

# Beta-elements and divided congruences

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## ABSTRACT

The  $f$ -invariant is an injective homomorphism from the 2-line of the Adams-Novikov spectral sequence to a group which is closely related to divided congruences of elliptic modular forms. We compute the  $f$ -invariant for two infinite families of  $\beta$ -elements and explain the relation of the arithmetic of divided congruences with the Kervaire invariant one problem.

## 1. Introduction

One of the most successful tools for studying the stable homotopy groups of spheres is the Adams-Novikov spectral sequence (ANSS)

$$E_2^{s,t} = \text{Ext}_{\text{MU}_* \text{MU}}^{s,t}(\text{MU}_*, \text{MU}_*) \Rightarrow \pi_{t-s}^s(S^0).$$

The corresponding filtration on  $\pi_*^s := \pi_*^s(S^0)$  defines a succession of invariants of framed bordism, each being defined whenever all of its predecessors vanish, the first one of which is simply the degree

$$d : F^{0,*} / F^{1,*} = \pi_0^s \longrightarrow E_2^{0,0} = \mathbb{Z}$$

which is an isomorphism. The next invariant, defined for all  $n > 0$ , is the  $e$ -invariant

$$e : \pi_n^s = F^{1,n+1} \longrightarrow E_2^{1,n+1} \subseteq \mathbb{Q}/\mathbb{Z},$$

c.f. [Sw, Chapter 19]. Though defined purely homotopy-theoretic here, the  $e$ -invariant is well-known to encode subtle geometric information. For its relation to index theory via the  $\eta$ -invariant see [APS, Theorem 4.14]. The  $e$ -invariant vanishes for all even  $n = 2k \geq 2$ , thus giving rise to the  $f$ -invariant

$$f : \pi_{2k}^s = F^{2,2k+2} \longrightarrow E_2^{2,2k+2}.$$

The understanding of this invariant is fragmentary at the moment. In particular, there is no index-theoretic interpretation of it comparable to the one available for the  $e$ -invariant.

As a first step towards understanding the  $f$ -invariant, G. Laures [L1] showed how elliptic homology can be used to consider the  $f$ -invariant

$$f : \pi_{2k}^s \longrightarrow E_2^{2,2k+2} \hookrightarrow \underline{D}_{k+1} \otimes \mathbb{Q}/\mathbb{Z}$$

as taking values in a group which is closely related to divided congruences of modular forms. Note that this is similar to the role taken by complex  $K$ -theory in the study of the e-invariant. Strictly speaking, at this point we had better switched from MU to BP. In fact, we will always work locally at a fixed prime  $p$  in the following.

This surprising connection of stable homotopy theory with something as genuinely arithmetic as divided congruences certainly motivates to ask for a thorough understanding of how these are related by the f-invariant.

The main purpose of this paper is to make this relation explicit.

We also include a fairly self-contained review of G. Laures' above version of the f-invariant to help the reader who might be interested in making his own computations. We now review the individual sections in more detail.

In section 2, we first remind the reader of the  $\beta$ -elements which generate the 2-line of the ANSS (with a little exception at the prime 2). We then construct, for suitable Hopf algebroids, a complex which is quasi-isomorphic to the cobar complex and which will facilitate later computations. Finally, we show how to use elliptic homology to obtain the f-invariant as above.

In section 3, we recall some fundamental results of N. Katz on the arithmetic of divided congruences and point out an interesting relation between BP-theory and the mod  $p$  Igusa tower (Theorem 5). Next, we give some specific computations of modular forms and divided congruences for  $\Gamma_1(3)$  which we will need to study the f-invariant of the Kervaire elements  $\beta_{2^n, 2^n} \in \text{Ext}^{2, 2^{n+2}}[\text{BP}]$  at the prime  $p = 2$ .

In section 4, we first compute the f-invariants of the infinite family of  $\beta$ -elements  $\beta_t$  for  $t \geq 1$  not divisible by  $p$  (Theorem 10). Then we explain how to approach the problem of computing the Chern numbers determining the  $\beta_{2^n, 2^n}$  (see [L2, Corollary 4.2.5] for the case of  $\beta_1$  at the prime 3). We do this by explicit computations in BP-theory for dimension 2 and 6 (Theorem 18). The computations get very complicated in higher dimensions. In order to use divided congruences, we then compute the f-invariants of the family  $\beta_{s2^n, 2^n}$  for  $n \geq 0$  and  $s \geq 1$  odd (Theorem 19). We hope that clever use of divided congruences will enable us to compute the Chern numbers determining the  $\beta_{2^n, 2^n}$  for all  $n$ . See Corollary 21 for a quick summary of what we can and cannot do at the moment.

## ACKNOWLEDGEMENTS

The idea that it might be possible to project to the element  $\beta_{2^n, 2^n}$  using divided congruences - and consequently rephrase the Kervaire invariant one problem - was communicated to us by G. Laures.

## 2. The construction of the f-invariant

We remind the reader of the construction of  $\beta$ -elements in section 2.1. In section 2.2, we construct a complex which is quasi-isomorphic to the cobar complex and in which we will compute representatives for some of the  $\beta$ -elements. This is used in section 2.3 where we explain how to express the f-invariant of elements in  $\text{Ext}^{2, 2k}[\text{BP}]$  in terms of divided congruences.

## 2.1 Beta-elements in stable homotopy

We review some facts on Brown-Peterson homology  $BP$  at the prime  $p$  and  $\beta$ -elements. See [MRW] and [R] for more details. The Brown-Peterson spectrum  $BP$  has coefficient ring  $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$  with  $v_i$  in dimension  $2(p^i - 1)$ . The universal  $p$ -typical formal group law is defined over this ring. The couple  $(A, \Gamma) := (BP_*, BP_*BP)$  becomes a Hopf algebroid in a standard way and we have  $BP_*BP = BP_* \otimes_{\mathbb{Z}_{(p)}}[t_1, t_2, \dots]$  such that the left unit  $\eta_L$  of the Hopf algebroid  $(A, \Gamma)$  is the standard inclusion. The right unit  $\eta_R$  is determined over  $\mathbb{Q}$  by the formula in [R, Theorem A.2.1.27]. Choosing the Hazewinkel generators [R, A.2.2.1] for the  $v_i$ , a short computation yields  $\eta_R(v_1) = v_1 + pt_1$  and  $\eta_R(v_2) = v_2 + v_1t_1^p - v_1^p t_1 \pmod{p}$ .

We have the chromatic resolution of  $BP_*$  as a left  $BP_*BP$ -comodule

$$BP_* \rightarrow M^0 \rightarrow M^1 \rightarrow \dots$$

which gives rise to the chromatic spectral sequence

$$\mathrm{Ext}_{BP_*BP}^{s,*}(BP_*, M^t) \Rightarrow \mathrm{Ext}_{BP_*BP}^{s+t,*}(BP_*, BP_*).$$

This allows the construction of elements in  $\mathrm{Ext}_{BP_*BP}^{*,*}(BP_*, BP_*)$ , the so called Greek-letter elements. Strictly speaking, these elements arise from comodule sequences  $0 \rightarrow N^n \rightarrow M^n \rightarrow N^{n+1} \rightarrow 0$ , but for our computations we will need the related comodule sequences (1) and (2) below. See [MRW, Lemma 3.7 and Remark 3.8] for the relationship between them. We abbreviate  $H^n(\cdot) := \mathrm{Ext}_{BP_*BP}^{n,*}(BP_*, \cdot)$  in the following.

To construct the  $\beta$ -elements [MRW, p. 476/477], choose integers  $t, s, r \geq 1$  such that  $(p^r, v_1^s, x_n^t) \subseteq BP_*$  is an invariant ideal where  $t = p^n t'$ ,  $(p, t') = 1$  and  $x_n$  is a homogeneous polynomial in  $v_1, v_2$  and  $v_3$  considered as an element of  $v_2^{-1}BP_*/(p^r, v_1^s)$  (see [R, p. 202] or [MRW, p. 476]), for example  $x_0 = v_2$ .

Consider the two short exact sequences of  $BP_*BP$ -comodules

$$(1) \quad 0 \rightarrow BP_* \xrightarrow{p^r} BP_* \rightarrow BP_*/(p^r) \rightarrow 0$$

$$(2) \quad 0 \rightarrow \Sigma^{2s(p-1)}BP_*/(p^r) \xrightarrow{v_1^s} BP_*/(p^r) \rightarrow BP_*/(p^r, v_1^s) \rightarrow 0.$$

Using the induced boundary maps

$$\delta : H^0(BP_*/(p^r, v_1^s)) \longrightarrow H^1(BP_*/(p^r)) \text{ and}$$

$$\delta' : H^1(BP_*/(p^r)) \longrightarrow H^2(BP_*)$$

we define  $\beta_{t,s,r} := \delta' \delta(x_n^t)$ .

It is known [MRW, Theorem 2.6], [Sh] for which indices  $(t, s, r)$  the elements  $\beta_{t,s,r}$  are non-zero in  $H^2(BP_*)$ . In this case the order of  $\beta_{t,s,r}$  is  $p^r$ . By construction, we have  $\beta_{t,s,r} \in H^{2, 2t(p^2-1)-2s(p-1)}(BP_*)$ .

**EXAMPLE 1.** For  $p = 2$  and  $n \geq 1$ , the element  $\beta_{2^n, 2^n} := \beta_{2^n, 2^n, 1} \in \mathrm{Ext}_{BP_*BP}^{2, 2^{n+2}}(BP_*, BP_*)$  is called the *Kervaire element*. It is mapped via the Thom reduction [R, Theorem 5.4.6] to the element  $h_{n+1}^2 \in \mathrm{Ext}_{\mathbb{H}\mathbb{Z}/2_*\mathbb{H}\mathbb{Z}/2}^{2, 2^{n+2}}(\mathbb{H}\mathbb{Z}/2_*, \mathbb{H}\mathbb{Z}/2_*)$ . The latter element survives to a non-zero element of  $\pi_{2^{n+2}-2}(S^0)$  in the Adams spectral sequence if and only if (as W. Browder [B, Theorem 7.1] has shown) there exists

a framed manifold of dimension  $2^{n+2} - 2$  with non-vanishing Kervaire invariant. Whether or not this is the case is unknown for  $n \geq 5$ , for  $n \leq 4$  see [BJM], [KM].

## 2.2 The rationalised cobar complex

The standard way of displaying elements in  $\text{Ext}^n := \text{Ext}_\Gamma^n(A, A)$  of a flat Hopf algebroid  $(A, \Gamma)$  is by means of the cobar complex. In this section we shall give another description of this Ext group as a subquotient of  $(A \otimes \mathbb{Q})^{\otimes n}$  needed to compute f-invariants (Proposition 3). The results of this section are a more algebraic version of [L1, Section 3.1].

Let  $(A, \Gamma)$  be a Hopf algebroid with structure maps  $\eta_L, \eta_R, \epsilon$  and  $\Delta$ . This determines a cosimplicial abelian group  $\Gamma^\cdot$  as follows: Set  $\Gamma^n := \Gamma^{\otimes n}$  with cofaces  $\partial^i : \Gamma^n \rightarrow \Gamma^{n+1}$ ;  $\partial^0(\gamma_1 \otimes \dots \otimes \gamma_n) := 1 \otimes \gamma_1 \otimes \dots \otimes \gamma_n$ ;  $\partial^i(\gamma_1 \otimes \dots \otimes \gamma_n) := \gamma_1 \otimes \dots \otimes \Delta(\gamma_i) \otimes \dots \otimes \gamma_n$  ( $1 \leq i \leq n$ ) and  $\partial^{n+1}(\gamma_1 \otimes \dots \otimes \gamma_n) := \gamma_1 \otimes \dots \otimes \gamma_n \otimes 1$  for  $n \geq 1$  and  $\partial^0 := \eta_R$ ,  $\partial^1 := \eta_L$  for  $n = 0$  and codegeneracies  $\sigma^i : \Gamma^{n+1} \rightarrow \Gamma^n$ ,  $\sigma^i(\gamma_0 \otimes \dots \otimes \gamma_n) := \gamma_0 \otimes \dots \otimes \epsilon(\gamma_i) \otimes \dots \otimes \gamma_n$ . We also denote by  $\Gamma^\cdot$  the associated cochain complex. Following [R, Definition A.1.2.11], we define the reduced cobar complex (usually denoted as  $C_\Gamma(A, A)$ ) as being the subcomplex  $\bar{\Gamma}^\cdot \subseteq \Gamma^\cdot$  with  $\bar{\Gamma}^n := \bar{\Gamma}^{\otimes n}$  for  $n \geq 1$  where  $\bar{\Gamma} := \ker(\epsilon)$  and  $\bar{\Gamma}^0 := A$ . This is a subcomplex because  $\Delta(\gamma) - \gamma \otimes 1 - 1 \otimes \gamma \in \bar{\Gamma} \otimes_A \bar{\Gamma}$  for any  $\gamma \in \bar{\Gamma}$ .

We now assume that  $(A, \Gamma)$  is a flat Hopf algebroid such that

i)  $A$  (and hence  $\Gamma$ ) is torsion free.

ii) The map  $A_{\mathbb{Q}}^{\otimes 2} := (A \otimes \mathbb{Q})^{\otimes 2} \xrightarrow{\phi} \Gamma_{\mathbb{Q}} := \Gamma \otimes \mathbb{Q}$ ,  $a \otimes b \mapsto a \cdot \eta_R(b)$  is an isomorphism ( $\eta_L$  is suppressed from notation).

iii) The  $\mathbb{Q}$ -algebra  $A_{\mathbb{Q}}$  is augmented by some  $\tau : A_{\mathbb{Q}} \rightarrow \mathbb{Q}$ .

**REMARK 2.** *The above assumptions i)-iii) are fulfilled by the flat Hopf algebroid  $(\text{BP}_*, \text{BP}_* \text{BP})$ : The proof of ii) follows from the fact that over any  $\mathbb{Q}$ -algebra any two ( $p$ -typical) formal group laws are isomorphic via a unique strict isomorphism. If  $\text{BP}_* \rightarrow E_*$  is a non-zero Landweber exact algebra [HS], then  $(E_*, E_* E)$  also fulfils all the above assumptions:  $E_*$  is a  $\mathbb{Z}_{(p)}$ -algebra on which  $p$  is not a zero divisor, hence  $E_*$  is torsion free. Conditions ii), iii) and the flatness of  $(E_*, E_* E)$  are inherited from  $(\text{BP}_*, \text{BP}_* \text{BP})$  because  $E_* E \simeq E_* \otimes_{\text{BP}_*} \text{BP}_* \text{BP} \otimes_{\text{BP}_*} E_*$ , c.f. [Na, Proposition 10] for the flatness.*

We define the cosimplicial abelian group  $D^\cdot$  by  $D^n := A_{\mathbb{Q}}^{\otimes(n+1)}$  ( $n \geq 0$ ) with cofaces  $\partial^i : D^n \rightarrow D^{n+1}$  to be given by  $\partial^i(a_0 \otimes \dots \otimes a_n) := a_0 \otimes \dots \otimes 1 \otimes \dots \otimes a_n$  with the 1 in the  $(i+1)^{\text{st}}$  position ( $i = 0, \dots, n+1$ ) and codegeneracies  $\sigma^i : D^n \rightarrow D^{n-1}$  ( $i = 0, \dots, n-1$ ) defined by  $\sigma^i(a_0 \otimes \dots \otimes a_n) := a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n$ . By ii) above we have for any  $n \geq 0$  an isomorphism

$$(3) \quad \phi^n : D^n = A_{\mathbb{Q}}^{\otimes(n+1)} \simeq (A_{\mathbb{Q}} \otimes_{\mathbb{Q}} A_{\mathbb{Q}})^{\otimes_{A_{\mathbb{Q}}} n} \xrightarrow{\phi^{\otimes n}} \Gamma_{\mathbb{Q}}^{\otimes_{A_{\mathbb{Q}}} n} = \Gamma^n \otimes \mathbb{Q}$$

which maps  $a_0 \otimes \dots \otimes a_n \mapsto a_0 \otimes \dots \otimes a_{n-2} \otimes a_{n-1} \cdot \eta_R(a_n)$  and one checks that  $\phi^\cdot$  is an isomorphism of cosimplicial groups and hence of cochain complexes.

We have  $H(D^\cdot) = H^0(D^\cdot) = \mathbb{Q}$ , a contracting homotopy being given by

$$(4) \quad H^n : D^n = A_{\mathbb{Q}}^{\otimes(n+1)} \rightarrow D^{n-1} = A_{\mathbb{Q}}^{\otimes n}; a_0 \otimes \dots \otimes a_n \mapsto \tau(a_0) a_1 \otimes \dots \otimes a_n.$$

Define a subcomplex  $\Sigma^\cdot \subseteq D^\cdot$  by

$$\Sigma^n := \sum_{i=1}^n \partial^i(D^{n-1}) = \sum_{i=1}^n A_{\mathbb{Q}}^{\otimes i} \otimes \mathbb{Q} \otimes A_{\mathbb{Q}}^{\otimes(n-i)} \subseteq D^n = A_{\mathbb{Q}}^{\otimes(n+1)},$$

for  $n \geq 1$  and  $\Sigma^0 := 0$ . One checks that the composition

$$\iota^n : \bar{\Gamma}^n \hookrightarrow \Gamma^n \hookrightarrow \Gamma^n \otimes \mathbb{Q} \xrightarrow{(\phi^n)^{-1}} D^n \longrightarrow D^n / \Sigma^n$$

is injective for all  $n \geq 0$ . This is obvious for  $n = 0$ , and for  $n \geq 1$  it follows from  $(\cap_{i=0}^{n-1} \ker(\sigma^i)) \cap (\sum_{i=1}^n \text{im}(\partial^i)) = 0$ , which in turn is an easy consequence of the cosimplicial identities: One shows by descending induction on  $1 \leq j \leq n$  that  $(\cap_{i=0}^{n-1} \ker(\sigma^i)) \cap (\sum_{i=j}^n \text{im}(\partial^i)) = 0$ . Observe that  $D^n / \Sigma^n$  is isomorphic to the group labelled  $E_* \otimes G_*^n$  in [L1, p. 404] for  $\mathbb{A} = \text{BP}$ .

We define a cochain complex  $Q^\cdot$  by the exactness of

$$(5) \quad 0 \longrightarrow \bar{\Gamma}^\cdot \xrightarrow{\iota^\cdot} D^\cdot / \Sigma^\cdot \xrightarrow{\pi^\cdot} Q^\cdot \longrightarrow 0,$$

hence  $Q^n \simeq A_{\mathbb{Q}}^{\otimes(n+1)} / \Sigma^n + \text{im}(\bar{\Gamma}^{\otimes A^n})$ .

From the definitions of the differential of  $D^\cdot$  and  $\Sigma^n$  one obtains ( $BQ^n \subseteq Q^n$  denoting the boundaries)

$$Q^n / BQ^n \simeq A_{\mathbb{Q}}^{\otimes(n+1)} / \tilde{\Sigma}^n + \text{im}(\bar{\Gamma}^{\otimes A^n}),$$

where  $\tilde{\Sigma}^n := \Sigma^n + \mathbb{Q} \otimes A_{\mathbb{Q}}^{\otimes n} = \sum_{i=1}^{n+1} A_{\mathbb{Q}}^{\otimes(i-1)} \otimes \mathbb{Q} \otimes A_{\mathbb{Q}}^{\otimes(n+1-i)}$ .

The alternative description of  $\text{Ext}$  we are aiming for is the following.

**PROPOSITION 3.** *For any  $n \geq 1$ , the connecting homomorphism  $\delta$  of (5) is an isomorphism*

$$H^n(Q^\cdot) \xrightarrow{\delta} H^{n+1}(\bar{\Gamma}^\cdot) = \text{Ext}^{n+1}.$$

*Proof.* One readily sees that the contracting homotopy (4) of  $D^\cdot$  respects the subcomplex  $\Sigma^\cdot$  in positive dimensions, hence the middle term of (5) is acyclic in these dimensions.  $\square$

To explicitly compute  $\delta$ , it is useful to note that the differential of  $D^\cdot / \Sigma^\cdot$  has the simple form

$$D^n / \Sigma^n \xrightarrow{d} D^{n+1} / \Sigma^{n+1}, [a_0 \otimes \dots \otimes a_n] \mapsto [1 \otimes a_0 \otimes \dots \otimes a_n],$$

as is immediate from the definitions. To compute  $\delta^{-1}$ , we consider the zig-zag

$$\begin{array}{ccccccc} \text{Ext}^{n+1} & \longleftarrow & Z(\bar{\Gamma}^{n+1}) & \xrightarrow{\iota^{n+1}} & D^{n+1} / \Sigma^{n+1} & \xrightarrow{H^{n+1}} & D^n / \Sigma^n \xrightarrow{\pi^n} Q^n \longrightarrow Q^n / BQ^n \\ & & & & & & \uparrow \\ & & & & & & H^n(Q^\cdot). \end{array}$$

$\delta^{-1}$  (dotted arrow from  $Z(\bar{\Gamma}^{n+1})$  to  $H^n(Q^\cdot)$ )

One checks that the dotted arrow exists and is the inverse of  $\delta$ .

Now let  $p$  be a prime,  $r \geq 1$  an integer, and let  $\delta' : \text{Ext}_\Gamma^n(A, A/p^r) \rightarrow \text{Ext}^{n+1}$  be the connecting homomorphism associated to the short exact sequence of  $\Gamma$ -comodules  $0 \rightarrow A \xrightarrow{p^r} A \rightarrow A/p^r \rightarrow 0$ . Consider the diagram

$$(6) \quad \begin{array}{ccccc} \delta' : \text{Ext}_\Gamma^n(A, A/p^r) & \longleftarrow & Z(\bar{\Gamma}^n \otimes_A A/p^r) & \longleftarrow & \{z \in \bar{\Gamma}^n \mid dz \in p^r \bar{\Gamma}^{n+1}\} & \xrightarrow{\alpha} & \text{Ext}^{n+1} \\ & & & & \nu \downarrow & \nearrow & \delta \uparrow \simeq \\ & & & & \mathbb{Q}^n / B\mathbb{Q}^n & \longleftarrow & H^n(\mathbb{Q}^{\cdot}) \end{array}$$

Here,  $\alpha(z) := [y]$  for any  $y \in \bar{\Gamma}^{n+1}$  satisfying  $dz = p^r y$  and  $\nu(z) := \pi^n(p^{-r} \iota^n(z)) \bmod B\mathbb{Q}^n$ . The upper horizontal line is  $\delta'$  by definition and one checks that  $\nu$  factors through the dotted arrow and makes the diagram commutative. Hence, when displaying an element of  $\text{im}(\delta')$  in  $H^*(\mathbb{Q}^{\cdot})$  rather than in the usual cobar complex, one does not have to compute the cobar differential implicit in  $\alpha$  but only the (easier) map  $\nu$ .

Finally, assume that everything in sight is graded where the grading on  $D^n, \Gamma^n$ , etc. is by total degree. For a fixed  $k \in \mathbb{Z}$ , consider the commutative diagram

$$\begin{array}{ccccc} \mathbb{Q}^n / B\mathbb{Q}^n \cong A_{\mathbb{Q}}^{\otimes(n+1)} / \tilde{\Sigma}^n + \text{im}(\bar{\Gamma}^{\otimes A^n}) & \longleftarrow & H^n(\mathbb{Q}^{\cdot}) & \xrightarrow[\simeq]{\delta} & \text{Ext}^{n+1} \\ \uparrow j & & \uparrow & & \uparrow \\ A_{\mathbb{Q}}^{\otimes(n+1),k} / \tilde{\Sigma}^{n,(k)} + \text{im}(\bar{\Gamma}^{\otimes A^n})^k & \longleftarrow & H^{n,k}(\mathbb{Q}^{\cdot}) & \xrightarrow[\simeq]{} & \text{Ext}^{n+1,k} \end{array}$$

Here  $\tilde{\Sigma}^{n,(k)} := \tilde{\Sigma}^n \cap A_{\mathbb{Q}}^{\otimes(n+1),k}$ . One checks that  $j$  is well defined and injective, and  $H^{n,k}(\mathbb{Q}^{\cdot})$  is defined to be the pull back of  $H^n(\mathbb{Q}^{\cdot})$  along  $j$ . The commutative diagram induces an isomorphism  $H^{n,k}(\mathbb{Q}^{\cdot}) \xrightarrow{\simeq} \text{Ext}^{n+1,k}$  as indicated.

For example, for  $(A, \Gamma) = (\text{BP}_*, \text{BP}_* \text{BP})$  and  $n = 1$  we obtain an inclusion

$$(7) \quad \text{Ext}^{2,k} \subseteq \frac{(\text{BP}_{\mathbb{Q}} \otimes \text{BP}_{\mathbb{Q}})^{(k)}}{\text{BP}_k \text{BP} + (\text{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \text{BP}_{\mathbb{Q}})^{(k)}}$$

which is important for us since the f-invariant is defined in terms of the group on the right hand side.

To effectively compute representatives of  $\beta$ -elements in the complex  $\mathbb{Q}^{\cdot}$  one proceeds as follows. Let  $t, s, r \geq 1$  be integers as in section 2.1 and  $\delta, \delta'$  the coboundary maps introduced there. Fix  $k \in \mathbb{Z}$  and  $x \in H^{0,k}(A/(p^r, v_1^s))$ , that is

$$x \in C_{\Gamma}^{0,k}(A/(p^r, v_1^s)) = (A/(p^r, v_1^s))^k$$

is an invariant element ( $C_{\Gamma}$  indicates the reduced cobar complex). As  $\delta$  is the connecting homomorphism determined by the short exact sequence of complexes obtained by applying  $C_{\Gamma}$  to (2), we compute  $\delta(x)$  as follows: Lift  $x$  to  $y \in (A/(p^r))^k$  and compute the cobar differential

$$d = \eta_R - \eta_L : C_\Gamma^{0,k}(A/p^r) = (A/p^r)^k \longrightarrow (A/p^r \otimes_A \bar{\Gamma})^k = (\bar{\Gamma}/p^r)^k = C_\Gamma^{1,k}(A/p^r)$$

obtaining  $d(y) \in (\bar{\Gamma}/p^r)^k = \bar{\Gamma}^k/p^r$ . Note that this computation requires knowledge of  $\eta_R(y) \bmod p^r$ . Now  $d(y)$  will be divisible by  $v_1^s$ , hence

$$d(y) = v_1^s z \text{ in } (\bar{\Gamma}/p^r)^k$$

with  $z \in (\bar{\Gamma}/p^r)^{k-2s(p-1)} = C_\Gamma^{1,k-2s(p-1)}(A/p^r)$  representing  $\delta(x)$ . Lift  $z$  to some  $w \in \bar{\Gamma}^{k-2s(p-1)}$ . To proceed, we use diagram (6):  $w$  lies in the last but one group of the top row, hence we compute  $\nu(w) \in H^2(Q)$  which is our representative for  $\delta'(\delta(x)) \in \text{Ext}^2$ . This requires to compute  $\phi^{-1}$ . Observe for example that  $\phi^{-1}(t_1) = \frac{1 \otimes v_1 - v_1 \otimes 1}{p}$ .

### 2.3 Elliptic homology theories and divided congruences: the f-invariant

We refer the reader to [L1] or [HBJ] for the notion of elliptic homology with respect to the congruence subgroup  $\Gamma_1(N)$ . In this section,  $E$  denotes the spectrum associated to the homology theory with coefficient ring  $E_* = M_*(\mathbb{Z}_{(p)}, \Gamma_1(N))$ , see section 3.2 for the notation. Finally,  $\alpha : \text{BP} \longrightarrow E$  denotes the orientation.

By the naturality of the constructions in section 2.2 we have a commutative diagram for any  $k \geq 0$

$$\begin{array}{ccc} \text{Ext}^{2,k}[\text{BP}] & \xrightarrow{\alpha} & \text{Ext}^{2,k}[E] \\ \downarrow (7) & & \downarrow (7) \iota \\ \frac{(\text{BP}_\mathbb{Q} \otimes \text{BP}_\mathbb{Q})^{(k)}}{\text{BP}_k \text{BP} + (\text{BP}_\mathbb{Q} \otimes \mathbb{Q} + \mathbb{Q} \otimes \text{BP}_\mathbb{Q})^{(k)}} & \xrightarrow{\alpha \otimes \alpha} & \frac{(\text{E}_\mathbb{Q} \otimes \text{E}_\mathbb{Q})^{(k)}}{\text{E}_k \text{E} + (\text{E}_\mathbb{Q} \otimes \mathbb{Q} + \mathbb{Q} \otimes \text{E}_\mathbb{Q})^{(k)}} \end{array}$$

The injectivity of  $\alpha$  holds for any Landweber exact theory  $E$  of height at least two, [L2, Proof of 4.3.2]. To proceed, however, we will use a more subtle property of  $E$ , namely the topological  $q$ -expansion principle. We put

$$\underline{\mathbb{D}}_k := \left\{ f = \sum_{i=0}^k f_i \in \bigoplus_{i=0}^k \text{E}_{\mathbb{Q},2i} \mid \text{there are } g_0 \in \mathbb{Q}, g_k \in \text{E}_{\mathbb{Q},2k} \text{ such that } (f + g_0 + g_k)(q) \in \mathbb{Z}_{(p)}^\Gamma[[q]] \right\}$$

where  $f(q)$  denotes the  $q$ -expansion of  $f$  at the cusp infinity, and for  $\Gamma = \Gamma_1(N)$  we set  $\mathbb{Z}^\Gamma := \mathbb{Z}[\frac{1}{N}, \zeta_N]$  if  $N > 1$  and  $\mathbb{Z}^{SL_2(\mathbb{Z})} := \mathbb{Z}[\frac{1}{6}]$  as in [L1]. We then define

$$(8) \quad \iota^2 : \frac{(\text{E}_\mathbb{Q} \otimes \text{E}_\mathbb{Q})^{(2k)}}{\text{E}_{2k} \text{E} + (\text{E}_\mathbb{Q} \otimes \mathbb{Q} + \mathbb{Q} \otimes \text{E}_\mathbb{Q})^{(2k)}} \longrightarrow \underline{\mathbb{D}}_k \otimes \mathbb{Q}/\mathbb{Z}, \quad \sum_{i+j=k} f_i \otimes g_j \mapsto \sum_{i+j=k} -q^0(f_i)g_j,$$

where  $q^0(f)$  is the constant term of the  $q$ -expansion of  $f$  at the cusp infinity. The composition  $\iota^2 \circ \iota$  is injective [L1, Proposition 3.9] and hence so is the f-invariant

$$f : \text{Ext}^{2,2k}[\text{BP}] \hookrightarrow \underline{\mathbb{D}}_k \otimes \mathbb{Q}/\mathbb{Z}.$$

We remark that our grading of  $E_*$  is the topological one, i.e. elements of dimension  $2k$  correspond to modular forms of weight  $k$ .

G. Laures describes  $\text{Ext}^2$  using the canonical Adams resolution [R, Definition 2.2.10] instead of the cobar resolution.

PROPOSITION 4. *For  $k > 0$  even, the above definition of the f-invariant  $\text{Ext}^{2,k} \longrightarrow \underline{\underline{D}}_{k/2} \otimes \mathbb{Q}/\mathbb{Z}$  coincides with the one given in [L1].*

*Proof.* We have to show that the map  $\text{Ext}^{2,k} \hookrightarrow \frac{(\text{BP}_{\mathbb{Q}} \otimes \text{BP}_{\mathbb{Q}})^{(k)}}{\text{BP}_k \text{BP} + (\text{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \text{BP}_{\mathbb{Q}})^{(k)}}$  we constructed in section 2.2,(7) coincides with the one of [L1]. We know [R, Lemma A.1.2.9 (b)] that, up to chain homotopy, there is a unique map from the unreduced cobar resolution  $\Gamma \otimes_{\mathbb{A}} \Gamma^{\otimes \mathbb{A} \cdot}$  to the canonical Adams resolution  $\text{BP}_*(\text{BP} \wedge \Sigma \bar{\text{BP}}^{\wedge \cdot})$  where  $\bar{\text{BP}} \rightarrow S^0 \xrightarrow{\eta} \text{BP} \xrightarrow{d} \Sigma \bar{\text{BP}}$  is an exact triangle in the stable homotopy category, see also [Br, Lemma 3.7] Now one checks that  $\Gamma^{\otimes_{\mathbb{A}}(n+1)} \cong \pi_*(\text{BP}^{\wedge n+2}) \xrightarrow{\pi_*(\text{id}_{\text{BP}}^{\wedge 2} \wedge d^{\wedge n})} \pi_*(\text{BP}^{\wedge 2} \wedge (\Sigma \bar{\text{BP}})^{\wedge n})$  is a map of chain complexes where the isomorphism follows from [R, Lemma 2.2.7] and induction. So the claim reduces to the fact that the triangle

$$\begin{array}{ccc} Z^{2,k}(\Gamma^{\otimes_{\mathbb{A} \cdot})} & \xrightarrow{\pi_*(\text{id}_{\text{BP}}^{\wedge 2} \wedge d)} & Z^{2,k}(\pi_*(\text{BP}^{\wedge 2} \wedge (\Sigma \bar{\text{BP}})^{\wedge \cdot})) \\ & \searrow & \swarrow \\ & \frac{(\text{BP}_{\mathbb{Q}} \otimes \text{BP}_{\mathbb{Q}})^{(k)}}{\text{BP}_k \text{BP} + (\text{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \text{BP}_{\mathbb{Q}})^{(k)}} & \end{array}$$

commutes. This follows using that our maps  $\tau, H^2$  and  $\pi_*(d \wedge d)$  correspond to  $r, \rho$  and the isomorphism  $D^1/\tilde{\Sigma}^1 \cong G^2 = \pi_*((\Sigma \bar{\text{BP}})^{\wedge 2}) \otimes \mathbb{Q}$  in [L1, section 3.1] (where we define  $\tau$  by mapping all  $v_i$  to zero).  $\square$

Note the degree shift and the factor 2 for the f-invariant  $\pi_{2k}^s \longrightarrow \underline{\underline{D}}_{k+1} \otimes \mathbb{Q}/\mathbb{Z}$  (both are missing in [L1, p. 411]).

### 3. Arithmetic computations

In section 3.1, we review results of N. Katz on divided congruences and establish a relation between BP-theory and the mod  $p$  Igusa tower (Theorem 5). In section 3.2 we give explicit computations for elliptic homology of level 3 and the corresponding divided congruences.

#### 3.1 Divided congruences

We review parts of [K1]. Some technical remarks are in order: In *loc. cit.* N. Katz works with level- $N$  structures of fixed determinant for  $N \geq 3$ . To confirm with general policy in algebraic topology we wish to consider  $\Gamma_1(N)$ -structures instead, which are representable only for  $N \geq 5$ . More seriously, one has to check that the relevant part of [K1] works for this different moduli problem. This we did, but we will not explain the details here and only remark that both the geometric irreducibility of the moduli spaces and the irreducibility of the Igusa tower are valid for  $\Gamma_1(N)$ . Furthermore, we will use these results for  $p \geq 5$  and  $N = 1$  and for  $p = 2$  and  $N = 3$ . These cases can be handled by using auxiliary rigid level structures and taking invariants under suitable finite groups as in [K1]. Fix a level  $N \geq 5$ , a prime  $p$  not dividing  $N$  and a primitive  $N$ -th root of unity  $\zeta \in \overline{\mathbb{F}}_p$ . We

put  $k := \mathbb{F}_p(\zeta)$ ,  $W := W(k)$  (Witt vectors) and also denote by  $\zeta \in W$  the Teichmüller lift of  $\zeta$ . Finally,  $K$  denotes the field of fractions of  $W$  and for any  $\mathbb{Z}_{(p)}$ -algebra  $R$  we denote by  $M_k(R, \Gamma_1(N))$  the  $R$ -module of holomorphic modular forms for  $\Gamma_1(N)$  of weight  $k$  and defined over  $R$ , see e.g. [K3] or [L1]. The  $\Gamma_1(N)$  is omitted from the notation if it is clear from the context. We fix a lift  $E_{p-1} \in M_{p-1}(W)$  of the Hasse invariant. The existence of such a lift puts further restrictions on  $p$  and  $N$  which are satisfied in our applications.

We define the ring of divided congruences  $D$  by

$$M_*(W) \subseteq D := \{f \in M_*(K) \mid f(q) \in W[[q]]\} \subseteq M_*(K),$$

where  $f(q)$  is the  $q$ -expansion of  $f$  at the cusp infinity. For  $n \geq 0$  we also define

$$D_n := D \cap \left( \bigoplus_{i=0}^n M_i(K) \right) \subseteq D \text{ and}$$

$$\underline{D}_n := D_n + K + M_n(K) \subseteq \bigoplus_{i=0}^n M_i(K)$$

which is consistent with the definition of the previous section. The group  $\underline{D}_k$  considered in [L1] differs from the  $\underline{D}_k$  above because the ring of holomorphic modular forms has been localised in [L1]. This difference is not serious because the  $f$ -invariant factors through holomorphic modular forms [L1, Proposition 3.13].

The ring  $D$  carries a uniformly continuous  $\mathbb{Z}_p^*$ -action (the diamond operators) defined by

$$[\alpha] \left( \sum_i f_i \right) := \sum_i \alpha^i f_i,$$

where  $\alpha \in \mathbb{Z}_p^*$  and  $f_i \in M_i(K)$ . We put  $\Gamma_0 := \mathbb{Z}_p^*$  and  $\Gamma_n := 1 + p^n \mathbb{Z}_p$  for  $n \geq 1$  and  $V_{1,n} := (D/pD)^{\Gamma_n}$  for  $n \geq 0$ . Then

$$V_{1,0} \subseteq V_{1,1} \subseteq \dots \subseteq D/pD$$

is an ind-étale  $\mathbb{Z}_p^*$ -Galois extension, the mod  $p$  Igusa tower. So,  $V_{1,0} \subseteq V_{1,1}$  is a  $(\mathbb{Z}/p)^*$ -Galois extension and for all  $n \geq 2$   $V_{1,n-1} \subseteq V_{1,n}$  is an étale  $\mathbb{Z}/p$ -extension and hence an Artin-Schreier extension. An immediate computation with diamond operators shows that the composition  $M_*(W) \rightarrow D \rightarrow D/pD$  factors through  $V_{1,1} \subseteq D/pD$ . It is a result of P. Swinnerton-Dyer that this induces an isomorphism

$$(9) \quad M_*(W)/(p, E_{p-1} - 1) \xrightarrow{\cong} V_{1,1},$$

see [K1, Corollary 2.2.8].

By Artin-Schreier theory, given  $n \geq 2$  and  $x \in D/pD$  satisfying  $[\alpha](x) = x$  for all  $\alpha \in \Gamma_n$  and  $[1 + p^{n-1}](x) = x + 1$  one has  $V_{1,n} = V_{1,n-1}[x]$  and the minimal polynomial of  $x$  over  $V_{1,n-1}$  is  $T^p - T - a$  for some  $a \in V_{1,n-1}$ .

At this point we can establish a first relation between BP-theory and divided congruences. Consider the ring extensions

$$M_*(W) \subseteq D \subseteq W[[q]].$$

We have a formal group  $\mathcal{F}$  over  $M_*(W)$  induced by the universal elliptic curve. The base change of  $\mathcal{F}$  to  $W[[q]]$  is the formal completion of a Tate elliptic curve and is thus isomorphic to  $\widehat{\mathbb{G}}_m$ . Implicit in [K1] is the fact that  $D$  is the *minimal* extension of  $M_*(W)$  over which  $\mathcal{F}$  becomes isomorphic to  $\widehat{\mathbb{G}}_m$ , i.e.  $D$  is obtained from  $M_*(W)$  by adjoining the coefficients of an isomorphism  $\mathcal{F} \simeq \widehat{\mathbb{G}}_m$  defined over  $W[[q]]$ . This is what underlies N. Katz' construction [K1, Section 5] of a sequence of elements  $d_n \in D$  which modulo  $p$  constitute a sequence of Artin-Schreier generators for the mod  $p$  Igusa tower.

Since the elements  $t_n \in \text{BP}_*\text{BP}$  are the coefficients of the universal isomorphism of a  $p$ -typical formal group law, one may expect a relation between the  $t_n$  and the  $d_n$ . To formulate this, denote by  $\alpha : \text{BP}_* \rightarrow M_*(W)$  the classifying map of  $\mathcal{F}$  and consider the composition

$$\phi : \text{BP}_*\text{BP} \subseteq \text{BP}_*\text{BP} \otimes \mathbb{Q} \xrightarrow{(3)} \text{BP}_{\mathbb{Q}}^{\otimes 2} \xrightarrow{\alpha \otimes \alpha} M_*(K)^{\otimes 2} \xrightarrow{-q^0 \otimes \text{id}} M_*(K).$$

Note that the map  $\iota_2$  in (8) composed with the orientation  $\alpha$  is a quotient of  $\phi$ .

The topological  $q$ -expansion principle guarantees that  $T_n := \phi(t_n) \in D$  for  $n \geq 1$  and we can thus define  $\overline{T}_n := (T_n \bmod pD) \in D/pD$ .

**THEOREM 5.** *For any  $n \geq 1$  we have  $[1 + p^k](\overline{T}_n) = \overline{T}_n$  for  $k > n$  and  $[1 + p^n](\overline{T}_n) = \overline{T}_n + 1$ . Hence  $\overline{T}_n$  is an Artin-Schreier generator for the extension  $V_{1,n} \subseteq V_{1,n+1}$ .*

*Proof.* Let  $\omega = (\sum_{n \geq 1} a_n t^{n-1}) dt$  be the expansion along infinity of a normalised (i.e.  $a_1 = 1$ ) invariant differential on the universal elliptic curve. Then  $a_n \in M_{n-1}(W)$  and the logarithm of the  $p$ -typification is  $\sum_{n \geq 0} \frac{a_{p^n}}{p^n} t^{p^n} \in M_*(K)[[t]]$ , i.e. the classifying map  $\alpha : \text{BP}_* \rightarrow M_*(W)$ , when tensored with  $\mathbb{Q}$ , sends  $l_n \in \text{BP}_{\mathbb{Q}, 2(p^n-1)}$  to  $\frac{a_{p^n}}{p^n} \in M_{p^n-1}(K)$ , see [R, Theorem A.2.1.27] for the definition of the  $l_n$  ( $=\lambda_n$  in the notation of *loc. cit.*).

Defining  $d_0 := 1$  and  $d_n$  ( $n \geq 1$ ) recursively by

$$(10) \quad \sum_{i=0}^n \frac{d_{n-i}^{p^i}}{p^i} = \frac{a_{p^n}}{p^n},$$

N. Katz shows in [K1, Corollary 5.7] that the  $\overline{d}_n := (d_n \bmod p) \in D/pD$  behave under the diamond operators as claimed for the  $\overline{T}_n$ . In  $\text{BP}_*\text{BP} \otimes \mathbb{Q}$  we have  $\eta_R(l_n) = \sum_{i=0}^n l_i t_{n-i}^{p^i}$  and we apply  $\phi$  to this relation to obtain

$$\frac{a_{p^n}}{p^n} = \sum_{i=0}^n \frac{q^0(a_{p^i})}{p^i} T_{n-i}^{p^i}$$

which, using (10), implies

$$(11) \quad \sum_{i=0}^n \frac{d_{n-i}^{p^i}}{p^i} = \sum_{i=0}^n \frac{q^0(a_{p^i})}{p^i} T_{n-i}^{p^i}.$$

We now proceed by induction on  $n \geq 1$ . For  $n = 1$  we have  $d_1 + 1/p = q^0(a_1)T_1 + q^0(a_p)/p$ . Also,  $q^0(a_1) = 1$  since  $a_1 = 1$  and  $q^0(a_p) \in 1 + pW$  because  $a_p$  reduces mod  $p$  to the Hasse invariant which has  $q$ -expansion equal to 1. We obtain  $\overline{T}_1 = \overline{d}_1 + \alpha$  for some  $\alpha \in k$ . As  $\alpha$  is invariant under all diamond operators, our claim for  $n = 1$  is obvious.

Assume that  $n \geq 2$ . From (11) and  $a_1 = 1$  we obtain

$$T_n = \sum_{i=0}^n \frac{d_{n-i}^{p^i}}{p^i} - \sum_{i=1}^n \frac{q^0(a_{p^i})}{p^i} T_{n-i}^{p^i}.$$

For  $k > n$  we know that the terms involving  $d_i$  are invariant mod  $p$  under  $[1 + p^k]$  whereas the remaining terms are likewise invariant by the induction hypothesis.

Finally, we have

$$[1 + p^n]T_n = [1 + p^n]d^n + [1 + p^n]\left(\sum_{i=1}^n \frac{d_{n-i}^{p^i}}{p^i}\right) - [1 + p^n]\left(\sum_{i=1}^n \frac{q^0(a_{p^i})}{p^i} T_{n-i}^{p^i}\right).$$

Here we have  $[1 + p^n]d^n \equiv d_n + 1(pD)$  and the remaining terms are invariant. Thus, indeed,  $[1 + p^n](\overline{T}_n) = \overline{T}_n + 1$ .  $\square$

### 3.2 Modular forms

For a prime  $p \geq 5$ , the following is well known [L1, Appendix]:

$$M_*(\mathbb{Z}_{(p)}, \Gamma_1(1)) = \mathbb{Z}_{(p)}[E_4, E_6],$$

where  $E_4$  and  $E_6$  are the Eisenstein series of level one of the indicated weight. For the discriminant  $\Delta$ , the ring of meromorphic modular forms is given by  $\mathbb{Z}_{(p)}[E_4, E_6, \Delta^{-1}]$  and the usual orientation

$$\text{BP}_* \longrightarrow \mathbb{Z}_{(p)}[E_4, E_6, \Delta^{-1}]$$

is Landweber exact of height 2 and factors through  $\mathbb{Z}_{(p)}[E_4, E_6]$ . A similar result holds for  $p \geq 3$  and  $M_*(\mathbb{Z}_{(p)}, \Gamma_1(2)) = \mathbb{Z}_{(p)}[\delta, \epsilon]$ .

The purpose of this section is to give analogous results for  $\Gamma_1(3)$  and  $p = 2$ , c.f. [St] for related results.

Consider the elliptic curve

$$E : y^2 + a_1xy + a_3y = x^3$$

defined over  $R := \mathbb{Z}[1/3][a_1, a_3, \Delta^{-1}]$  where  $\Delta = a_3^3(a_1^3 - 27a_3)$  is the discriminant of the given Weierstrass equation. Note that, unlike in level one,  $\Delta$  is not irreducible as a polynomial in  $a_1$  and

$a_3$  and we put  $f := a_3$ ,  $g := a_1^3 - 27a_3$ , hence  $\Delta = f^3g$ .

The section  $P := (0, 0) \in E(R)$  is of exact order 3 in every geometric fibre as follows from [Si1, III,2.3] and  $\omega := dx/(2y + a_1x + a_3)$  is an invariant differential on  $E$ .

The following may be compared with [St, Lemma 11]:

**PROPOSITION 6.** *The above tuple  $(E/R, \omega, P)$  is the universal example of an elliptic curve over a  $\mathbb{Z}[1/3]$ -scheme together with a point of order 3 and a non-zero invariant differential.*

*Proof.* We have to show that whenever  $T$  is a  $\mathbb{Z}[1/3]$ -scheme and  $E'/T$  is an elliptic curve with non-zero invariant differential  $\omega'$  and  $P' \in E'(T)$  of exact order 3, there is a unique map  $\phi : T \rightarrow \text{Spec}(R)$  such that  $\phi^*(E, P, \omega) = (E', P', \omega')$ . We show the uniqueness of  $\phi$  first. This amounts to seeing that the only change of coordinates

$$x = u^2x' + r, \quad y = u^3y' + u^2sx' + t$$

with  $r, s, t \in R$  and  $u \in R^*$  (see [Si1, III Table 1.2]) preserving  $(E, P, \omega)$  is the identity, i.e.  $r = s = t = 0$  and  $u = 1$ .

From  $x'(P) = y'(P) = 0$  we obtain  $r = t = 0$ . Next,  $a_4 = a_4'$  implies  $-sa_3 = 0$ , hence  $s = 0$  because  $\Delta = f^3g$  and thus also  $f = a_3$  is a unit in  $R$ . Finally,  $\omega' = u\omega$  forces  $u = 1$ .

Given the uniqueness of  $\phi$  in general, its existence is a local problem on  $T$  and we can assume that  $T = \text{Spec}(S)$  is affine and  $E'/T$  is given by a Weierstrass equation with coefficients  $a'_i \in S$ . Moving  $P'$  to  $(0, 0)$  gives  $a'_6 = 0$ . We claim that  $a'_3 \in S^*$ : This can be checked on geometric fibres where it follows from [Si1, III,2.3] and the fact that  $(0, 0)$  has order 3 (if  $a'_3$  vanished on some geometric fibre the point  $(0, 0)$  would have order 2 in that fibre). Using this, one finds a transformation such that  $(dy)_{P'} = 0$  in  $\Omega_{E'/T, P'}$ , hence  $a'_2 = a'_4 = 0$ . We thus have some  $\psi : T \rightarrow \text{Spec}(R)$  such that  $\psi^*(E, P) = (E', P')$  and  $\psi^*(\omega) = u\omega'$  for some  $u \in S^*$ . Adjusting  $\psi$  using  $u$ , i.e. multiplying the  $a'_i$  by  $u^{-i}$ , we obtain the desired  $\phi$ .  $\square$

We conclude that the ring of meromorphic modular forms is given as

$$M_*^{mer}(\mathbb{Z}_{(2)}, \Gamma_1(3)) = \mathbb{Z}_{(2)}[a_1, a_3, \Delta^{-1}]$$

with  $a_i$  of weight  $i$ , and likewise for any other prime different from 3 in place of 2.

As usual,  $t = -x/y$  is a local parameter at infinity for  $E/R$  which is normalised for  $\omega$  and hence determines a 2-typical formal group law over  $M_*^{mer}(\mathbb{Z}_{(2)}, \Gamma_1(3))$ . Using [Si1, p. 113] one checks that the corresponding classifying map

$$\alpha : \text{BP}_* \rightarrow M_*^{mer}(\mathbb{Z}_{(2)}, \Gamma_1(3))$$

satisfies  $\alpha(v_1) = a_1$  and  $\alpha(v_2) = a_3$  for the Hazewinkel generators  $v_i$ . Thus  $\alpha$  makes  $M_*^{mer}(\mathbb{Z}_{(2)}, \Gamma_1(3))$  a Landweber exact BP algebra of height 2.

Using the orders of  $f$  and  $g$  at the two cusps 0 and  $\infty$  of  $X_1(3)$ , one can check that the ring of holomorphic modular forms is given by

$$(12) \quad M_*(\mathbb{Z}_{(2)}, \Gamma_1(3)) = \mathbb{Z}_{(2)}[a_1, a_3].$$

We stick to the notations of section 3.1 for  $p = 2$  and  $N = 3$ . For example,  $\zeta$  denotes a primitive cube-root of unity and  $W = W(\mathbb{F}_2(\zeta)) = W(\mathbb{F}_4) = \mathbb{Z}_2[\zeta]$  is the unique unramified quadratic extension of  $\mathbb{Z}_2$ .

To study divided congruences we will need to know the  $q$ -expansions of  $a_1$  and  $a_3$ . Given a Dirichlet character  $\chi$ , we consider it as a function on  $\mathbb{Z}$  as usual and define for  $k \geq 0$  and  $n \geq 1$

$$\sigma_k^\chi(n) := \sum_{1 \leq d|n} \chi(d)d^k.$$

In the following,  $\chi$  will always denote the unique non-trivial character mod 3

$$\chi : (\mathbb{Z}/3\mathbb{Z})^* \longrightarrow \mathbb{C}^*.$$

PROPOSITION 7. *The  $q$ -expansions of  $a_1$  and  $a_3$  at the cusp infinity are given as follows.*

$$a_1(q) = (1 + 2\zeta)(1 + 6 \sum_{n \geq 1} \sigma_0^\chi(n)q^n) \text{ and}$$

$$a_3(q) = (1 + 2\zeta)(-\frac{1}{9} + \sum_{n \geq 1} \sigma_2^\chi(n)q^n) \text{ in } W[[q]].$$

*Proof.* From (12) we know that  $\text{rk } M_1(\mathbb{Z}_{(2)}) = 1$  and  $\text{rk } M_3(\mathbb{Z}_{(2)}) = 2$ . Using [K2, section 2.1.1] we see that

$$(13) \quad 6G_{1,\chi}(q) = 1 + 6 \sum_{n \geq 1} \sigma_0^\chi(n)q^n \in M_1(\mathbb{Z}_{(2)}) \text{ and}$$

$$G_{3,\chi}(q) = -\frac{1}{9} + \sum_{n \geq 1} \sigma_2^\chi(n)q^n \in M_3(\mathbb{Z}_{(2)}).$$

We have evaluated  $L(0, \chi) = 1/3$  and  $L(-2, \chi) = -2/9$  using [Ne, Theorem VII.2.9] and [Wa, formula following Proposition 4.1 and Exercise 4.2(b)].

It is easy to see that  $G_{3,\chi}(0) = 0$ , i.e.  $G_{3,\chi}$  vanishes at the cusp 0. Below, we explain how to compute the following values of  $a_1$  and  $a_3$  at the cusps zero and infinity.

$$(14) \quad a_1(\infty) = 1 + 2\zeta$$

$$(15) \quad a_3(\infty) = -\frac{1}{9}(1 + 2\zeta)$$

$$(16) \quad a_3(0) = 0.$$

Using these values and the dimensions of the spaces of modular forms of weight 1 and 3, we conclude that  $a_1 = 6(1 + 2\zeta)G_{1,\chi}$  and  $a_3 = (1 + 2\zeta)G_{3,\chi}$ , hence that  $a_1$  and  $a_3$  have desired  $q$ -expansions by (13). We are using the fact that the map  $M_3(\mathbb{Z}_{(2)}) \otimes \mathbb{C} = M_3(\mathbb{C}) \longrightarrow \mathbb{C}^2$ ,  $f \mapsto (f(\infty), f(0))$  is an isomorphism, as follows from the theory of Eisenstein series.

To establish (14) and (15) one has to evaluate  $a_1$  and  $a_3$  at the tuple  $(T(q), \omega_{can}, P)$  consisting of the Tate curve  $T(q)/\mathbb{Z}((q))$ , its canonical invariant differential  $\omega_{can}$  and a specific section  $P \in T(q)(\mathbb{Z}[\zeta]((q)))[3]$ . To do so, one may use J. Tate's uniformisation [Si2, p. 426] to

write  $T(q)/(\mathbb{Z}[[q]]/(q^3))$  in Weierstrass form, the point  $P$  having coordinates  $(X(q, \zeta), Y(q, \zeta))$ . One then uses Weierstrass transformations to bring  $(T(q), \omega_{can}, P)/\mathbb{Z}[[q]]/(q^3)$  to the standard form of Proposition 6. The coefficients  $a_1$  and  $a_3$  of the Weierstrass equation thus obtained are by definition  $a_1(\infty)$  and  $a_3(\infty)$ . The computation for (16) is similar, the point  $P$  has to be replaced by  $Q = (X(q, q^{1/3}), Y(q, q^{1/3}))$ .  $\square$

REMARK 8. In E. Hecke's notation [He], we have  $a_1 = \frac{9i}{\pi} G_1(\tau, 0, 1, 3)$  and  $a_3 = \frac{27i}{4\pi^3} G_3(\tau, 0, 1, 3)$ .

Note that  $a_1(q) \equiv 1 \pmod{2}$ , hence  $a_1 \in M_1(\mathbb{Z}_{(2)}, \Gamma_1(3))$  is a lift of the Hasse invariant for  $p = 2$ . From section 3.1 we know that

$$V_{1,0} = V_{1,1} \stackrel{(9)}{\simeq} M_*(W, \Gamma_1(3))/(2, a_1 - 1) = k[a_3]$$

( $k := W/2W = \mathbb{F}_4$ ) and that, for  $T := \frac{q^0(a_1) - a_1}{2} \in D$ ,  $\bar{T} := (T \pmod{2}) \in D/2D$  is an Artin-Schreier generator for  $V_{1,1} \subseteq V_{1,2}$ , in particular  $\bar{T}^2 + \bar{T} \in V_{1,1} \simeq k[a_3]$  and for later use we will need the following more precise result.

PROPOSITION 9.  $\bar{T}^2 + \bar{T} = 1 + a_3$ .

*Proof.* Recall that the  $q$ -expansion map  $V_{1,1} \subseteq D/pD \hookrightarrow k[[q]]$  is injective [K1, (1.4.6) for  $m = 1$ ]. In  $k[[q]]$  we have  $T = \sum_{n \geq 1} \sigma_0^\chi(n) q^n$  and  $T^2 = \sum_{n \geq 1} \sigma_0^\chi(n) q^{2n}$ , hence

$$T^2 + T = \sum_{n \geq 1} (\sigma_0^\chi(n/2) + \sigma_0^\chi(n)) q^n,$$

where we understand that  $\sigma_0^\chi(n/2) = 0$  for  $n$  odd. To complete the proof, one needs to check that for all  $n \geq 1$  one has

$$\sigma_0^\chi(n/2) + \sigma_0^\chi(n) \equiv \sigma_2^\chi(n) \pmod{2},$$

and we leave this exercise in elementary number theory to the reader.  $\square$

#### 4. f-invariants and Kervaire invariant one

In this section, we compute the f-invariants of two infinite families of  $\beta$ -elements including the Kervaire elements  $\beta_{2^n, 2^n}$  and explain the relation of our results with the Kervaire invariant one problem.

##### 4.1 $f(\beta_t)$ for $t$ not divisible by $p$

Fix a prime  $p$  and the level  $N$  as  $N = 1$  for  $p \geq 5$ ,  $N = 5$  for  $p = 3$  and  $N = 3$  for  $p = 2$ . We keep the notations of section 3.1 for this choice of  $p$  and  $N$ . Given an integer  $t \geq 1$  not divisible by  $p$ , recall that  $\beta_t = \delta' \delta(v_2^t) \in \text{Ext}^{2, 2t(p^2-1)-2(p-1)}[\text{BP}]$  has its f-invariant in  $\underline{\underline{D}}_n \otimes \mathbb{Q}/\mathbb{Z}$ ,  $n := t(p^2 - 1) - (p - 1)$ . When trying to express  $f(\beta_t)$  in terms of divided congruences, we encounter what is in fact the major obstacle at the moment for using the arithmetic of divided congruences in homotopy theory: The group  $\underline{\underline{D}}_n \otimes \mathbb{Q}/\mathbb{Z}$  is not *directly* related to  $D$ . Instead, we have  $\underline{\underline{D}}_n = D + K + M_n(K)$  by definition and there is a canonical surjection

$$\pi : D_n \otimes \mathbb{Q}/\mathbb{Z} \simeq \left( \bigoplus_{i=0}^n M_i(K) \right) / D_n \longrightarrow \underline{D}_n \otimes \mathbb{Q}/\mathbb{Z} \simeq \left( \bigoplus_{i=0}^n M_i(K) \right) / \underline{D}_n$$

which is split because its kernel is divisible, hence  $W$ -injective. In particular,  $\pi$  remains surjective when restricted to  $p$ -torsion

$$(17) \quad \pi : D_n \otimes \mathbb{Q}/\mathbb{Z}[p] \longrightarrow \underline{D}_n \otimes \mathbb{Q}/\mathbb{Z}[p],$$

note that  $f(\beta_t) \in \underline{D}_n \otimes \mathbb{Q}/\mathbb{Z}[p]$ . The group  $D_n \otimes \mathbb{Q}/\mathbb{Z}[p]$  is related to the ring of divided congruences as follows:

$$(18) \quad \psi : D_n \otimes \mathbb{Q}/\mathbb{Z}[p] \xrightarrow{\simeq} D_n/pD_n \hookrightarrow D/pD,$$

where the first arrow is multiplication by  $p$  and the injectivity of the last map is immediate. What we will do is to compute some element in  $D/pD$  in the image of  $\psi$  which under  $\pi$  projects to  $f(\beta_t)$ . At the low risk of confusion we will continue to label such an element, which is in general not unique, as  $f(\beta_t)$ . Recall that we have fixed an elliptic orientation  $\alpha : \text{BP}_* \longrightarrow M_*(W)$  and denote  $T := \frac{\alpha(v_1) - q^0(\alpha(v_1))}{p} \in D/pD$ . We also put  $b := ((q^0(\alpha(v_2))) \bmod p) \in k$ .

**THEOREM 10.** *For an integer  $t \geq 1$  not divisible by the fixed prime  $p$ , we have*

$$f(\beta_t) = b^t - (T^p - T + b)^t \in V_{1,0} \subseteq D/pD.$$

*Proof.* Note first that from section 3.1 we know that  $T$  is an Artin-Schreier generator for  $V_{1,1} \subseteq V_{1,2}$ , hence  $T^p - T \in V_{1,1}$ . A short computation with diamond operators, which we leave to the reader, shows that in fact  $T^p - T \in V_{1,0}$ , hence also  $b^t - (T^p - T + b)^t \in V_{1,0}$ .

We introduce  $a := q^0(\phi(v_1))$  and compute as explained at the end of section 2.2 using the notations introduced there. From  $\eta_R v_2 \equiv v_2 + v_1 t_1^p - v_1^p t_1 \bmod p$  we obtain

$$\eta_R v_2^t \equiv v_2^t + \sum_{i=1}^t \binom{t}{i} v_2^{t-i} v_1^i t_1^i (t_1^{p-1} - v_1^{p-1})^i \bmod p,$$

hence

$$w = \sum_{i=1}^t \binom{t}{i} v_2^{t-i} v_1^{i-1} t_1^i (t_1^{p-1} - v_1^{p-1})^i$$

and

$$\begin{aligned} \nu(w) &= \frac{1}{p} \sum_{i=1}^t \binom{t}{i} (v_2^{t-i} v_1^{i-1} \otimes 1) \left( \frac{1 \otimes v_1 - v_1 \otimes 1}{p} \right)^i. \\ &\left( \left( \frac{1 \otimes v_1 - v_1 \otimes 1}{p} \right)^{p-1} - v_1^{p-1} \otimes 1 \right)^i \in \frac{(\text{BP}_{\mathbb{Q}} \otimes \text{BP}_{\mathbb{Q}})^{(2n)}}{\text{BP}_{2n} \text{BP} + (\text{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \text{BP}_{\mathbb{Q}})^{(2n)}}. \end{aligned}$$

As in section 2.3 we apply  $\iota^2 \circ (\alpha \otimes \alpha)$  to this expression to obtain, denoting  $\alpha(v_1) \in M_{p-1}(W)$  as  $v_1$  for simplicity,

$$\begin{aligned}
& -\frac{1}{p} \sum_{i=1}^t \binom{t}{i} b^{t-i} a^{i-1} \left(\frac{v_1 - a}{p}\right)^i \left(\left(\frac{v_1 - a}{p}\right)^{p-1} - a^{p-1}\right)^i = \\
& -\frac{1}{pa} \left[ -b^t + \left( a \left(\frac{v_1 - a}{p}\right) \left(\left(\frac{v_1 - a}{p}\right)^{p-1} - a^{p-1}\right) + b \right)^t \right] = \\
& \frac{-1}{pa} \left( \left( a \left(\frac{v_1 - a}{p}\right) \left(\left(\frac{v_1 - a}{p}\right)^{p-1} - a^{p-1}\right) + b \right)^t - b^t \right) \in \bigoplus_{i=0}^n M_i(K).
\end{aligned}$$

This is a representative for  $f(\beta_i)$  in  $D_n \otimes \mathbb{Q}/\mathbb{Z}[p]$  to which we have to apply the map  $\psi$  from (18) to obtain an element in  $D/pD$ . For this, note that  $a \equiv 1 \pmod p$  because  $v_1$  reduces to the Hasse invariant mod  $p$ . This allows us to put  $a = 1$  in the above expression (but not to replace  $\frac{v_1 - a}{p}$  by  $\frac{v_1 - 1}{p}$ ; this would require the congruence  $a \equiv 1 \pmod{p^2}$ , which does not hold in general). We then obtain indeed

$$-((T(T^{p-1} - 1) + b)^t - b^t) = b^t - (T^p - T + b)^t.$$

□

REMARK 11. Assume that  $p \geq 5$  in the situation of Theorem 10. In general, the elliptic orientation will not map  $v_1$  to the Eisenstein series  $E_{p-1}$  of weight  $p - 1$  and level one. But  $\alpha(v_1)$  and  $E_{p-1}$  can only differ by a modular form divisible by  $p$  and we may thus change the orientation to force  $\alpha(v_1) = E_{p-1}$ . Assuming this, we see that  $f(\beta_{1,1,1}) = \frac{E_{p-1}-1}{p^2} - \frac{1}{p}(\frac{E_{p-1}-1}{p})^p$ , as first computed by G. Laures [L1, p. 414] (where the second summand is missing).

REMARK 12. The injectivity of the  $f$ -invariant together with the known structure of  $\text{Ext}^2[\text{BP}]$  provides some non-trivial information about the arithmetic of divided congruences as follows. Fix some  $x \in \text{Ext}^{2,k}[\text{BP}]$  of order  $p^r$ . Then  $f(x) \in \underline{\mathbb{D}}_k \otimes \mathbb{Q}/\mathbb{Z}$  will be of order  $p^r$ , hence a representative of  $f(x)$  in  $D_k \otimes \mathbb{Q}/\mathbb{Z}$  will be of order  $p^s$  for some  $s \geq r$ . Thus the  $f$ -invariant relates the order of a  $\beta$ -element to the (non-)existence of a certain divided congruence.

Let us assume that  $r = 1$  as is the case for all  $\beta_{t,s,r}$  considered in this article. Then our results show that our representatives in  $D \otimes \mathbb{Q}/\mathbb{Z}$  have order  $p$  and the non-trivial additional information on divided congruences is then that they do not lie in the kernel of  $D \otimes \mathbb{Q}/\mathbb{Z} \rightarrow \underline{\mathbb{D}} \otimes \mathbb{Q}/\mathbb{Z}$ .

To give an example, assume that we are in the situation of Remark 11. The arithmetic of divided congruences shows that  $F := \frac{E_{p-1}-1}{p^2} - \frac{1}{p}(\frac{E_{p-1}-1}{p})^p \in D_{p(p-1)} \otimes \mathbb{Q}/\mathbb{Z}$  is of order  $p$ , i.e.  $F \in \bigoplus_{i=0}^{p(p-1)} M_i(K)$  has a  $q$ -expansion with denominator exactly  $p$ . The additional information is then that for any  $\alpha \in K$  and  $f \in M_{p(p-1)}(K)$  the  $q$ -expansion of  $F + \alpha + f$  will still have exact denominator  $p$ .

EXAMPLE 13. Fix  $p = 5$  and set  $g_2 := \frac{1}{12}E_4$  and  $g_3 := \frac{-1}{216}E_6$  as in [K1]. The comparison of the logarithm of the universal  $p$ -typical formal group law [R] and the corresponding coefficients of the logarithm of the elliptic curve  $(E, \omega)$  [K1, (5.0.3)] ( $p$ -typification does not change these coefficients) shows that the orientation  $a$  maps  $v_1$  to  $a_p$  and  $v_2$  to  $\frac{a_{p^2} - a_p^{p+1}}{p}$ , the  $a_i$  denoting the normalised (multiplied with  $-1/2$ )  $a_i$  of [K1, p. 351]. One deduces that  $v_1$  maps to  $-8g_2$  and  $v_2$  maps to  $\frac{a_{25} - a_5^6}{5}$ . A computation with Maple shows that  $a_{25} = 129761280g_2^3g_3^2 + 32440320g_3^4 + 3784704g_2^6$  (and also that the correct value for the unnormalised  $a_{11}$  is  $-2520g_2g_3$  and not  $-512g_2g_3$ ). It follows that  $q^0(v_1) = -\frac{2}{3}$  and  $q^0(v_2) = -\frac{4900}{3^{10}}$ , so Theorem 10 may be rephrased in terms of the Eisenstein series  $g_2$  and  $g_3$ .

## 4.2 Projecting to the Kervaire element

In this section, we compute  $f(\beta_{s2^n, 2^n})$  for  $n \geq 0$  and  $s \geq 1$  odd at the prime  $p = 2$ . Using this, we are able to determine a single coefficient in the f-invariant of a  $(U, fr)^2$ -manifold of dimension  $2^n$  the non-vanishing of which is necessary and sufficient for the corner of  $X$  to be a Kervaire manifold, that is having Kervaire invariant one. See [L2] for the notion of cobordism of manifolds with corners. We begin by recalling the well-known relation of the Kervaire invariant to certain  $\beta$ -elements, due to W. Browder [B]. Fix some  $n \geq 3$ . We have a homomorphism

$$K : \pi_{2^n-2}^s \longrightarrow \mathbb{Z}/2$$

which sends the class of a stably framed manifold to its Kervaire invariant. Consider on the other hand the composition

$$K' : \pi_{2^n-2}^s \longrightarrow \pi_{2^n-2}^s[2] \longrightarrow E_\infty^{2,2^n}[\mathbb{H}\mathbb{Z}/2] \hookrightarrow E_2^{2,2^n}[\mathbb{H}\mathbb{Z}/2] = \mathbb{Z}/2 \cdot h_{n-1}^2.$$

Here, the first map is the projection to the 2-primary part, the second is the projection onto  $F^2/F^3$  in the (classical) Adams spectral sequence at  $p = 2$ , the third is an edge homomorphism and the final equality is due to J. Adams, [R, 3.4.1, c)].

PROPOSITION 14.  $K = K'$ .

*Proof.* For any  $y \in \pi_{2^n-2}^s[2]$  we have  $K(y) = 1$  if and only if  $y$  has Adams filtration 2. This is implicit in [B], c.f. [BJM2, p. 144].  $\square$

We can easily obtain a similar homotopy theoretic description of  $K$  using BP instead of  $\mathbb{H}\mathbb{Z}/2$ .

PROPOSITION 15. *Let  $n \geq 2$ . Then  $\text{Ext}^{2,2^n}$  is a direct sum of cyclic groups of order 2. It is generated by the element  $\alpha_1 \cdot \alpha_{2^n-1-1}$  and the elements  $\beta_{s2^i, 2^i}$  with  $s$  odd and  $i \geq 0$  such that  $(3s-1)2^{i+1} = 2^n$  and the case  $(s, i) = (1, 0)$  has to be omitted.*

*Proof.* This follows from [R, Corollary 5.4.5]. Observe that the  $\bar{\alpha}_t$  in loc. cit equals  $\alpha_t$  as  $t$  is odd, see [Sh, Theorem 1.5] or [R, Theorem 5.2.6].  $\square$

REMARK 16. *The Lemma shows that the number of generators of  $\text{Ext}^{2,2^n}$  is  $[n/2] + 1$  for  $n \geq 3$ . The low dimensional cases are as follows.*

$$\begin{aligned} \text{Ext}^{2,4} & : \alpha_1^2 \\ \text{Ext}^{2,8} & : \alpha_1 \alpha_3, \beta_{2,2} \\ \text{Ext}^{2,16} & : \alpha_1 \alpha_7, \beta_{4,4}, \beta_{3,1} \\ \text{Ext}^{2,32} & : \alpha_1 \alpha_{15}, \beta_{8,8}, \beta_{6,2} \\ \text{Ext}^{2,64} & : \alpha_1 \alpha_{31}, \beta_{16,16}, \beta_{12,4}, \beta_{11,1} \\ \text{Ext}^{2,128} & : \alpha_1 \alpha_{63}, \beta_{32,32}, \beta_{24,8}, \beta_{22,2} \\ \text{Ext}^{2,256} & : \alpha_1 \alpha_{127}, \beta_{64,64}, \beta_{48,16}, \beta_{44,4}, \beta_{43,1}. \end{aligned}$$

Now we consider the composition

$$K'' : \pi_{2^n-2}^s \longrightarrow \pi_{2^n-2}^s[2] \longrightarrow E_\infty^{2,2^n}[\text{BP}] \hookrightarrow E_2^{2,2^n}[\text{BP}] \longrightarrow \mathbb{Z}/2 \cdot \beta_{2^n-2, 2^n-2}$$

which is defined in analogy with  $K'$ , the final map being the projection to the  $\mathbb{Z}/2$ -summand generated by  $\beta_{2^n-2, 2^n-2}$ .

PROPOSITION 17.  $K'' = K$ .

*Proof.* We have the Thom reduction  $\Phi : \text{Ext}^*[\text{BP}] \longrightarrow \text{Ext}^*[\text{HZ}/2]$  which satisfies  $\Phi(\beta_{2^n-2, 2^n-2}) = h_{n-1}^2$  and is zero on all other generators of  $\text{Ext}^{2,2^n}[\text{BP}]$ , see [R, 5.4.6, a)] and Proposition 15. The result then follows from Proposition 14.  $\square$

Let  $X$  be a  $(U, fr)^2$ -manifold of dimension  $2^n$ . From the above, we see that the corner of  $X$  is a Kervaire manifold if and only if the f-invariant of  $X$  contains  $\beta_{2^n, 2^n}$  as a summand. Thus, one certainly wants a more geometric description of the f-invariant (or just its projection to  $\beta_{2^n-2, 2^n-2}$ ). In principle, it is possible to obtain such a description in terms of Chern numbers of  $X$ , simply because they determine the  $(U, fr)^2$ -bordism class of  $X$  [L2], but the necessary computations become quite complicated already in low dimensions. At the end of this section, we will explain how divided congruences might simplify such computations. We then would like to generalise Theorem 18 below to higher dimensions.

Recall [L2, section 4.1] that if  $X$  is a  $(U, fr)^2$ -manifold then there is a decomposition of its stable tangent bundle  $TX = TX^{(0)} \oplus TX^{(1)}$  and we have Chern classes  $c_i^{(j)} \in H^{2i}(X, \mathbb{Z})$  accordingly ( $i \geq 0, j = 0, 1$ ).

THEOREM 18. a) Let  $X$  be a  $(U, fr)^2$ -manifold of dimension 4 and put  $q := \langle c_1^{(0)} c_1^{(1)}, [X] \rangle \in \mathbb{Z}$ . Then  $q$  is odd if and only if the corner of  $X$  has Kervaire invariant 1. If  $q$  is even, then the corner of  $X$  is the boundary of a framed manifold.

b) Let  $X$  be a  $(U, fr)^2$ -manifold of dimension 8 and put  $q := \langle c_1^{(0)}(c_1^{(1)3} + c_1^{(1)}c_2^{(1)} + c_3^{(1)}) + (c_2^{(0)} + c_1^{(0)2})(c_2^{(1)} + c_1^{(1)2}), [X] \rangle \in \mathbb{Z}$ . Then  $q$  is odd if and only if the corner of  $X$  has Kervaire invariant 1. If  $q$  is even, then the corner of  $X$  is the boundary of a framed manifold.

*Proof.* If  $\sum_{i \geq 0} l_i x^{2i}$  is the logarithm of the universal 2-typical formal group law, then (see [L2, Example 4.2.4])

$$\exp(x) = x - l_1 x^2 + 2l_1^2 x^3 - (5l_1^3 + l_2) x^4 \pmod{x^5}$$

and thus

$$Q(x) := \frac{x}{\exp(x)} = 1 + l_1 x - l_1^2 x^2 + (2l_1^3 + l_2) x^3 \pmod{x^4}.$$

For indeterminates  $x_i$  of dimension 2 we set  $\Pi := \prod_i Q(x_i)$ . Denoting by  $c_i$  the  $i$ -th elementary symmetric function in the  $x_i$  one gets (using the definition of the Hazewinkel generators)

$$\Pi^{(2)} = \frac{v_1}{2} c_1$$

$$\Pi^{(4)} = \frac{v_1^2}{4} (3c_2 - c_1^2) \text{ and}$$

$$\Pi^{(6)} = \frac{v_1^3}{8} (4c_1^3 - 13c_1 c_2 + 16c_3) + \frac{v_2}{2} (c_1^3 - 3c_1 c_2 + 3c_3).$$

To prove part a) one has that  $\mathrm{BP}_{\mathbb{Q}}^{\otimes 2,4}/(\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})$  is a one-dimensional  $\mathbb{Q}$ -vector space generated by  $v_1 \otimes v_1$ . Moreover, one checks that the image of  $\mathrm{BP}_4\mathrm{BP}$  is generated (over  $\mathbb{Z}_{(2)}$ ) by  $\frac{v_1 \otimes v_1}{2}$  and that  $\frac{v_1 \otimes v_1}{4}$  is a representative of  $\alpha_1^2$ . To see the latter, observe that  $\alpha_1$  is represented by  $t_1$ , hence [R, A.1.2.15]  $\alpha_1^2$  is represented in the cobar complex by  $t_1 \otimes t_1 = (1 \otimes t_1)(t_1 \otimes 1)$ , and then one applies the description of  $\delta^{-1}$  given in section 2.2. Using the notations introduced in [L2], one computes that

$$K_{\mathrm{BP}\langle 2 \rangle}(TX)^{(4)} = c_1^{(0)} c_1^{(1)} \frac{v_1 \otimes v_1}{4} \text{ in } \mathrm{BP}_{\mathbb{Q}}^{\otimes 2,4}/(\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(4)},$$

hence the image of the corner of  $X$  in  $\mathrm{Ext}^{2,4}$  is represented by  $\frac{q}{2} \cdot \frac{v_1 \otimes v_1}{2} = q \cdot \alpha_1^2$ . The final assertion follows because the only non-trivial element of  $\pi_2^s$  has Adams-Novikov filtration precisely 2.

For part b), we know that  $\mathrm{Ext}^{2,8}$  is generated by  $\alpha_1 \bar{\alpha}_3$  and  $\beta_{2,2}$ . As  $\beta_{2,2}$  is a permanent cycle in the ANSS whereas  $\alpha_1 \bar{\alpha}_3$  is not we know that the image of  $X$  in  $\mathrm{Ext}^{2,8}$  is a multiple of  $\beta_{2,2}$ . One computes that in the notation of section 2.2  $\delta(v_2^2)$  is represented by  $z = t_1^4 + v_1^2 t_1^2$  in  $C^1(A/2)$ . Hence  $\beta_{2,2} = \delta' \delta(v_2^2)$  is represented in  $\mathrm{BP}_{\mathbb{Q}}^{\otimes 2,8}/(\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(8)}$  by  $-\frac{1}{8}(v_1 \otimes v_1^3) + \frac{5}{16}(v_1^2 \otimes v_1^2) - \frac{3}{8}(v_1^3 \otimes v_1)$ . Computing enough of the image of  $v_1^3 t_1, v_1^2 t_1^2, v_1^3 t_1$  and  $t_1^4$  under  $\mathrm{BP}_8\mathrm{BP} \hookrightarrow \mathrm{BP}_8\mathrm{BP} \otimes \mathbb{Q} \simeq \mathrm{BP}_{\mathbb{Q}}^{\otimes 2,8} \rightarrow \mathrm{BP}_{\mathbb{Q}}^{\otimes 2,8}/(\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(8)}$ , one sees that  $\frac{v_1^2 \otimes v_1^2}{8} \in \mathrm{im}(\mathrm{BP}_8\mathrm{BP})$  and  $\beta_{2,2}$  is represented by  $\frac{v_1^2 \otimes v_1^2}{16}$ . We provide the following argument for general  $n$ , for the proof here we need the case  $n = 3$ . Observe that by the computations of the previous sections, the image of  $\mathrm{Ext}^{2,2^n}$  in  $\mathrm{BP}_{\mathbb{Q}}^{\otimes 2,2^n}/(\mathrm{BP}_{\mathbb{Q},2^n} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q},2^n} + \mathrm{BP}_{2^n}\mathrm{BP})$  is given by representatives consisting of summands of the form  $v_1^i v_2^k \otimes v_1^j$  with rational coefficients. Moreover, these elements map under the isomorphism  $\phi$  of section 2.2 to polynomials in  $v_1, v_2$  and  $t_1$ . In other words, the f-invariant in bidegree  $(2, 2^n)$  factors through the subgroup generated by elements  $v_1^i v_2^k \otimes v_1^j$  modulo elements of the form  $\phi^{-1}(v_1^i v_2^j t_1^k)$ ,  $1 \otimes v_1^j$  and  $v_1^i v_2^j \otimes 1$ . Denote this quotient by  $B_{2^n}$ . A computation using the results of the previous section shows that  $\frac{c}{2^{2^n-1}} v_1^{2^n-2} \otimes v_1^{2^n-2}$  is not zero in  $B_{2^n}$  for  $c$  an odd integer. More precisely, no element in the relations defining the quotient  $B_{2^n}$  contains such a summand. Among the elements in  $\mathrm{Ext}^{2,2^n}$ , the image of  $\beta_{2^{n-2}, 2^{n-2}}$  in  $B_{2^n}$  contains such a summand and the other generators exhibited in Proposition 15 do not. Thus we have a well-defined map  $\mathrm{Ext}^{2,2^n} \rightarrow \mathbb{Z}/2$  given by mapping an element to 1 if it admits a representative in  $B_{2^n}$  having a summand  $\frac{c}{2^{2^n-1}} v_1^{2^n-2} \otimes v_1^{2^n-2}$  with  $c$  odd. This map is a projection to  $\beta_{2^{n-2}, 2^{n-2}}$ . We would like to consider the summands of

$$K_{\mathrm{BP}\langle 2 \rangle}(TX)^{(8)} \equiv \Pi^{(2)} \otimes \Pi^{(6)} + \Pi^{(4)} \otimes \Pi^{(4)} + \Pi^{(6)} \otimes \Pi^{(2)} \text{ mod } (\mathrm{BP}_{\mathbb{Q}} \otimes \mathbb{Q} + \mathbb{Q} \otimes \mathrm{BP}_{\mathbb{Q}})^{(8)}$$

as elements in  $B_{2^3}$ . Once this is achieved, we have to consider only the summands involving  $v_1^2 \otimes v_1^2$ . The only summand which is not already given by a representative in  $B_{2^3}$  is  $c_1^{(0)}(c_1^{(1)3} - 3c_1^{(1)}c_2^{(1)} + 3c_3^{(1)})\frac{v_1 \otimes v_2}{4}$  in  $\Pi^{(2)} \otimes \Pi^{(6)}$ . One computes (for  $p = 2$  and the Hazewinkel generators as before) that  $\phi^{-1}(t_2) = \frac{1 \otimes v_2}{2} - \frac{v_2 \otimes 1}{1} + \frac{1 \otimes v_1^3}{4} - \frac{v_1 \otimes v_1^2}{8} + \frac{v_1^2 \otimes v_1}{4} - 3\frac{v_1^3 \otimes 1}{8}$ . Hence we have  $\phi^{-1}(t_1 t_2) = -\frac{v_1 \otimes v_2}{4} - \frac{v_1^2 \otimes v_1^2}{16} + \dots$ , so we have to look at the coefficients of  $\frac{v_1 \otimes v_2}{4}$  and  $\frac{v_1^2 \otimes v_1^2}{16}$  which by the above equal  $c_1^{(0)}(c_1^{(1)3} - 3c_1^{(1)}c_2^{(1)} + 3c_3^{(1)})$  and  $(3c_2^{(0)} - c_1^{(0)2})(3c_2^{(1)} - c_1^{(1)2})$ . We also have that  $c_1^{(0)}(c_1^{(1)3} - 3c_1^{(1)}c_2^{(1)} + 3c_3^{(1)})$  equals  $c_1^{(0)}(c_1^{(1)3} + c_1^{(1)}c_2^{(1)} + c_3^{(1)})$  and  $(3c_2^{(0)} - c_1^{(0)2})(3c_2^{(1)} - c_1^{(1)2})$  equals  $(c_2^{(0)} + c_1^{(0)2})(c_2^{(1)} + c_1^{(1)2})$  modulo 2 when evaluated on  $[X]$ . Now the assertion follows as in part a).  $\square$

Of course, it is possible to do similar but more complicated computations for  $2^n$ -dimensional  $(U, fr)^2$ -manifolds in case  $n \geq 4$ . We always have a projection to  $\mathbb{Z}/2$  looking at the power of 2 in the denominator of the coefficient of the summand  $v_1^{2^n-2} \otimes v_1^{2^n-2}$ . The computation then reduces to compute those  $\Pi^{(2^i)}$  which contribute to this summand. The diligent reader may thus

prove statements of the following form: The element  $h_{n-1}^2$  survives (equivalently: there is a framed manifold in dimension  $2^n - 2$  having Kervaire invariant 1) if and only if  $\langle F_n(c_i^{(0)}, c_j^{(1)}), [X] \rangle$  is odd for a certain explicit polynomial  $F_n$ . The main problem in the computation of  $F_n$  is to find representatives in  $B_{2^n}$  (that is sums of  $v_1^i v_2^k \otimes v_1^j$ ) for elements arising in the  $\Pi^{(2^n-i)} \otimes \Pi^{(i)}$ . For  $n = 3$  this was done using  $t_1 t_2$ . In the case  $n > 3$ , it will be necessary to find suitable elements in  $\text{BP}_{2^n} \text{BP}$  which will involve  $t_i$  for larger  $i$  and the computation of  $\phi^{-1}$  of these elements.

The rest of this section is devoted to the computation of the f-invariant in dimension  $2^n$  at the prime 2. More generally, we compute the f-invariant of  $\beta_{s2^n, 2^n} \in \text{Ext}^{2, (3s-1)2^{n+1}}$  for all  $n \geq 0$  and  $s \geq 1$  odd.

We use the notations of section 3.1 for  $p = 2$  and  $N = 3$  and those of section 3.2 and write  $T := \frac{a_1-1}{2} \in D/2D$  which is an Artin-Schreier generator for the extension  $V_{1,0} = V_{1,1} \simeq k[a_3] \subseteq V_{1,2}$ .

**THEOREM 19.** *The image of the f-invariant in  $V_{1,2}$  is given by*

$$\begin{aligned} f(\alpha_1 \alpha_{2^{n+1}-1}) &= T \text{ for } n \geq 0, \\ f(\beta_s) &= 1 + a_3^s \text{ for } s \geq 3 \text{ odd}, \\ f(\beta_{s2,2}) &= 1 + a_3^{2s} \text{ for } s \geq 1 \text{ odd and} \\ f(\beta_{s2^n, 2^n}) &= (a_3^{2^n} + a_3^{3 \cdot 2^{n-2}})^s \text{ for } s \geq 1 \text{ odd and } n \geq 2. \end{aligned}$$

*Proof.* For the first line, recall that mod 2 we have  $\alpha_t := \alpha_{t,1} := \delta(v_1^t)$ . One computes that in the cobar complex  $\alpha_1 \alpha_t$  is represented by  $t_1 \otimes \frac{1}{2}[(2t_1 + v_1)^t - v_1^t]$ , use the description of the product in the cobar complex of [R, A.1.2.15]. Using that  $\phi$  is a ring isomorphism and the description of  $\delta^{-1}$ , see section 2.2, one further computes that  $\alpha_1 \alpha_t$  is represented by  $-\frac{1}{4}v_1 \otimes v_1^t$  in the usual quotient of  $\text{BP}_{\mathbb{Q}}^{\otimes 2}$ , c.f. (7).

The second line is a special case of Theorem 10 (use Proposition 9) and implies the third line because  $x_1 \equiv x_0^2 = v_2^2 \pmod{v_1^2}$ , recall the invariant sequences  $(2, v_1^{2^n}, x_n)$  from section 2.1.

The only case requiring a longer computation is  $f(\beta_{4,4})$ :

In the notation of section 2.1 we have  $r = 1, s = t = 4$  and

$$x_2 = v_2^4 - v_1^3 v_2^3 \in \text{H}^{0,24}(\text{BP}/(2, v_1^4)).$$

This value of  $x_2$  follows from the definition in [MRW, p. 476] or [R, Theorem 5.2.13]) after cancelling all possible multiples of  $v_1^4$ . One computes that, in the notation of section 2.2,  $z \in (\overline{\Gamma}/2)^{16}$  is given as

$$\begin{aligned} z &= t_1^8 + v_1^4 t_1^4 + v_2^2 t_1(t_1 + v_1) + v_1 v_2 t_1^2(t_1^2 + v_1^2) + v_1^2 t_1^3(t_1 + v_1)^3 \\ &= v_1 v_2^2 t_1 + v_2^2 t_1^2 + v_1^3 v_2 t_1^2 + v_1^5 t_1^3 + v_1 v_2 t_1^4 + v_1^3 t_1^5 + v_1^2 t_1^6 + t_1^8. \end{aligned}$$

One then computes that  $f(\beta_{4,4}) = \nu(w)$ , where  $w \in \overline{\Gamma}$  is a lift of  $z$  as in section 2.2, is as claimed, using the relation in Proposition 9 and that  $q^0(a_1) = q^0(a_3) = 1 \pmod{2}$  (see Proposition 7).

Now the value for  $f(\beta_{s4,4})$  for any  $s$  follows immediately from the derivation property of the connecting homomorphism  $\delta$ , namely  $\delta(x^n) = \delta(x)(\sum_{i=0}^{n-1} \eta_R(x)^i \eta_L(x)^{n-1-i})$ . Alternatively,  $f(\beta_{s4,4})$  may be computed directly for any odd  $s$  using that  $(q^0 \otimes \text{id})\eta_L(x_2) = 0 \pmod{2}$ ,  $\eta_R$  and  $\phi^{-1}$  are ring homomorphisms and Proposition 9. We obtain  $f(\beta_{s2^n, 2^n}) = f(\beta_{s4,4})^{n-2}$  for all  $s$  and  $n \geq 3$  because  $x_n = x_{n-1}^2$  for all  $n \geq 3$ .  $\square$

Fix some  $n \geq 3$ . To explain the relevance of the above computation for the problem of projecting the f-invariant to  $\beta_{2^{n-2}, 2^{n-2}}$  we contemplate the following diagram.

$$(19) \quad \begin{array}{ccccc} & & & \xrightarrow{\pi'} & \mathbb{Z}/2 \\ & & & \nearrow & \downarrow \\ \text{Ext}^{2,2^n}[\text{BP}] & \xrightarrow{f} & \underline{\underline{D}}_{2^{n-1}} \otimes \mathbb{Q}/\mathbb{Z}[2] & & \\ & & \uparrow (17) & & \\ & & D_{2^{n-1}} \otimes \mathbb{Q}/\mathbb{Z}[2] & \xrightarrow{(18)} & D/2D \\ & \searrow \iota & \uparrow & \uparrow \text{(section 3.1)} & \\ & & V_{1,2} & \xrightarrow{\pi} & k = \mathbb{F}_4 \end{array}$$

By the results in section 3.1,  $V_{1,2}$  is  $k$ -free on the set  $\{a_3^i T^j | i \geq 0, j = 0, 1\}$  and  $\pi$  is defined to be the projection to the coefficient of  $a_3^{2^{n-2}}$ . The map  $\pi'$  is defined to be the projection to the generator  $\beta_{2^{n-2}, 2^{n-2}}$ , c.f. Proposition 15. Theorem 19 determines representatives in  $\tilde{V}_{1,2} := D_{2^{n-1}} \otimes \mathbb{Q}/\mathbb{Z}[2] \cap V_{1,2}$  for all generators of  $\text{Ext}^{2,2^n}[\text{BP}]$  and thus defines the map  $\iota$ . We know that (19) commutes when  $\pi$  and  $\pi'$  are omitted.

**THEOREM 20.** *The diagram (19) is commutative.*

*Proof.* By inspection of Proposition 15 and Theorem 19, the only generator of  $\text{Ext}^{2,2^n}[\text{BP}]$  whose  $f$ -invariant contains  $a_3^{2^{n-2}}$  is the Kervaire element  $\beta_{2^{n-2}, 2^{n-2}}$ .  $\square$

**COROLLARY 21.** *Let  $n \geq 3$  and  $X$  a  $(U, fr)^2$ -manifold of dimension  $2^n$ . Then the corner of  $X$  has Kervaire invariant one if and only if the  $f$ -invariant of  $X$  admits a representative in  $\tilde{V}_{1,2}$  which contains the summand  $a_3^{2^{n-2}}$ .*

Note that the coefficient of  $a_3^{2^{n-2}}$  in the  $f$ -invariant of  $X$  can rather easily be expressed in terms of Chern numbers of  $X$ . The very reason that this does not give us the Chern numbers determining the Kervaire elements is the indeterminacy in the above constructions caused by the projection  $D_{2^{n-1}} \otimes \mathbb{Q}/\mathbb{Z}[2] \longrightarrow \underline{\underline{D}}_{2^{n-1}} \otimes \mathbb{Q}/\mathbb{Z}[2]$ .

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