LOCAL COHOMOLOGY OF BP_*BP-COMODULES

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Abstract. Given a spectrum X, we construct a spectral sequence of BP_*BP-comodules that converges to $BP_*(L_nX)$, where L_nX is the Bousfield localization of X with respect to the Johnson-Wilson theory $E(n)_*$. The E_2-term of this spectral sequence consists of the derived functors of an algebraic version of L_n. We show how to calculate these derived functors, which are closely related to local cohomology of BP_*-modules with respect to the ideal I_{n+1}.

Introduction

The most common approach to understanding stable homotopy theory involves localization. One first localizes at a fixed prime p; after doing so there is a tower of localization functors

$$\cdots L_n \to L_{n-1} \to \cdots \to L_1 \to L_0.$$

called the chromatic tower [Rav92, Section 7.5]. Each functor L_n retains a little more information than the previous one L_{n-1}; the chromatic convergence theorem [Rav92, Theorem 7.5.7] says that the homotopy inverse limit of the L_nX is X itself for a finite p-local spectrum X. These localization functors come from the Brown-Peterson homology theory BP, where

$$BP_*(S^0) \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

with $|v_i| = 2(p^i - 1)$. There are different possible choices for the generators v_n, but the ideals $I_n = (p, v_1, \dots, v_{n-1})$ for $0 \leq n \leq \infty$ are canonical. The functor L_n is Bousfield localization with respect to $v_n^{-1}BP$; all the different choices for v_n give the same localization functor.

In previous work [HS03], the authors constructed an algebraic endofunctor L_n on the category of BP_*BP-comodules, analogous to the chromatic localization L_n on spectra. This functor L_n is the localization obtained by inverting all maps of comodules whose kernel and cokernel are v_n-torsion (or, equivalently, I_{n+1}-torsion). The L_n-local comodules are equivalent to the category of $E(n)_*E(n)$-comodules, or to the category of E_*E-comodules for any Landweber exact commutative ring spectrum with $E_*/I_{n+1} = 0$ but $E_*/I_n \neq 0$.

In [HS03], our main interest was algebraic. In this paper, we compare our algebraic version of L_n with the topological one. As always, when one has a topological version of an algebraic construction, one expects a spectral sequence converging to the topological construction whose E_2-term involves the derived functors of the algebraic construction. Since the algebraic L_n is left exact, it has right derived functors L_n^i. We prove the following theorem.

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Theorem A. *Let X be a spectrum. There is a natural spectral sequence $E_*^{**}(X)$ with $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ and E_2-term $E_2^{s,t}(X) \cong (L_n^s BP_*X)_t$, converging strongly to $BP_{t-s}(L_nX)$. This is a spectral sequence of BP_*BP-comodules, in the sense that $E_r^{s,*}$ is a graded BP_*BP-comodule for all $r \geq 2$ and $d_r\colon E_r^{s,*} \to E_r^{s+r,*}$ is a BP_*BP-comodule map of degree $r-1$. Furthermore, every element in $E_2^{0,*}$ that comes from BP_*X is a permanent cycle.*

For this spectral sequence to be useful, we need to be able to compute the E_2-term. The derived functors L_n^i turn out to be closely related to local cohomology, which is well-known in commutative algebra and was introduced to algebraic topology by Greenlees [Gre93]. Recall from [GM95] that, given an ideal I in a ring R, one can form the local cohomology $H_I^*(-)$ and the Cech cohomology $\check{C}H_I^*(-)$ of an R-module M. Although it is not phrased this way in [GM95], the functor $\check{C}H^0_{I_{n+1}}$ on the category of BP_*-modules is the localization functor that inverts all maps of modules whose kernel and cokernel are v_n-torsion. Thus $\check{C}H^0_{I_{n+1}}$ is the analog of L_n in the category of BP_*-modules, and hence Cech cohomology is simply the derived functors of this localization functor on the category of BP_*-modules.

The following theorem describes the behavior of L_n itself.

Theorem B. *Let M be a BP_*BP-comodule.*

(1) $L_nM \cong \check{C}H^0_{I_{n+1}}M$.
(2) *If v_j acts isomorphically on M for some $0 \leq j \leq n$, then $L_nM = M$.*
(3)
$$L_n(BP_*/I_k) = \begin{cases} BP_* \otimes \mathbb{Q} & \text{if } k = n = 0; \\ v_n^{-1}BP_*/I_n & \text{if } k = n > 0; \\ BP_*/I_k & \text{if } k < n; \\ 0 & \text{if } k > n. \end{cases}$$
(4) *L_n commutes with filtered colimits, arbitrary direct sums, and finite limits.*

Part (1) of this theorem is proved in Theorem 4.5, part (2) in Proposition 1.6, part (3) in Proposition 1.1(c), Corollary 1.5, and Corollary 1.7, and part (4) in Proposition 1.8.

The derived functors of L_n are described in the following theorem.

Theorem C. *Let M be a BP_*BP-comodule.*

(1) *We have $L_n^iM \cong \check{C}H^i_{I_{n+1}}M$.*
(2) *$L_n^i(M) = 0$ for $i > n$.*
(3) *$L_n^i(M)$ is I_{n+1}-torsion for all $i > 0$.*
(4) *If v_j acts isomorphically on M for some j, then $L_n^iM = 0$ for $i > 0$.*
(5) *If $i > 0$, then $L_n^i(BP_*/I_k) = 0$ unless $k < n$ and $i = n-k$, in which case we have*
$$L_n^{n-k}(BP_*/I_k) = BP_*/(p, v_1, \dots, v_{k-1}, v_k^\infty, \dots, v_n^\infty).$$
(6) *For $i > 0$, L_n^i commutes with filtered colimits and arbitrary direct sums.*

Part (1) of Theorem C is proved in Theorem 4.5, part (2) in Theorem 3.7, part (3) in Proposition 3.4, part (4) in Theorem 3.5, part (5) in Corollary 3.6, and part (6) in Theorem 3.8.

Most of Theorem C, except part (6), would follow from part (1) of it, and known facts about local cohomology. However, local cohomology is generally considered only for Noetherian rings, and BP_* is not Noetherian. This turns out not to be a problem, but because there is no discussion of non-Noetherian local cohomology in the literature, and because it is not very hard, we offer direct proofs of the remaining parts of Theorem C.

In the light of Theorem C the reader may naturally wonder whether there is a connection between the local cohomology spectral sequence of [Gre93] and [GM95] and our spectral sequence. Recall that Greenlees and May begin with the category of modules over a strictly commutative ring spectrum; since BP is not known to be such, we must begin with MU. Combining Theorems 5.1 and 6.1 of [GM95], and applying them to the MU-module spectrum $MU \wedge X$, then gives a spectral sequence converging to $MU_*(L_nX)$ whose E_2-term is $\check{C}H^{-s,-t}_{I_{n+1}}(MU_*X)$. Our spectral sequence would have the same E_2-term as this Greenlees-May spectral sequence if we used MU instead of BP (and we reindexed the spectral sequence). However, our construction allows us to conclude that we have a spectral sequence **of comodules**, which the Greenlees-May construction does not. This significantly restricts the possible differentials and extensions that can occur in the spectral sequence.

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1. The functor L_n

In this section, we define our localization functor L_n and prove almost all of Theorem B, though we postpone Theorem B(1) to Section 4.

In this section, and throughout the paper, n will be a fixed nonnegative integer, and

$$\Phi\colon (BP_*, BP_*BP) \to (E(n)_*, E(n)_*E(n))$$

will denote the evident map of Hopf algebroids. There is an induced exact functor

$$\Phi_*\colon BP_*BP\text{-comod} \to E(n)_*E(n)\text{-comod}$$

of the categories of graded comodules that takes M to $E(n)_* \otimes_{BP_*} M$. As explained in [Hov04, Proposition 1.2.3], Φ_* has a left exact right adjoint

$$\Phi^*\colon E(n)_*E(n)\text{-comod} \to BP_*BP\text{-comod}.$$

On relatively injective comodules, Φ^* is defined by

$$\Phi^*(E(n)_*E(n) \otimes_{E(n)_*} N) = BP_*BP \otimes_{BP_*} N.$$

Since every comodule is a kernel of a map between relatively injective comodules, and Φ^* is left exact, this determines Φ^* in general. Explicitly, as pointed out by Mark Behrens,

$$\Phi^*(N) = (BP_*BP \otimes_{BP_*} E(n)_*)\square_{E(n)_*E(n)} N,$$

where the symbol $\square$ denotes the cotensor product (see [Rav86, Lemma A1.1.8]).

We prove in Section 2 of [HS03] that Φ^* is a fully faithful embedding with

$$\Phi_*\Phi^* M \cong M,$$

and in Section 4 of that paper that the composite functor $\Phi^*\Phi_*$ is localization with respect to the hereditary torsion theory consisting of v_n-torsion comodules. It is this composite $\Phi^*\Phi_*$ that we denote by L_n. For a quick review of the theory of localization with respect to hereditary torsion theories, see the discussion immediately following Corollary 4.3. Since the v_n-torsion comodules are the smallest hereditary torsion theory containing BP_*/I_{n+1} [HS03], L_n is localization away from BP_*/I_{n+1} so is analogous to L_n^f on the category of spectra. On the other hand, the v_n-torsion comodules are precisely the kernel of Φ_* [HS03], so L_n is also analogous to the functor L_n on the category of spectra.

Because the collection of v_n-torsion comodules is a hereditary torsion theory, the submodule T_nM of v_n-torsion elements in a comodule M is in fact a subcomodule. This also follows from [JY80, Corollary 2.4] or [Lan79, Corollary 2].

The most basic facts about L_n are contained in the following proposition.

Proposition 1.1. (a) *L_n is left exact and idempotent.*
(b) *Given a map f of comodules, L_nf is an isomorphism if and only if the kernel and cokernel of f are v_n-torsion.*
(c) *For a comodule M, $L_nM = 0$ if and only if M is v_n-torsion. In particular, $L_n(BP_*/I_k) = 0$ for $k > n$.*
(d) *For any comodule M, there is an exact sequence of comodules*

$$0 \to T_nM \to M \to L_nM \to T' \to 0$$

where T' is a v_n-torsion comodule.
(e) *A comodule M is L_n-local if and only if*

$$\mathrm{Hom}^*_{BP_*}(BP_*/I_{n+1}, M) = \mathrm{Ext}^{1,*}_{BP_*BP}(BP_*/I_{n+1}, M) = 0.$$

Remark. In part (e), note that the Hom group is defined in the category of BP_*-modules, whereas the Ext group is defined in the category of BP_*BP-comodules. However, we know from [Lan79, Corollary 4] that $\mathrm{Hom}^*_{BP_*BP}(BP_*/I_{n+1}, M) = 0$ iff $\mathrm{Hom}^*_{BP_*}(BP_*/I_{n+1}, M) = 0$ iff M has no v_n-torsion. We will see in Lemma 1.3 that the Ext condition can also be reformulated in the category of modules.

Proof. Parts (a), (b), and (c) are immediate consequences of the fact that L_n is localization with respect to the v_n-torsion comodules. Part (e) is proven in Corollary 4.3 of [HS03]. Part (d) follows from the other parts; since L_n is idempotent, the map $\iota\colon M \to L_nM$ is an L_n-equivalence, so its kernel and cokernel are v_n-torsion. Part (e) implies that L_nM has no v_n-torsion, so the kernel of ι is T_nM. □

We now identify some L_n-local comodules.

Proposition 1.2. *Suppose $j < n$ and M is a v_{j-1}-torsion comodule on which (v_j, v_{j+1}) is a regular sequence. Then M is L_n-local.*

For this proposition, we need the following lemma. We will prove the converse of this lemma in Corollary 4.6.

Lemma 1.3. *Suppose M is a BP_*BP-comodule with no v_n-torsion such that*

$$\mathrm{Ext}^{1,*}_{BP_*}(BP_*/I_{n+1}, M) = 0.$$

Then M is L_n-local.

Proof. We must show that $\mathrm{Ext}^1_{BP_*BP}(BP_*/I_{n+1}, M) = 0$. So suppose we have a short exact sequence

$$0 \to M \to X \xrightarrow{p} BP_*/I_{n+1} \to 0$$

of comodules. We know that $X \cong M \bigoplus BP_*/I_{n+1}$ as BP_*-modules, and $T_nM = 0$, so p induces an isomorphism $T_nX \to T_n(BP_*/I_{n+1}) = BP_*/I_{n+1}$ (of comodules). The inverse of this isomorphism splits the sequence. □

Proof of Proposition 1.2. First note that M has no v_j-torsion, so M has no v_n-torsion either. In light of Lemma 1.3, we must show that

$$\mathrm{Ext}^{1,*}_{BP_*}(BP_*/I_{n+1}, M) = 0.$$

For notational simplicity, we will assume that $* = 0$, but the proof works for any value of $*$. Suppose we have a short exact sequence

$$0 \to M \xrightarrow{f} X \xrightarrow{g} BP_*/I_{n+1} \to 0 \tag{1.4}$$

of BP_*-modules. As in Lemma 1.3, we observe that g induces an injective map $T_{n+1}X \to BP_*/I_{n+1}$; if we can show that this is also surjective, then the inverse will split the sequence, as required. Choose an $x \in X$ such that $g(x) = 1$. Then

$$g(v_jx) = g(v_{j+1}x) = 0,$$

so there are elements y and z in M such that $f(y) = v_jx$ and $f(z) = v_{j+1}x$. Then $f(v_{j+1}y) = f(v_jz)$, so $= v_{j+1}y = v_jz$. Since (v_j, v_{j+1}) is a regular sequence on M, we conclude that $y = v_jw$ for some w. This means $v_jv_{j+1}w = v_jz$, so, since v_j is not a zero-divisor on M, $z = v_{j+1}w$. Now consider the element $x' = x - f(w)$. We claim that this element defines a splitting of the short exact sequence 1.4. Certainly $g(x') = 1$ and $v_jx' = v_{j+1}x' = 0$. Now suppose $k \leq n$. We claim that $v_kx' = 0$. Certainly $g(v_kx') = 0$, so $v_kx' = f(t)$ for some t. But then $f(v_jt) = 0$, so $v_jt = 0$. This forces t to be 0, as required. □

Proposition 1.2 immediately gives us part of Theorem B(3).

Corollary 1.5. *Suppose $k < n$. Then BP_*/I_k is L_n-local.* □

Another class of local comodules is given by the following proposition, which is Theorem B(2).

Proposition 1.6. *If v_j acts invertibly on a BP_*BP-comodule M for some $j \leq n$, then M is L_n-local.*

Proof. If v_j acts invertibly on M and $j < n$, then (v_j, v_{j+1}) is a regular sequence on M, so Proposition 1.2 implies M is L_n-local. If v_n acts invertibly on M, then certainly M has no v_n-torsion. By Lemma 1.3, it suffices to show that $\mathrm{Ext}^{1,*}_{BP_*}(BP_*/I_{n+1}, M) = 0$. Since v_n acts by 0 on BP_*/I_{n+1}, it also acts by 0 on this $\mathrm{Ext}^{1,*}$ group. On the other hand, since v_n acts isomorphically on M, it acts isomorphically on this $\mathrm{Ext}^{1,*}$ group as well. Hence $\mathrm{Ext}^{1,*}(BP_*/I_{n+1}, M) = 0$ as required. □

The following corollary completes the proof of Theorem B(3).

Corollary 1.7. *Suppose M is a v_{n-1}-torsion BP_*BP-comodule. Then $L_nM \cong v_n^{-1}M$.*

Note that this includes the case $n = 0$, where we interpret $v_0 = p$ and $v_{-1} = 0$.

Proof. Note that $v_n^{-1}M$ is a BP_*BP-comodule by [JY80, Proposition 2.9]. The map $M \to v_n^{-1}M$ obviously has v_n-torsion kernel and cokernel, so is an L_n-equivalence. Proposition 1.6 implies that $v_n^{-1}M$ is L_n-local, so it must be L_nM. □

We conclude this section with the proof of Theorem B(4).

Proposition 1.8. *The functors Φ^* and L_n commute with filtered colimits, arbitrary direct sums, and finite limits.*

Proof. The functor Φ^* is a right adjoint, so preserves all limits, and L_n is left exact, so preserves finite limits. To complete the proof, we show that Φ^* preserves filtered colimits. It will then follow that Φ^* preserves arbitrary direct sums, which are filtered colimits of finite direct sums, and that $L_n = \Phi^*\Phi_*$ preserves filtered colimits and arbitrary direct sums, completing the proof.

So suppose X_i is a filtered diagram of $E(n)_*E(n)$-comodules. There is certainly a natural map

$$\alpha\colon \operatorname{colim} \Phi^* X_i \to \Phi^*(\operatorname{colim} X_i).$$

We need to recall that a BP_*BP-comodule (resp. $E(n)_*E(n)$-comodule) P is called **dualizable** if it is finitely generated and projective over BP_* (resp. $E(n)_*$), and that the dualizable comodules generate the category of BP_*BP-comodules (resp. $E(n)_*E(n)$-comodules). See [Hov04, Section 1.4]. Thus, to show α is an isomorphism, it suffices to check that it is an isomorphism upon applying $BP_*BP\text{-comod}(P, -)$ for any dualizable BP_*BP-comodule P, since the dualizable comodules generate. The main point is that Φ_* preserves dualizable comodules, and dualizable comodules are finitely presented. This implies

$$\begin{aligned} \operatorname{Hom}_{BP_*BP}(P, \operatorname{colim} \Phi^* X_i) &\cong \operatorname{colim} \operatorname{Hom}_{BP_*BP}(P, \Phi^* X_i) \\ \cong \operatorname{colim} \operatorname{Hom}_{E(n)_*E(n)}(\Phi_* P, X_i) &\cong \operatorname{Hom}_{E(n)_*E(n)}(\Phi_* P, \operatorname{colim} X_i) \\ &\cong \operatorname{Hom}_{BP_*BP}(P, \Phi^*(\operatorname{colim} X_i)), \end{aligned}$$

so Φ^* preserves filtered colimits. □

2. Injective BP_*BP-comodules

In order to construct the spectral sequence of Theorem A and in order to compute the right derived functors L_n^i of L_n, we need to known something about injective objects in the category of BP_*BP-comodules. Very little seems to have been written about these absolute injectives; relative injectives are easier to understand and have been used much more often. The object of this section is to learn a little more; in particular, we prove that L_n, Φ_*, Φ^*, and T_n all preserve injectives.

The most basic fact about injective BP_*BP-comodules is the following well-known lemma. Recall that a Hopf algebroid (A, Γ) is said to be **flat** if Γ is flat as a left (or, equivalently, right) A-module.

Lemma 2.1. *Let (A, Γ) be a flat Hopf algebroid.*

(a) *If I is an injective A-module, then the extended Γ-comodule $\Gamma \otimes_A I$ is an injective Γ-comodule.*
(b) *There are enough injective Γ-comodules.*
(c) *A Γ-comodule is injective if and only if it is a comodule retract of $\Gamma \otimes_A I$ for some injective A-module I.*

Proof. Because the extended comodule functor is right adjoint to the forgetful functor from Γ-comodules to A-modules, we have

$$\operatorname{Hom}_\Gamma(-, \Gamma \otimes_A M) \cong \operatorname{Hom}_A(-, M)$$

from which part (a) follows.

Now, if M is an arbitrary Γ-comodule, choose an injective A-module J so that there is an embedding $M \xrightarrow{j} J$. The composite

$$M \xrightarrow{\psi} \Gamma \otimes_A M \xrightarrow{1\otimes j} \Gamma \otimes_A J$$

is a comodule embedding of M into an injective Γ-comodule, proving part (b). If M is itself injective, this embedding must have a retraction, proving part (c). □

This lemma is of little practical assistance, since injective BP_*-modules are extremely complex. They must not only be v_n-divisible for all n, but also x-divisible for every nonzero homogeneous element x in BP_*. This is the reason one generally uses relatively injective BP_*BP-comodules, as they are much simpler. However, to compute right derived functors of L_n, we must use absolute injectives. Indeed, as explained following the proof of Corollary 3.6, $L_n^i(BP_*BP)$ is not always zero for positive i.

The first step is to understand the v_n-torsion in an injective comodule.

Proposition 2.2. *Suppose M is a v_n-torsion BP_*BP-comodule and N is an essential extension of M in the category of BP_*BP-comodules. Then N is v_n-torsion. In particular, the injective hull of M is v_n-torsion.*

Proof. Suppose N is not v_n-torsion. Let x be an element of N that is not v_n-torsion, and let $I = \sqrt{\operatorname{Ann} x}$. Since x is not v_n-torsion, v_n is not in I. Theorem 1 of [Lan79] guarantees that I is an invariant ideal of BP_*, so we must have $I = I_k$ for some $k \leq n$. Theorem 2 of [Lan79] tells us that there is some primitive y in N such that $\operatorname{Ann}(y) = I_k$. Hence BP_*/I_k is isomorphic to a subcomodule of N. This subcomodule has no v_n-torsion, so cannot intersect M nontrivially. This contradicts our assumption that N is an essential extension of M. □

This proposition leads to the following useful theorem.

Theorem 2.3. *Suppose I is an injective BP_*BP-comodule, and let T_nI denote the v_n-torsion in I. Then T_nI and I/T_nI are injective, and $I \cong T_nI \oplus I/T_nI$.*

Proof. The injective hull of T_nI must be a subcomodule of I, since I is injective, and it must be v_n-torsion by Proposition 2.2. Hence it must be T_nI itself. □

Corollary 2.4. *Suppose I is an injective BP_*BP-comodule. Then $L_nI = I/T_nI$. In particular, L_n preserves injectives.*

Proof. Certainly the map $I \to I/T_nI$ is an L_n-equivalence. But I/T_nI is an injective comodule by Theorem 2.3, and has no v_n-torsion. It is therefore L_n-local, by Proposition 1.1(e). □

Corollary 2.5. *The functor Φ^* preserves and reflects injectivity, and the functor Φ_* preserves injectives.*

Proof. The functor Φ^* is right adjoint to the exact functor Φ_*, so preserves injectives. Conversely, suppose Φ^*I is injective, $j\colon M \xrightarrow{j} N$ is an inclusion of $E(n)_*E(n)$-comodules, and $f\colon M \to I$ is a map. Applying Φ^*, we find a map $h\colon \Phi^*N \to \Phi^*I$ such that $h \circ \Phi^*j = \Phi^*f$. Since Φ^* is fully faithful, we conclude that $h = \Phi^*g$ for some extension g of f. Hence I is injective.

Now $L_n = \Phi^*\Phi_*$ preserves injectives by Corollary 2.4. Since Φ^* reflects injectives, we conclude that Φ_* must preserve injectives. □

Theorem 2.3 divides the study of injective BP_*BP-comodules into those with no v_n-torsion and those which are all v_n-torsion. About all we know about injective comodules which are all v_n-torsion is the following proposition.

Proposition 2.6. *Suppose I is an injective BP_*BP-comodule that is all v_n-torsion. Then I is v_{n+1}-divisible.*

Proof. Suppose $x \in I$. Because every BP_*BP-comodule is a filtered colimit of finitely presented BP_*BP-comodules, there is a map $P \to I$ from a comodule P that is a free finitely generated BP_*-module, whose image contains x. Since I is v_n-torsion, and therefore v_i-torsion for all $i \leq n$, this map factors through

$$Q = P/JP \xrightarrow{g} I$$

for some invariant ideal $J = (p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$. There is some k such that v_{n+1}^k is invariant modulo J. Thus multiplication by v_{n+1}^k defines a monomorphism of comodules $Q \to \Sigma^{-t} Q$. Because I is injective, g must extend to a map $\Sigma^{-t} Q \to I$, showing that x is divisible by v_{n+1}. □

We now turn our attention to injectives that have no v_n-torsion.

Theorem 2.7. *Suppose M is a BP_*BP-comodule with no v_n-torsion. Then there is an embedding of M into an injective BP_*BP-comodule with no v_n-torsion, and this embedding can be chosen to be functorial on the category of BP_*BP-comodules with no v_n-torsion.*

Proof. Since the category of $E(n)_*E(n)$-comodules is a Grothendieck category (see Section 1.4 of [Hov04]), there is a functorial embedding of any $E(n)_*E(n)$-comodule into an injective $E(n)_*E(n)$-comodule. (Apply Quillen's small object argument to the set of subobjects of a generator). In particular, for M a BP_*BP-comodule, we get a functorial embedding $\Phi_* M \to I$. Applying Φ^* gives us a functorial embedding $L_n M \to \Phi^* I$, and $\Phi^* I$ is an injective comodule (by Corollary 2.5), and has no v_n-torsion (since it is L_n-local). Since M has no v_n-torsion, M embeds in $L_n M$. □

We expect that injective BP_*BP-comodules are not closed under filtered colimits, though we do not have a counterexample. Those with no v_n-torsion, on the other hand, are better behaved.

Proposition 2.8. *Injective BP_*BP-comodules with no v_n-torsion are closed under filtered colimits.*

This proposition depends on the following lemma.

Lemma 2.9. *Injective $E(n)_*E(n)$-comodules are closed under filtered colimits.*

Proof. Recall that the category of $E(n)_*E(n)$-comodules is a Grothendieck category; a set of generators is given by the comodules which are finitely generated and projective over $E(n)_*$ [Hov04, Section 1.4]. There is a version of Baer's criterion for injectivity that works for any Grothendieck category [Ste75]. Let $\{G_j\}$ be a set of generators for the Grothendieck category in question; then an object I is injective if and only if $\operatorname{Hom}(G_j, I) \to \operatorname{Hom}(N_j, I)$ is surjective for all j and all subobjects N_j of G_j. In particular, if $\operatorname{Hom}(G_j, -)$ and $\operatorname{Hom}(N_j, -)$ commute with filtered colimits (that is, if G_j and N_j are finitely presented), then injectives are closed under filtered colimits. In our case, the generators G_j are finitely generated and projective over $E(n)_*$. Since $E(n)_*$ is Noetherian, the objects N_j are also finitely presented over $E(n)_*$. This means the objects N_j and G_j are also finitely presented as $E(n)_*E(n)$-comodules by Proposition 1.3.3 of [Hov04], completing the proof. □

Proof of Proposition 2.8. Suppose $F\colon \mathcal{J} \to BP_*BP$-comod is a functor from a filtered category $\mathcal{J}$ to injective comodules with no v_n-torsion. Then $F(j)$ is L_n-local for all $j \in J$, so we have

$$\operatorname{colim} F \cong \operatorname{colim} \Phi^*\Phi_*F \cong \Phi^*(\operatorname{colim} \Phi_*F),$$

by Proposition 1.8. Now $\Phi_*F(j)$ is an injective $E(n)_*E(n)$-comodule for all $j \in \mathcal{J}$ by Corollary 2.5, so Lemma 2.9 tells us that $\operatorname{colim} \Phi_*F$ is injective. Since Φ^* preserves injectives, we conclude that $\operatorname{colim} F$ is injective. □

We can now give a partial structure theorem for injective BP_*BP-comodules.

Proposition 2.10. *Suppose I is an injective BP_*BP-comodule, and $n \geq 0$. Then*

$$I \cong I_0 \oplus I_1 \oplus \cdots I_n \oplus T_nI$$

where

(a) *Each I_j is an injective BP_*BP-comodule with $v_j^{-1}I_j = I_j$ (and thus I_j is v_{j-1}-torsion).*
(b) *T_nI is injective and v_n-torsion.*

In particular, if I is indecomoposable then either $I = v_j^{-1}I$ for some j or I is v_j-torsion for all j.

Proof. Put $I_j = T_{j-1}I/T_jI$ (where $T_{-1}I = I$). As the comodules T_jI are injective (by Theorem 2.3), the filtration

$$T_nI \leq T_{n-1}I \leq \ldots \leq T_0I \leq I$$

must split, giving $I = T_nI \oplus \bigoplus_{j=0}^n I_j$. The comodule I_j is a summand of I and thus is injective. By construction it is v_{j-1}-torsion, and thus v_j-divisible by Proposition 2.6. The definition also implies that there is no v_j-torsion, so $I_j = v_j^{-1}I_j$. □

It would be nice to have some explicit knowledge of injective BP_*BP-comodules I with $v_n^{-1}I = I$. When $n = 0$, at least, this is easy.

Proposition 2.11. *Suppose M is a BP_*BP-comodule with no p-torsion. Then M is injective if and only if M is a rational vector space.*

Proof. Proposition 2.6 shows that if M is injective, then it must be a rational vector space. Conversely, if M is rational, then $M = L_0M = \Phi^*\Phi_*M$. The category of $E(0)_*E(0)$-comodules is the category of rational vector spaces, so Φ_*M is injective. Since Φ^* preserves injectives, we conclude that M is injective. □

The analogue of this proposition is definitely false when $n > 0$. Still, this gives a rationale for why the chromatic resolution is useful. Indeed, suppose we want to find an injective resolution of BP_* as a BP_*BP-comodule. Proposition 2.11 implies that $M^0 = p^{-1}BP_*$ is the injective hull of BP_* as a BP_*BP-comodule. The cokernel N^1 is usually written $BP_*/(p^\infty)$. The injective hull of N^1 must be a p-torsion essential extension of N^1 on which v_1 acts invertibly. The simplest way to do this is to form $M^1 = v_1^{-1}N^1$, which is the next term in the chromatic resolution. Sadly, N^1 is not actually injective, but it seems to be the closest one can get to the injective hull of M^1 in a fairly simple way. Iterating this idea leads to the chromatic resolution.

3. The derived functors of L_n

Now that we have some knowledge of injective BP_*BP-comodules, we can begin to compute derived functors. The goal of this section is to prove Theorem C except for part (1), which we deal with in the next section.

Recall that L_n^i denotes the ith right derived functor of L_n. We also let T_n^i denote the ith right derived functor of T_n, where $T_n(M)$ is the subcomodule of v_n-torsion elements in M.

The first thing to point out is that L_n^i and T_n^i are closely related.

Theorem 3.1. *If M is a BP_*BP-comodule, we have a natural short exact sequence*

$$0 \to T_n M \to M \to L_n M \to T_n^1 M \to 0.$$

and natural isomorphisms

$$L_n^i M \cong T_n^{i+1} M$$

for $i > 0$.

Proof. Let I_* be an injective resolution of M. Then $T_n^i M \cong H_{-i}(T_n I_*)$, and $L_n^i M \cong H_{-i}(L_n I_*)$. But $L_n I_* \cong I_*/T_n I_*$ by Corollary 2.4. Hence we have a natural short exact sequence of complexes

$$0 \to T_n I_* \to I_* \to L_n I_* \to 0.$$

The long exact sequence in homology gives the desired result. □

We also point out that computing L_n^i is equivalent to computing the right derived functors of Φ^*.

Proposition 3.2. *Let M be a BP_*BP-comodule, and let $R^i\Phi^*$ denote the ith right derived functor of Φ^*. Then we have a natural isomorphism*

$$(R^i\Phi^*)(\Phi_* M) \cong L_n^i M.$$

Note that, since $\Phi_*\Phi^* N \cong N$, we can also write this isomorphism as

$$(R^i\Phi^*)(N) \cong L_n^i(\Phi^* N).$$

Proof. Let I_* be an injective resolution of M. Then $\Phi_* I_*$ is an injective resolution of $\Phi_* M$, since Φ_* is exact and preserves injectives by Corollary 2.5. Hence

$$(R^i\Phi^*)(\Phi_* M) \cong H_{-i}(\Phi^*\Phi_* I_*) \cong H_{-i}(L_n I_*) \cong L_n^i M.$$

□

We now begin the computation of L_n^i.

Proposition 3.3. *If T is a v_n-torsion BP_*BP-comodule, then $L_n^i T = 0$ for all $i \geq 0$. Furthermore, for an arbitrary comodule M, the map $M \to L_n M$ induces an isomorphism $L_n^i M \to L_n^i L_n M$.*

Proof. Using Proposition 2.2, one can construct an injective resolution I_* of T that is all v_n-torsion. Hence $L_n I_* = 0$, so $L_n^i T = 0$ for all i. For the second statement, recall that we have short exact sequences

$$0 \to T \to M \to M/T \to 0$$

and

$$0 \to M/T \to L_n M \to T' \to 0$$

where T and T' are v_n-torsion. Applying L_n gives the desired result. □

The following proposition is part (3) of Theorem C.

Proposition 3.4. *Suppose M is a BP_*BP-comodule. Then L_n^iM is v_n-torsion for $i > 0$.*

Proof. Let I_* be an injective resolution of M. Then $T_n^i(M) = H_{-i}T_nI_*$ is obviously v_n-torsion. The result follows from Theorem 3.1. □

We now show that the chromatic resolution is as good as an injective resolution for computing L_n^i. The following theorem also proves part (4) of Theorem C.

Theorem 3.5. *Suppose M is a BP_*BP-comodule on which v_j acts isomorphically for some j. Then $L_n^iM = 0$ for all $i > 0$ and all n. Moreover, we have $L_nM = 0$ if $j > n$ and $L_nM = M$ if $j \leq n$.*

Proof. We claim that we can choose an injective resolution I_* of M for which $I_* = v_j^{-1}I_*$. To see this, it suffices by induction to show that if N is a BP_*BP-comodule for which $N = v_j^{-1}N$, there there is a short exact sequence

$$0 \to N \to I \to N' \to 0$$

of comodules for which I is injective, $v_j^{-1}I = I$, and $v_j^{-1}N' = N'$. Since $N = v_j^{-1}N$, N is all v_{j-1}-torsion by Proposition 2.9 of [JY80], and of course N has no v_j-torsion. Proposition 2.2 and Theorem 2.7 together imply that the injective hull I of N is v_{j-1}-torsion and has no v_j-torsion. Proposition 2.6 then implies that $I = v_j^{-1}I$. It follows easily that multiplication by v_j is surjective on N', but we claim it is injective as well. Indeed, suppose $x \in N'$ has $v_jx = 0$. Choose a y in I whose image in N' is x, so that v_jy is in N. Since $N = v_j^{-1}N$, there is a z in N such that $v_jz = v_jy$. It follows that $z = y$, and so $x = 0$.

We now have our desired injective resolution I_* of M for which $I_* = v_j^{-1}I_*$. The argument now breaks into two cases. If $j \leq n$, we apply the v_n-torsion functor T_n. Since there is no v_j-torsion in I_*, there is also no v_n-torsion by Lemma 2.3 of [JY80]. Thus $T_nI_* = 0$, and so

$$L_n^iM \cong T_n^{i+1}M \cong H_{-i-1}T_nI_* = 0$$

for all $i > 0$. For $i = 0$, use Proposition 1.6.

Now suppose $j > n$. Since $v_j^{-1}I_* = I_*$, I_* is all v_{j-1}-torsion, so also all v_n-torsion. Hence $L_nI_* = 0$, so

$$L_n^iM = H_{-i}L_nI_* = 0$$

for $i \geq 0$. □

Theorem 3.5 allows us to compute L_n^iM for some important BP_*BP-comodules M. The following corollary is Theorem C(5).

Corollary 3.6. (a) *Suppose $k > n$. Then $L_n^i(BP_*/I_k) = 0$ for all i.*

(b) *$L_k^i(BP_*/I_k) = 0$ for $i > 0$, whereas $L_k^0(BP_*/I_k) = v_k^{-1}BP_*/I_k$.*

(c) *Suppose $k < n$. Then $L_n^i(BP_*/I_k) = 0$ unless $i = 0$ or $n - k$. We have*

$$L_n^0(BP_*/I_k) = BP_*/I_k$$

and

$$L_n^{n-k}(BP_*/I_k) = BP_*/(p, v_1, \ldots, v_{k-1}, v_k^\infty, \ldots, v_n^\infty).$$

Proof. Let $M = BP_*/I_k$, and consider the chromatic resolution $M \to J_*$ of M, where

$$J_t = v_{t+k}^{-1} BP_*/(p, v_1, \dots, v_{k-1}, v_k^\infty, \dots, v_{t+k-1}^\infty).$$

By Theorem 3.5, we have $L_n^i J_t = 0$ for all $i > 0$. Hence $L_n^i M \cong H_{-i} L_n J_*$. Now, each of the comodules J_t is v_{k-1}-torsion, so $L_n J_* = 0$ if $k > n$. This completes the proof of part (a).

If $k = n$, then $L_n J_t = 0$ for $t > 0$, from which part (b) follows easily. If $k < n$, on the other hand, $L_n J_t = J_t$ for $t < n - k + 1$, and is 0 for $t \geq n - k + 1$, from which part (c) follows. □

Now suppose M is a BP_*BP-comodule that is flat over BP_*, and let J_* denote the chromatic resolution of BP_*. Then $M \otimes_{BP_*} J_*$ is the chromatic resolution of M. Furthermore, v_t still acts invertibly on $M \otimes_{BP_*} J_t$, so $L_n^i(M \otimes_{BP_*} J_t) = 0$ for all $i > 0$. Just as in the proof of Corollary 3.6, then, we conclude that

$$L_n^n(M) = M/(p^\infty, v_1^\infty, \dots, v_n^\infty).$$

In particular, $L_n^n(BP_*BP)$ is non-zero, showing that relative injectives do not suffice to compute L_n^i.

We also discover that L_n has only finitely many right derived functors.

Theorem 3.7. *Suppose M is a v_k-torsion comodule for some $-1 \leq k \leq n$. Then $L_n^i M = 0$ for $i \geq n - k$. In particular, $L_n^i N = 0$ for $i > n$ for any comodule N.*

For the purposes of this theorem, we take $v_{-1} = 0$, so that every comodule is v_{-1}-torsion. This theorem proves part (2) of Theorem C.

Proof. We proceed by downwards induction on k. The base case $k = n$ is Proposition 3.3. So suppose we know the theorem for k, and M is a v_{k-1}-torsion comodule. Let $T_k M$ denote the v_k-torsion in M. We have a short exact sequence

$$0 \to T_k M \to M \to N \to 0$$

where N has no v_k-torsion. By our induction hypothesis, $L_n^i(T_k M) = 0$ for $i \geq n-k$. It therefore suffices to show that $L_n^i(N) = 0$ for $i > n - k$.

Now, since N is v_{k-1}-torsion but has no v_k-torsion, we have a short exact sequence

$$0 \to N \to v_k^{-1} N \to T \to 0,$$

where T is v_k-torsion. Our induction hypothesis guarantees that $L_n^i T = 0$ for $i \geq n - k$, and Theorem 3.5 guarantees that $L_n^i T \cong L_n^{i+1} N$ for $i > 0$. Hence $L_n^i N = 0$ for $i > n - k$, as required. □

Corollary 3.6 together with the Landweber filtration theorem gives a method for computing $L_n^i M$ for finitely presented BP_*BP-comodules M. To compute $L_n^i M$ for more general comodules M, we use the following theorem, which is part (6) of Theorem C.

Theorem 3.8. *The functors L_n^i preserve filtered colimits and arbitrary direct sums of BP_*BP-comodules.*

Since L_n itself preserves filtered colimits, this theorem would be easy if filtered colimits of injective comodules were injective, but we believe that this is false in general. However, to compute L_n^i the only injectives that matter are injectives with no v_n-torsion, and these we know are closed under filtered colimits by Proposition 2.8.

Proof. It suffices to show that L_n^i preserves filtered colimits, since arbitrary direct sums are filtered colimits of finite direct sums. We use induction on i. When $i = 0$ we have seen this already in Proposition 1.8. Now suppose L_n^i preserves filtered colimits for some $i \geq 0$, and let $\{M_t\}$ be a filtered diagram of comodules. Then $\{L_n M_t\}$ is a filtered diagram of comodules with no v_n-torsion, so we can use Theorem 2.7 to find a filtered diagram of injectives $\{I_t\}$ with no v_n-torsion and a short exact sequence of filtered diagrams

$$\{0\} \to \{L_n M_t\} \to \{I_t\} \to \{N_t\} \to \{0\}.$$

This gives us a short exact sequence

$$0 \to \operatorname{colim} L_n M_t \to \operatorname{colim} I_t \to \operatorname{colim} N_t \to 0, \tag{3.9}$$

and $\operatorname{colim} I_t$ is injective by Proposition 2.8.

We must now separate the case $i = 0$ from the case $i > 0$. If $i = 0$, by taking the colimit of the exact sequences

$$0 \to L_n M_t \to I_t \to L_n N_t \to L_n^1 M_t \to 0,$$

we get an exact sequence

$$0 \to \operatorname{colim} L_n M_t \to \operatorname{colim} I_t \to \operatorname{colim} L_n N_t \to \operatorname{colim} L_n^1 M_t \to 0.$$

On the other hand, by applying L_n to the short exact sequence 3.9, we get the exact sequence

$$0 \to \operatorname{colim} L_n M_t \to \operatorname{colim} I_t \to L_n(\operatorname{colim} N_t) \to L_n^1(\operatorname{colim} L_n M_t) \to 0.$$

There is a map from the first of these sequences to the second, which is an isomorphism on every nonzero term except the last one, so we get an isomorphism

$$\operatorname{colim} L_n^1 M_t \cong L_n^1(\operatorname{colim} L_n M_t).$$

On the other hand, using Proposition 3.3, and the fact that L_n commutes with filtered colimits, we get

$$L_n^1(\operatorname{colim} L_n M_t) \cong L_n^1 L_n(\operatorname{colim} M_t) \cong L_n^1(\operatorname{colim} M_t),$$

as required.

If $i > 0$, the situation is easier. Indeed, using Proposition 3.3 and the fact that L_n commutes with filtered colimits, we have

$$\begin{aligned} \operatorname{colim} L_n^{i+1} M_t &\cong \operatorname{colim} L_n^{i+1}(L_n M_t) \cong \operatorname{colim} L_n^i N_t \\ \cong L_n^i(\operatorname{colim} N_t) &\cong L_n^{i+1}(\operatorname{colim} L_n M_t) \cong L_n^{i+1} L_n(\operatorname{colim} M_t) \cong L_n^{i+1}(\operatorname{colim} M_t), \end{aligned}$$

completing the proof. □

4. Comparison with Čech cohomology

The object of this section is to prove part (1) of Theorem B and Theorem C, showing that, for a comodule M, $L_n^i(M)$ is the same as the ith Čech cohomology group $\check{C}H^i_{I_{n+1}} M$ of M with respect to I_{n+1}. We also show that Čech cohomology $\check{C}H^*_{I_{n+1}}(-)$ is the derived functors of localization in the category of BP_*-modules with respect to the hereditary torsion theory of I_{n+1}-torsion modules.

We first remind the reader of the definition of Čech cohomology from [GM95]. Given an element α in a commutative ring R, which we will always take to be BP_*, we form the cochain complex $K^\bullet(\alpha)$ which is R in degree 0 and $R[1/\alpha]$ in degree 1, with the differential being the obvious map $R \to R[1/\alpha]$. Given an ideal $I = (\alpha_0, \dots, \alpha_n)$, we define $K^\bullet(I)$ to be the cochain complex

$$K^\bullet(I) = K^\bullet(\alpha_0) \otimes_R K^\bullet(\alpha_1) \otimes_R \cdots \otimes_R K^\bullet(\alpha_n).$$

This stable Koszul complex of course depends on the choice of generators α_i, but its quasi-isomorphism class does not [GM95, Corollary 1.2]. There is an obvious surjection $K^\bullet(\alpha_i) \to R$ of complexes, where R is the complex consisting of R concentrated in degree 0. Tensoring these together gives us a map

$$\epsilon\colon\ K^\bullet(I) \to R.$$

We define the flat Čech complex $\check{C}^\bullet(I)$ by

$$\check{C}^\bullet(I) = \Sigma(\ker \epsilon),$$

where $(\Sigma B)^n = B^{n+1}$ for a cochain complex B. Thus

$$\check{C}^k(I) = \bigoplus_{|S|=k+1} R[1/\alpha_S]$$

for $0 \leq k \leq n$, where S runs through the $k+1$-element subsets of $(0, 1, \dots, n)$ and $\alpha_S = \prod_{i \in S} \alpha_i$.

Definition 4.1. The **local cohomology** $H_I^*(M)$ of an R-module M with respect to a finitely generated ideal $I = (\alpha_0, \dots, \alpha_n)$ is

$$H_I^*(M) = H^*(K^\bullet(I) \otimes_R M).$$

The **Čech cohomology** $\check{C}H_I^*(M)$ of M with respect to I is

$$\check{C}H_I^*(M) = H^*(\check{C}^\bullet(I) \otimes_R M).$$

Some of the basic properties of local and Cech cohomology are summarized in the following proposition.

Proposition 4.2. *Suppose $I = (\alpha_0, \dots, \alpha_n)$ is a finitely generated ideal in a commutative ring R, and M is an R-module.*

(a) *We have a natural exact sequence*

$$0 \to H_I^0(M) \to M \to \check{C}H_I^0(M) \to H_I^1(M) \to 0,$$

and natural isomorphisms $\check{C}H_I^k(M) \cong H_I^{k+1}(M)$ for $k > 0$.

(b) $\check{C}H_I^k(M) = 0$ *unless $0 \leq k \leq n$.*

(c) *$H_I^k(M)$ is I-torsion for all k, and $\check{C}H_I^k(M)$ is I-torsion for all $k > 0$. On the other hand, $\check{C}H_I^0(M)$ has no I-torsion.*

(d) *$H_I^0(M)$ is the submodule of I-torsion elements in M.*

(e) *$\check{C}H_I^k(M) = 0$ for all k if and only if M is I-torsion, and this is true if and only if $\check{C}H_I^0(M) = 0$.*

(f) *A short exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

of R-modules gives rise to natural long exact sequences

$$0 \to H_I^0(M') \to H_I^0(M) \to H_I^0(M'') \\ \to H_I^1(M') \to \cdots \to H_I^{n+1}(M) \to H_I^{n+1}(M'') \to 0,$$

and

$$0 \to \check{C}H_I^0(M') \to \check{C}H_I^0(M) \to \check{C}H_I^0(M'') \\ \to \check{C}H_I^1(M') \to \cdots \to \check{C}H_I^n(M) \to \check{C}H_I^n(M'') \to 0.$$

(g) *Both H_I^0 and $\check{C}H_I^0$ are left exact idempotent functors.*

Note that part (f) certainly suggests that H_I^k is the kth right derived functor of H_I^0, but it does not prove it, since we also need to know that H_I^k sends injective modules to 0 for all $k > 0$. We will have to deal with this issue later.

Proof. Most of this proposition follows from [GM95]. Part (a) appears in Section 1 of that paper, and part (b) is obvious from the definition of $\check{C}^\bullet(I)$. The first sentence of part (c) is also in Section 1 of [GM95]. For the second part of part (c), simply note that $\check{C}H_I^0(M)$ is a submodule of $\bigoplus_i M[1/\alpha_i]$. As $M[1/\alpha_i]$ has no α_i-torsion, it follows that $\bigoplus_i M[1/\alpha_i]$ has no I-torsion. Part (d) is clear from the fact that $H_I^0(M)$ is the kernel of the map

$$M \to \bigoplus_i M[1/\alpha_i].$$

For part (e), first suppose that M is I-torsion. Then the complex $\check{C}^\bullet(I) \otimes_R M$ is the zero complex, and so of course $\check{C}H_I^k(M) = 0$ for all k. Conversely, suppose $\check{C}H_I^0(M) = 0$. Then parts (a) and (d) show that M is I-torsion. For part (f), simply note that the complexes $K^\bullet(I)$ and $\check{C}^\bullet(I)$ are complexes of flat modules. Hence, a short exact sequence of modules gives rise to a short exact sequence of complexes on applying either $K^\bullet(I) \otimes_R (-)$ or $\check{C}^\bullet(I) \otimes_R (-)$. The resulting long exact sequence in cohomology gives us part (f). For part (g), note that part (f) immediately implies that both H_I^0 and $\check{C}H_I^0$ are left exact. Part (d) shows that H_I^0 is idempotent. To see that $\check{C}H_I^0$ is idempotent, apply part (f) to the short exact sequences

$$0 \to H_I^0(M) \to M \to M/H_I^0(M) \to 0$$

and

$$0 \to M/H_I^0(M) \to \check{C}H_I^0(M) \to H_I^1(M) \to 0.$$

Since both $H_I^0(M)$ and $H_I^1(M)$ are I-torsion by part (c), part (e) tells us that we get isomorphisms

$$\check{C}H_I^0(M) \cong \check{C}H_I^0(M/H_I^0(M)) \cong \check{C}H_I^0(\check{C}H_I^0 M),$$

completing the proof. □

Corollary 4.3. *Suppose $I = (\alpha_0, \alpha_1, \dots, \alpha_n)$ is a finitely generated ideal in a commutative ring R. The functor $\check{C}H_I^0$ is localization in the category of R-modules with respect to the hereditary torsion theory of I-torsion modules.*

For this corollary to make sense, recall that a class of objects in an abelian category $\mathcal{A}$ is a **hereditary torsion theory** if it is closed under subobjects, extensions, quotient objects, and arbitrary coproducts. If $\mathcal{T}$ is a hereditary torsion theory, we define a map f to be a **$\mathcal{T}$-equivalence** if its kernel and cokernel are in $\mathcal{T}$. An object X is called **$\mathcal{T}$-local** if $\mathcal{A}(f, X)$ is an isomorphism for all $\mathcal{T}$-equivalences f. A **$\mathcal{T}$-localization** of an object M is a $\mathcal{T}$-local object X together with a $\mathcal{T}$-equivalence $M \to X$. When $\mathcal{T}$-localizations exist, they are unique up to unique isomorphism and are functorial.

Proof. One can easily check that the class $\mathcal{T}$ of I-torsion modules is a hereditary torsion theory. It is clear from parts (a) and (c) of Proposition 4.2 that the map $M \to \check{C}H_I^0(M)$ has I-torsion kernel and cokernel, so is a $\mathcal{T}$-equivalence. It remains to show that $\check{C}H_I^0(M)$ is $\mathcal{T}$-local. By factoring a $\mathcal{T}$-equivalence into an injection followed by a surjection, we see that this boils down to showing that $\check{C}H_I^0(M)$ has no I-torsion and that $\operatorname{Ext}_R^1(T, \check{C}H_I^0(M)) = 0$ for all I-torsion modules T. The first part is part (c) of Proposition 4.2. For the second part, suppose we have an extension

$$0 \to \check{C}H_I^0(M) \to X \to T \to 0$$

where T is I-torsion. Applying the left exact idempotent functor $\check{C}H_I^0$, we get an isomorphism $\check{C}H_I^0(M) \to \check{C}H_I^0(X)$. Thus the composite

$$X \to \check{C}H_I^0(X) \cong \check{C}H_I^0(M)$$

defines a splitting of our extension. Thus $\check{C}H_I^0(M)$ is $\mathcal{T}$-local as required. □

We also need to know that $\check{C}H_I^*$ are the right derived functors of $\check{C}H_I^0$. This seems to require some hypotheses on I.

Theorem 4.4. *Suppose I is an ideal in a commutative ring R, generated by a regular sequence $(\alpha_0, \dots, \alpha_n)$, in which each element α_i is not a zero-divisor. Then*

$$H_I^k(M) = \operatorname{colim}_J \operatorname{Ext}_R^k(R/J, M),$$

where J runs over ideals $J \leq I$ with $\sqrt{J} = \sqrt{I}$. Moreover, H_I^k is the k'th right derived functor of H_I^0 and $\check{C}H_I^k$ is the k'th right derived functor of $\check{C}H_I^0$.

Proof. Put $I_r = (\alpha_0^r, \dots, \alpha_n^r)$; these ideals are evidently cofinal among the J's. Let $K_r^\bullet$ be the usual (unstable) Koszul complex for I_r, which is the tensor product over j of the complexes $(R \xrightarrow{\alpha_j^r} R)$. As our sequence of generators is regular, this is a finite resolution of R/I_r by finitely generated free modules. Now let $DK_r^\bullet$ be the dual of $K_r^\bullet$, which is naturally thought of as the tensor product of the complexes $R \to R \cdot \alpha_j^{-r}$. (In fact $DK_r^\bullet$ is isomorphic to $K_r^\bullet$, up to a degree shift in the graded case.) It is clear that the stable Koszul complex $K^\bullet(I)$ is the colimit of the complexes $DK_r^\bullet$, so

$$\begin{aligned} H_I^*(M) &= \operatorname{colim}_r H^*(DK_r^\bullet \otimes_R M) \\ &= \operatorname{colim}_r H^* \operatorname{Hom}_R(K_r^\bullet, M) \\ &= \operatorname{colim}_r \operatorname{Ext}_R^*(R/I_r, M) \\ &= \operatorname{colim}_J \operatorname{Ext}_R^*(R/J, M). \end{aligned}$$

It is immediate from this that $H_I^i(M) = 0$ if $i > 0$ and M is injective. Using part (a) of Proposition 4.2, we see that $\check{C}H_I^i(M) = 0$ as well. It now follows formally from the long exact sequences in Proposition 4.2 that H_I^k is the k'th right derived functor of H_I^0, and $\check{C}H_I^k$ is the k'th right derived functor of $\check{C}H_I^0$. □

We can now investigate the functors H_I^* and $\check{C}H_I^*$ restricted to the category of comodules, proving part (1) of Theorem B amd Theorem C.

Theorem 4.5. *Suppose M is a BP_*BP-comodule. Then there are natural isomorphisms $T_n^k(M) \cong H_{I_{n+1}}^k(M)$ and $L_n^k M \cong \check{C}H_{I_{n+1}}^k(M)$.*

Proof. We first show that $H^k_{I_n}(M) = 0$ for all injective BP_*BP-comodules M and $k > 0$. This does not follow from Theorem 4.4 because injective comodules need not be injective as BP_*-modules. We proceed by induction on n, using the spectral sequence

$$H^s_{v_n} H^t_{I_n}(M) \Rightarrow H^{s+t}_{I_{n+1}}(M).$$

discussed in [GM95, Section 2]. By induction, the E_2-term of this spectral sequence is $H^s_{v_n} H^0_{I_n}(M)$. In degree $s = 1$, this is

$$v_n^{-1} H^0_{I_n}(M)/H^0_{I_n}(M).$$

But Theorem 2.3 shows that $H^0_{I_n} M = T_{n-1} M$ is an injective BP_*BP-comodule, which of course is v_{n-1}-torsion. Proposition 2.6 then shows that $H^0_{I_n} M$ is v_n-divisible. Hence the E_2-term of our spectral sequence is $H^0_{v_n} H^0_{I_n}(M) = H^0_{I_{n+1}}(M)$ concentrated in bidegree $(0, 0)$, completing the proof.

Now suppose M is an arbitrary BP_*BP-comodule. Take an resolution I_* of M by injective BP_*BP-comodules. By definition, $T^k_n(M) \cong H_{-k}(T_n I_*)$. On the other hand, applying $H^*_{I_{n+1}}$, which we have just seen vanishes on injective comodules, shows that

$$H^k_{I_{n+1}}(M) \cong H_{-k}(H^0_{I_{n+1}} I_*) \cong H_{-k}(T_n I_*)$$

as well.

Similarly, $L^k_n(M) \cong H_{-k}(L_n I_*)$, which is isomorphic to $H_{-k}(I_*/T_n I_*)$ by Corollary 2.4. Now suppose N is an injective BP_*BP-comodule. The exact sequence

$$0 \to H^0_{I_{n+1}}(N) \to N \to \check{C}H^0_{I_{n+1}}(N) \to H^1_{I_{n+1}}(N) \to 0$$

of Proposition 4.2 together with the fact that $H^k_{I_{n+1}}(N) = 0$ for $k > 0$ implies that $\check{C}H^0_{I_{n+1}}(N) \cong N/T_n N$. Also,

$$\check{C}H^k_{I_{n+1}}(N) \cong H^{k+1}_{I_{n+1}}(N) = 0$$

for $k > 0$. Hence, applying $\check{C}H^*_{I_{n+1}}$ to I_*, we find that $\check{C}H^k_{I_{n+1}}(M) \cong H_{-k}(I_*/T_n I_*)$, completing the proof. □

We can now give the promised converse to Lemma 1.3.

Corollary 4.6. *A BP_*BP-comodule M is L_n-local if and only if*

$$\mathrm{Hom}^*_{BP_*}(BP_*/I_{n+1}, M) = \mathrm{Ext}^{1,*}_{BP_*}(BP_*/I_{n+1}, M) = 0.$$

Proof. The if direction is Lemma 1.3. For the only if direction, suppose M is L_n-local. Then M is also local with respect to the hereditary torsion theory of I_{n+1}-torsion BP_*-modules, in view of Theorem 4.5 and Corollary 4.3. This means that, for any I_{n+1}-torsion module T, we have

$$\mathrm{Hom}_{BP_*}(T, M) = \mathrm{Ext}^1_{BP_*}(T, M) = 0.$$

Applying this to BP_*/I_{n+1} and all its suspensions gives the desired result. □

5. The spectral sequence

The object of this section is to prove Theorem A. That is, we construct a spectral sequence converging to BP_*L_nX whose E_2-term consists of the derived functors $L_n^s(BP_*X)$. Analogously, let C_nX denote the fiber of $X \to L_nX$. We construct a spectral sequence converging to BP_*C_nX whose E_2-term consists of the derived functors $T_n^s(BP_*X)$. Our method is based on the construction of the modified Adams spectral sequence due to Devinatz and Hopkins [Dev97, Section 1].

Definition 5.1. Define a functor D from injective BP_*BP-comodules to (the homotopy category of) spectra as follows. Given an injective BP_*BP-comodule I, consider the functor D_I from spectra to abelian groups defined by

$$D_I(X) = \operatorname{Hom}_{BP_*BP}(BP_*X, I).$$

Then D_I is a cohomology functor, so there is a unique spectrum $D(I)$ such that there is a natural isomorphism

$$D_I(X) \cong [X, D(I)].$$

The reason for the letter D is that D_I is a sort of duality functor, built along the lines of Brown-Comenetz duality [BC76]. Also note that we are considering cohomology functors as exact functors to ungraded abelian groups; we recover the usual graded cohomology functor by $D_I^t(X) = D_I(\Sigma^t X)$.

The following theorem is a special case of Theorem 1.5 of [Dev97].

Theorem 5.2. *Suppose I is an injective BP_*BP-comodule. Then there is a natural isomorphism $BP_*D(I) \cong I$.*

This isomorphism of course corresponds to the identity map of $D(I)$ under the isomorphism

$$[D(I), D(I)] \cong \operatorname{Hom}_{BP_*BP}(BP_*D(I), I).$$

We need to know how the $D(I)$ behave under localization.

Proposition 5.3. *Suppose I is an injective BP_*BP-comodule. Then the natural map $I \to L_nI$ induces an isomorphism*

$$L_nD(I) \to D(L_nI).$$

Proof. Recall that L_nI is again injective, by Corollary 2.4. We first note that $D(L_nI)$ is $E(n)$-local. Indeed, if $E(n)_*(X) = 0$, then $BP_*(X)$ is all v_n-torsion. Since L_nI has no v_n-torsion, we have

$$[X, D(L_nI)] \cong \operatorname{Hom}_{BP_*BP}(BP_*X, L_nI) = 0.$$

Thus $D(L_nI)$ is indeed L_n-local.

On the other hand, the map $D(I) \to D(L_nI)$ induces the map $I \to L_nI$ on BP_*-homology, by Theorem 5.2. Since $L_nI \cong I/T_nI$ by Corollary 2.4, this map becomes an isomorphism after applying Φ_*, and so $D(I) \to D(L_nI)$ is an $E(n)$-equivalence. $\square$

Corollary 5.4. *Suppose I is an injective BP_*BP-comodule. Then the natural map $T_nI \to I$ induces an isomorphism*

$$D(T_nI) \to C_nD(I).$$

Proof. Note that $D(T_nI)$ makes sense since T_nI is an injective comodule by Theorem 2.3. Since $BP_*(D(T_nI)) \cong T_nI$, one easily sees that $D(T_nI)$ is $E(n)$-acyclic. Therefore, the map $D(T_nI) \to D(I)$ induced by the inclusion $T_nI \to I$ induces the desired map

$$D(T_nI) \to C_nD(I).$$

This map is an isomorphism on $BP_*(-)$ by Proposition 5.3, and one can check that both sides are BP-local, so it is an isomorphism. □

We can now build our spectral sequences, following the standard approach used by Ravenel in [Rav86, Section 2.1]. Suppose X is a spectrum, and let $C = BP_*X$. Choose an injective resolution

$$0 \to C \xrightarrow{\eta} I_0 \xrightarrow{\tau_0} I_1 \xrightarrow{\tau_1} \cdots$$

of C in the category of BP_*BP-comodules. Let $\eta_s\colon C_s \to I_s$ denote the kernel of τ_s, so that $\eta_0 = \eta$.

The following lemma is easily proved by induction on s, and is implicit in [Dev97, Section 1].

Lemma 5.5. *Let X be a spectrum and choose an injective resolution of BP_*X as above. Then there is a tower*

$$\begin{array}{ccccccc} X = X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \cdots \\ & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\ & & K_0 & & K_1 & & \end{array}$$

over X satisfying the following properties.

(a) $K_s = \Sigma^{-s}D(I_s)$.
(b) X_{s+1} *is the fiber of* f_s.
(c) $BP_*X_s \cong \Sigma^{-s}C_s$.
(d) *The map f_s is induced by the inclusion $C_s \to I_s$.*
(e) $BP_*g_s = 0$, *and the boundary map $K_s \to \Sigma X_{s+1}$ induces the surjection $\Sigma^{-s}I_s \to \Sigma^{-s}C_{s+1}$ on BP_*-homology.*

We can now construct our spectral sequences. The following theorem is Theorem A except for the statements about convergence.

Theorem 5.6. *Let X be a spectrum. There is a natural spectral sequence $E_*^{**}(X)$ with $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ and E_2-term $E_2^{s,t}(X) \cong (L_n^sBP_*X)_t$. This is a spectral sequence of BP_*BP-comodules, in the sense that $E_r^{s,*}$ is a graded BP_*BP-comodule for all $r \geq 2$ and $d_r\colon E_r^{s,*} \to E_r^{s+r,*}$ is a BP_*BP-comodule map of degree $r-1$. Furthermore, every element in $E_2^{0,*}$ that comes from BP_*X is a permanent cycle.*

Proof. Begin with the tower of Lemma 5.5 and apply L_n. We get the tower below.

$$\begin{array}{ccccccc} L_nX = L_nX_0 & \xleftarrow{L_ng_0} & L_nX_1 & \xleftarrow{L_ng_1} & L_nX_2 & \xleftarrow{L_ng_2} & \cdots \\ & & \downarrow{\scriptstyle L_nf_0} & & \downarrow{\scriptstyle L_nf_1} & & \\ & & L_nK_0 & & L_nK_1 & & \end{array} \tag{5.7}$$

By applying BP_*-homology, we get an associated exact couple and spectral sequence. That is, we let $D_1^{s,t} = BP_{t-s}L_nX_s$ and $E_1^{s,t} = BP_{t-s}L_nK_s$. We take

$$i_1 = BP_{t-s}L_ng_s\colon D_1^{s+1,t+1} \to D_1^{s,t} \text{ and } j_1 = BP_{t-s}L_nf_s\colon D_1^{s,t} \to E_1^{s,t}$$

and we take

$$k_1\colon\ E_1^{s,t} \to D_1^{s+1,t}$$

to be BP_{t-s} of the boundary map $L_nK_s \to \Sigma L_nX_{s+1}$.

Note that this is an exact couple in the category of BP_*BP-comodules, in that each $D_1^{s,*}$ and $E_1^{s,*}$ is a graded BP_*BP-comodule and the maps i_1, j_1, k_1 are maps of comodules. It follows that the spectral sequence is a spectral sequence of BP_*BP-comodules.

Now, by combining Theorem 5.2 and Proposition 5.3, we find

$$E_1^{s,t} \cong BP_{t-s}(L_n\Sigma^{-s}D(I_s)) \cong BP_tD(L_nI_s) \cong (L_nI_s)_t.$$

To compute the first differential d_1, note that we have the commutative diagram below.

$$\begin{array}{ccccc} K_s & \longrightarrow & \Sigma X_{s+1} & \longrightarrow & \Sigma K_{s+1} \\ \downarrow & & \downarrow & & \downarrow \\ L_nK_s & \longrightarrow & L_n(\Sigma X_{s+1}) & \longrightarrow & L_n(\Sigma K_{s+1}) \end{array}$$

The map on BP_{t-s}-homology induced by the bottom composite is d_1. The map on BP_{t-s}-homology induced by the top composite is τ_s, by Lemma 5.5. The outside vertical maps are surjective in BP_*-homology, by Proposition 5.3 and Corollary 2.4. It follows that $d_1 = L_n\tau_s$. Therefore, the E_2-term of our spectral sequence is

$$E_2^{s,t} \cong H^s(L_nI_*)_t \cong (L_n^sBP_*X)_t,$$

as required.

The naturality of the spectral sequence follows in the usual way. That is, a map of spectra $X \to Y$ induces a map $BP_*X \to BP_*Y$. This can be lifted, nonuniquely, to a map of injective resolutions and so to a map of the towers of Lemma 5.5. This map induces a map of spectral sequences which is the evident map

$$L_n^s(BP_*X) \to L_n^s(BP_*Y)$$

on the E_2-terms. This map is independent of the choice of map of injective resolutions, and so is functorial. This also shows that our spectral sequence is independent of the choice of injective resolution (from E_2 on).

To complete the proof, we must show that every element in $E_2^{0,*}$ that comes from BP_*X is a permanent cycle. To see this, note that there is a map from the tower of Lemma 5.5 to the tower 5.7 induced by L_n. Applying BP_*-homology to the tower of Lemma 5.5 gives us a spectral sequence with $E_2^{s,t} = 0$ if $s > 0$ and $E_2^{0,t} = BP_tX$. The map from this spectral sequence to our spectral sequence immediately gives the desired result. □

We have an analogous theorem for C_n.

Theorem 5.8. *Let X be a spectrum. There is a natural spectral sequence $E_*^{**}(X)$ with $d_r\colon\ E_r^{s,t} \to E_r^{s+r,t+r-1}$ and E_2-term $E_2^{s,t}(X) \cong (T_n^sBP_*X)_t$. This is a spectral sequence of BP_*BP-comodules, in the sense that $E_r^{s,*}$ is a graded BP_*BP-comodule for all $r \geq 2$ and $d_r\colon\ E_r^{s,*} \to E_r^{s+r,*}$ is a BP_*BP-comodule map of degree $r-1$.*

Proof. Begin with the tower of Lemma 5.5 and apply C_n to get the tower below.

$$\begin{array}{ccccccc} C_nX = C_nX_0 & \xleftarrow{C_n g_0} & C_nX_1 & \xleftarrow{C_n g_1} & C_nX_2 & \xleftarrow{C_n g_2} & \cdots \\ & & \downarrow{\scriptstyle C_n f_0} & & \downarrow{\scriptstyle C_n f_1} & & \\ & & C_nK_0 & & C_nK_1 & & \end{array} \tag{5.9}$$

Apply BP_*-homology to get an associated exact couple and spectral sequence, as in the proof of Theorem 5.6. This time the E_1 term will be

$$E_1^{s,t} \cong BP_{t-s}C_nK_s \cong (T_nI_s)_t,$$

using Corollary 5.4. The identification of the E_2-term uses the commutative diagram below.

$$\begin{array}{ccccc} C_nK_s & \longrightarrow & C_n(\Sigma X_{s+1}) & \longrightarrow & C_n(\Sigma K_{s+1}) \\ \downarrow & & \downarrow & & \downarrow \\ K_s & \longrightarrow & \Sigma X_{s+1} & \longrightarrow & \Sigma K_{s+1} \end{array}$$

The vertical maps are injective on BP_*-homology by Corollary 5.4 and Theorem 2.3. Thus d_1, which is the effect on BP_{t-s}-homology of the top horizontal composite, is $T_n\tau_s$. Hence we get the desired E_2-term and naturality, as in Theorem 5.6. □

We must now prove that our spectral sequences converge strongly. This essentially boils down to showing that the homotopy inverse limits of the towers 5.7 and 5.9 are trivial. The plan of the proof is very simple; in the original tower of Lemma 5.5, we have $BP_*g_s = 0$. Hence $E(n)_*(L_ng_s) = E(n)_*g_s = 0$ as well by Landweber exactness. Now we just apply the following theorem.

Theorem 5.10. *Given $n \geq 0$, there exists an N such that every composite*

$$g = f_N \circ f_{N-1} \circ \cdots \circ f_1$$

of maps of spectra such that $E(n)_*f_i = 0$ for all i has $L_ng = 0$.

This theorem was certainly known to Hopkins and probably others.

Proof. Use the modified Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{E(n)_*E(n)}(E(n)_*X, E(n)_*Y) \Rightarrow [X, L_nY]_{t-s}$$

of Devinatz [Dev97]. It was proved in [HS99, Proposition 6.5] that there are integers r_0 and s_0, independent of X and Y, such that $E_r^{s,t} = 0$ whenever $r \geq r_0$ and $s \geq s_0$. Take $N = s_0$. Then the composite g is represented by an element in $E_2^{s,t}$ for some $s \geq N$. Therefore g must be represented by some element in $E_\infty^{s,t}$ for some $s \geq N$, so $g = 0$. □

The following corollary completes the proof of Theorem A.

Corollary 5.11. *The spectral sequence of Theorem* 5.6 *converges strongly to BP_*L_nX.*

Proof. In view of Theorem 5.10, the composites $L_nX_{k+s} \to L_nX_k$ in the tower 5.7 are trivial for large s. Hence $\lim_s BP_*L_nX_s = \lim^1_s BP_*L_nX_s = 0$, and so the spectral sequence converges conditionally [Boa99, Definition 5.10]. On the other hand, it is clear that $\lim^1_r E_r^{s,t} = 0$, since we have a horizontal vanishing line. Thus, the spectral sequence converges strongly to BP_*L_nX [Boa99, Theorem 7.3]. □

We also want to know that the other spectral sequence we have constructed converges.

Corollary 5.12. *The spectral sequence of Theorem* 5.8 *converges strongly to* BP_*C_nX*.*

Proof. We have a cofiber sequence $C_nX_s \to X_s \to L_nX_s$ of towers, where X_s denotes the tower of Lemma 5.5. By applying BP_*, we get an exact sequence of towers

$$BP_{*+1}L_nX_s \to BP_*C_nX_s \to BP_*X_s \to BP_*L_nX_s.$$

We have just seen, in Corollary 5.11, that the towers $BP_{*+1}L_nX_s$ and $BP_*L_nX_s$ are pro-trivial. It follows that the tower $BP_*C_nX_s$ is pro-isomorphic to the tower BP_*X_s. But the tower BP_*X_s is obviously pro-trivial by Lemma 5.5, so the tower $BP_*C_nX_s$ is also pro-trivial. Hence $\lim_s BP_*C_nX_s \cong \lim^1_s BP_*C_nX_s = 0$, and so the spectral sequence of Theorem 5.8 converges conditionally. Since it has a horizontal vanishing line, it converges strongly to BP_*C_nX [Boa99, Theorem 7.3]. □

We close the paper by considering the spectral sequence of Theorem A in case $X = S^0$ and $n > 0$. In that case, we have $E_2^{0,*} \cong BP_*$ and $E_2^{n,*} = BP_*/I_{n+1}^\infty$, by Corollary 3.6. The only possible differential is d_n, but this must be trivial since $E_2^{0,*}$ must consist of permanent cycles by Theorem A. Thus our spectral sequence degenerates to the short exact sequence of comodules

$$0 \to \Sigma^{-n}BP_*/I_{n+1}^\infty \to BP_*L_nS^0 \to BP_* \to 0.$$

A splitting of this sequence is given by the map $BP_* \to BP_*L_nS^0$ induced by $S^0 \to L_nS^0$. Hence we recover Ravenel's computation of $BP_*L_nS^0$ [Rav84, Theorem 6.2].

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