HOMOTOPY THEORY OF COMODULES OVER A HOPF ALGEBROID

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Abstract. Given a good homology theory E and a topological space X, E_*X is not just an E_*-module but also a comodule over the Hopf algebroid (E_*, E_*E). We establish a framework for studying the homological algebra of comodules over a well-behaved Hopf algebroid (A, Γ). That is, we construct the derived category $\mathrm{Stable}(\Gamma)$ of (A, Γ) as the homotopy category of a Quillen model structure on $\mathrm{Ch}(\Gamma)$, the category of unbounded chain complexes of Γ-comodules. This derived category is obtained by inverting the homotopy isomorphisms, **NOT** the homology isomorphisms. We establish the basic properties of $\mathrm{Stable}(\Gamma)$, showing that it is a compactly generated tensor triangulated category.

Introduction

Given a commutative ring k, a *Hopf algebroid* over k is a cogroupoid object in the category of commutative k-algebras. That is, a Hopf algebroid is a pair (A, Γ) of commutative k-algebras, so that, given a commutative k-algebra R, the set $k\text{-alg}(A, R)$ is naturally the objects of a groupoid with morphisms $k\text{-alg}(\Gamma, R)$. This gives several structure maps of which we remined the reader below. The reason for our interest in Hopf algebroids is that, if $E_*(-)$ is a well-behaved homology theory on topological spaces, then E_*X is naturally a comodule over the Hopf algebroid (E_*, E_*E). In particular, the study of comodules over the Hopf algebroid BP_*BP led to the Landweber exact functor theorem [Lan76], a result of fundamental importance in algebraic topology. When (E_*, E_*E) is a Hopf algebroid, the E_2-term of the Adams spectral sequence based on E is the bigraded Ext in the category of E_*E-comodules.

Thus we would like to understand the homological algebra of comodules over a Hopf algebroid. The simplest kind of Hopf algebroid is a *discrete* Hopf algebroid (A, A). The associated groupoid has no non-identity maps, and a comodule over (A, A) is the same thing as an A-module. One of the most useful tools in studying the homological algebra of A-modules is the unbounded derived category $\mathcal{D}(A)$, obtained by inverting the homology isomorphisms in the category $\mathrm{Ch}(A)$ of unbounded chain complexes of A-modules. The goal of this paper is to construct $\mathcal{D}(A, \Gamma)$, the derived category of a Hopf algebroid (A, Γ). We stress that homology isomorphisms are **NOT** the right thing to invert to form $\mathcal{D}(A, \Gamma)$. This is already clear in case (A, Γ) is a Hopf algebra over a field k, such as the Steenrod algebra. In this case, $\mathcal{D}(A, \Gamma)$ was constructed by the author in [Hov99, Section 2.5] and studied by Palmieri [Pal01]. The idea is that chain complexes of comodules are like topological spaces; they have homotopy as well as homology, and it is the

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homotopy isomorphisms we should invert, not the homology isomorphisms. For a discrete Hopf algebroid (A, A), homotopy and homology coincide, but not in general. To avoid confusion, we refer to $\mathcal{D}(A, \Gamma)$ as the *stable homotopy category of* Γ*-comodules*, and denote it by $\mathrm{Stable}(\Gamma)$.

We want $\mathrm{Stable}(\Gamma)$ to have all the usual properties of $\mathcal{D}(A)$ or the stable homotopy category; it should be a triangulated category with a compatible closed symmetric monoidal structure, and there should be a good set of generators. In fact, we want $\mathrm{Stable}(\Gamma)$ to be a stable homotopy category in the sense of [HPS97]. We also want

$$\mathrm{Stable}(\Gamma)(S^0 A, S^0 M)_* \cong \mathrm{Ext}^*_\Gamma(A, M)$$

for a comodule M, where $S^0 N$ denotes the complex consisting of N in degree 0 and 0 everywhere else. This will guarantee that we recover the E_2-term of the Adams spectral sequence based on E.

The axioms for a stable homotopy category, or, indeed, even for a triangulated category, are so painful to check that the best way to construct such a category is as the homotopy category of a Quillen model structure [Qui67]. One of the main goals of [Hov99] was to enumerate the conditions we need on a model structure so that its homotopy category is a stable homotopy category. We also point out that there are many advantages of a model structure over its associated homotopy category; the model structure allows one to perform constructions, such as homotopy limits and colimits, that are inaccessible in the homotopy category, and allows one to make comparisons with other model categories.

Thus, the bulk of this paper is devoted to constructing a model structure on $\mathrm{Ch}(\Gamma)$ in which the weak equivalences are the homotopy isomorphisms. We expect that the associated stable homotopy category $\mathrm{Stable}(\Gamma)$ will have many good properties and will provide insight into homotopy theory, as Palmieri's work [Pal01, Pal99] on the Steenrod algebra has done. We have in fact shown that $\mathrm{Stable}(E_*E)$ is a Bousfield localization of $\mathrm{Stable}(BP_*BP)$ for any Landweber exact commutative ring spectrum E in [Hov02a]; this then gives rise to a general change of rings theorem containing the change of rings theorem of Miller-Ravenel [MR77] and the author and Sadofsky [HS99a]. The construction of the model structure is complicated enough that we do not discuss such applications in this foundational paper. We do establish some beginning properties of $\mathrm{Stable}(\Gamma)$ in Section 6. In particular, we show that $\mathrm{Stable}(\Gamma)$ is monogenic (that is, bigraded suspensions of the sphere weakly generate the category) when $\Gamma = BP_*BP$ or $\Gamma = E_*E$ for E any Landweber exact homology theory over BP.

In order to establish our model structure, we need to first study the structure of the abelian category Γ-comod of Γ-comodules, which we do in Section 1. Most of the results in this section seem to be new, at least in the generality in which we give them, and of independent interest. For example, we study duality in Γ-comod, showing that a comodule is dualizable if and only if it is finitely generated and projective as an A-module.

We follow the usual plan to construct our model structure. That is, we start by building an auxiliary model structure in Section 2 called the projective model structure. This model structure is easy to construct, but has too few weak equivalences (unless the Hopf algebroid is discrete). So we must localize it by making the homotopy isomorphisms weak equivalences. This first necessitates a study of the homotopy isomorphisms in Section 3 and a reminder, with a few new results, about localization of model categories in 4. We finally construct the desired model structure in 5, and study some of the basic properties of Stable(Γ) in the aforementioned Section 6.

We should note that our results do not apply to an arbitrary Hopf algebroid. We need our Hopf algebroid to be *amenable*, defined precisely in Definition 2.3.2. All of the amenable Hopf algebroids we know are in fact Adams Hopf algebroids, defined in [GH00] but implicit in Adams' blue book [Ada74, Section III.13]. If E is a ring spectrum that satisfies Adams' condition, which we call *topologically flat*, that E be a minimal weak colimit of finite spectra X_α such that E_*X_α is finitely generated and projective over E_*, then (E_*, E_*E) is an Adams Hopf algebroid (Section 1.4). To make sure our results apply in cases of interest, we must check that interesting ring spectra E are topologically flat. We do this in Section 1.4, building on results of Adams and of Hopkins.

Hopf algebroids also arise in algebraic geometry in connection with stacks [FC90]. More precisely, a Hopf algebroid (A, Γ) is the same thing as a representable sheaf of groupoids Spec(A, Γ) with respect to the flat topology on the category of affine schemes. A Γ-comodule is equivalent to a quasi-coherent sheaf over Spec(A, Γ), as is proved in [Hov02b]. It is then natural to ask whether the approach we take in this paper to study the homological algebra of comodules can be generalized to quasi-coherent sheaves over possibly non-representable sheaves of groupoids, such as stacks. We do not know how to do this. The best we can say is that the Quillen equivalence class of our model structure on chain complexes of comodules depends only on the homotopy type of the associated stack of the given Hopf algebroid. The stacks that arise in algebraic topology, such as the stack of formal groups, are the associated stacks of Hopf algebroids, so this does give us a good construction of the unbounded derived category for such stacks.

The author has been trying to prove the results in this paper since 1997, when Doug Ravenel strongly encouraged him to build a stable homotopy category of BP_*BP-comodules. It is a pleasure to acknowledge the author's debt to Neil Strickland, who constructed Stable(BP_*BP) in a fairly ad hoc way, without a model structure, about 1997. The crucial input that finally enabled the author to build the model structure came from the paper of Paul Goerss and Mike Hopkins [GH00].

1. The abelian category of comodules

We begin with a fairly comprehensive study of the category Γ-comod of comodules over a Hopf algebroid (A, Γ). Some of these results are well-known, but others are apparently new.

Before we begin, we establish notation and remind the reader of some of the basic structure maps of Hopf algebroids. The symbol (A, Γ) will always denote a Hopf algebroid [Rav86, Appendix 1], and the symbol $\otimes$ always denotes $\otimes_A$, the tensor product of A-**bimodules**. Given an A-bimodule M, $\widetilde{M}$ denotes M with the A-actions reversed.

With these conventions, the structure maps of (A, Γ) include maps of commutative k-algebras $\eta_L\colon A \to \Gamma$ corepresenting the source of a morphism, $\eta_R\colon A \to \Gamma$ corepresenting the target of a morphism, and $\varepsilon\colon \Gamma \to A$ corepresenting the identity maps of the groupoid. This makes Γ into an A-**bimodule**, with η_L giving the left A-action and η_R giving the right A-action. There are then additional structure maps of k-algebras $\chi\colon \Gamma \to \widetilde{\Gamma}$ corepresenting the inverse of a morphism, and

$\Delta\colon \Gamma \to \Gamma \otimes \Gamma$ corepresenting the composition of a pair of morphisms. Of course, these maps must satisfy some relations assuring that we get a groupoid. For example, $\varepsilon\eta_R = \varepsilon\eta_L = 1_A$, since the source and target of the identity map at x are both x. The remaining relations can be found in [Rav86, Appendix 1].

1.1. **Basic structure.** Recall that a *left* Γ*-comodule* is a left A-module M equipped with a map $\psi\colon M \to \Gamma \otimes M$ satisfying a coassociativity and counit condition. There is an obvious notion of a map of comodules.

Lemma 1.1.1. *Suppose* Γ *is flat as a right* A*-module. Then the category* Γ*-comod is a cocomplete abelian subcategory of* A*-mod.*

Proof. Since the tensor product commutes with colimits, the A-module colimit of a diagram of comodules is again a comodule, and is the colimit in Γ-comod. That Γ-comod is abelian when Γ is flat is proved in [Rav86, Theorem A1.1.3]; we require flatness in order to conclude that the A-module kernel of a comodule map is again a comodule. □

Because of this lemma, we will assume throughout the paper that (A, Γ) is a *flat* Hopf algebroid; that is, that Γ is flat as a right A-module. Note that the conjugation χ defines an isomorphism between the left A-module Γ and the right A-module Γ, so Γ is also flat as a left A-module.

Lemma 1.1.2. *The category* Γ*-comod is a symmetric monoidal category. We denote the symmetric monoidal product by* $M \wedge N$*.*

The symmetric monoidal product is of course given by the tensor product, but the author, following Margolis [Mar83], thinks it is better to reserve the notation $M \otimes N$ for the tensor product of A-bimodules.

Proof. We define $M \wedge N = M \otimes N$, the tensor product of **left** A-modules, with comodule structure given by the composite

$$M \otimes N \xrightarrow{\psi\otimes\psi} (\Gamma \otimes M) \otimes (\Gamma \otimes N) \xrightarrow{g} \Gamma \otimes M \otimes N,$$

where $g(x \otimes m \otimes y \otimes n) = xy \otimes m \otimes n$. Note that g involves both multiplication and the twist isomorphism, and we must do both of these together to get the necessary bilinearity. We leave it to the reader to check that this does define a map from the tensor product, and that the composition above is a comodule structure. The unit of the tensor product is A, with comodule structure given by η_L. □

We now point out that the category of comodules is natural. Recall that a map $\Phi\colon (A, \Gamma) \to (B, \Sigma)$ of Hopf algebroids is a pair of ring homomorphisms $\Phi_0\colon A \to B$ and $\Phi_1\colon \Gamma \to \Sigma$ that corepresents a natural morphism of groupoids. This means that $\Phi_0\epsilon = \epsilon\Phi_1$, $\Phi_1\eta_L = \eta_L\Phi_0$, $\Phi_1\eta_R = \eta_R\Phi_0$, and $(\Phi_1 \otimes \Phi_1)\Delta = \Delta\Phi_1$.

Lemma 1.1.3. *A map* $\Phi\colon (A, \Gamma) \to (B, \Sigma)$ *induces a symmetric monoidal functor* $\Phi_*\colon \Gamma$*-comod* $\to \Sigma$*-comod.*

Proof. Define $\Phi_* M = B \otimes_A M$. The Σ-comodule structure on $B \otimes_A M$ is given by the composite

$$B \otimes M \xrightarrow{1\otimes\psi} B \otimes \Gamma \otimes M \to \Sigma \otimes M \cong \Sigma \otimes_B (B \otimes M),$$

where the map $B \otimes \Gamma \to \Sigma$ takes $b \otimes x$ to $b\Phi_1(x)$. □

In light of this lemma, the following definition is natural.

Definition 1.1.4. A map $\Phi\colon (A,\Gamma) \to (B,\Sigma)$ is a *weak equivalence* if Φ_* is an equivalence of categories.

The notion of weak equivalence is a fundamentally new feature that arises in studying Hopf algebroids; any weak equivalence of discrete Hopf algebroids is necessarily an isomorphism, but there are many examples of weak equivalences of Hopf algebroids that are not isomorphisms given in [Hov02b] and [HS02]. For example, if $E = v_n^{-1}BP$ and $F = E(n)$, it is proved in [HS02] that the evident map $(E_*, E_*E) \to (F_*, F_*F)$ is a weak equivalence of Hopf algebroids. The author used a different definition of weak equivalence in [Hov02b], but the two definitions are in fact equivalent [HS02]. In fact, Φ is a weak equivalence of Hopf algebroids if and only if it induces a homotopy equivalence of the associated stacks (that is, if and only if Φ induces a weak equivalence in the Hollander model structure on sheaves of groupoids [Hol01]).

A particular example of a map of Hopf algebroids is the map $\epsilon\colon (A,\Gamma) \to (A,A)$ that is the identity on A and the counit ϵ on Γ. Geometrically, this is the inclusion of the identity maps of a groupoid into the whole groupoid. The functor ϵ_* is just the forgetful functor from Γ-comodules to A-modules. As is well known [Rav86, A1.2.1], this functor has a right adjoint that takes an A-module M to the Γ-comodule $\Gamma\otimes M$, with structure map $\Delta \otimes 1$. This is called the *extended comodule* on M; in case M is itself a free A-module on the set S, then $\Gamma \otimes M$ is called the *cofree* comodule on M. We have a natural isomorphism

$$A\text{-mod}(M, N) \to \Gamma\text{-comod}(M, \Gamma \otimes N)$$

for Γ-comodules M and A-modules N. This natural isomorphism takes a map $f\colon M \to N$ of A-modules to the map of comodules $(1\otimes f)\psi$, and a map of comodules $g\colon M \to \Gamma \otimes N$ to the map $(\epsilon \otimes 1)g$ of A-modules.

It is less well-known that the extended comodule functor $M \mapsto \Gamma \otimes M$ itself has a right adjoint $R\colon \Gamma\text{-comod} \to A\text{-mod}$, defined by $RN = \Gamma\text{-comod}(\Gamma, N)$. The A-module action on RN is defined by $(af)(x) = f(x\eta_R(a))$.

Note that, if M is itself a comodule, then we can form the extended comodule $\Gamma \otimes M$ and the tensor product $\Gamma \wedge M$. The following lemma is well-known.

Lemma 1.1.5. *Suppose (A,Γ) is a flat Hopf algebroid, M is an A-module, and N is a Γ-comodule. Then there is a natural isomorphism of comodules*

$$(\Gamma \otimes M) \wedge N \to \Gamma \otimes (M \otimes N).$$

In particular, when $M = A$, we get a natural isomorphism of comodules

$$\Gamma \wedge N \to \Gamma \otimes N.$$

Proof. We first note that $(\Gamma \otimes M) \wedge N$ is the tensor product of the **left** A-modules $\Gamma \otimes M$ and N. There is a natural comodule map

$$f_{MN}\colon (\Gamma \otimes M) \wedge N \to \Gamma \otimes (M \otimes N)$$

adjoint to $\varepsilon \otimes 1 \otimes 1$. For fixed N, this is a natural transformation of right exact functors of M that commutes with direct sums. Since every A-module is a quotient of a map of free A-modules, it suffices to show that f_{AN} is an isomorphism.

In fact, we construct an inverse $g = g_{AN}$ to f_{AN}. We define g to be the composite

$$\Gamma \otimes M \xrightarrow{1\otimes\psi} \Gamma \otimes \Gamma \otimes M \xrightarrow{(\mu\circ(1\otimes\chi))\otimes 1} \Gamma \wedge M,$$

which is, *a priori*, only a map of A-modules. Note that, though the multiplication μ does not factor through $\Gamma\otimes\Gamma$, the composite $\mu\circ(1\otimes\chi)$ does do so, since χ switches the left and right units. A diagram chase shows that g and f_{AN} are inverses (and therefore that g is a comodule map). □

It is tempting to think that, given an A-module M, one can think of M as a trivial Γ-comodule, via the map $\eta_L\otimes 1\colon M\to\Gamma\otimes M$. This is wrong; for example, $v_n^{-1}BP_*$, for $n>0$, cannot be given the structure of a BP_*BP-comodule [JY80, Proposition 2.9]. The difficulty is that η_L is not a map of A-bimodules. However, there is a symmetric monoidal trivial comodule functor from the category of abelian groups to Γ-comodules that takes the abelian group M to $A\otimes_{\mathbb{Z}} M$ with the trivial comodule structure given by $\eta_L\otimes_{\mathbb{Z}} 1$. This functor has a right adjoint that takes the comodule N to the abelian group of primitive elements in N.

1.2. **Limits.** In general, right adjoints such as limits are difficult to construct for comodules, because the forgetful functor from Γ-comodules to A-modules does not preserve products, though it does preserve kernels. We give a general method for constructing right adjoints, involving resolutions by extended comodules. This is the same method, using right adjoints rather than left adjoints, used in [BW85] to construct colimits in the category of algebras over a triple. It is really an application of the special adjoint functor theorem.

For a comodule M, the adjoint to the identity map is the map $\psi\colon M\to\Gamma\otimes M$, which we now think of as a map of comodules, giving $\Gamma\otimes M$ the extended comodule structure. The map ψ is of course an embedding, since it is split over A by $\epsilon\otimes 1$. In particular, if $p\colon\Gamma\otimes M\to N$ denotes the cokernel of ψ, which is itself a comodule, then we have a natural diagram

$$M\xrightarrow{\psi}\Gamma\otimes M\xrightarrow{\psi p}\Gamma\otimes N \tag{1.2.1}$$

expressing M as the kernel of a map of extended comodules.

Now, if R is a right adjoint, then R will have to preserve kernels, so R is completely determined by its restriction to the full subcategory of extended comodules.

We first use this idea to show that Γ-comod is complete.

Proposition 1.2.2. *Suppose (A,Γ) is a flat Hopf algebroid. Then Γ-comod has products, and so is complete.*

Proof. Let us denote the comodule product we are trying to construct by $\prod_i^{\Gamma} M_i$. Adjointness shows that, if $\{M_i\}$ is a set of A-modules, then

$$\prod_i^{\Gamma}(\Gamma\otimes M_i)\cong\Gamma\otimes\prod_i M_i.$$

Now suppose we have a set of comodule maps $f_i\colon\Gamma\otimes M_i\to\Gamma\otimes N_i$. We need to define the comodule product $\prod_i^{\Gamma} f_i$. We define it to be the composite

$$\begin{aligned}\Gamma\otimes\prod M_i\xrightarrow{\Delta\otimes 1}\Gamma\otimes\Gamma\otimes\prod M_i\xrightarrow{1\otimes\alpha}\Gamma\otimes\prod(\Gamma\otimes M_i)\\ \xrightarrow{1\otimes\prod f_i}\Gamma\otimes\prod(\Gamma\otimes N_i)\xrightarrow{1\otimes\prod(\epsilon\otimes 1)}\Gamma\otimes\prod N_i,\end{aligned}$$

where α is the evident natural transformation. The reader can then check that this is a good definition of the product on the full subcategory of extended comodules, and, in particular, that $\prod_i^{\Gamma}(1\otimes g_i)=1\otimes\prod g_i$ for A-module maps g_i.

The definition of the product for a family of general comodules M_i is now forced on us, as explained in the paragraph preceding this proposition. To wit, given a set of comodules M_i, we have left exact sequences

$$0 \to M_i \xrightarrow{\psi} \Gamma \otimes M_i \xrightarrow{\psi p_i} \Gamma \otimes N_i,$$

and so we define $\prod_i^\Gamma M_i$ by the left exact sequence

$$0 \to \prod_i^{\Gamma} M_i \to \Gamma \otimes \prod_i M_i \xrightarrow{\prod_i^\Gamma (\psi p_i)} \Gamma \otimes \prod_i N_i.$$

We leave to the reader the proof that $\prod_i^\Gamma M_i$ is indeed the product in Γ-comod. □

Remark. An alternative approach to the category of Γ-comodules that is sometimes used (e.g., by Boardman [Boa82]) is to establish an equivalence of categories between Γ-comodules and a subcategory of Γ^*-modules. Here $\Gamma^* = \mathrm{Hom}_A^r(\Gamma, A)$, the A-bimodule of right A-module maps from Γ to A. It turns out that Γ^* is a (noncommutative) algebra over k and there is a map of algebras $A \to \Gamma^*$. There is a map $\Gamma \otimes M \xrightarrow{\alpha} \mathrm{Hom}_A(\Gamma^*, M)$. Using this map, any Γ-comodule becomes a Γ^*-module, and this clearly defines a faithful functor from Γ-comodules to Γ^*-modules. However, this functor will not in general be full, because the map α need not be injective. If Γ is projective over A, then α is injective, but Γ is not projective over A for many of the Hopf algebroids we are interested in. If Γ is projective over A, one can establish an equivalence between Γ-comodules and a full coreflective subcategory of Γ^*-modules. This means that the inclusion functor from Γ-comodules to Γ^*-modules has a right adjoint R. Indeed, if M is a Γ^*-module, let us denote by $\mu^*\colon M \to \mathrm{Hom}_A(\Gamma^*, M)$ the adjoint to the structure map of M. Then

$$RM = \{x \in M | \mu^*(x) = \alpha(y) \text{ for some } y\}.$$

We refer to RM as the largest subcomodule of M. One can then define the product of a set of comodules $\{M_i\}$ to be the largest subcomodule of the A-module product.

Building on the remark above, note that, for a set of comodules $\{M_i\}$, there is the natural commutative diagram of A-modules below.

$$\begin{array}{ccccc} \prod_i^\Gamma M_i & \longrightarrow & \Gamma \otimes \prod_i M_i & \longrightarrow & \Gamma \otimes \prod_i N_i \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ \prod_i M_i & \xrightarrow[\prod_i \psi]{} & \prod_i(\Gamma \otimes M_i) & \xrightarrow[\prod_i \psi p_i]{} & \prod_i (\Gamma \otimes N_i). \end{array}$$

This means that there is a natural induced map of A-modules $\prod_i^\Gamma M_i \to \prod_i M_i$. This map is injective when α is injective, which is certainly true if Γ is projective over A. It is an isomorphism when α is so, which is true if Γ is finitely generated and projective over A.

Since the product is right adjoint to the exact diagonal functor, the product is left exact. But it need not be exact in general. Indeed, let $A = \mathbf{Q}$ and $\Gamma = A[x]$, thought of as a primitively generated Hopf algebra over A. Let $X_n = \Gamma/(x^n)$ for $n \geq 1$, and let $Y_n = A$. There is a surjection $X_n \to Y_n$ that sends x^{n-1} to 1 and every other power of x to 0. But one can check that $\prod^\Gamma Y_n \cong \prod_n Y_n$, and that there is no element of $\prod^\Gamma X_n$ that hits $(1, 1, \dots, 1, \dots)$. Indeed, $\prod^\Gamma X_n$ consists of those elements $(f_1, f_2, \dots)$ of $\prod X_n$ such that the degrees of f_i are bounded.

We can also use this technique of constructing right adjoints to prove the following proposition.

Proposition 1.2.3. *Suppose* $\Phi\colon (A,\Gamma)\to(B,\Sigma)$ *is a map of Hopf algebroids. Then the functor* $\Phi_*\colon \Gamma\text{-comod}\to\Sigma\text{-comod}$ *has a right adjoint* Φ^*.

Proof. An adjointness argument shows that we must define

$$\Phi^*(\Sigma\otimes_B N)=\Gamma\otimes N$$

when N is a B-module. Given a comodule map $f\colon \Sigma\otimes_B N\to\Sigma\otimes_B N'$, we define $\Phi^*(f)\colon \Gamma\otimes N\to\Gamma\otimes N'$ as the following composite.

$$\Gamma\otimes N\xrightarrow{\Delta\otimes 1}\Gamma\otimes\Gamma\otimes N\xrightarrow{1\otimes\alpha}\Gamma\otimes\Sigma\otimes_B N\xrightarrow{1\otimes f}\Gamma\otimes\Sigma\otimes_B N'\xrightarrow{1\otimes\epsilon\otimes 1}\Gamma\otimes N'.$$

Here the map α is defined by $\alpha(x\otimes n)=\Phi_1(x)\otimes n$. We leave it to the reader to check that this definition is functorial, so that we have defined Φ^* on the full subcategory of extended comodules.

As usual, given an arbitrary Σ-comodule N, we write N as the kernel of

$$\psi p\colon \Sigma\otimes_B N\to\Sigma\otimes_B N',$$

where $p\colon \Sigma\otimes_B N\to N'$ is the cokernel of ψ. We then define $\Phi^*(N)$ as the kernel of

$$\Gamma\otimes N\xrightarrow{\Phi^*(\psi p)}\Gamma\otimes N'.$$

We leave to the reader the check that Φ^* is right adjoint to Φ_*. □

In fact, as pointed out to the author by Mark Behrens, $\Phi^*(N)\cong(\Gamma\otimes_A B)\square_\Sigma N$, where the symbol $\square$ denotes the cotensor product. This construction is studied briefly in [Rav86, Lemma A1.1.8], though the adjointness is not proved there.

1.3. **Duality and finite presentation.** We now show that Γ-comod is in fact a closed symmetric monoidal category, and we characterize the dualizable comodules.

Theorem 1.3.1. *If* (A,Γ) *is a flat Hopf algebroid, then the category* Γ*-comod is closed symmetric monoidal. Furthermore, the closed structure* $F(M,N)$ *is left exact in* N *and right exact in* M.

Proof. An adjointness argument shows that we must define

$$F(M,\Gamma\otimes N)=\Gamma\otimes\operatorname{Hom}_A(M,N).$$

Suppose we have a map $f\colon \Gamma\otimes N\xrightarrow{f}\Gamma\otimes N'$ of extended comodules. We need to define the map $F(M,f)\colon \Gamma\otimes\operatorname{Hom}_A(M,N)\to\Gamma\otimes\operatorname{Hom}_A(M,N')$. This map will be adjoint to a map

$$\Gamma\otimes\operatorname{Hom}_A(M,N)\to\operatorname{Hom}_A(M,N')$$

of A-modules, which will in turn be adjoint to the composite

$$\Gamma\otimes\operatorname{Hom}_A(M,N)\otimes M\xrightarrow{1\otimes\mathrm{Ev}}\Gamma\otimes N\xrightarrow{f}\Gamma\otimes N'\xrightarrow{\epsilon\otimes 1}N'.$$

We leave to the reader the check that this definition is functorial, so that we have defined $F(M,-)$ on the full subcategory of extended comodules. We also leave to the reader the check that there is a natural isomorphism

$$\Gamma\text{-comod}(L,F(M,\Gamma\otimes N))\cong\Gamma\text{-comod}(L\wedge M,\Gamma\otimes N),$$

where naturality refers to an arbitrary map of extended comodules $\Gamma\otimes N\to\Gamma\otimes N'$.

We then have no choice but to define $F(M,N)$ as the kernel of

$$\Gamma \otimes \operatorname{Hom}(M,N) \xrightarrow{F(M,\psi p)} \Gamma \otimes \operatorname{Hom}(M,N')$$

where $p\colon \Gamma \otimes N \to N'$ is the cokernel of ψ. The necessary adjunction isomorphism follows immediately.

Since $F(M,-)$ is right adjoint to the right exact functor $M \otimes -$, it is left exact. Now suppose we have a right exact sequence

$$M' \to M \to M'' \to 0.$$

Then we have a right exact sequence

$$L \wedge M' \to L \wedge M \to L \wedge M'' \to 0,$$

and so a left exact sequence of abelian groups

$$0 \to \Gamma\text{-comod}(L \wedge M'', N) \to \Gamma\text{-comod}(L \wedge M, N) \to \Gamma\text{-comod}(L \wedge M', N).$$

Applying adjointness, we find that $F(M'',N)$ has the universal property characterizing the kernel of $F(M,N) \to F(M',N)$. □

It would be nice to have a better understanding of $F(M,N)$. The following proposition is helpful.

Proposition 1.3.2. *Suppose (A,Γ) is a flat Hopf algebroid, and M and N are Γ-comodules.*

(a) *There is a natural map $F(M,N) \xrightarrow{\tau_{MN}} \operatorname{Hom}_A(M,N)$ of A-modules.*
(b) *If M is finitely generated over A, then τ_{MN} is injective.*
(c) *If M is finitely presented over A, then τ_{MN} is an isomorphism.*

Proof. Consider the natural diagram below.

$$\begin{array}{ccccc} F(M,N) & \longrightarrow & \Gamma \otimes \operatorname{Hom}_A(M,N) & \longrightarrow & \Gamma \otimes \operatorname{Hom}_A(M,N') \\ & & \downarrow & & \downarrow \\ \operatorname{Hom}_A(M,N) & \longrightarrow & \operatorname{Hom}_A(M,\Gamma \otimes N) & \longrightarrow & \operatorname{Hom}_A(M,\Gamma \otimes N') \end{array}$$

The vertical arrows take $x \otimes f$ to the map that takes m to $x \otimes f(m)$. It is not obvious that this diagram is commutative, but a careful diagram chase shows that it is. The rows both express their left-hand entry as a kernel, the first row by definition, and the second row by applying $\operatorname{Hom}_A(M,-)$ to diagram 1.2.1. Thus, there is a natural induced map $F(M,N) \to \operatorname{Hom}_A(M,N)$, proving part (a).

Parts (b) and (c) will follow if we can show that the vertical maps are injections when M is finitely generated and isomorphisms when M is finitely presented. Since Γ is flat as a right A-module, we can write $\Gamma = \operatorname{colim} C_i$, where the C_i are finitely generated projective A-modules. The natural map

$$C_i \otimes \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_A(M, C_i \otimes N)$$

is therefore an isomorphism. Hence, the map

$$\Gamma \otimes \operatorname{Hom}_A(M,N) \cong \operatorname{colim}_i C_i \otimes \operatorname{Hom}_A(M,N) \to \operatorname{colim}_i \operatorname{Hom}_A(M, C_i \otimes N)$$

is also an isomorphism. When M is finitely generated over A, the map

$$\operatorname{colim}_i \operatorname{Hom}_A(M, C_i \otimes N) \to \operatorname{Hom}_A(M, \Gamma \otimes N)$$

is injective; when M is finitely presented over A, it is an isomorphism. Parts (b) and (c) follow. □

We now recall that an object M in a cocomplete category $\mathcal{C}$ is called *λ-presented*, for a regular cardinal λ, if $\mathcal{C}(M,-)$ commutes with λ-filtered colimits (See [Bor94, Section 6.4]). When $\lambda=\omega$, we get the usual notion of a *finitely presented* object. An A-module M is λ-presented if and only if it is a quotient of a map of free modules, each of which has rank $<\lambda$.

Proposition 1.3.3. *Suppose (A,Γ) is a flat Hopf algebroid, M is a Γ-comodule, and λ is a regular cardinal. Consider the following three statements.*

(a) *M is λ-presented as a Γ-comodule.*
(b) *M is λ-presented as an A-module.*
(c) *The functor $F(M,-)$ commutes with λ-filtered colimits.*

Then (a) and (b) are equivalent, and (c) implies (a). In particular, M is κ-presented for some κ.

Proof. We first show that (a) implies (b). So suppose that M is λ-presented as a Γ-comodule, and we have a λ-filtered diagram of A-modules N_i. Then

$$\begin{aligned}\operatorname{colim}\operatorname{Hom}_A(M,N_i)&\cong\operatorname{colim}\Gamma\text{-comod}(M,\Gamma\otimes N_i)\\ \cong\Gamma\text{-comod}(M,\operatorname{colim}(\Gamma\otimes N_i))&\cong\Gamma\text{-comod}(M,\Gamma\otimes\operatorname{colim}N_i)\\ &\cong\operatorname{Hom}_A(M,\operatorname{colim}N_i).\end{aligned}$$

We now show that (b) implies (a). Suppose that M is λ-presented as an A-module, and we have a λ-filtered diagram of comodules N_i. We must show that the map

$$\operatorname{colim}\Gamma\text{-comod}(M,N_i)\to\Gamma\text{-comod}(M,\operatorname{colim}N_i)$$

is an isomorphism. It is obviously injective, since the forgetful functor to A-modules is faithful, and M is λ-presented as an A-module. On the other hand, suppose we have a map $f\colon M\to\operatorname{colim}N_i$ of comodules. As a map of A-modules f factors through some map $g\colon M\to N_i$ for some i. The difficulty is that g may not be a map of comodules, since ψg may not be equal to $(1\otimes g)\psi$. But they are equal as maps to

$$\Gamma\otimes\operatorname{colim}N_j\cong\operatorname{colim}(\Gamma\otimes N_j),$$

so they must be equal in some $\Gamma\otimes N_j$. It follows that the composite

$$M\xrightarrow{g}N_i\to N_j$$

is the desired factorization of f.

We now show that (c) implies (a). So suppose that $F(M,-)$ commutes with λ-filtered colimits, and we have a λ-filtered diagram of comodules N_i. Then

$$\begin{aligned}\operatorname{colim}\Gamma\text{-comod}(M,N_i)&\cong\operatorname{colim}\Gamma\text{-comod}(A,F(M,N_i))\\ \cong\Gamma\text{-comod}(A,\operatorname{colim}F(M,N_i))&\cong\Gamma\text{-comod}(A,F(M,\operatorname{colim}N_i))\\ &\cong\Gamma\text{-comod}(M,\operatorname{colim}N_i),\end{aligned}$$

where the second isomorphism holds because A is finitely presented. □

In any closed symmetric monoidal category with unit A, we define $DM=F(M,A)$. There is always a natural map $DM\wedge N\to F(M,N)$. When this map is an isomorphism for all N, M is called *strongly dualizable*, which we generally abbreviate to *dualizable*. The author does not know to whom this concept is due; perhaps Puppe [Pup79]. An excellent reference is [LMSM86, Chapter III], and the basic properties of dualizable objects are summarized in [HPS97, Theorem A.2.5].

Proposition 1.3.4. *Suppose (A,Γ) is a flat Hopf algebroid. Then a Γ-comodule M is dualizable in Γ-comod if and only if M is finitely generated and projective as an A-module.*

Proof. First suppose that M is finitely generated and projective as an A-module. We need to check that the map $F(M,A)\wedge N \to F(M,N)$ is an isomorphism. But $F(M,A)\cong \operatorname{Hom}_A(M,A)$ and $F(M,N)\cong \operatorname{Hom}_A(M,N)$ by Proposition 1.3.2. It is well known and easy to check that the natural map $\operatorname{Hom}_A(M,A)\otimes N \to \operatorname{Hom}_A(M,N)$ is an isomorphism when M is a finitely generated projective.

Now suppose that M is dualizable. Then the functor $F(M,-)$ commutes with colimits, since it is isomorphic to $F(M,A)\wedge(-)$. Proposition 1.3.3 then implies that M is finitely presented as an A-module. We now show that M must in fact be projective over A. Indeed, the functor $F(M,-)$ is always left exact, and because M is dualizable, $F(M,-)\cong F(M,A)\wedge(-)$ is also right exact. Hence $F(M,-)$ is an exact functor on the category of Γ-comodules. But Proposition 1.3.2 tells us that $F(M,-)\cong \operatorname{Hom}_A(M,-)$ since M is finitely presented over A. Now suppose E is an exact sequence of A-modules. Then $\operatorname{Hom}_A(M,\Gamma\otimes E)$ is again exact. But, since M is finitely presented,

$$\operatorname{Hom}_A(M,\Gamma\otimes E)\cong \Gamma\otimes \operatorname{Hom}_A(M,E)$$

by the argument of Proposition 1.3.2. Since Γ is faithfully flat over A, we conclude that $\operatorname{Hom}_A(M,E)$ is exact, so M is projective over A. □

1.4. **Generators and Adams Hopf algebroids.** We have just seen that the category of Γ-comodules has many good properties when (A,Γ) is a flat Hopf algebroid. But those properties are still not enough for us, because we need a good set of generators for the category of Γ-comodules. Recall that a set of objects $\mathcal{G}$ in an abelian category $\mathcal{C}$ is said to *generate* $\mathcal{C}$ when $\mathcal{C}(P,f)=0$ for all $P\in\mathcal{G}$ implies that $f=0$.

For much of the sequel, we will require that the dualizable comodules generate the category of Γ-comodules. The main advantage of this hypothesis is the following proposition.

Proposition 1.4.1. *Suppose (A,Γ) is a flat Hopf algebroid for which the dualizable comodules generate the category of Γ-comodules. Then the category of Γ-comodules is a locally finitely presentable Grothendieck category. Furthermore:*

(a) *Every comodule is a quotient of a direct sum of dualizable comodules.*
(b) *If M is a comodule and $x\in M$, then there is a dualizable comodule P and a map $P\to M$ of comodules whose image contains x.*
(c) *Every comodule that is finitely generated over A is a quotient of a dualizable comodule.*
(d) *Every comodule is the union of its subcomodules that are finitely generated over A.*
(e) *Every comodule is a filtered colimit of finitely presented comodules.*

Most of these facts are true in a general locally finitely presentable Grothendieck category; see [Ste75] for details. We therefore give only a sketch of the proof.

Proof. There is only a set of isomorphism classes of dualizable comodules, and dualizable comodules are finitely presented. Hence Γ-comod is a locally finitely presentable Grothendieck category. For part (a), let $\mathcal{G}$ denote a set containing one element from each isomorphism class of dualizable comodules, and let T be the set of all maps f with $\operatorname{dom} f \in \mathcal{G}$ and $\operatorname{codom} f = M$. Consider the map

$$\alpha\colon \bigoplus_{f\in T} \operatorname{dom} f \to M,$$

and let β denote the cokernel of this map. Then $\Gamma\text{-comod}(P, \beta) = 0$ for all dualizable P, so $\beta = 0$. Hence α is surjective.

Part (b) is an immediate corollary of part (a), and part (c) and part (d) follow easily from part (b). For part (e), we choose a small skeleton $\mathcal{F}$ of the category of finitely presented comodules and consider the category $\mathcal{F}/M$ consisting of all maps from an element of $\mathcal{F}$ to M. There is an obvious map

$$\operatorname{colim}_{f\in\mathcal{F}/M} \operatorname{dom} f \to M.$$

One can readily verify that $\mathcal{F}$ is filtered, and that this map is a monomorphism, for any flat Hopf algebroid (A, Γ). If the dualizable comodules generate, then it is an epimorphism by part (b). □

We also have the following corollary, which answers the question left open by Proposition 1.3.3.

Corollary 1.4.2. *Suppose (A, Γ) is a flat Hopf algebroid for which the dualizable comodules generate the category of Γ-comodules, M is a Γ-comodule, and λ is a regular cardinal. If M is λ-presented, then $F(M, -)$ commutes with λ-filtered colimits.*

Proof. Suppose we have a λ-filtered system of comodules N_i. We need to show that

$$\operatorname{colim} F(M, N_i) \xrightarrow{\alpha} F(M, \operatorname{colim} N_i)$$

is an isomorphism. Because the dualizable comodules generate, it suffices to show that $\Gamma\text{-comod}(P, f)$ is an isomorphism for all dualizable comodules P. Since dualizable comodules are in particular finitely presented, we have

$$\begin{aligned}
\Gamma\text{-comod}(P, \operatorname{colim} F(M, N_i)) &\cong \operatorname{colim} \Gamma\text{-comod}(P, F(M, N_i)) \\
\cong \operatorname{colim} \Gamma\text{-comod}(P \wedge M, N_i) &\cong \operatorname{colim} \Gamma\text{-comod}(M, DP \wedge N_i) \\
\cong \Gamma\text{-comod}(M, \operatorname{colim}(DP \wedge N_i)) &\cong \Gamma\text{-comod}(M, DP \wedge \operatorname{colim} N_i) \\
\cong \Gamma\text{-comod}(M \wedge P, \operatorname{colim} N_i) &\cong \Gamma\text{-comod}(P, F(M, \operatorname{colim} N_i)).
\end{aligned}$$

□

We now owe the reader some examples of flat Hopf algebroids for which the dualizable comodules generate the category of Γ-comodules. We learned the following definition from [GH00], but it is implicit in [Ada74, Section III.13].

Definition 1.4.3. A Hopf algebroid (A, Γ) is said to be an *Adams Hopf algebroid* when Γ is the colimit of a filtered system of comodules Γ_i, where Γ_i is finitely generated and projective over A.

In particular, any Adams Hopf algebroid is flat, since the colimit of projective modules is flat. Thus the Γ_i are dualizable comodules. The following proposition is a restatement of [GH00, Lemma 3.4].

Proposition 1.4.4. *If (A, Γ) is an Adams Hopf algebroid, then the dualizable comodules generate the category of Γ-comodules.*

Proof. Suppose (A,Γ) is Adams, and M is a Γ-comodule. Then we have:

$$\begin{aligned} M &\cong \operatorname{Hom}_A(A,M) \cong \Gamma\text{-comod}(A,\Gamma\otimes M) \\ &\cong \Gamma\text{-comod}(A,\Gamma\wedge M) \cong \Gamma\text{-comod}(A,\operatorname{colim}\Gamma_i\wedge M) \\ &\cong \operatorname{colim}\Gamma\text{-comod}(A,\Gamma_i\wedge M) \cong \operatorname{colim}\Gamma\text{-comod}(D\Gamma_i, M). \end{aligned}$$

The result follows. □

We now give some examples of Adams Hopf algebroids. Most of the ones we are interested in come from algebraic topology. Recall the notion of minimal weak colimit from [HPS97, Section 2.2].

Definition 1.4.5. A ring spectrum R is called *topologically flat* if R is the minimal weak colimit of a filtered diagram of finite spectra X_i such that R_*X_i is a finitely generated projective R_*-module.

This definition is based on [Ada74, Condition III.13.3].

Lemma 1.4.6. *Suppose R is a ring spectrum that is topologically flat and such that R_*R is commutative. Then (R_*,R_*R) is an Adams Hopf algebroid.*

Proof. Write R as the minimal weak colimit of the X_i. Then $R_*R=\operatorname{colim} R_*X_i$. In particular, R_*R is flat over R_* (and this is the reason for the term "topologically flat") and satisfies the Adams condition. □

The reason for the hypothesis that R_*R be commutative is that there could well be non-commutative ring spectra R, such as Morava K-theory $K(n)$ at the prime 2, where R_*R is nevertheless commutative.

Adams gave several examples of topologically flat ring spectra in [Ada74, Proposition III.13.4], which we restate here.

Theorem 1.4.7. *The ring spectra MU, MSp, K, KO, $H\mathbb{F}_p$, and $K(n)$ are topologically flat.*

Adams did not of course consider $K(n)$, since it had not been discovered yet, but his proof for $H\mathbb{F}_p$ works for any field spectrum.

We can add another case to this list as well. Recall that BP is the Brown-Peterson spectrum, and $BP_*\cong\mathbb{Z}_{(p)}[v_1,v_2,\dots]$. Given an invariant regular sequence $J=(p^{i_0},v_1^{i_1},\dots,v_{k-1}^{i_{k-1}})$ in BP_*, there is a spectrum BPJ with $BPJ_*\cong BP_*/J$ studied in [JY80].

Proposition 1.4.8. *Let $J=(p^{i_0},v_1^{i_1},\dots,v_{k-1}^{i_{k-1}})$ be an invariant regular sequence of length k in BP_*. Then BPJ is topologically flat.*

Proof. Write BP as a minimal weak colimit of spectra X_α, where X_α is a finite spectrum with cells in only even degrees. Then $BPJ\wedge BP$ is the minimal weak colimit of the $BPJ\wedge X_\alpha$. On the other hand, we claim that $BPJ\wedge BPJ$ is a wedge of 2^k copies of $BPJ\wedge BP$. Indeed, BPJ_*BPJ is a free module over BPJ_*BP, and so we choose generators for the free module and use them to construct the desired splitting.

Hence $BPJ\wedge BPJ$ is the minimal weak colimit of $BPJ\wedge Y_\alpha$, where Y_α is a finite wedge of copies of X_α. Since each $BPJ_*(Y_\alpha)$ is a free module, this completes the proof. □

The following theorem, generalizing Proposition 2.12 of [HS99b], gives us many other examples of topologically flat ring spectra. Recall that, if R is a ring spectrum and E is an R-module spectrum, then E is said to be *Landweber exact* over R if the natural map

$$E_* \otimes_{R_*} R_*X \to R_*X$$

is an isomorphism for all spectra X.

Theorem 1.4.9. *Suppose R is a topologically flat ring spectrum, and E is a Landweber exact R-module spectrum. Then E is topologically flat.*

As mentioned in [HS99b], a version of this theorem was certainly known to Hopkins. Rezk also proves a version of this theorem as Proposition 15.3 of [Rez98].

Proof. The proof is much like that of Proposition 2.12 of [HS99b]. Write R as the minimal weak colimit of finite spectra X_i such that R_*X_i is finitely generated and projective over R_*. Then E_*X_i is finitely generated and projective over E_*. We show that E is the filtered colimit of a diagram of finite wedges of suspensions of X_i. To do so, consider the category $\mathcal{F}/E$ of all maps of finite spectra to E, and consider the full subcategory $\mathcal{F}'/E$ of such maps whose domain is a (variable) finite wedge of suspensions of the X_j. We claim that this is cofinal in $\mathcal{F}/E$. Since E is the minimal weak colimit of the obvious functor from $\mathcal{F}/E$ to spectra, it will follow that E is the minimal weak colimit of the restriction of this functor to $\mathcal{F}'/E$.

To show that $\mathcal{F}'/E$ is cofinal in $\mathcal{F}/E$, it suffices to show that any map f from a finite spectrum Z to E factors through a finite wedge of suspensions of the X_i. By Spanier-Whitehead duality

$$E^*(Z) \cong E^* \otimes_{R^*} R^*Z.$$

We can thus write $f = \sum_{i=1}^m b_i \otimes c_i$. Because R is the minimal weak colimit of the X_j, each map c_i has a factorization

$$c_i = (Z \xrightarrow{g_i} \Sigma^{-|c_i|}X_i \xrightarrow{e_i} \Sigma^{-|c_i|}R).$$

Let $Y = \bigvee_{i=1}^m \Sigma^{-|c_i|}X_i$, let $g\colon Z \to Y$ be the map with components g_i, and let $h\colon Y \to E$ be the map with components $b_i \otimes e_i \in E^* \otimes_{R^*} R^*X_i \cong E^*(X_i)$. This gives the desired factorization. □

Of course, there are algebraic examples of Adams Hopf algebroids as well.

Proposition 1.4.10. *Suppose Γ is a Hopf algebra over a field k. Then (k, Γ) is an Adams Hopf algebroid.*

Proof. By Lemma 9.5.3 of [HPS97], every Γ-comodule is the filtered colimit of its finite-dimensional sub-comodules. In particular, this is true for Γ itself. □

Proposition 1.4.11. *Suppose (A, Γ) is an Adams Hopf algebroid, I is an invariant ideal in A, and v is a primitive element in A. Then $(A/I, \Gamma/I)$ and $(v^{-1}A, v^{-1}\Gamma)$ are Adams Hopf algebroids.*

Proof. Suppose that $\Gamma \cong \operatorname{colim} \Gamma_j$, where each Γ_j is finitely generated and projective over A. Then $\Gamma/I = \operatorname{colim} \Gamma_j/I$ and Γ_j/I is finitely generated and projective over A/I. Similarly, $v^{-1}\Gamma = \operatorname{colim} v^{-1}\Gamma_j$. □

Despite all these examples of Adams Hopf algebroids, there is a theoretical dif-ficuly with the notion.

Question 1.4.12. *Suppose* $\Phi\colon (A,\Gamma)\to(B,\Sigma)$ *is a weak equivalence of Hopf algebroids. Is it true that* (A,Γ) *is Adams if and only if* (B,Σ) *is Adams?*

Note that if Φ is a weak equivalence, then the dualizable Γ-comodules generate if and only if the dualizable Σ-comodules generate.

2. The projective model structure

In this section, we establish a preliminary model structure on $\mathrm{Ch}(\Gamma)$, the category of unbounded chain complexes of Γ-comodules.

2.1. **Construction and basic properties.** We recall the results of [CH02]. Beginning with a set of objects $\mathcal{S}$ in a cocomplete abelian category $\mathcal{A}$, there is a projective class $(\mathcal{P},\mathcal{E})$, where $\mathcal{E}$ consists of all maps f such that $\mathcal{A}(P,f)$ is onto for all P in $\mathcal{S}$, and $\mathcal{P}$ consists of all retracts of direct sums of elements of $\mathcal{S}$. The elements of $\mathcal{P}$ are called *relative projectives*, and the maps of $\mathcal{E}$ are called *relative epimorphisms*. This is [CH02, Lemma 1.5], but it is also easy to see.

The main result of [CH02] associates a model structure on $\mathrm{Ch}(\mathcal{A})$, the category of unbounded chain complexes in $\mathcal{A}$, to a projective class $(\mathcal{P},\mathcal{E})$, given some hypotheses. We recall that a chain map ϕ is a fibration in this model structure when $\mathcal{A}(P,\phi)$ is a degreewise surjection for all $P\in\mathcal{P}$, and a weak equivalence when $\mathcal{A}(P,\phi)$ is a homology isomorphism for all $P\in\mathcal{P}$.

The hypothesis needed is that functorial cofibrant replacements exist. This is automatic, by [CH02, Proposition 4.2], when $\mathcal{A}$ is complete and cocomplete, there are enough κ-small $\mathcal{P}$-projectives for some cardinal κ, and functorial $\mathcal{P}$-resolutions exist. When $\mathcal{P}$ is generated by a set $\mathcal{S}$ as above, then functorial $\mathcal{P}$-resolutions obviously exist, since there is a functorial $\mathcal{P}$-epic

$$\bigoplus_{P\in\mathcal{S}}\ \bigoplus_{f\in\mathcal{A}(P,M)} P\to M$$

for any $M\in\mathcal{A}$. When each object of $\mathcal{S}$ is λ-small for some λ, then there are enough κ-small $\mathcal{P}$-projectives (take κ to be the supremum of the λ's).

Now, if $\mathcal{A}$ happens to be a Grothendieck abelian category, then it is automatically complete and cocomplete, and every object in $\mathcal{A}$ is κ-presented, and so *a fortiori* κ-small, for some κ. This latter statement is an immediate corollary of the fact that Grothendieck abelian categories are locally presentable [Bek00, Proposition 3.10], but a direct proof can be found in the Appendix to [Hov01].

We thus have the following result, which was inexplicably not stated in [CH02].

Theorem 2.1.1. *Suppose* $\mathcal{A}$ *is a Grothendieck abelian category, and* $\mathcal{S}$ *is a set of objects in* $\mathcal{A}$*. Then there is a model structure on* $Ch(\mathcal{A})$ *in which the fibrations are the maps* ϕ *such that* $\mathcal{A}(P,\phi)$ *is a surjection for all* $P\in\mathcal{S}$ *and the weak equivalences are the maps* ϕ *such that* $\mathcal{A}(P,\phi)$ *is a homology isomorphism for all* $P\in\mathcal{S}$*. More generally, this model structure exists when* $\mathcal{A}$ *is complete and cocomplete, but not necessarily Grothendieck, as long as every object of* $\mathcal{S}$ *is* κ*-small for some* κ*.*

Now we return to the case at hand, when $\mathcal{A}$ is the category of Γ-comodules and (A,Γ) is a Hopf algebroid. In the light of the results of Section 1.4, we should take $\mathcal{S}$ to be the set of dualizable Γ-comodules.

Definition 2.1.2. Suppose (A,Γ) is a flat Hopf algebroid. Let $\mathcal{S}$ be a set containing one comodule from each isomorphism class of dualizable Γ-comodules. We refer to the retracts of direct sums of elements of $\mathcal{S}$ as *relatively projective* comodules, and to the maps f of comodules such that Γ-comod(P, f) is surjective for all $P \in \mathcal{S}$ as *relative epimorphisms*. The resulting model structure on $\mathrm{Ch}(\Gamma)$ obtained from Theorem 2.1.1 is called the *projective model structure*. Thus, a map ϕ is a *projective fibration* if ϕ is a degreewise relative epimorphism, and ϕ is a *projective equivalence* if Γ-comod(P, ϕ) is a homology isomorphism for all $P \in \mathcal{S}$. The map ϕ is a *projective cofibration*, or simply a *cofibration*, if ϕ has the left lifting property with respect to all projective trivial fibrations. We refer to an chain complex F as *projectively trivial* if $0 \to F$ is a projective equivalence.

Goerss and Hopkins [GH00] put a model structure on the category of nonnegatively graded chain complexes over an Adams Hopf algebroid. Their model structure gave us the idea for the projective model structure, but it is not the same, as they took $\mathcal{S}$ to be the set of all the $D\Gamma_i$ (under the assumption that $\Gamma = \operatorname{colim} \Gamma_i$). We do not know whether the projective model structure (when restricted to nonnegatively graded complexes) is Quillen equivalent to the Goerss-Hopkins model structure. The answer would seem to depend on questions about the structure of dualizable comodules that we are unable to answer. One obvious advantage of our definition is that the dualizable comodules are canonically attached to the symmetric monoidal category of Γ-comodules, while the $D\Gamma_i$ are not.

We point out that we do not need (A, Γ) to be an Adams Hopf algebroid, or even for the dualizable comodules to generate, for the projective model structure to exist. Also note that every relatively projective comodule is projective as an A-module, but we do not know if the converse holds.

Note also that because the elements of $\mathcal{S}$ are finitely presented in the category Γ-comod (see Proposition 1.3.3), filtered colimits of projective equivalences (resp. projective fibrations) are again projective equivalences (resp. projective fibrations).

We then have the following theorem describing some of the properties of the projective model structure.

Theorem 2.1.3. *Suppose (A, Γ) is a flat Hopf algebroid. Then the projective model structure on $Ch(\Gamma)$ is proper, finitely generated, stable, and symmetric monoidal. A map ϕ is a cofibration if and only if it is a degreewise split monomorphism whose cokernel is cofibrant. A chain complex X is cofibrant if and only if it is a retract of a colimit of complexes*

$$X_0 \to X_1 \to \cdots X_\alpha \cdots$$

where each $X_\alpha \to X_{\alpha+1}$ is a degreewise split monomorphism whose cokernel is a complex of relative projectives with no differential. The homotopy relation between cofibrant objects is the usual chain homotopy relation.

Proof. This all follows from the results of [CH02]. The characterization of cofibrant objects follows from Corollary 4.4 of [CH02], and the characterization of cofibrations follows from Proposition 2.5 of [CH02]. The fact that the model structure is proper is [CH02, Proposition 2.18], and stability is [CH02, Corollary 2.17] and obvious. The fact that homotopy is the usual chain homotopy is [CH02, Lemma 2.13].

The generating cofibrations for the projective model structure are $S^{n-1}P \to D^nP$ for $P \in \mathcal{S}$, and the generating trivial cofibrations are $0 \to D^nP$ for $P \in \mathcal{S}$. Here $S^{n-1}P$ denotes the complex which is P in degree $n-1$ and zero elsewhere, and D^nP denotes the complex which is P in degrees n and $n-1$ and 0 elsewhere.

This is proved in Section 5 of [CH02]. Each of $S^{n-1}P$, 0, and D^nP are finitely presented, so the projective model structure is finitely generated.

Finally, the fact that the projective model structure is symmetric monoidal follows from Corollary 2.21 of [CH02], the fact that A, the unit of $\wedge$, is a relative projective, and the fact that relative projectives are closed under $\wedge$. □

Note that, when (A, Γ) is discrete, a Γ-comodule is the same thing as an A-module. In this case, the relative projectives are just the projective A-modules, and we see that the projective model structure agrees with the usual projective model structure on $\mathrm{Ch}(A)$, in which the fibrations are the surjections and the weak equivalences are the homology isomorphisms.

We point out that there is another model structure on $\mathrm{Ch}(\Gamma)$ given by [CH02, Example 3.4] called the *absolute model structure*. In this model structure, the weak equivalences are the chain homotopy equivalences, the cofibrations are the degreewise split monomorphisms, and the fibrations are the degreewise split epimorphisms. Since the generating cofibrations of the projective model structure are degreewise split monomorphisms, and the generating trivial cofibrations are chain homotopy equivalences, we conclude that the identity functor is a left Quillen functor from the projective model structure to the absolute model structure. In particular, a trivial cofibration in the projective model structure is a chain homotopy equivalence, and all chain homotopy equivalences are projective equivalences.

The symmetric monoidal product behaves particularly well with respect to the projective model structure.

Proposition 2.1.4. *Suppose (A, Γ) is a flat Hopf algebroid. Then the projective model structure satisfies the monoid axiom. Furthermore, if X is cofibrant and f is a projective equivalence, then $X \wedge f$ is a projective equivalence.*

This proposition is important because of the work of Schwede and Shipley [SS00], who introduced the monoid axiom. As a consequence of their work and Proposition 2.1.4, given a monoid R in $\mathrm{Ch}(\Gamma)$, which is just a differential graded comodule algebra, there is a model structure on (differential graded) R-modules in which the fibrations are underlying projective fibrations and the weak equivalences are underlying projective equivalences. There is also a similar model structure on differential graded comodule algebras, and a projective equivalence $R \to R'$ of differential graded comodule algebras induces a Quillen equivalence from R-modules to R'-modules.

Proof. The monoid axiom, introduced by Schwede and Shipley in [SS00], asserts that, if K is the class of maps $\{j \wedge X\}$ where j is a generating trivial cofibration and X is arbitrary, then all transfinite compositions of pushouts of maps of K are projective equivalences. In the case at hand, j is one of the maps $0 \to D^nP$, where $P \in \mathcal{S}$. It follows easily that $j \wedge X$ is a dimensionwise split monomorphism and a chain homotopy equivalence, so a trivial cofibration in the absolute model structure. Thus, all transfinite compositions of pushouts of maps of K are also trivial cofibrations in the absolute model structure, and so in particular chain homotopy equivalences. Hence they are also projective equivalences.

Now suppose $f\colon Y \to Z$ is a projective equivalence and X is cofibrant. We want to show that $X \wedge f$ is a projective equivalence. Since X is cofibrant, X is a retract of a colimit of a sequence of complexes $\{X_i\}_{i<\lambda}$, where $X_i \to X_{i+1}$ is a degreewise split monomorphism whose cokernel C_i is a complex of relative projectives with zero differential. Since projective equivalences are closed under filtered colimits, it suffices to show that $X_i \wedge f$ is a projective equivalence for all $i \leq \lambda$, where $X_\lambda = \operatorname{colim} X_i$. We prove this by transfinite induction on i. We assume the base case $i = 0$ for the moment. The limit ordinal case follows from the fact the filtered colimits of projective equivalences are projective equivalences. For the sucessor ordinal case, we have the commutative diagram below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_i \wedge Y & \longrightarrow & X_{i+1} \wedge Y & \longrightarrow & C_i \wedge Y & \longrightarrow & 0 \\ & & {\scriptstyle X_i \wedge f}\downarrow & & {\scriptstyle X_{i+1} \wedge f}\downarrow & & \downarrow{\scriptstyle C_i \wedge f} & & \\ 0 & \longrightarrow & X_i \wedge Z & \longrightarrow & X_{i+1} \wedge Z & \longrightarrow & C_i \wedge Z & \longrightarrow & 0 \end{array}$$

The rows of this diagram are degreewise split and short exact, since $X_i \to X_{i+1}$ is degreewise split. Now apply the functor $\Gamma\text{-comod}(P, -)$ to this diagram for a fixed $P \in \mathcal{S}$. The rows of the resulting diagram will still be short exact. The induction hypothesis tells us that $\Gamma\text{-comod}(P, X_i \wedge f)$ is a homology isomorphism, and the base case of the induction (which we have postponed) tells us that $\Gamma\text{-comod}(P, C_i \wedge f)$ is a homology isomorphism. The long exact sequence in homology then implies that $\Gamma\text{-comod}(P, X_{i+1} \wedge f)$ is a homology isomorphism.

We are left with showing that $C \wedge f$ is a projective equivalence, where C is a complex of relative projectives with zero differential. Then $C \cong \bigoplus_n S^n C_n$. Again using the fact that the objects in $\mathcal{S}$ are finitely presented, we find that it suffices to show that $S^n C_n \wedge f$ is a projective equivalence. But C_n is a retract of a direct sum of elements of $\mathcal{S}$. Another use of the fact that objects in S are finitely presented reduces us to showing that $S^n Q \wedge f$ is a projective equivalence, for $Q \in \mathcal{S}$. Assume $P \in \mathcal{S}$. Then, since Q is strongly dualizable in Γ-comod by Proposition 1.3.4, we have

$$\Gamma\text{-comod}(P, S^n Q \wedge f) \cong \Sigma^{-n} \Gamma\text{-comod}(P \wedge DQ, f)$$

which is a homology isomorphism since $P \wedge DQ$ is also (isomorphic to something) in $\mathcal{S}$. □

An obvious drawback with the projective model structure is that is difficult to tell what the weak equivalences look like. We do have the following proposition.

Proposition 2.1.5. *Suppose (A, Γ) is a flat Hopf algebroid for which the dualizable Γ-comodules generate the category of Γ-comodules. Then every projective fibration is surjective, and every projective equivalence is a homology isomorphism.*

Proof. Suppose $p\colon X \to Y$ is a projective fibration, and $y \in Y_n$. By Proposition 1.4.1, there is a comodule P in $\mathcal{S}$ and a map $f\colon P \to Y_n$ whose image contains y. Suppose $f(t) = y$. Since p is a projective fibration, there is a map $g\colon P \to X_n$ such that $pg = f$. In particular, $pg(t) = y$, so p is surjective.

Now suppose p is a projective equivalence. We wish to show that p is a homology isomorphism. Every projective trivial cofibration is a chain homotopy equivalence, so a homology isomorphism. We can thus assume that p is a projective trivial fibration. In particular, p is surjective. Thus it suffices to show that $\ker p$ is exact. We know that $\ker p \to 0$ is a projective trivial fibration, so $\Gamma\text{-comod}(P, \ker p)$ is exact for all $P \in \mathcal{S}$. Suppose that x is a cycle in $\ker p_n$. Let Z_n denote the comodule of cycles in $\ker p_n$. Then there is a $P \in \mathcal{S}$, a $t \in P$, and a comodule map $f\colon P \to Z_n$ such that $f(t) = x$, by Proposition 1.4.1. The map f is a cycle in Γ-comod$(P, \ker p)$, so there is a map $g\colon P \to \ker p_{n+1}$ such that $dg = f$. In particular, $dg(t) = x$, so x is a boundary. □

2.2. **Naturality.** We now show that the projective model structure is natural in (A, Γ).

Proposition 2.2.1. *Suppose $\Phi\colon (A,\Gamma) \to (B,\Sigma)$ is a map of flat Hopf algebroids. Then Φ induces a left Quillen functor $\Phi_*\colon Ch(\Gamma) \to Ch(\Sigma)$ of the projective model structures.*

Proof. We have seen in Proposition 1.2.3 that Φ induces an adjunction

$$(\Phi_*, \Phi^*)\colon \Gamma\text{-comod} \to \Sigma\text{-comod}.$$

This prolongs to an adjunction $(\Phi_*, \Phi^*)\colon \mathrm{Ch}(\Gamma) \to \mathrm{Ch}(\Sigma)$ by defining Φ_* and Φ^* degreewise. Since Φ_* is symmetric monoidal, it preserves dualizable comodules. This is easy to see directly in this case, since if M is finitely generated and projective over A, then $B \otimes M$ is finitely generated and projective over B. It follows easily that Φ_* takes the generating (trivial) cofibrations of the projective model structure on $\mathrm{Ch}(\Gamma)$ to (trivial) cofibrations in the projective model structure on $\mathrm{Ch}(\Sigma)$. □

Note that there is a map of Hopf algebroids $\Phi\colon (A,\Gamma) \to (A,A)$ which is the identity on A and ϵ on Γ. The functor Φ_* is just the forgetful functor from Γ-comodules to A-modules, and the right adjoint Φ^* is the extended comodule functor. Hence, if f is a homology isomorphism of complexes of A-modules, then $\Gamma \otimes f$ is a projective equivalence.

Theorem 2.2.2. *Suppose $\Phi\colon (A,\Gamma) \to (B,\Sigma)$ is a weak equivalence of flat Hopf algebroids. Then $\Phi_*\colon Ch(\Gamma) \to Ch(\Sigma)$ is a Quillen equivalence of the projective model structures. In fact, both Φ_* and Φ^* preserve and reflect projective equivalences.*

Proof. Since Φ^* is a right Quillen functor, it preserves weak equivalences (between fibrant objects, but everything is fibrant). Since Φ_* is an equivalence, the unit $X \to \Phi^*\Phi_*X$ is an isomorphism, so Φ_* reflects projective equivalences. On the other hand, Φ^* is a symmetric monoidal left adjoint since Φ_* is an equivalence of categories. In particular, Φ^* preserves dualizable comodules. Thus Φ^* is a left (and right) Quillen functor of the projective model structures. Hence Φ_* is also a left and right Quillen functor, so Φ_* preserves projective equivalences. Thus Φ^* reflects projective equivalences.

To show that Φ_* is a Quillen equivalence, we need to show that, for X cofibrant and Y fibrant, a map $f\colon \Phi_*X \to Y$ is a projective equivalence if and only if its adjoint $g\colon X \to \Phi^*Y$ is a projective equivalence. Recall that g is obtained from f as the composite

$$X \to \Phi^*\Phi_*X \xrightarrow{\Phi^* f} \Phi^*Y.$$

The first map in this composite is an isomorphism. Thus g is a projective equivalence if and only if Φ^*f is so. But Φ^* preserves and reflects projective equivalences, so Φ^*f is a projective equivalence if and only if f is so. □

2.3. **The cobar resolution.** The projective model structure is clearly not the model structure we want, because

$$\mathrm{ho}\,\mathrm{Ch}(\Gamma)(S^0A, S^0A)_* \cong A$$

concentrated in degree 0, because A is both cofibrant and fibrant in the projective model structure. Recall that we want

$$\mathrm{Stable}(\Gamma)(S^0A, S^0A)_* \cong \mathrm{Ext}^*_\Gamma(A, A).$$

Therefore, we have to get an injective, or at least relatively injective, resolution of A involved. See Section 3.1 for a description of relatively injective comodules.

The resolution we choose is the (reduced) cobar resolution [Rav86, A1.2.11], though we offer a simpler construction of it. Suppose M is a Γ-comodule. Then ψ is a natural comodule embedding $M \to \Gamma \otimes M$ of M into an extended comodule, which is split over A by $\epsilon \otimes 1$. We can iterate this to construct a resolution of M by extended A-comodules. The most important case is when $M = A$. We begin with the A-split short exact sequence of comodules

$$0 \to A \xrightarrow{\eta_L} \Gamma \to \overline{\Gamma} \to 0.$$

Here $\overline{\Gamma}$ is of course the cokernel of η_L, but it is easily seen to be isomorphic to $\ker \epsilon$. When we think of it as $\ker \epsilon$, the coaction is defined by $\psi(x) = \Delta(x) - x \otimes 1$. We can then tensor this sequence with $\overline{\Gamma}^{\wedge s}$ to get the A-split short exact sequence of comodules

$$0 \to \overline{\Gamma}^{\wedge s} \to \Gamma \wedge \overline{\Gamma}^{\wedge s} \to \overline{\Gamma}^{\wedge(s+1)} \to 0.$$

We splice these short exact sequences together to obtain a complex LA, where $(LA)_{-n} = \Gamma \wedge \overline{\Gamma}^{\wedge n}$ for $n \geq 0$ and $(LA)_{-n} = 0$ for $n < 0$, and the differential is the composite

$$\Gamma \wedge \overline{\Gamma}^{\wedge n} \to \overline{\Gamma}^{\wedge(n+1)} \to \Gamma \wedge \overline{\Gamma}^{\wedge(n+1)}.$$

In particular, there is a homology isomorphism $S^0A \to LA$ induced by η_L, so that LA is a resolution of A, and the cycle comodule $Z_{-n}(LA)$ is isomorphic to $\overline{\Gamma}^{\wedge n}$ for $n > 0$ (and A for $n = 0$). Furthermore, the A-splittings patch together to show that $S^0A \to LA$ is a chain homotopy equivalence of complexes of A-modules.

The complex LA will be very important in the rest of this paper, but LA is not cofibrant in the projective model structure, since $(LA)_0 = \Gamma$ is not even projective over A in general. The following proposition is then crucial for us.

Proposition 2.3.1. *Suppose that (A, Γ) is a flat Hopf algebroid, that the dualizable Γ-comodules generate the category of Γ-comodules, and that $\overline{\Gamma} \wedge X$ is projectively trivial when X is so. Let LA denote the cobar resolution of A.*

(a) *If p is a projective fibration, then $LA \wedge p$ is a projective fibration.*
(b) *If p is a projective equivalence, then $LA \wedge p$ is a projective equivalence.*

Note that, for part (a), it is sufficient to assume that dualizable comodules generate.

In view of Proposition 2.3.1, we make the following definition.

Definition 2.3.2. A Hopf algebroid (A, Γ) is *amenable* when it is flat, the dualizable Γ-comodules generate the category of Γ-comodules, and $\overline{\Gamma} \wedge (-)$ preserves projectively trivial complexes.

For Proposition 2.3.1 to be of use, we need to know that amenable Hopf algebroids do exist.

Proposition 2.3.3. *Every Adams Hopf algebroid is amenable.*

The rest of this section will be devoted to proving Propositions 2.3.1 and 2.3.3. Proposition 2.3.3 is an immediate consequence of the following lemma and Proposition 1.4.4.

Lemma 2.3.4. *Suppose (A, Γ) is a flat Hopf algebroid.*

(a) *If M is a filtered colimit of dualizable comodules, then $M \wedge (-)$ preserves projectively trivial complexes.*
(b) *If (A, Γ) is Adams, then $\overline{\Gamma}$ is a filtered colimit of dualizable comodules.*

Proof. For part (a), recall that filtered colimits of projective equivalences are projective equivalences. We can therefore assume that M itself is a dualizable comodule. Suppose then that X is projectively trivial. We must show that $\Gamma\text{-comod}(P, M \wedge X)$ has no homology for all dualizable comodules P. But

$$\Gamma\text{-comod}(P, M \wedge X) \cong \Gamma\text{-comod}(P \wedge DM, X)$$

since M is dualizable. Furthermore, $P \wedge DM$ is again dualizable, so since X is projectively trivial, we are done.

For part (b), since (A, Γ) is an Adams Hopf algebroid, we have $\Gamma = \operatorname{colim}_{i \in \mathcal{I}} \Gamma_i$ for a filtered small category $\mathcal{I}$ of arrows $i\colon \Gamma_i \to \Gamma$ such that Γ_i is dualizable. Let $\mathcal{J}$ denote the category of factorizations $A \to \Gamma_i \xrightarrow{i} \Gamma$ of η_L through an arrow of $\mathcal{I}$. By abuse of notation, we write the map $A \to \Gamma_i$ as η_L as well; note that this η_L must be a split monomorphism of A-modules, since the usual η_L is so. We claim that $\mathcal{J}$ is filtered and that the obvious functor from $\mathcal{J}$ to $\mathcal{I}$ is cofinal (see Definition 2.3.8 of [HPS97] for a reminder of what this means). This is a straightforwad consequence of the fact that A is itself finitely presented as a Γ-comodule. It follows then that $\operatorname{colim}_{j \in \mathcal{J}} \Gamma_j \cong \Gamma$, and therefore that $\operatorname{colim} \overline{\Gamma_j} \cong \overline{\Gamma}$, where $\overline{\Gamma_j} = \operatorname{coker} \eta_L$. Each $\overline{\Gamma_j}$ is finitely generated and projective over A, and hence dualizable. □

Part (a) of Proposition 2.3.1 is an immediate consequence of the following lemma and Proposition 2.1.5.

Lemma 2.3.5. *Let (A, Γ) be a flat Hopf algebroid.*

(a) *If M is an extended comodule, then $M \wedge (-)$ takes surjections of comodules to relative epimorphisms.*
(b) *If X is a complex of extended comodules, then $X \wedge (-)$ takes surjections of complexes to projective fibrations.*

Proof. Suppose f is a surjection of complexes, and $M \cong \Gamma \otimes N$ is an extended comodule. Let P be a dualizable comodule. Then, using Lemma 1.1.5, we have

$$\begin{aligned} \Gamma\text{-comod}(P, M \wedge f) &\cong \Gamma\text{-comod}(P, (\Gamma \otimes N) \wedge f) \\ &\cong \Gamma\text{-comod}(P, \Gamma \otimes (N \otimes f)) \cong \operatorname{Hom}_A(P, N \otimes f). \end{aligned}$$

Since f is surjective, so is $N \otimes f$. Since P is projective over A, $\operatorname{Hom}_A(P, N \otimes f)$ is also surjective, so $M \wedge f$ is a relative epimorphism.

Now suppose X is a complex of extended comodules. Then, in degree n, we have

$$(X \wedge f)_n \cong \bigoplus_m X_m \wedge f_{n-m}.$$

Each map f_{n-m} is surjective, so part (a) assures us that $X_m \wedge f_{n-m}$ is a relative epimorphism. Since the dualizable complexes are finitely presented, direct sums of relative epimorphisms are again relative epimorphisms. Hence $X \wedge f$ is a projective fibration. □

We are left with proving part (b) of Proposition 2.3.1. Our approach is similar to that of Lemma 2.3.5.

Lemma 2.3.6. *Suppose (A, Γ) is a flat Hopf algebroid.*

(a) *Suppose N is a flat A-module. Then $(\Gamma \otimes N) \wedge (-)$ takes exact complexes to projectively trivial complexes.*
(b) *Suppose X is a bounded below complex such that $X_n \wedge (-)$ preserves projectively trivial complexes for all n. Then $X \wedge (-)$ preserves projectively trivial complexes.*
(c) *Suppose X is a complex of comodules such that $X_n \wedge (-)$ and $Z_n X \wedge (-)$ preserve projectively trivial complexes for all n. Then $X \wedge (-)$ preserves projectively trivial complexes.*

Here $Z_n X$ denotes the cycles in degree n, as usual.

Proof. For part (a), suppose Y is a projectively trivial complex and P is a dualizable comodule. Then, using Lemma 1.1.5, we have

$$\Gamma\text{-comod}(P, (\Gamma \otimes N) \wedge Y) \cong \Gamma\text{-comod}(P, \Gamma \otimes (N \otimes Y)) \cong \operatorname{Hom}_A(P, N \otimes Y).$$

Since Y is exact and N is flat, $N \otimes Y$ is also exact. Since P is projective over A, $\operatorname{Hom}_A(P, N \otimes Y)$ is also exact, as required.

For part (b), let Y be a projectively trivial complex. Without loss of generality, we can assume that $X_n = 0$ for $n < 0$. Suppose $P \in \mathcal{S}$, and $z \colon P \to (X \wedge Y)_n$ is a cycle in $\Gamma\text{-comod}(P, X \wedge Y)$. Since P is finitely presented as a Γ-comodule,

$$\Gamma\text{-comod}(P, (X \wedge Y)_n) \cong \bigoplus_{i=0}^{\infty} \Gamma\text{-comod}(P, X_i \wedge Y_{n-i}).$$

We can therefore write $z = (z_0, z_1, \dots, z_i, \dots)$, where $z_i \colon P \to X_i \wedge Y_{n-i}$ and $z_i = 0$ for large i. Define the *degree* of z to be the largest i such that z_i is nonzero. We will show that every cycle z is homologous to a cycle of smaller degree; since there are no cycles of degree -1 this will complete the proof. Indeed, suppose z has degree k. Then z_k has to be a cycle in the complex $X_k \wedge Y$. By assumption, $X_k \wedge Y$ is projectively trivial, so z_k must be a boundary in this complex. This means that there is a $w \colon P \to X_k \wedge Y_{n-k+1}$ such that $(1 \wedge d)w = z_k$. But then z is homologous to $z' = z + (-1)^{k+1} dw$, and one can easily check that w has degree $< k$.

For part (c), again assume that Y is projectively trivial. Let X^i be the subcomplex of X such that $X^i_n = X_n$ for $n > -i$, $X^i_n = 0$ for $n < -i$, and $X^i_{-i} = Z_{-i} X$. By part (b), each of the complexes $X^i \wedge Y$ is projectively trivial. But $X = \operatorname{colim} X^i$, so $X \wedge Y = \operatorname{colim} X^i \wedge Y$. Since filtered colimits of projective equivalences are projective equivalences, $X \wedge Y$ is therefore projectively trivial. □

We can now prove Proposition 2.3.1(b).

Proof of Proposition 2.3.1(*b*). We need to show that $LA \wedge (-)$ preserves projective equivalences. Since the projective trivial cofibrations are in particular chain homotopy equivalences, $LA \wedge (-)$ certainly takes them to projective equivalences. It therefore suffices to show that $LA \wedge (-)$ preserves projective trivial fibrations. By part (a), $LA \wedge (-)$ preserves projective fibrations, so it suffices to show that $LA \wedge (-)$ preserves projectively trivial complexes.

In view of Lemma 2.3.6, it suffices to show that $(LA)_n \wedge (-)$ and $Z_n LA \wedge (-)$ preserve projectively trivial complexes. Since $(LA)_{-n} = \Gamma \wedge \overline{\Gamma}^{\wedge n}$ for $n \geq 0$, and $\overline{\Gamma}$

is flat as an A-module since Γ is so, Lemma 2.3.6(a) guarantees that $(LA)_n \wedge (-)$ preserves projectively trivial complexes. On the other hand, $Z_{-n}(LA) \cong \overline{\Gamma}^{\wedge n}$ for $n \geq 0$. The amenable assumption guarantees that $\overline{\Gamma} \wedge (-)$ preserves projectively trivial complexes, and then iteration shows that $Z_{-n}(LA) \wedge (-)$ does so as well. □

3. Homotopy groups

When the dualizable Γ-comodules generate the category of Γ-comodules, we know from Proposition 2.1.5 that projective equivalences are homology isomorphisms. But homology is not the most important functor of complexes of comodules; homotopy is. In this section we define and study the homotopy groups of a chain complex of comodules. We show that these homotopy groups are closely related to Ext in the category of Γ-comodules and have similar properties. When (A, Γ) is amenable, every projective equivalence is a homotopy isomorphism and every homotopy isomorphism is a homology isomorphism. The object of Section 5 will then be to construct a model structure on $\mathrm{Ch}(\Gamma)$ in which the weak equivalences are the homotopy isomorphisms.

3.1. **Relatively injective comodules.** To explain homotopy groups, we need to remind the reader of some of the basic results on relatively injective comodules. Some of this can be found in [Rav86, Appendix 1].

Definition 3.1.1. Suppose (A, Γ) is a flat Hopf algebroid. A comodule I is called *relatively injective* if Γ-comod$(-, I)$ takes A-split short exact sequences to short exact sequences.

Lemma 3.1.2. *Suppose (A, Γ) is a flat Hopf algebroid. The relatively injective comodules are the retracts of extended comodules. In particular, there is a natural A-split embedding of any comodule into a relatively injective comodule.*

Proof. We have $\Gamma\text{-comod}(-, \Gamma \otimes N) \cong \mathrm{Hom}_A(-, N)$. Thus extended comodules, and so also retracts of extended comodules, are relatively injective. Conversely, if I is a relative injective, the map $I \xrightarrow{\psi} \Gamma \otimes I$ must have a retraction, since it is a map of comodules that is split over A by $\epsilon \otimes 1$. Thus I is a retract of $\Gamma \otimes I$. The natural A-split embedding of the statement of the lemma is just $\psi\colon M \to \Gamma \otimes M$. □

Of course, there are (absolutely) injective comodules as well. A similar argument shows that the injective comodules are retracts of extended comodules $\Gamma \otimes I$, where I is an injective A-module. But relatively injective comodules are much easier to work with than injective comodules, partly because injective A-modules are complicated, and partly because of the following lemma.

Lemma 3.1.3. *Suppose (A, Γ) is a flat Hopf algebroid.*

(a) *Relatively injective comodules are closed under coproducts and products.*
(b) *If M is an arbitrary comodule and I is relatively injective, then $I \wedge M$ and $F(M, I)$ are relatively injective.*

Proof. For part (a), it suffices to show that extended comodules are closed under coproducts and products. But we have

$$\bigoplus (\Gamma \otimes M_i) \cong \Gamma \otimes (\bigoplus M_i) \text{ and } \prod^{\Gamma} (\Gamma \otimes M_i) \cong \Gamma \otimes (\prod M_i),$$

the latter by the construction of products in Proposition 1.2.2.

For part (b), we first prove that $F(M,I)$ is relatively injective. We must show that Γ-comod$(-,F(M,I))$ takes A-split short exact sequences to short exact sequences. But Γ-comod$(-,F(M,I))$ is naturally isomorphic to Γ-comod$(-\wedge M,I)$, so this is clear.

To show that $I\wedge M$ is relatively injective, note that $I\wedge M$ is a retract of $(\Gamma\otimes I)\wedge M$, which is isomorphic to $\Gamma\wedge I\wedge M$ by Lemma 1.1.5. On the other hand, another use of Lemma 1.1.5 shows that $\Gamma\wedge I\wedge M$ is isomorphic to the extended comodule $\Gamma\otimes(I\otimes M)$, completing the proof. □

Relatively injective comodules can be used to compute Ext when the source is projective over A.

Lemma 3.1.4. *Suppose (A,Γ) is a flat Hopf algebroid, P is a Γ-comodule that is projective over A, and I is a relatively injective comodule. Then $\operatorname{Ext}^n_\Gamma(P,I)=0$ for all $n>0$. Hence, if I_* is a resolution of M by relatively injective comodules,*

$$\operatorname{Ext}^n_\Gamma(P,M)\cong H_{-n}(\Gamma\text{-}comod(P,I_*)).$$

This lemma is proved as Lemma A1.2.8(b) of [Rav86].

3.2. **Homotopy groups.**

Now recall that LA denotes a specific resolution of A by relatively injective comodules, defined in Section 2.3, such that the map $S^0A\to LA$ is a chain homotopy equivalence over A. It follows that $LA\wedge M$ is a resolution of M for any comodule M. It is in fact a resolution by relative injectives by Lemma 3.1.3. Thus we have

$$\operatorname{Ext}^n_\Gamma(P,M)\cong H_{-n}(\Gamma\text{-comod}(P,LA\wedge M))$$

for any comodule P that is projective over A.

We now extend this definition, replacing M by a complex X.

Definition 3.2.1. Suppose (A,Γ) is a flat Hopf algebroid, $X\in \operatorname{Ch}(\Gamma)$, $P\in\mathcal{S}$, and $n\in\mathbb{Z}$. Define the *nth homotopy group of X with coefficients in P*, $\pi^P_n(X)$, by $\pi^P_n(X)=H_{-n}(\Gamma\text{-comod}(P,LA\wedge X))$.

These homotopy groups will be equal to the graded maps from P to X in the stable homotopy category of Γ-comodules that we are trying to construct. Since that category will be closed symmetric monoidal, it will also turn out that $\pi^P_n(X)\cong H_n(RF(P,X))$, where $RF(-,-)$ is the right derived functor, using the homotopy model structure we will develop in Section 5, of the chain complex Hom functor $F(-,-)$.

We need to say a few words about grading. We have essentially two choices; we can grade homotopy as if it were the homotopy groups of a space, or we can grade it as if it were the Ext groups of a comodule. Either way has problems; grading it like Ext means the exact sequences on homotopy go up instead of down in dimension, but grading it like homotopy means the homotopy groups of A will be concentrated in negative degrees. Following Palmieri's work on the Steenrod algebra [Pal01], we choose to grade it like Ext. This extends to bigrading as well; if (A,Γ) is a graded Hopf algebroid, as it always is in algebraic topology, we define

$$\pi^P_{s,t}(X)=H_{-s,t}(\Gamma\text{-comod}(P,LA\wedge X)).$$

The homotopy groups are of course functorial in X, and they satisfy the expected properties, correcting for the strange grading.

Lemma 3.2.2. *Suppose (A,Γ) is a flat Hopf algebroid.*

(a) *A short exact sequence of complexes*

$$0 \to X \to Y \to Z \to 0$$

induces a natural long exact sequence

$$\cdots \to \pi_n^P(X) \to \pi_n^P(Y) \to \pi_n^P(Z) \to \pi_{n+1}^P(X) \to \cdots$$

(b) *If X is a filtered colimit of complexes X^i, then $\pi_n^P(X) \cong \operatorname{colim} \pi_n^P(X^i)$.*

Proof. For part (a), since LA is degreewise flat over A, the sequence

$$0 \to LA \wedge X \to LA \wedge Y \to LA \wedge Z \to 0$$

remains exact. By Lemma 3.1.3, $LA \wedge X$ is a complex of relative injectives. Therefore, the sequence of complexes

$$0 \to \Gamma\text{-comod}(P, LA \wedge X) \to \Gamma\text{-comod}(P, LA \wedge Y) \to \Gamma\text{-comod}(P, LA \wedge Z) \to 0$$

remains exact, by Lemma 3.1.4. The long exact sequence in homology of this short exact sequence finishes the proof of part (a).

For part (b), we simply note that $LA \wedge -$, Γ-comod$(P, -)$, and homology all commute with filtered colimits. □

3.3. **Homotopy isomorphisms.** A chain map ϕ is called a *homotopy isomorphism* if $\pi_n^P(\phi)$ is an isomorphism for all $n \in \mathbb{Z}$ and all $P \in \mathcal{S}$. Note that ϕ is a homotopy isomorphism if and only if $LA \wedge \phi$ is a projective equivalence. We claim that homotopy isomorphisms are the natural notion of weak equivalence in $\mathrm{Ch}(\Gamma)$.

Proposition 3.3.1. *Suppose (A, Γ) is an amenable Hopf algebroid. Then every projective equivalence is a homotopy isomorphism, and every homotopy isomorphism is a homology isomorphism.*

Proof. Suppose p is a projective equivalence. Then Proposition 2.3.1 tells us that $LA \wedge p$ is also a projective equivalence, so p is a homotopy isomorphism. Now suppose $p\colon X \to Y$ is a homotopy isomorphism. Then $LA \wedge p$ is a projective equivalence, and hence a homology isomorphism by Proposition 2.1.5. But $A \to LA$ is a chain homotopy equivalence over A, so $X \to LA \wedge X$ and $Y \to LA \wedge Y$ are also chain homotopy equivalences over A, and in particular homology isomorphisms. Hence p is a homology isomorphism. □

Homotopy isomorphisms have the properties one would hope for in a collection of weak equivalences.

Proposition 3.3.2. *Suppose (A, Γ) is a flat Hopf algebroid.*

(a) *Homotopy isomorphisms are closed under retracts and have the two out of three property.*
(b) *Homotopy isomorphisms are closed under filtered colimits.*
(c) *If f is an injective homotopy isomorphism, and g is a pushout of f, then g is an injective homotopy isomorphism. Dually, if f is surjective homotopy isomorphism, and g is a pullback of f, then g is a surjective homotopy isomorphism.*
(d) *If f is a homotopy isomorphism, then any pushout of f through an injective map is again a homotopy isomorphism. Dually, any pullback of f through a surjective map is again a homotopy isomorphism.*

(e) *Suppose* $f\colon X \to Y$ *is an injective homotopy isomorphism, and* $g\colon A \to B$ *is a cofibration. Then the pushout product*

$$f \,\square\, g\colon (X \wedge B) \amalg_{X \wedge A} (Y \wedge A) \to Y \wedge B$$

is an injective homotopy isomorphism.

Proof. We leave part (a) to the reader. Part (b) is immediate from the fact that homotopy groups commute with filtered colimits. For part (c), suppose g is a pushout of the injective homotopy isomorphism f. Then g is injective, with cokernel $\operatorname{coker} f$. Since f is a homotopy isomorphism, the long exact sequence of Lemma 3.2.2 shows that $\operatorname{coker} f$ has zero homotopy. Another use of that long exact sequence shows that g is a homotopy isomorphism. The dual case is similar.

For part (d), suppose that $g\colon B \to D$ is the pushout of the homotopy isomorphism $f\colon A \to C$ through the injection $i\colon A \to B$. Then we have the map of short exact sequences below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \longrightarrow & X & \longrightarrow & 0 \\ & & {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

The long exact sequence in homotopy and the five lemma show that g is a homotopy isomorphism. The dual case is similar.

For part (e), note that parts (b) and (c) imply that injective homotopy isomorphisms are closed under pushouts and filtered colimits, hence transfinite compositions. Thus it suffices to prove part (e) when g is one of the generating cofibrations $S^{n-1}P \to D^nP$ of the projective model structure, by Lemma 4.2.4 of [Hov99]. We leave to the reader the check that $f \,\square\, g$ is injective in this case, and just prove it is a homotopy isomorphism. Since f is a homotopy isomorphism, $LA \wedge f$ is a projective equivalence. Therefore, $(LA \wedge f) \wedge S^{n-1}P$ is a projective equivalence by Proposition 2.1.4, and so $f \wedge S^{n-1}P$ is a homotopy isomorphism. Similarly, $f \wedge D^nP$ is a homotopy isomorphism. Both such maps are also injective, since P is flat over A. Part (c) implies that the pushout

$$X \wedge D^nP \to (X \wedge D^nP) \amalg_{X \wedge S^{n-1}P} (Y \wedge S^{n-1}P)$$

is also an injective homotopy isomorphism. The two out of three property for homotopy isomorphisms then implies that $f \,\square\, g$ is a homotopy isomorphism. $\square$

Our next goal is to give some useful examples of homotopy isomorphisms that are not projective equivalences. We begin with the following lemma.

Lemma 3.3.3. *Let* (A, Γ) *be a flat Hopf algebroid. Suppose* $X \in Ch(\Gamma)$ *is bounded above and contractible as a complex of* A*-modules, and* $Y \in Ch(\Gamma)$ *is a complex of relatively injective comodules. Then every chain map* $f\colon X \to Y$ *is chain homotopic to* 0.

Proof. We construct a chain homotopy $D_n\colon X_n \to Y_{n+1}$ by downwards induction on n. Getting started is easy, since $X_n = 0$ for large n. Suppose we have constructed D_{n+1} and D_{n+2} such that $dD_{n+2} + D_{n+1}d = f_{n+2}$. We need to construct D_n such that $dD_{n+1} + D_nd = f_{n+1}$. One can readily verify that

$$(f_{n+1} - dD_{n+1})d = 0$$

and so $f_{n+1} - dD_{n+1}$ defines a map $g\colon X_{n+1}/\operatorname{im} d = X_{n+1}/\ker d \to Y_{n+1}$. On the other hand, we are given that X is A-contractible, so there are A-module maps $s_n\colon X_n \to X_{n+1}$ such that $ds + sd = 1$. In particular, $d\colon X_{n+1}/\ker d \to X_n$ is an A-split monomorphism. Since Y_{n+1} is relatively injective, there is a map $D_n\colon X_n \to Y_{n+1}$ such that $D_n d = f_{n+1} - dD_{n+1}$. This completes the induction step and the proof. □

This gives the following proposition.

Proposition 3.3.4. *Let (A, Γ) be a flat Hopf algebroid, and suppose $f\colon X \to Y$ is a map of bounded above complexes in Ch(Γ) that is an A-split monomorphism in each degree and a chain homotopy equivalence of complexes of A-modules. Then $LA \wedge f$ is a chain homotopy equivalence. In particular, f is a homotopy isomorphism.*

Proof. Let Z denote the cokernel of f. Then Z is bounded above and contractible as a complex of A-modules (one can check this directly, but it also follows because f is a trivial cofibration in the absolute model structure on Ch(A) [CH02, Example 3.4]). Therefore $LA \wedge Z$ is a bounded above complex of relatively injective comodules that is contractible over A. Lemma 3.3.3 implies that $LA \wedge Z$ is contractible. Since LA is degreewise flat over A, $LA \wedge Z$ is the cokernel of $LA \wedge f$. Furthermore, $LA \wedge f$ is a degreewise A-split monomorphism of relatively injective comodules, so it is a degreewise split monomorphism. It follows that $LA \wedge f$ is a chain homotopy equivalence. □

Corollary 3.3.5. *Suppose (A, Γ) is a flat Hopf algebroid. Then the map*

$$\eta_L \wedge X\colon X \to LA \wedge X$$

is a homotopy isomorphism for all complexes X. In fact, $LA \wedge \eta_L \wedge X$ is a chain homotopy equivalence.

Proof. The map $\eta_L\colon A \to LA$ is a map of bounded above complexes that is a degreewise A-split monomorphism and an A-chain homotopy equivalence. Proposition 3.3.4 implies that $LA \wedge \eta_L$ is a chain homotopy equivalence. One can easily check that this forces $LA \wedge \eta_L \wedge X$ to be a chain homotopy equivalence for any X, and so $\eta_L \wedge X$ is a homotopy isomorphism. □

4. Localization

In the next section, we will localize the projective model structure to obtain a model structure on Ch(Γ) in which the weak equivalences are the homotopy isomorphisms. To prove that this construction works, we need some general results about Bousfield localization of model categories. The basic reference for Bousfield localization is [Hir03]. The results we prove in this section all follow in reasonably straightforward fashion from the techniques of [Hir03], but they seem not to have been noticed before.

Suppose we have a model category $\mathcal{M}$ and a class of maps $\mathcal{T}$. The Bousfield localization $L_{\mathcal{T}}\mathcal{M}$ of $\mathcal{M}$ with respect to $\mathcal{T}$ is a new model structure on $\mathcal{M}$, with the same cofibrations as the given one, in which the maps of $\mathcal{T}$ are weak equivalences. Futhermore, it is the initial such model category, in the sense that if $F\colon \mathcal{M} \to \mathcal{N}$ is a (left) Quillen functor that sends the maps of $\mathcal{T}$ to weak equivalences, then $F\colon L_{\mathcal{T}}\mathcal{M} \to \mathcal{N}$ is also a Quillen functor.

The Bousfield localization is known to exist when $\mathcal{T}$ is a set, $\mathcal{M}$ is left proper, and, in addition, $\mathcal{M}$ is either cellular [Hir03] or combinatorial (unpublished work of Jeff Smith). The cellular condition is technical, but has the virtue of being written down and of applying to topological spaces. The combinatorial condition is simpler; it just means that $\mathcal{M}$ is cofibrantly generated and locally presentable as a category.

To describe the localized model structure, it is necessary to recall that any model category $\mathcal{M}$ possesses a unital action by the category **SSet** of simplicial sets. That is, there is a bifunctor $\mathcal{M} \times \mathbf{SSet} \to \mathcal{M}$ that takes (X, K) to $X \otimes K$ described in [Hov99, Chapter 5]. This is a unital action but is not associative; it induces an associative action of $\operatorname{ho}\mathbf{SSet}$ on $\operatorname{ho}\mathcal{M}$. In fact, $\operatorname{ho}\mathcal{M}$ is not only tensored over $\operatorname{ho}\mathbf{SSet}$, but also cotensored and enriched over $\operatorname{ho}\mathbf{SSet}$ [Hov99, Chapter 5]. The enrichment is denoted by $\operatorname{map}(X, Y) \in \mathbf{SSet}$.

Now, a fibrant object X in $L_{\mathcal{T}}\mathcal{M}$, called a *$\mathcal{T}$-local fibrant object*, is a fibrant object X in $\mathcal{M}$ such that $\operatorname{map}(f, X)$ is a weak equivalence of simplicial sets for all $f \in \mathcal{T}$. Adjointness gives an equivalent description, as follows. Given a map $f \colon X \to Y$ in $\mathcal{M}$, let $\widetilde{f} \colon \widetilde{X} \to \widetilde{Y}$ denote a cofibration that is a cofibrant approximation to f. This means that $\widetilde{X}$ and $\widetilde{Y}$ are cofibrant, $\widetilde{f}$ is a cofibration, and we have the commutative diagram below

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

where the vertical arrows are weak equivalences. Then a *horn on f* is one of the maps $\widetilde{f} \,\square\, i_n$ for $n \geq 0$, where $i_n \colon \partial\Delta[n] \to \Delta[n]$ is the standard inclusion of simplicial sets, and $\widetilde{f} \,\square\, i$ is the pushout product map

$$(\widetilde{X} \otimes \Delta[n]) \amalg_{\widetilde{X} \otimes \partial\Delta[n]} (\widetilde{Y} \otimes \partial\Delta[n]) \to Y \otimes \Delta[n].$$

These maps are all cofibrations [Hov99, Proposition 5.4.1]. Then Proposition 4.2.4 of [Hir03] says that a fibrant object X is a $\mathcal{T}$-local fibrant object if and only if $X \to *$ has the right lifting property with respect to the horns on all the maps of $\mathcal{T}$.

Having obtained the fibrant objects, one defines a map f to be a *$\mathcal{T}$-local equivalence* if $\operatorname{map}(f, X)$ is a weak equivalence of simplicial sets for all $\mathcal{T}$-local fibrant objects X. These are the weak equivalences in $L_{\mathcal{T}}\mathcal{M}$. The fibrations are then the maps that have the right lifting property with respect to all maps that are both cofibrations and $\mathcal{T}$-local equivalences.

If $\mathcal{M}$ is a simplicial model category, then one can understand the horns of f by using the simplicial structure. But unbounded chain complexes are not simplicial. It is still easy to understand the horns, however. Let $\overline{\Delta[n]}$ be the chain complex of abelian groups defined by letting $\overline{\Delta[n]}_k$ be the free abelian group on the $\binom{n+1}{k+1}$ $k+1$-element subsets of $\{0, 1, \dots, n\}$, for $k \geq 0$. If 1_S denotes the generator corresponding to the set $S = \{s_0 < s_1 < \cdots < s_k\}$, we define

$$d(1_S) = \sum_{i=0}^{k} (-1)^i 1_{S - \{s_i\}}.$$

This is the obvious chain complex corresponding to the nondegenerate simplices of $\Delta[n]$. Then $\overline{\partial\Delta[n]}_k$ denotes the subcomplex containing all the 1_S except the one in degree n corresponding to $S = \{0, 1, \ldots, n\}$. Let $\overline{i_n}$ denote the obvious inclusion $\overline{\partial\Delta[n]} \to \overline{\Delta[n]}$.

The following lemma is a consequence of the naturality of the action of ho **SSet** on ho $\mathcal{M}$, and can be deduced from [Hov99, Chapter 5].

Lemma 4.1. *Suppose $\mathcal{M}$ is a Ch($\mathbb{Z}$)-model category, and $f \in \mathcal{M}$ with a cofibrant approximation $\widetilde{f}$ that is a cofibration. Then in the description of Bousfield localization above, one can replace the horns on f with the maps $\widetilde{f} \,\Box\, \overline{i_n}$.*

In general, it is difficult to understand the weak equivalences in $L_{\mathcal{T}}\mathcal{M}$; certainly the maps of $\mathcal{T}$ become weak equivalences, but many other maps do as well. The following theorem, which is obtained by pulling together different results in [Hir03], is of some help.

Theorem 4.2. *Suppose $\mathcal{M}$ is a left proper model category that is either cellular or combinatorial, and $\mathcal{T}$ is a set of maps in $\mathcal{M}$. Let $\mathcal{W}$ be a class of maps in $\mathcal{M}$ satisfying the two out of three property, containing the horns on the maps of $\mathcal{T}$ and every weak equivalence in $\mathcal{M}$, and such that maps that are both cofibrations and in $\mathcal{W}$ are closed under transfinite compositions and pushouts. Then every weak equivalence in $L_{\mathcal{T}}\mathcal{M}$ is in $\mathcal{W}$.*

Proof. Suppose $f\colon X \to Y$ is a weak equivalence in the Bousfield localization. Let $L\colon \mathcal{M} \to \mathcal{M}$ denote the functor obtained by applying the small object argument based on $\mathcal{T} \cup J$, where J is the set of generating trivial cofibrations of $\mathcal{M}$, to the map $X \to *$. Then the maps $X \to LX$ and $Y \to LY$ are transfinite compositions of pushouts of maps of $\mathcal{T} \cup J$. This means that they are weak equivalences in $L_{\mathcal{T}}\mathcal{M}$, by Propositions 3.3.10 and 4.2.3 of [Hir03], and also in $\mathcal{W}$ by our hypotheses. Hence Lf is a weak equivalence in $L_{\mathcal{T}}\mathcal{M}$ whose domain and codomain are fibrant (by Proposition 4.2.4 of [Hir03]) in $L_{\mathcal{T}}\mathcal{M}$. Thus, by Theorem 3.2.13 of [Hir03], Lf is a weak equivalence in $\mathcal{M}$ itself, and hence is in $\mathcal{W}$. The two out of three property for $\mathcal{W}$ now guarantees that f is in $\mathcal{W}$. □

In general, Bousfield localization causes one to lose control of the set of generating trivial cofibrations. Even if $\mathcal{M}$ itself has a very nice set of generating trivial cofibrations, all the theory tells you is that $L_{\mathcal{T}}\mathcal{M}$ has some, possibly gigantic, set of generating trivial cofibrations. The following proposition is at least of some help in dealing with this.

Proposition 4.3. *Suppose $\mathcal{M}$ is a left proper, cellular or combinatorial model category, and $\mathcal{T}$ is a set of maps in $\mathcal{M}$. Assume that $\mathcal{M}$ has a set of generating trivial cofibrations whose domains are cofibrant. Then $L_{\mathcal{T}}\mathcal{M}$ has a set of generating trivial cofibrations whose domains are cofibrant.*

Proof. Let J be a set of generating trivial cofibrations of $\mathcal{M}$ whose domains (and hence codomains) are cofibrant, and let J' be some set of generating trivial cofibrations of $L_{\mathcal{T}}\mathcal{M}$. For each map $j \in J'$, choose a cofibration $\hat{j}$ of cofibrant objects that is a cofibrant approximation to j, so that we have a commutative square

$$\begin{array}{ccc} \operatorname{dom} \hat{\jmath} & \longrightarrow & \operatorname{dom} j \\ {\scriptstyle \hat{\jmath}}\downarrow & & \downarrow{\scriptstyle j} \\ \operatorname{codom} \hat{\jmath} & \longrightarrow & \operatorname{codom} j \end{array}$$

where the horizontal maps are weak equivalences in $\mathcal{M}$. Let $\hat{J}'$ denote the set of those $\hat{\jmath}$, and let $K = J \cup \hat{J}'$. Then K is a set of trivial cofibrations in $L_{\mathcal{T}}\mathcal{M}$ with cofibrant domains. We claim that K is a generating set of trivial cofibrations. Indeed, suppose p has the right lifting property with respect to K. Then p has the right lifting property with respect to J, so p is a fibration in $\mathcal{M}$. Since p also has the right lifting property with respect to $\hat{J}'$ and $\mathcal{M}$ is left proper, Proposition 13.2.1 of [Hir03] implies that p has the right lifting property with respect to J', and hence that p is a fibration in $L_{\mathcal{T}}\mathcal{M}$. □

5. The homotopy model structure

The object of this section is to construct a model structure on $\mathrm{Ch}(\Gamma)$, when (A, Γ) is an amenable Hopf algebroid, in which the weak equivalences are the homotopy isomorphisms. Proposition 3.3.1 tells us that we need to add more weak equivalences to the projective model structure. We do this by using Bousfield localization, described in the previous section.

5.1. Construction and basic properties.

Definition 5.1.1. Suppose (A, Γ) is an amenable Hopf algebroid. Let $\mathcal{S}$ denote a set containing one element from each isomorphism class of dualizable comodules. Define the *homotopy model structure* on $\mathrm{Ch}(\Gamma)$ to be the Bousfield localization of the projective model structure with respect to the maps

$$\eta_L \wedge S^n P \colon S^n P \to LA \wedge S^n P$$

for $P \in \mathcal{S}$ and $n \in \mathbb{Z}$.

We have already seen that Γ-comod is a locally (finitely) presentable category 1.4.1. It follows easily that $\mathrm{Ch}(\Gamma)$ is also locally (finitely) presentable, so that $\mathrm{Ch}(\Gamma)$ is a combinatorial model category. Thus the (unpublished) work of Jeff Smith guarantees that the homotopy model structure exists. In fact, $\mathrm{Ch}(\Gamma)$ is also cellular, so one can use Hirschhorn's theory [Hir03] as well.

Note that the cofibrations do not change under Bousfield localization, though the fibrations and weak equivalences will change. This means that the trivial fibrations also do not change under Bousfield localization, and therefore that a cofibrant replacement functor in the projective model structure is also a cofibrant replacement functor in the homotopy model structure. Since Bousfield localization preserves left properness [Hir03, Theorem 4.1.1], the homotopy model structure is left proper.

Our first goal is to prove that the weak equivalences in the homotopy model structure are the homotopy isomorphisms, explaining the name.

Proposition 5.1.2. *Let (A, Γ) be an amenable Hopf algebroid. Then every weak equivalence in the homotopy model structure is a homotopy isomorphism.*

Proof. Proposition 3.3.1 and Proposition 3.3.2 tell us that the class of homotopy isomorphisms has all the properties necessary for Theorem 4.2 to apply. It remains to check that the horns on $\eta_L \wedge S^n P$ are homotopy isomorphisms. Now $\mathrm{Ch}(\Gamma)$ is a $\mathrm{Ch}(\mathbb{Z})$ model category; in fact, there is a symmetric monoidal left Quillen functor $\mathrm{Ch}(\mathbb{Z}) \to \mathrm{Ch}(\Gamma)$, induced by the trivial comodule functor $M \mapsto A \otimes_{\mathbb{Z}} M$. Lemma 4.1 implies that the horns on f can be taken to be the maps $f \,\square\, (A \otimes_{\mathbb{Z}} \overline{i_n})$. One can easily check that each map $A \otimes_{\mathbb{Z}} \overline{i_n}$ is a projective cofibration. The lemma then follows from Proposition 3.3.2(e). □

To prove the converse, we need the following proposition.

Proposition 5.1.3. *Let (A, Γ) be a flat Hopf algebroid, and suppose C is cofibrant in $\mathrm{Ch}(\Gamma)$. Then $\eta_L \wedge C \colon C \to LA \wedge C$ is a weak equivalence in the homotopy model structure.*

Proof. Factor $\eta_L \colon S^0 A \to LA$ into a cofibration $i\colon S^0 A \to QLA$ followed by a trivial fibration q. It suffices to show that $i \wedge C$ is a trivial cofibration in the homotopy model structure, because $q \wedge C$ is a projective equivalence by Proposition 2.1.4. For dualizable P, $\eta_L \wedge S^n P$ is a weak equivalence in the homotopy model structure by construction, so $i \wedge S^n P$ is a trivial cofibration in the homotopy model structure. Since C is cofibrant, $0 \to C$ is a retract of a transfinite composition

$$0 = C_0 \to C_1 \to \cdots C_i \to \cdots$$

where each map $C_i \to C_{i+1}$ is a pushout of a map $S^{n-1}P \to D^n P$, where $P \in \mathcal{S}$ and $n \in \mathbb{Z}$. It suffices to show that $i \wedge C_i$ is a trivial cofibration in the homotopy model structure for all i, which we do by transfinite induction. The base case is trivial, since $C_0 = 0$. For the successor ordinal step, suppose $i \wedge C_i$ is a trivial cofibration. We have pushout diagrams

$$\begin{array}{ccc} S^{n-1}P & \longrightarrow & D^n P \\ \downarrow & & \downarrow \\ C_i & \longrightarrow & C_{i+1} \end{array}$$

and

$$\begin{array}{ccc} QLA \wedge S^{n-1}P & \longrightarrow & QLA \wedge D^n P \\ \downarrow & & \downarrow \\ QLA \wedge C_i & \longrightarrow & QLA \wedge C_{i+1} \end{array}$$

and i induces a map from one of these to the next. This map is a weak equivalence on the upper left corners as mentioned above, a chain homotopy equivalence on the upper right corners because $D^n P$ is contractible, and a weak equivalence on the lower left corners by the induction hypothesis. It follows from the cube lemma [Hov99, Lemma 5.2.6] that $i \wedge C_{i+1}$ is a weak equivalence as well. Because the projective structure is monoidal, $i \wedge C_{i+1}$ is also a cofibration.

We are left with the limit ordinal step of the induction. So suppose that $i \wedge C_i$ is a trivial cofibration in the homotopy model structure for all $i < \alpha$ for some limit ordinal α. Then Proposition 17.9.1 of [Hir03] implies that $i \wedge C_\alpha$ is a weak equivalence as well, and hence a trivial cofibration. □

Theorem 5.1.4. *Suppose (A, Γ) is an amenable Hopf algebroid. Then the weak equivalences in the homotopy model structure are the homotopy isomorphisms.*

Proof. We have already seen that every weak equivalence in the homotopy model structure is a homotopy isomorphism. Conversely, suppose $f\colon X \to Y$ is a homotopy isomorphism. By using a cofibrant replacement functor Q in the projective model structure, we can construct the commutative diagram below,

$$\begin{array}{ccc} QX & \xrightarrow{Qf} & QY \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where q_X and q_Y are projective equivalences, and QX and QY are cofibrant. In particular, since projective equivalences are homotopy isomorphisms by Proposition 3.3.1, Qf is a homotopy isomorphism. Since every projective equivalence is a weak equivalence in the homotopy model structure, it suffices to show that the homotopy isomorphism Qf is a weak equivalence. Consider the commutative square below.

$$\begin{array}{ccc} QX & \xrightarrow{\eta_L \wedge QX} & LA \wedge QX \\ {\scriptstyle Qf}\downarrow & & \downarrow{\scriptstyle LA\wedge Qf} \\ QY & \xrightarrow[\eta_L \wedge QY]{} & LA \wedge QY \end{array}$$

Both of the horizontal maps are weak equivalences in the homotopy model structure by Proposition 5.1.3. Since Qf is a homotopy isomorphism, $LA \wedge Qf$ is a projective equivalence, and therefore a weak equivalence in the homotopy model structure. It follows that Qf is a weak equivalence in the homotopy model structure as well. □

Many properties of the homotopy model structure follow immediately from Theorem 5.1.4.

Theorem 5.1.5. *Suppose (A, Γ) is an amenable Hopf algebroid. Then the homotopy model structure is proper, symmetric monoidal, and satisfies the monoid axiom. Moreover, if C is cofibrant, then $C \wedge -$ preserves weak equivalences.*

The monoid axiom and the last statement of this theorem are important because they guarantee the existence of homotopy invariant model categories of modules and monoids, as explained following the statement of Proposition 2.1.4.

Proof. Bousfield localization preserves left properness, as has already been observed. The fact that the homotopy model structure is right proper follows immediately from Theorem 5.1.4 and Proposition 3.3.2. The homotopy model structure is symmetric monoidal by Proposition 3.3.2(e).

Now for the monoid axiom, which we recall states that any transfinite composition of pushouts of maps of the form $f \wedge X$, where f is a trivial cofibration and X is arbitrary, is a weak equivalence. Let us suppose we know that $f \wedge X$ itself is a weak equivalence for all trivial cofibrations f and all X. Since cofibrations are degreewise Γ-split monomorphisms, it follows that $f \wedge X$ is also injective. Since injective homotopy isomorphisms are closed under pushouts and filtered colimits by Propostion 3.3.2, the monoid axiom will follow.

We are left with showing that $f \wedge X$ is a homotopy isomorphism for all trivial cofibrations f in the homotopy model structure and all X. It suffices to show this for a set of generating trivial cofibrations f, and Proposition 4.3 allows us to assume those generating trivial cofibrations f have cofibrant domains and codomains. Let $q\colon QX \to X$ be a cofibrant replacement of X, so that q is a projective equvalence and QX is cofibrant. We have the commutative square below.

$$\begin{array}{ccc} \operatorname{dom} f \wedge QX & \xrightarrow{\operatorname{dom} f\wedge q} & \operatorname{dom} f \wedge X \\ {\scriptstyle f\wedge QX}\downarrow & & \downarrow{\scriptstyle f\wedge X} \\ \operatorname{codom} f \wedge QX & \xrightarrow[\operatorname{codom} f\wedge q]{} & \operatorname{codom} f \wedge X \end{array}$$

By Proposition 2.1.4, both the horizontal maps are projective equivalences, and hence homotopy isomorphisms. Since the homotopy model structure is symmetric monoidal, $f \wedge QX$ is a homotopy isomorphism. It follows that $f \wedge X$ is a homotopy isomorphism as well, completing the proof of the monoid axiom.

Now suppose C is cofibrant, and f is a weak equivalence. Then $LA \wedge f$ is a projective equivalence. By Proposition 2.1.4, it follows that $C \wedge LA \wedge f$ is still a projective equivalence. Hence $C \wedge f$ is a homotopy isomorphism, so a weak equivalence. □

5.2. **Fibrations.** We now characterize the fibrations in the homotopy model structure.

Proposition 5.2.1. *Suppose (A, Γ) is an amenable Hopf algebroid. Then a map p is a fibration in the homotopy model structure if and only if p is a projective fibration and $\ker p$ is fibrant in the homotopy model structure.*

Proof. Certainly, if p is a homotopy fibration, then p must be a projective fibration and $\ker p$ must be homotopy fibrant. Conversely, suppose $p\colon X \to Y$ is a projective fibration and $\ker p$ is homotopy fibrant. Form the commutative square below by using factorization,

$$\begin{array}{ccc} X & \xrightarrow{i_X} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow[i_Y]{} & Y' \end{array} \tag{5.2.2}$$

where i_X and i_Y are trivial cofibrations in the homotopy model structure, X' and Y' are homotopy fibrant, and q is a homotopy fibration. We claim that this square is a homotopy pullback square in the projective model structure. Proposition 3.4.7 of [Hir03] then implies that p is a fibration in the homotopy model structure.

To see that the square 5.2.2 is a homotopy pullback square, let $P \xrightarrow{q'} Y$ be the pullback of q through i_Y. Then q' is a projective fibration, and there is an induced factorization

$$X \xrightarrow{s} P \xrightarrow{t} X'$$

of i_X. Since t is the pullback of the homotopy isomorphism i_Y through the surjection q, Proposition 3.3.2 implies that t is a homotopy isomorphism. Hence s is a homotopy isomorphism as well. Consider the commutative diagram below

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker p & \longrightarrow & X & \xrightarrow{p} & Y & \longrightarrow & 0 \\ & & {\scriptstyle r}\downarrow & & {\scriptstyle s}\downarrow & & \| & & \\ 0 & \longrightarrow & \ker q & \longrightarrow & P & \xrightarrow{q'} & Y & \longrightarrow & 0 \end{array}$$

whose rows are short exact (since projective fibrations are surjective). The long exact sequence in homotopy implies that r is a homotopy isomorphism. But $\ker p$ is homotopy fibrant by assumption, and $\ker q$ is homotopy fibrant since q is a homotopy fibration. Theorem 3.2.13 of [Hir03] implies that r is a projective equivalence. Applying Γ-comod$(Q,-)$ for $Q \in \mathcal{S}$ and considering the long exact homology sequence shows that s is a projective equivalence as well. This means that the square 5.2.2 is a homotopy pullback square, completing the proof. □

The characterization of fibrations we have just given would be more helpful if we knew what the fibrant objects in the homotopy model structure are.

Theorem 5.2.3. *Suppose (A, Γ) is an amenable Hopf algebroid. Then the following are equivalent.*

(a) $\eta_L \wedge X \colon X \to LA \wedge X$ *is a projective equivalence.*
(b) X *is projectively equivalent to some complex of relative injectives.*
(c) X *is fibrant in the homotopy model structure.*

Proof. It is clear that (a) implies (b). To see that (b) implies (c), our first goal is to show that if X is projectively equivalent to some complex of relative injectives, then $\eta_L \wedge X$ is a projective equivalence (incidentally proving (b) implies (a)). It obviously suffices to show this for actual complexes of relative injectives X. Any such complex can be written as the colimit of the bounded above complexes X^n, where $X^n_i = 0$ for $i > n$ and $X^n_i = X_i$ for $i \leq n$. Since colimits of projective equivalences are projective equivalences, we can assume that X is a bounded above complex of relative injectives. In this case, we will show that $\eta_L \wedge X$ is in fact a chain homotopy equivalence. Indeed, since η_L is a degreewise A-split monomorphism, $\eta_L \wedge X$ is a degreewise A-split monomorphism between complexes of relative injectives. It follows that $\eta_L \wedge X$ is a degreewise split monomorphism of relative injectives. Let Y denote the cokernel of $\eta_L \wedge X$. Then Y is also a bounded above complex of relative injectives, and Y is contractible as a complex of A-modules since η_L is a chain homotopy equivalence of complexes of A-modules. Lemma 3.3.3 then implies that Y is contractible as a complex of comodules. Given this, an elementary argument then shows that $\eta_L \wedge X$ is a chain homotopy equivalence.

Now suppose that X is projectively equivalent to a complex of relative injectives, C is cofibrant and $\pi_* C = 0$. We claim that every chain map $f \colon C \to X$ is chain homotopic to 0. Indeed, the composite

$$C \xrightarrow{f} X \xrightarrow{\eta_L \wedge X} LA \wedge X$$

factors through $LA \wedge C$, which is projectively trivial. Hence the map $(\eta_L \wedge X) \circ f$ is 0 in the homotopy category of the projective model structure. Since $\eta_L \wedge X$ is a projective equivalence by the previous paragraph, we conclude that f is 0 in the homotopy category of the projective model structure. Since C is cofibrant and everything is fibrant in the projective model structure, it follows that f is chain homotopic to 0.

We can now complete the proof that (b) implies (c) by showing that $X \to 0$ has the right lifting property with respect to every cofibration $i \colon A \to B$ that is a homotopy isomorphism. Indeed, suppose $f \colon A \to X$ and let C be the cokernel of i. Since i is a cofibration, it is a split monomorphism in each degree, and so $B_n \cong A_n \oplus C_n$. The differential on B must then be given by $d(a, c) = (da + \tau c, dc)$, where $\tau d = -d\tau$. Thus τ is a chain map from the desuspension $\Sigma^{-1}C$ of C to A. The composition

$$\Sigma^{-1}C \xrightarrow{\tau} A \xrightarrow{f} X$$

must be chain homotopic to 0, since $\Sigma^{-1}C$ is cofibrant and $\pi_*\Sigma^{-1}C = 0$. Hence there are maps $D_n\colon C_n \to A_n$ such that $-D_{n-1}d + dD_n = f\tau$. Define

$$g(a,c) = fa + D_n c.$$

Then $g\colon B \to C$ is a chain map extending f, so $X \to 0$ has the right lifting property with respect to i as required.

We now show that (c) implies (a). So suppose X is fibrant in the homotopy model structure. The map $\eta_L \wedge X\colon X \to LA \wedge X$ is a homotopy isomorphism by Corollary 3.3.5. Since X is fibrant, and $LA \wedge X$ is also fibrant since (b) implies (c), it follows from Theorem 3.2.13 of [Hir03] that $\eta_L \wedge X$ is a projective equivalence. □

Corollary 5.2.4. *Suppose (A,Γ) is an amenable Hopf algebroid, and give $Ch(\Gamma)$ the homotopy model structure. For any $X \in Ch(\Gamma)$, the map*

$$\eta_L \wedge X\colon X \to LA \wedge X$$

is a natural weak equivalence whose target is fibrant.

This follows immediately from Corollary 3.3.5 and Theorem 5.2.3. Note that $\eta_L \wedge X$ is not normally a cofibration, however.

We also note the following corollary.

Corollary 5.2.5. *Suppose (A,Γ) is an amenable Hopf algebroid, and given $Ch(\Gamma)$ the homotopy model structure. Then weak equivalences and fibrations are closed under filtered colimits.*

This corollary is saying that the homotopy model structure behaves as if it were finitely generated. We do not know if it is in fact finitely generated for a general amenable Hopf algebroid, though for an Adams Hopf algebroid it is.

Proof. We have seen in Proposition 3.3.2 that homotopy isomorphisms are closed under filtered colimits. Since homotopy fibrations are just projective fibrations with homotopy fibrant kernel, and projective fibrations are closed under filtered colimits, it suffices to show that homotopy fibrant objects are closed under filtered colimits. This follows from the characterization of homotopy fibrant objects in part (a) of Theorem 5.2.3, since projective equivalences are closed under filtered colimits. □

5.3. **Naturality.** Like the projective model structure, the homotopy model structure is natural.

Proposition 5.3.1. *Suppose $\Phi\colon (A,\Gamma) \to (B,\Sigma)$ is a map of amenable Hopf algebroids. Then Φ induces a left Quillen functor $\Phi_*\colon Ch(\Gamma) \to Ch(\Sigma)$ of the homotopy model structures.*

Proof. By Proposition 2.2.1, Φ^* is a right Quillen functor of the projective model structures. Thus Φ^* preserves projective fibrations and projective equivalences (since everything is fibrant in the projective model structure), and so will also preserves trivial fibrations in the homotopy model structure, since these coincide with projective trivial fibrations. Suppose p is a homotopy fibration. Then p is a projective fibration such that $\ker p$ is projectively equivalent to a complex of relative injectives K, by Proposition 5.2.1 and Theorem 5.2.3. Hence $\Phi^* p$ is also a projective fibration, and $\ker \Phi^* p$ is projectively equivalent to $\Phi^* K$. We will show that $\Phi^* K$ is a complex of relative injectives. Hence $\Phi^* p$ is a homotopy fibration by Theorem 5.2.3 and Proposition 5.2.1, and so Φ^* is a right Quillen functor as required.

We are now reduced to showing that Φ^* preserves relative injectives. Suppose I is a relatively injective Σ-comodule, and E is an A-split short exact sequence of Γ-comodules. Then

$$\Gamma\text{-comod}(E, \Phi^* I) \cong \Sigma\text{-comod}(\Phi_* E, I).$$

Since $\Phi_* E = B \otimes_A E$, $\Phi_* E$ is a B-split short exact sequence, so $\Sigma\text{-comod}(\Phi_* E, I)$ is exact. □

The homotopy model structure is also invariant under weak equivalences, but this is considerably harder to prove. We begin with a definition.

Definition 5.3.2. Suppose (A, Γ) is an amenable Hopf algebroid. Define a Γ-comodule I to be *pseudo-injective* if $\operatorname{Ext}^n_\Gamma(P, I) = 0$ for all dualizable comodules P and all $n > 0$.

Every relative injective is pseudo-injective, by Lemma 3.1.4. The reason for introducing pseudo-injectives is the following lemma, which would be false for relative injectives.

Lemma 5.3.3. *Suppose $\Phi\colon (A, \Gamma) \to (B, \Sigma)$ is a weak equivalence of flat Hopf algebroids. If I is a pseudo-injective Γ-comodule, then $\Phi_* I$ is a pseudo-injective Σ-comodule.*

Proof. Suppose P is a dualizable Σ-comodule. Because Φ^* is an equivalence of categories whose right adjoint is naturally isomorphic to Φ_*, we have

$$\operatorname{Ext}^n_\Sigma(P, \Phi_* I) \cong \operatorname{Ext}^n_\Gamma(\Phi^* P, I).$$

As explained in the proof of Theorem 2.2.2, $\Phi^* P$ is a dualizable Γ-comodule. The lemma follows. □

Lemma 5.3.4. *Suppose (A, Γ) is a flat Hopf algebroid. Any bounded above complex of pseudo-injectives with no homology is projectively trivial.*

Proof. Suppose X is a bounded above complex of pseudo-injectives with no homology. We claim that the cycle comodule $Z_n X$ is pseudo-injective for all n. This is obvious for large n, since X is bounded above. We have a short exact sequence

$$0 \to Z_n X \to X_n \to Z_{n-1} X \to 0$$

since X has no homology. The long exact sequence in $\operatorname{Ext}^*_\Gamma(P, -)$ shows that $Z_{n-1} X$ is pseudo-injective. By induction, $Z_n X$ is pseudo-injective for all n. One can then easily check that $\Gamma\text{-comod}(P, X)$ is exact for all dualizable comodules P. □

Corollary 5.3.5. *Suppose (A, Γ) is a flat Hopf algebroid, and $f\colon X \to Y$ is a homology isomorphism of complexes of bounded above pseudo-injectives. Then f is a projective equivalence.*

Proof. Let C denote the mapping cylinder of f, and let $Z = C/X$ denote the mapping cone of f. Then $f = pi$, where $i\colon X \to C$ is a degreewise split monomorphism and p is a chain homotopy equivalence. It therefore suffices to show that the homology isomorphism i is a projective equivalence. Since i is degreewise split, it suffices to show that Z is projectively trivial. But $Z_n = Y_n \oplus X_{n-1}$, so Z is a bounded above complex of pseudo-injectives with no homology. Lemma 5.3.4 implies that Z is projectively trivial. □

Theorem 5.3.6. *Suppose $\Phi\colon (A,\Gamma) \to (B,\Sigma)$ is a weak equivalence of amenable Hopf algebroids. Then $\Phi_*\colon Ch(\Gamma) \to Ch(\Sigma)$ is a Quillen equivalence of the homotopy model structures. In fact, both Φ_* and Φ^* preserve and reflect homotopy isomorphisms.*

Proof. We have seen in Theorem 2.2.2 that Φ_* is a Quillen equivalence of the projective model structures, and that Φ_* and Φ^* preserve and reflect projective equivalences. We first show that Φ_* preserves homotopy isomorphisms, and hence that Φ^* reflects homotopy isomorphisms. Indeed, it follows from Proposition 5.3.1 that Φ_* preserves homotopy trivial cofibrations. Thus it suffices to show that $\Phi_* p$ is a homotopy isomorphism when p is a homotopy trivial fibration. But then p is a projective equivalence, so $\Phi_* p$ is also a projective equivalence.

It is more difficult to show that Φ_* reflects homotopy isomorphisms. To see this, we first construct a factorization

$$B = \Phi_* A \xrightarrow{\Phi_*\eta_L} \Phi_* LA \xrightarrow{\alpha} LB$$

of $\eta_L\colon B \to LB$. We have

$$(\Phi_* LA)_n \cong \Phi_*\Gamma \wedge (\Phi_*\overline{\Gamma})^{\wedge n} \text{ and } (LB)_n = \Sigma \wedge \overline{\Sigma}^{\wedge n}.$$

There is a natural map of comodules $\Phi_*\Gamma = B \otimes \Gamma \to \Sigma$ that takes $b \otimes x$ to $\eta_L(b)\Phi_1(x)$. This map induces a map $\Phi_*\overline{\Gamma} \to \overline{\Sigma}$, which in turn induces the desired map $\alpha\colon \Phi_* LA \to LB$ of complexes.

Now $\Phi_* LA$ and LB are both complexes of pseudo-injectives, by Lemma 5.3.3. The map $\Phi_*\eta_L$ is a homology isomorphism, since Φ_*, like any equivalence of abelian categories, is exact. The map $\eta_L\colon B \to LB$ is a homology isomorphism by construction. Thus, $\alpha\colon \Phi_* LA \to LB$ is a homology isomorphism of bounded above complexes of pseudo-injectives, and so a projective equivalence, by Corollary 5.3.5.

We can now show that Φ_* reflects homotopy isomorphisms between cofibrant objects. Indeed, suppose that $f\colon X \to Y$ is a map of cofibrant objects such that $\Phi_* f$ is a homotopy isomorphism. Then $LB \wedge \Phi_* f$ is a projective equivalence. But we have the commutative square below.

$$\begin{array}{ccc}
\Phi_* LA \wedge \Phi_* X & \xrightarrow{\Phi_* LA \wedge \Phi_* f} & \Phi_* LA \wedge \Phi_* Y \\
{\scriptstyle \alpha\wedge\Phi_* X}\downarrow & & \downarrow{\scriptstyle \alpha\wedge\Phi_* Y} \\
LB \wedge \Phi_* X & \xrightarrow[LB\wedge\Phi_* f]{} & LB \wedge \Phi_* Y
\end{array}$$

Proposition 2.1.4 implies that the vertical maps in this square are projective equivalences. Hence $\Phi_* LA \wedge \Phi_* f \cong \Phi_*(LA \wedge f)$ is a projective equivalence. But Φ_* reflects projective equivalences, so $LA \wedge f$ is a projective equivalence. Hence f is a homotopy isomorphism as required.

We now claim that Φ_* reflects all homotopy isomorphisms, from which it follows easily that Φ^* preserves all homotopy isomorphisms. So suppose $f\colon X \to Y$ is a map such that $\Phi_* f$ is a homotopy isomorphism. We have the commutative square below, in which the vertical maps are projective equivalences and QX and QY are cofibrant.

$$\begin{array}{ccc} QX & \xrightarrow{Qf} & QY \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Since Φ_* preserves projective equivalences, we conclude that $\Phi_* Qf$ is a homotopy isomorphism. Since Φ_* reflects homotopy isomorphisms between cofibrant objects, Qf is a homotopy isomorphism, and hence f is a homotopy isomorphism as required.

It now follows easily that Φ_* is a Quillen equivalence, as in the proof of Theorem 2.2.2. □

5.4. **Comparison with injective model structure.** When $A = k$ is a field, there is a model structure on $\mathrm{Ch}(\Gamma)$ in which $\operatorname{ho}\mathrm{Ch}(\Gamma)(A, A)_* \cong \operatorname{Ext}_{\Gamma}^{-*}(A, A)$ developed in [Hov99, Section 2.5]. In this model structure, which we call the *injective model structure*, the cofibrations are just the monomorphisms, and the fibrations are the degreewise surjections with degreewise injective kernels. (Remember that relatively injective and injective coincide in case A is a field). The weak equivalences are the homotopy isomorphisms, where homotopy is defined as in Definition 3.2.1 but using only simple comodules (that is, those comodules with no nontrivial subcomodule) as the source. For years, the author searched for a generalization of this model structure to Hopf algebroids without success. The injective model structure is **NOT** a special case of the homotopy model structure. Indeed, cofibrations in the homotopy model structure are degreewise split over Γ, whereas cofibrations in the injective model structure are split over A, but not necessarily Γ. However, we have the following theorem.

Theorem 5.4.1. *Suppose Γ is a Hopf algebra over a field $k = A$. Then the identity functor defines a Quillen equivalence from the homotopy model structure to the injective model structure.*

Proof. We claim that the two model structures have the same weak equivalences. If we can prove this, then the identity functor will be a left Quillen functor from the homotopy model structure to the injective model structure, since every projective cofibration is a monomorphism. It must be a Quillen equivalence since the weak equivalences are the same.

Note that the dualizable comodules coincide with the finite-dimensional comodules. Every simple comodule is finite-dimensional by Lemma 9.5.5 of [HPS97]. Thus every homotopy isomorphism is a weak equivalence in the injective model structure. To prove the converse, it suffices to prove that if $\Gamma\text{-comod}(P, LA \wedge f)$ is a homology isomorphism for all simple comodules P, then it is a homology isomorphism for all finite-dimensional comodules P. We do this by induction on the dimension of P. Since every one-dimensional comodule is simple, the base case is easy. Now suppose we know $\Gamma\text{-comod}(P, LA \wedge f)$ is an isomorphism for all P of dimension $< n$, and P has dimension n. If P is simple, there is nothing to prove. If P is not simple, there is a short exact sequence of comodules

$$0 \to Q \to P \to P/Q \to 0$$

with $\dim Q < n$. This sequence is necessarily split over k, since k is a field. Therefore, the sequence

$$0 \to \Gamma\text{-comod}(P/Q, LA \wedge X) \to \Gamma\text{-comod}(P, LA \wedge X) \to \Gamma\text{-comod}(Q, LA \wedge X) \to 0$$

is still short exact, as is the corresponding sequence with Y replacing X. The map f induces a map between the corresponding long exact sequences in homology. By the induction hypothesis, Γ-comod$(Q, LA \wedge f)$ and Γ-comod$(P/Q, LA \wedge f)$ are isommorphism. The five lemma implies that Γ-comod$(P, LA \wedge f)$ is an isomorphism as well, completing the proof of the induction step. □

6. The stable category

We define the homotopy category of the homotopy model structure on Ch(Γ) to be the *stable homotopy category of* (A, Γ), and we denote it by Stable(Γ), following Palmieri [Pal01] in the case of the Steenrod algebra. The category Stable(Γ) is what we should mean by the *derived category* $\mathcal{D}(A, \Gamma)$ of the Hopf algebroid (A, Γ). This is consistent with the usual notation, since $\mathcal{D}(A, A) = \mathcal{D}(A)$, the usual unbounded derived category of A. However, we must remember that to form the derived category, we invert the **homotopy isomorphisms**, not the homology isomorphisms. It is just that in the case of a discrete Hopf algebroid (A, A), the homotopy isomorphisms coincide with the homology isomorphisms.

We conclude the paper by establishing some basic properties of Stable(Γ). We show that it is a unital algebraic stable homotopy category [HPS97]. This means that it shares most of the formal properties of the derived category of a commutative ring, or the ordinary stable homotopy category, except that it has several generators rather than just one. In certain cases of interest in algebraic topology, such as $\Gamma = BP_*BP$, we show that Stable(Γ) is monogenic, so that (bigraded) suspensions of BP_* weakly generate the category. We also show that

$$\text{Stable}(\Gamma)(S^0 M, S^k N) \cong \text{Ext}^k_\Gamma(M, N)$$

for certain Γ-comodules M and N.

We begin with the following lemma.

Lemma 6.1. *Suppose (A, Γ) is an amenable Hopf algebroid, and P is a dualizable Γ-comodule. Then $S^n P$ is dualizable in the homotopy category of the projective model structure on Ch(Γ) for all n.*

Proof. Recall that the symmetric monoidal product in the homotopy category is the derived smash product $X \wedge^L Y = QX \wedge QY$, where Q denotes a cofibrant replacement functor. Similarly, the closed structure is $RF(X, Y) = F(QX, RY)$, where R is a fibrant replacement functor. To show that X is dualizable, we must show that the unit

$$S^0 A \to RF(X, X)$$

factors through the composition map

$$RF(X, S^0 A) \wedge^L X \to RF(X, X).$$

In the projective model structure, everything is fibrant, so we may as well take R to be the identity functor. Furthermore, $S^n P$ is cofibrant, so we conclude that

$$RF(S^n P, S^n P) \cong S^0 F(P, P), RF(S^n P, S^0 A) \cong S^{-n} F(P, A),$$

and

$$RF(S^nP.S^0A) \wedge^L S^nP \cong S^0(F(P,A) \wedge P).$$

It is now clear that S^nP is dualizable, since P is so. □

Theorem 6.2. *Suppose (A,Γ) is an amenable Hopf algebroid. Then the homotopy category of the projective model structure and Stable(Γ) are unital algebraic stable homotopy categories. A set of small, dualizable, weak generators is given by the set of all S^nP for P a dualizable comodule and $n \in \mathbb{Z}$.*

Proof. It is easy to check that the ordinary suspension, defined by $(\Sigma X)_n = X_{n-1}$ with $d_{\Sigma X} = -d_X$, is a Quillen equivalence of both the projective and homotopy model structures. One can also check that it induces the model category theoretic suspension on the homotopy categories. This means that both the projective and homotopy model structures are stable in the sense of [Hov99, Section 7.1], and therefore that the homotopy category of the projective model structure and Stable(Γ) are triangulated.

Since the projective and homotopy model structures are symmetric monoidal, their homotopy categories are also symmetric monoidal in a way that is compatible with the triangulation (see Chapters 4 and 6 of [Hov99]). In fact, they satisfy much stronger compatibility relations than those demanded in [HPS97]; see [May01].

The projective model structure is finitely generated, so the results of Sections 7.3 and 7.4 of [Hov99] guarantee that the cofibers of the generating cofibrations, namely the S^nP, form a set of small weak generators for the homotopy category. The homotopy model structure may not be finitely generated, but fibrations and weak equivalences are closed under filtered colimits by Corollary 5.2.5, and this is all that is needed for the arguments of Section 7.4 of [Hov99] to apply. Thus the S^nP also form a set of small weak generators for Stable(Γ). Lemma 6.1 guarantees that they are dualizable in the homotopy category of the projective model structure; since the functor from this category to Stable(Γ) is symmetric monoidal, they are also dualizable in Stable(Γ). □

We now investigate when the homotopy category of the homotopy model structure is monogenic. We first recall a definition from abelian categories.

Definition 6.3. A *thick subcategory* of an abelian category $\mathcal{C}$ is a full subcategory $\mathcal{T}$ that is closed under retracts and has the two-out-of-three property. This means that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence, and two out of M', M, M'' are in $\mathcal{T}$, so is the third. If $\mathcal{C}$ is graded, we also insist that thick subcategories be closed under aribtrary shifts.

The reader used to BP_*BP-comodules will be familiar with the following definition.

Definition 6.4. Suppose (A,Γ) is an amenable Hopf algebroid, and M is a Γ-comodule. We say that M *has a Landweber filtration* if there is a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$$

of M by subcomodules such that each quotient $M_j/M_{j-1} \cong A/I_j$ for some ideal I_j of A that is generated by an invariant finite regular sequence.

We recall that a sequence $x_1, \dots, x_n$ is an invariant regular sequence if x_i is a primitive nonzero divisor in $A/(x_1, \dots, x_{i-1})$ for all i.

In case (A, Γ) is graded, we allow the filtration quotients M_i/M_{i-1} to be isomorphic to some shift of A/I_j rather than A/I_j itself.

From a structural point of view, whether or not M has a Landweber filtration is not important. What matters is whether M is in the thick subcategory generated by A.

Lemma 6.5. *Suppose (A, Γ) is an amenable Hopf algebroid, and M is a Γ-comodule with a Landweber filtration. Then M is in the thick subcategory generated by A.*

Proof. Since thick subcategories are closed under extensions, it suffices to check that A/I is in the thick subcategory generated by A, where I is generated by a finite invariant regular sequence $x_1, \dots, x_n$. This follows from the short exact sequences of comodules

$$0 \to A/(x_1, \dots, x_{i-1}) \xrightarrow{x_i} A/(x_1, \dots, x_{i-1}) \to A/(x_1, \dots, x_i) \to 0$$

and induction. □

Theorem 6.6. *Suppose (A, Γ) is an amenable Hopf algebroid, and every dualizable comodule P is in the thick subcategory generated by A. Then Stable(Γ) is monogenic, in the sense that $\{S^n A\}$ form a set of small weak generators.*

In the graded case, we would instead get that $\{S^{n,m} A\}$ would form a set of weak generators.

Corollary 6.7. *Let E be a ring spectrum that is Landweber exact over MU or BPJ for some finite invariant regular sequence J, and suppose that E_*E is commutative. Then Stable(E_*E) is a bigraded monogenic stable homotopy category.*

Proof. It is shown in [HS02] that every finitely presented E_*E-comodule is a retract of a comodule with a Landweber filtration, and hence in the thick subcategory generated by E_*. □

To prove Theorem 6.6, we first need a lemma.

Lemma 6.8. *Suppose (A, Γ) is an amenable Hopf algebroid, and*

$$0 \to M' \xrightarrow{f} M \to M'' \to 0$$

is a short exact sequence of comodules. Then

$$S^0 M' \to S^0 M \to S^0 M''$$

is a cofiber sequence in Stable(Γ).

Proof. Factor $S^0 f$ into a projective cofibration $i \colon S^0 M' \to X$ followed by a projective trivial fibration p. Then we have the commutative diagram below, whose rows are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^0 M' & \xrightarrow{i} & S^0 M & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow p & & \downarrow q & & \\ 0 & \longrightarrow & S^0 M' & \xrightarrow[S^0 f]{} & S^0 M & \longrightarrow & S^0 M'' & \longrightarrow & 0 \end{array}$$

The long exact sequence in homotopy (Lemma 3.2.2) and the five lemma imply that q is a homotopy isomorphism. Therefore the bottom row is isomorphic in Stable(Γ) to the top row, which is a cofiber sequence. □

Proof of Theorem 6.6. Suppose that $\pi_*^A(X)=0$. Let us denote maps in the homotopy category of the homotopy model structure by $[Y,Z]_*$, so that $[S^0A,X]_*=0$. Let $\mathcal{T}$ denote the full subcategory of all comodules M such that $[S^0M,X]_*=0$. We claim that $\mathcal{T}$ is a thick subcategory, and therefore contains the dualizable comodules. Theorem 6.2 then completes the proof.

It is clear that $\mathcal{T}$ is closed under retracts. To show that $\mathcal{T}$ is thick, suppose we have a short exact sequence

$$0\to M'\to M\to M''\to 0$$

such that two out of M',M,M'' are in $\mathcal{T}$. Lemma 6.8 then implies that

$$S^0M'\to S^0M\to S^0M''$$

is a cofiber sequence in Stable(Γ). The long exact sequence obtained by applying $[,X]_*$ then shows that the other one is also in $\mathcal{T}$. □

Finally, we study Stable(Γ)(S^0M,S^0N) for comodules M and N.

Definition 6.9. A full subcategory of an abelian category $\mathcal{C}$ is called *localizing* if it is a thick subcategory closed under coproducts.

Proposition 6.10. *Let (A,Γ) be an amenable Hopf algebroid, and M and N be Γ-comodules. Then there is a natural map*

$$\operatorname{Ext}^k_\Gamma(M,N)\xrightarrow{\alpha_{MN}} \mathit{Stable}(\Gamma)(S^0M,S^kN).$$

This map is an isomorphism if M is in the localizing subcategory generated by the dualizable comodules.

Note that the Ext groups that appear in this proposition are Ext groups in the category of Γ-comodules, not relative Ext groups.

Proof. A class in $\operatorname{Ext}^k_\Gamma(M,N)$ is represented by an exact sequence of comodules

$$0\to N=E_0\xrightarrow{f_0}E_1\xrightarrow{f_1}E_2\xrightarrow{f_2}\dots\xrightarrow{f_{k-1}}E_k\xrightarrow{f_k}E_{k+1}=M\to 0.$$

We can split this into the short exact sequences

$$0\to\ker f_i\to E_i\to\operatorname{coker} f_i\to 0.$$

Each such short exact sequence is gives rise to a cofiber sequence

$$S^0(\ker f_i)\to S^0E_i\to S^0(\operatorname{coker} f_i)\to S^1(\ker f_i)$$

in Stable(Γ) by Lemma 6.8. By composing the maps $S^0(\operatorname{coker} f_i)\to S^1(\ker f_i)$, we get a map $S^0M\to S^kN$ in Stable(Γ). One can check that this respects the equivalence relation that defines $\operatorname{Ext}^k_\Gamma(M,N)$, and is natural.

Note that this map is an isomorphism when $M=P$ is dualizable, for then

$$\operatorname{Stable}(\Gamma)(S^0P,S^kN)\cong\pi_k^P(S^0N)\cong\operatorname{Ext}^k_\Gamma(P,N)$$

by Lemma 3.1.4. Let $\mathcal{T}$ be the full subcategory consisting of all M such that

$$\alpha_{MN}\colon\ \operatorname{Ext}^k_\Gamma(M,N)\to\operatorname{Stable}(\Gamma)(S^0M,S^kN)$$

is an isomorphism for all N and all $k \geq 0$. We claim that $\mathcal{T}$ is a localizing subcategory. Indeed, it is clear that $\mathcal{T}$ is closed under retracts and coproducts. To check that $\mathcal{T}$ is thick, we note that a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

induces a long exact sequence in $\operatorname{Ext}^*_\Gamma(-, N)$. Because short exact sequences are also cofiber sequences in Stable(Γ) by 6.8, we also get a long exact sequence in Stable(Γ)$(-, S^*N)$. There is a map between these two long exact sequences (one must check that α_{MN} is compatible with the map $\operatorname{Ext}^k_\Gamma(M'', N) \to \operatorname{Ext}^{k+1}_\Gamma(M', N)$ but the construction of α_{MN} makes this easy to check). The five lemma tells that that if two out of M', M, M'' are in $\mathcal{T}$, so is the third. □

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