lax - P(\mathcal{C}, \mathcal{G}rpd)$ is analogous to the relation between two different ways of defining principal G-bundles. One can define a bundle on X as a space over X which is locally trivial, or one can define the bundle to be the space over X *together with* a set of local trivializations. When the trivializations are part of the definition, one has to add morphisms to the category which give equivalences between the different choices of trivializations.

Definition B.1. [Bry, Brn] *The objects of $lax - P(\mathcal{C}, \mathcal{G}rpd)$ are the assignments:*

- *for each object $X \in \mathcal{C}$, a groupoid $\mathcal{F}(X)$,*
- *for each morphism $Y \xrightarrow{f} X \in \mathcal{C}$, a functor $\mathcal{F}(X) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(Y)$,*
- *for each pair of composable morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X \in \mathcal{C}$, a natural transformation $\mathcal{F}(g) \circ \mathcal{F}(f) \xrightarrow{\theta_{g,f}} \mathcal{F}(f \circ g)$,*

such that

- *for every triple of composable morphisms $W \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X \in \mathcal{C}$, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{F}(h) \circ \mathcal{F}(g) \circ \mathcal{F}(f) & \xrightarrow{\mathcal{F}(h)\circ\theta_{g,f}} & \mathcal{F}(h) \circ \mathcal{F}(f \circ g) \\ {\scriptstyle \theta_{h,g}\circ\mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle \theta_{h,f\circ g}} \\ \mathcal{F}(g \circ h) \circ \mathcal{F}(f) & \xrightarrow{\theta_{g\circ h,f}} & \mathcal{F}(f \circ g \circ h). \end{array}$$

A morphism $\phi : \mathcal{F} \longrightarrow \mathcal{F}' \in lax - P(\mathcal{C}, \mathcal{G}rpd)$ is an assignment:

- *for each object $X \in \mathcal{C}$, a map $\mathcal{F}(X) \xrightarrow{\phi(X)} \mathcal{F}'(X)$,*
- *for each morphism $Y \xrightarrow{f} X \in \mathcal{C}$, a natural isomorphism $\phi(Y) \circ \mathcal{F}(f) \xrightarrow{\phi(f)} \mathcal{F}'(f) \circ \phi(X)$,*

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\phi(X)} & \mathcal{F}'(X) \\ {\scriptstyle \mathcal{F}(f)}\downarrow & \overset{\phi(f)}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle \mathcal{F}'(f)} \\ \mathcal{F}(Y) & \xrightarrow[\phi(Y)]{} & \mathcal{F}'(Y), \end{array}$$

such that

- *for each pair of composable morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X \in \mathcal{C}$, the following diagram commutes*

$$\begin{array}{ccc} & \phi(Z)\circ\mathcal{F}(f\circ g) & \\ {\scriptstyle \phi(Z)\circ\theta_{g,f}}\swarrow & & \searrow{\scriptstyle \phi(f\circ g)} \\ \phi(Z)\circ\mathcal{F}(g)\circ\mathcal{F}(f) & & \mathcal{F}'(f\circ g)\circ\phi(X) \\ \downarrow{\scriptstyle \phi(g)\circ\mathcal{F}(f)} & & \uparrow{\scriptstyle \theta'_{g,f}\circ\phi(X)} \\ \mathcal{F}'(g)\circ\phi(Y)\circ\mathcal{F}(f) & \xrightarrow[\mathcal{F}'(g)\circ\phi(f)]{} & \mathcal{F}'(g)\circ\mathcal{F}'(f)\circ\phi(X). \end{array}$$

There is a natural groupoid action on $lax - P(\mathcal{C}, \mathcal{G}rpd)$, in which:

- the groupoid of maps has objects maps, and morphisms the coherent natural isomorphisms,
- the tensor and cotensor are defined objectwise.

There is an obvious inclusion $i : P(\mathcal{C}, \mathcal{G}rpd) \longrightarrow lax - P(\mathcal{C}, \mathcal{G}rpd)$ which preserves the groupoid action.

Example B.2 (Vector Bundles on $\mathcal{T}op$ Revisited)**.** Consider the assignment $\mathcal{T}op \to \mathcal{G}rpd$ which sends Y to the groupoid of vector bundles over Y, and a map f to the pullback function $f^\star$. This assignment is not a functor because given $Z \xrightarrow{g} Y \xrightarrow{f} X \in \mathcal{T}op$ and $E \longrightarrow X$ a vector bundle, the pullbacks g^*f^*E and $(f\circ g)^*E$ are not equal. There is, however, a canonical isomorphism $g^*f^*E \to (f\circ g)^*E$ so the assignment above together with the canonical isomorphisms is an example of a lax presheaf on $\mathcal{T}op$.

Instead of working with this lax presheaf, we can consider its associated category of pairs, or Grothendieck construction. This has objects the pairs $(Y, E \longrightarrow Y)$, where E is a vector bundle over $Y \in \mathcal{T}op$, and morphisms $(Y, E) \to (Z, E')$, the pairs formed by a map $f : Y \longrightarrow Z$, and an isomorphism $\alpha : E \longrightarrow f^*E'$. It is easy to check that this category is isomorphic to the category $Vec(\mathcal{T}op) \in \mathcal{G}rpd/\mathcal{C}$ of Example 5.1.

Just as in the bundle case, there is a "forgetful functor" $lax - P(\mathcal{C}, \mathcal{G}rpd) \longrightarrow \mathcal{G}rpd/\mathcal{C}$ which sends lax presheaves corresponding to different choices of pullback functors for $\mathcal{E} \to \mathcal{C}$, to objects in $\mathcal{G}rpd/\mathcal{C}$ which are canonically isomorphic to $\mathcal{E} \to \mathcal{C}$.

Definition B.3. *Given $\mathcal{F} \in lax - P(\mathcal{C}, \mathcal{G}rpd)$, let $p\mathcal{F} \in \mathrm{Cat}/\mathcal{C}$ be the category with*

- *objects, the pairs (X, a) with $X \in \mathcal{C}$ and $a \in \mathcal{F}(X)$,*
- *morphisms $(X, a) \to (Y, b)$, the pairs (f, α) where $f : X \longrightarrow Y$ is a morphism in $\mathcal{C}$ and $\alpha : a \longrightarrow \mathcal{F}(f)b$ is an isomorphism in $\mathcal{F}(X)$.*

The composition of two morphisms $(X, a) \xrightarrow{(f,\alpha)} (Y, b) \xrightarrow{(g,\beta)} (Z, c)$ is the pair $(g \circ f, \theta_{f,g} \circ \mathcal{F}(f)(\beta) \circ \alpha)$.

It is not hard to show that $p\mathcal{F}$ is a category fibered in groupoids over $\mathcal{C}$, and that p defines a functor $lax - P(\mathcal{C}, \mathcal{G}rpd) \to \mathcal{G}rpd/\mathcal{C}$.

Theorem B.4. *The functor $p : lax - P(\mathcal{C}, \mathcal{G}rpd) \leftrightarrow \mathcal{G}rpd/\mathcal{C}$ is an equivalence of categories.*

Proof. Let $\mathcal{E} \in \mathcal{G}rpd/\mathcal{C}$. A choice of a pullback functor $f^* : \mathcal{E}_X \to \mathcal{E}_Y$ for each $Y \xrightarrow{f} X \in \mathcal{C}$, determines a lax presheaf $\mathcal{F}$ with $\mathcal{F}(X) := \mathcal{E}_X$, and $\mathcal{F}(f) = f^*$. Given two such choices of lax presheaves $\mathcal{F}, \mathcal{F}'$, there is a canonical isomorphism

$\phi : \mathcal{F} \longrightarrow \mathcal{F}'$, where $\phi(X) = id_{\mathcal{E}_X}$ for each $X \in \mathcal{C}$, and $\phi(f)$ is the canonical natural isomorphism from $\mathcal{F}(f) \to \mathcal{F}'(f)$. For each $\mathcal{E} \in \mathcal{G}rpd/\mathcal{C}$ make an arbitrary choice of pullback functors, and let $L(\mathcal{E})$ denote the resulting lax presheaf.

For each $X \xrightarrow{f} Y \in \mathcal{C}$, a map $\mathcal{E} \xrightarrow{F} \mathcal{E}'$ determines squares

$$\begin{array}{ccc} \mathcal{E}_X & \xrightarrow{f^*} & \mathcal{E}_Y \\ {\scriptstyle F}\downarrow & \overset{!}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle F} \\ \mathcal{E}'_X & \xrightarrow[f^*]{} & \mathcal{E}'_Y . \end{array}$$

where the unique natural isomorphism follows from condition 2. of Definition 5.2. The uniqueness of the natural isomorphism in the square above guarantees that these squares patch together to give a morphism $L(F) : L(\mathcal{E}) \to L(\mathcal{E}') \in lax - P(\mathcal{C}, \mathcal{G}rpd)$ and that L is indeed a functor.

It is now easy to check that there are canonical natural isomorphisms $L \circ p \cong id_{lax-P}$ and $p \circ L \cong id_{\mathcal{G}rpd/\mathcal{C}}$. □

Note B.5. It is easy to check directly from the definition of stacks in lax presheaves [Brn, Pg.5] that $\mathcal{F} \in lax-P(\mathcal{C}, \mathcal{G}rpd)$ is a stack if and only if $p\mathcal{F}$ is a stack in $\mathcal{G}rpd/\mathcal{C}$. Thus, the equivalence of categories between $lax - P(\mathcal{C}, \mathcal{G}rpd)$ and $\mathcal{G}rpd/\mathcal{C}$ restricts to an equivalence between the subcategories of stacks.

REFERENCES

[An] D.W. Anderson, *Fibrations and Geometric Realizations*, Bull. Am. Math Soc. 84, 765-786, (1978), 765-786.

[Bo] A. K. Bousfield, *Homotopy Spectral Sequences and Obstructions*, Israel Journal of Math., Vol.66, Nos.1-3, (1989), 54-105.

[BK] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions, and Localizations*, Lecture Notes in Math. 304, Springer, (1972).

[Brn] L. Breen, *On the Classification of 2-Gerbes and 2-Stacks*, Asterisque 225, (1994).

[Bry] J.-L. Brylinski, *Loop Spaces, Characteristic Classes,and Geometric Quantization*, Progress in Mathematics 107, Birkhaüser, Basel, (1993).

[DM] P. Deligne and D. Mumford, *The Irreducibility of the Space of Curves of Given Genus*, Publ. Math. IHES 36, (1969), 75-110.

[Db] Eduardo J. Dubuc. *Kan Extensions in Enriched Category Theory*, Lecture Notes in Math. 145, Springer, (1970).

[DF] Emmanuel Dror-Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics 1635, Springer, (1995).

[Dg] Dan Dugger, *Universal homotopy theories*, preprint (2000), to appear in Advances in Mathematics.

[Dg2] Dan Dugger, *Simplicial Presheaves, Revisited*, preprint (2000).

[DK] W. G. Dwyer and D. M. Kan, *Homotopy Commutative Diagrams and their Realizations*, J. of Pure and Applied Algebra 57, (1989), 5-24.

[DS] W. G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of Algebraic Topology, Elsevier Science, (1995).

[EK] S. Eilenberg and G. M. Kelly, *Closed categories*, Proc. Conf. on Categorical Algebra (La Jolla 1965), Springer-Verlag.

[Gi] J. Giraud, *Cohomologie non-abelienne*, Springer Verlag, Berlin Heidelberg New York, (1971).

[GJ] P. Goerss and J. F. Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics 174, Birkhaüser, Basel, (1999).

[GZ] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer Verlag, New York, (1967).

[He] A. Heller, *Homotopy Theories*, Memoirs Amer. Math. Soc. Vol. 71, No. 383, (1988).

[Hi] P. S. Hirschhorn, *Localization of model categories*, 2000 Preprint.

[Ho] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs 63, American Mathematical Society, (1999).

[Holl] S. Hollander, work in progress.

[Ja] J.F. Jardine, *Simplicial Presheaves*, J. of Pure and Applied Algebra 47 (1987), 35-87.

[Ja2] J.F. Jardine, *Stacks and the Homotopy Theory of Simplicial Presheaves*, preprint.

[Ml] Saunders MacLane, *Categories for the Working Mathematician*, Grad. Texts in Math., Vol.5, Springer Verlag, New York, (1971).

[MM] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer Verlag, Berlin Heidelberg New York, (1992).

[May] J. Peter May, *Simplicial objects in Algebraic Topology*, Chicago Lectures in Mathematics, (1969).

[MV] F. Morel and V. Voevodsky, $\mathbb{A}^1$*-homotopy theory of schemes*, preprint.

[Q] Daniel J. Quillen *Homotopical Algebra*, Lecture Notes in Math. 43, Springer, (1967).

[Sm] Jeff Smith *Combinatorial Model Categories*, preprint.

[St] Neil Strickland $K(n)$*-local duality for finite groups and groupoids*, Topology 39, (2000).

E-mail address: `sharon@math.mit.edu+`

DEPARTMENT OF MATHEMATICS, MASSACHUSETTES INSTITUTE OF TECHNOLOGY