

# ON FREYD'S GENERATING HYPOTHESIS

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ABSTRACT. We revisit Freyd's generating hypothesis in stable homotopy theory. We derive new equivalent forms of the generating hypothesis and some new consequences of it. A surprising one is that  $I$ , the Brown-Comenetz dual of the sphere and the source of many counterexamples in stable homotopy, is the cofiber of a self map of a wedge of spheres. We also show that a consequence of the generating hypothesis, that the homotopy of a finite spectrum that is not a wedge of spheres can never be finitely generated as a module over  $\pi_*S$ , is in fact true for finite torsion spectra.

## INTRODUCTION

Freyd's generating hypothesis [Fre66] is perhaps the most important question in stable homotopy theory. A precise statement of it follows.

**Conjecture A** (Freyd's generating hypothesis). *If  $X$  and  $Y$  are finite spectra, and  $S$  is the sphere spectrum, then the natural map*

$$[X, Y] \rightarrow \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*Y)$$

*is a monomorphism.*

If we fix  $Y$  (perhaps not finite) and allow  $X$  to vary, we get a special case of the generating hypothesis which I will refer to as **Freyd's generating hypothesis with target  $Y$** . Here  $[X, Y]$  denotes maps from  $X$  to  $Y$  in the stable homotopy category, and  $\pi_*X = [S, X]_*$  denotes the homotopy groups of  $X$ . In practice, we implicitly assume that we are actually in the  $p$ -local stable homotopy category for some fixed integer prime  $p$ .

Freyd proves that the generating hypothesis actually implies that the map

$$[X, Y] \rightarrow \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*Y)$$

is an isomorphism for all finite spectra  $X, Y$ . Kahn derived other consequences of the truth or falsity of the generating hypothesis in a series of papers, including [Kah69, Kah72, Kah73].

Devinatz and Hopkins [Dev98] have a program for proving the generating hypothesis with target  $S$  using chromatic technology. This approach generalizes Devinatz' work in [Dev90], where he proves that if  $f: X \rightarrow S$  has  $\pi_*f = 0$ , and  $p$  is odd, then the composite  $X \rightarrow S \rightarrow L_1S$  is null. The program depends on the truth of either the telescope conjecture (currently believed to be likely false) or a weak form of the chromatic splitting conjecture and several other conjectures.

In this paper, we prove the following theorem. Let  $X_p$  denote the  $p$ -completion of a spectrum  $X$ .

**Theorem B.** *Let  $Y$  be a finite spectrum. The following are equivalent:*

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- (1) *Freyd's generating hypothesis with target  $Y$* ;
- (2)  $\pi_*Y_p$  *is an injective  $\pi_*S$ -module*;
- (3)  $\pi_*Y_p$  *is an injective  $\pi_*S_p$ -module*;
- (4) *The natural map*

$$[X, Y_p] \rightarrow \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*Y_p)$$

*is an isomorphism for all spectra  $X$ .*

Freyd [Fre66, Theorem 9.9] proved that the generating hypothesis is equivalent to  $\pi_*Y$  being an injective  $\pi_*S$ -module for all finite torsion spectra  $Y$ . T. Y. Lin [Lin76] showed that  $\pi_*Y$  is not an injective  $\pi_*S$ -module if  $Y$  is not torsion, but did not realize that completion would solve this problem if the generating hypothesis is true. Our approach is different from Freyd's, and yields a more precise result. In addition, Freyd does not mention part (4) of Theorem B, which focuses attention on maps from infinite spectra  $X$  to  $Y_p$ . Infinite spectra  $X$  that might be worth studying in this context include the rational Eilenberg-MacLane spectrum  $H\mathbb{Q}$  and  $\Sigma^\infty BG_+$ , the classifying space of a finite group, where the Segal conjecture tells us  $[X, S_p]$ .

Of course, even if the generating hypothesis is false,  $\pi_*Y_p$  has some injective hull  $J_Y$  as a  $\pi_*S$ -module, so one can attempt to study the map  $\pi_*Y_p \rightarrow J_Y$ . We show in this paper that  $\pi_*S_p \rightarrow J_S$  is a split monomorphism of abelian groups in degree 0, for example.

The methods of this paper may also be helpful in investigating the generating hypothesis in other stable homotopy categories. Lockridge [Loc05] has investigated this question; he shows that the generating hypothesis holds in the unbounded derived category  $\mathcal{D}(R)$  of a commutative ring  $R$  if and only if  $R$  is von Neumann regular, for example.

Theorem B has a number of corollaries. Perhaps the most surprising of them is the following. Let  $IY$  denote the Brown-Comenetz dual of  $Y$ , so that

$$[X, IY] = \mathrm{Hom}_{\mathbb{Z}/(p)}(\pi_*(X \wedge Y), \mathbb{Q}/\mathbb{Z}/(p))$$

for all  $X$ .

**Corollary C.** *Let  $Y$  be a finite spectrum. Freyd's generating hypothesis with target  $Y$  holds if and only if  $\pi_*(IY)$  is a flat  $\pi_*S$ -module. In particular, this implies that the natural map*

$$\pi_*(IY) \otimes_{\pi_*S} \pi_*X \rightarrow \pi_*(IY \wedge X)$$

*is an isomorphism for all  $X$ . Furthermore, in this case  $\pi_*(IY)$  has projective dimension 1 as a  $\pi_*S$ -module and, if Freyd's generating hypothesis with target  $S$  holds as well, there is a cofiber sequence*

$$\Sigma^{-1}IY \xrightarrow{\delta} W \rightarrow W \rightarrow IY$$

*for which  $W$  is a coproduct of spheres of varying dimension and  $\delta$  is a phantom map.*

Note that the map  $\delta$  cannot be 0, for then  $IY$  would be a coproduct of spheres itself. Since  $IY = DY \wedge I$  is  $BP$ -acyclic, this is impossible unless  $IY = 0$ , which is false.

On the other hand, there are several reasons to think that  $\delta$  should be 0, and so the generating hypothesis should be false. For example,  $\delta$  is a map from a  $BP$ -acyclic spectrum to a coproduct of spheres, and one might expect that a coproduct

of spheres would be  $BP$ -local. Any bounded below coproduct of spheres is  $BP$ -local, as is any suspension spectrum [HR92], but we do not know about arbitrary coproducts of spheres. Similarly, one might think that there are no phantom maps to  $W_p$ , which should also lead to a disproof of the generating hypothesis. Again, however, the fact that the spheres in  $W$  occur in infinitely many dimensions makes us unable to prove this.

But we can also give some weak evidence that the generating hypothesis might be true. One of the most simple corollaries of the generating hypothesis is that  $\pi_*Y$  is not a finitely generated  $\pi_*S$ -module for any finite  $Y$  except for finite coproducts of spheres.

**Theorem D.** *Suppose  $Y$  is a finite torsion spectrum. Then  $\pi_*Y$  is not a finitely generated  $\pi_*S$ -module.*

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## 1. PROOF OF THEOREM B

We begin with a basic result about injective  $\pi_*S$ -modules.

**Lemma 1.1.** *Suppose  $E$  is a spectrum such that  $\pi_*E$  is an injective  $\pi_*S$ -module. Then the natural map*

$$[X, E] \rightarrow \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*E)$$

*is an isomorphism for all spectra  $X$ .*

This lemma shows that condition (2) of Theorem B implies condition (4). We note that Lemma 1.1 holds, with the proof given below, in any monogenic stable homotopy category in the sense of [HPS97].

*Proof.* Since  $\pi_*E$  is injective, the functor  $\mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*E)$  is exact. The Brown representability theorem then implies that there is a spectrum  $J$  and a natural isomorphism

$$[X, J] \cong \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*E)$$

for all  $X$ . Taking  $X = S$  tells us that there is an isomorphism  $r: \pi_*J \cong \pi_*E$  of  $\pi_*S$ -modules, but not that the natural isomorphism takes  $f \in [X, J]$  to  $f_*$ . In fact, if  $1 \in [J, J]$  corresponds to  $\alpha: \pi_*J \rightarrow \pi_*E$ , then naturality tells us that  $f \in [X, J]$  corresponds to  $\alpha \circ f_*$ . In particular, there is a map  $f: E \rightarrow J$  such that  $\alpha \circ f_*$  is the isomorphism  $r$ . Hence  $\alpha$  is a split epimorphism. On the other hand, if  $x \in \pi_n J \cong [S^n, J]$  is a homotopy class of  $J$ , then  $x$  corresponds to  $\alpha \circ x_* \in \mathrm{Hom}_{\pi_*S}(\pi_*S^n, \pi_*J)$ , which is multiplication by the class  $\alpha(x)$ . Thus  $\alpha(x)$  can't be 0, and so  $\alpha$  is a monomorphism as well. Thus  $f_*$  is an isomorphism as well, and so  $E \cong J$ . The lemma follows.  $\square$

The following proposition is the heart of the argument proving Theorem B.

**Proposition 1.2.** *Suppose  $Y$  is a spectrum such that there are no nonzero phantom maps to  $Y$ . Then Freyd's generating hypothesis with target  $Y$  holds if and only if  $\pi_*Y$  is an injective  $\pi_*S$ -module.*

This proposition will also hold in any monogenic stable homotopy category.

*Proof.* The “if” half of this proposition follows immediately from Lemma 1.1 and does not require the assumption about phantom maps. For the “only if” half, assume

$$[X, Y] \rightarrow \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*Y)$$

is injective for all finite  $X$ . Let  $J$  denote the injective hull of  $\pi_*Y$  as a  $\pi_*S$ -module. Then Brown representability and Lemma 1.1 imply that there is a spectrum  $I$  with  $\pi_*I = J$  and such that the natural map

$$[X, I] \rightarrow \mathrm{Hom}_{\pi_*S}(\pi_*X, J)$$

is an isomorphism for all  $X$ . In particular, there is a map  $Y \rightarrow I$  corresponding to the inclusion  $\pi_*Y \rightarrow J$ . Consider the commutative diagram below.

$$\begin{array}{ccc} [X, Y] & \longrightarrow & [X, I] \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*Y) & \longrightarrow & \mathrm{Hom}_{\pi_*S}(\pi_*X, J) \end{array}$$

The left-hand vertical map is injective for all finite  $X$ , and the bottom horizontal map is always injective. It follows that the top horizontal map is injective for all finite  $X$ , and hence that the fiber  $F \rightarrow Y$  of the map  $Y \rightarrow I$  is a phantom map. Since there are no nonzero phantom maps to  $Y$ , we see that  $Y$  is a retract of  $I$ . Hence  $\pi_*Y$  is a retract of  $J$ , and so is an injective  $\pi_*S$ -module.  $\square$

To apply Proposition 1.2, we need to know the relationship between the generating hypothesis with target  $Y$  and the generating hypothesis with target  $Y_p$ . Since the fiber of the map  $Y \rightarrow Y_p$  is rational, we need the following proposition.

**Proposition 1.3.** *Suppose  $V$  is a graded rational vector space. Then there is a unique  $\pi_*S$ -module structure on  $V$  extending the abelian group structure, and  $V$  is an injective  $\pi_*S$ -module with this structure.*

This proposition will hold if  $\pi_*S$  is replaced by any ring  $R$  such that  $R/\mathfrak{p} \cong \mathbb{Z}_{(p)}$ , where  $\mathfrak{p}$  is the ideal of  $p$ -torsion elements.

*Proof.* Let  $\mathfrak{p}$  denote the ideal of  $p$ -torsion elements in  $\pi_*S$ . Then, since  $V$  is torsion-free, the only way to make  $\pi_*S$  act on  $V$  is through the homomorphism  $\pi_*S \rightarrow \pi_*S/\mathfrak{p} \cong \mathbb{Z}_{(p)}$ .

Now suppose  $f: M \rightarrow V$  is a map of  $\pi_*S$ -modules and  $i: M \rightarrow N$  is an inclusion of  $\pi_*S$ -modules. Let  $\mathrm{Tor}(M)$  denote the  $p$ -torsion in  $M$ , which is a  $\pi_*S$ -submodule. Then  $f$  factors through  $\bar{f}: M/\mathrm{Tor}(M) \rightarrow V$ , and furthermore  $i$  induces an inclusion  $\bar{i}: M/\mathrm{Tor}(M) \rightarrow N/\mathrm{Tor}(N)$ . Since  $V$  is an injective abelian group, there is then a map  $\bar{g}: N/\mathrm{Tor}(N) \rightarrow V$  of abelian groups extending  $\bar{f}$ . But  $\bar{g}$  is in fact a map of  $\pi_*S$ -modules, since  $N/\mathrm{Tor}(N)$  and  $V$  are torsion-free so  $\mathfrak{p}$  acts trivially. Hence

$$N \rightarrow N/\mathrm{Tor}(N) \xrightarrow{\bar{g}} V$$

gives us the desired extension of  $f$ .  $\square$

We then have the following proposition.

**Proposition 1.4.** *Suppose  $Y$  is a finite spectrum. Then Freyd’s generating hypothesis with target  $Y$  holds if and only if Freyd’s generating hypothesis with target  $Y_p$  holds.*

*Proof.* Let  $C$  denote the cofiber of the map  $Y \rightarrow Y_p$ , so that  $C$  is rational. Recall that the fiber  $\Sigma^{-1}C \rightarrow Y$  is a phantom map (see [Mar83, Theorem 9.5], for example). Then we have the commutative diagram below.

$$\begin{array}{ccccc} [X, Y] & \longrightarrow & [X, Y_p] & \longrightarrow & [X, C] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*Y) & \longrightarrow & \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*Y_p) & \longrightarrow & \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*C) \end{array}$$

The top row of this diagram is short exact for finite  $X$ , since  $\Sigma^{-1}C \rightarrow Y$  is phantom. The bottom row is left exact for all  $X$ . The right-hand vertical map is an isomorphism. A simple diagram chase now shows that the left-hand vertical map is a monomorphism for finite  $X$  if and only if the middle vertical map is a monomorphism for finite  $X$ .  $\square$

Analysis of the proof of Proposition 1.4 in fact gives us the following proposition, independent of the generating hypothesis.

**Proposition 1.5.** *Suppose  $X$  and  $Y$  are finite spectra. Then the natural map*

$$\mathrm{Ext}_{\pi_*S}^n(\pi_*X, \pi_*Y) \rightarrow \mathrm{Ext}_{\pi_*S}^n(\pi_*X, \pi_*Y_p)$$

*is an isomorphism for all  $n \geq 1$ .*

*Proof.* In the commutative diagram used in the proof of Proposition 1.4, we pointed out that the bottom row

$$\mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*Y) \rightarrow \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*Y_p) \rightarrow \mathrm{Hom}_{\pi_*S}(\pi_*X, \pi_*C)$$

was left exact. But in fact a diagram chase shows that it is exact, since the right-hand vertical map is an isomorphism. From this and the fact that  $\pi_*C$  is an injective  $\pi_*S$  module, the proposition follows.  $\square$

The last ingredient we need for Theorem B is the simple proof that conditions (2) and (3) are equivalent.

**Lemma 1.6.** *Let  $Y$  be a finite spectrum. Then  $\pi_*Y_p$  is an injective  $\pi_*S$ -module if and only if it is an injective  $\pi_*S_p$ -module.*

*Proof.* Since  $\pi_*S_p$  is a flat  $\pi_*S$ -module, the forgetful functor from  $\pi_*S_p$ -modules to  $\pi_*S$ -modules preserves injectives. Conversely, assume  $\pi_*Y_p$  is an injective  $\pi_*S$ -module. To see that  $\pi_*Y_p$  is injective over  $\pi_*S_p$ , we use Baer's criterion, which tells us we need only check that, given an ideal  $\mathfrak{a}$  of  $\pi_*S_p$  and a map  $f: \mathfrak{a} \rightarrow \pi_*Y_p$ , there is an extension  $\pi_*S_p \rightarrow \pi_*Y_p$  of  $\pi_*S_p$ -modules. Let  $\mathfrak{b} = \mathfrak{a} \cap \pi_*S$ . Then  $\mathfrak{b}$  is an ideal of  $\pi_*S$ , so we have an extension  $\pi_*S \rightarrow \pi_*Y_p$  of  $\pi_*S$ -modules. This gives a map  $\pi_*S_p = \pi_*S \otimes_{\mathbb{Z}(p)} \rightarrow \pi_*Y_p$  of  $\pi_*S_p$ -modules. When restricted to  $\mathfrak{b}$ , this map extends  $f$ . But since  $\mathfrak{a} = \mathfrak{b} \otimes_{\mathbb{Z}(p)}$ , it follows that it is an extension of  $f$  on  $\mathfrak{a}$  as well.  $\square$

We can now prove Theorem B.

*Proof of Theorem B.* Suppose Freyd's generating hypothesis with target  $Y$  holds. Then Proposition 1.4 implies that the generating hypothesis with target  $Y_p$  holds. Since there are no nonzero phantom maps to  $Y_p$ , Proposition 1.2 tells us that  $\pi_*Y_p$  is an injective  $\pi_*S$ -module. Thus condition (1) implies condition (2).

Lemma 1.6 tells us that conditions (2) and (3) are equivalent, and Lemma 1.1 tells us that condition (2) implies condition (4). Condition (4) obviously implies that Freyd's generating hypothesis holds with target  $Y_p$ , and then Proposition 1.4 implies that it holds with target  $Y$  as well.  $\square$

## 2. BROWN-COMENETZ DUALITY

In this section, we investigate the consequences of the generating hypothesis for Brown-Comenetz duals of finite spectra, proving Corollary C.

*Proof of Corollary C.* Suppose  $Y$  is finite. Then  $Y_p = I^2Y$ , as is well-known. Hence  $\pi_*Y_p = \text{Hom}_{\mathbb{Z}_{(p)}}(\pi_*(IY), \mathbb{Q}/\mathbb{Z}_{(p)})$ . Now apply Lambek's theorem [Lam99, Theorem 4.9] to conclude that  $\pi_*Y_p$  is injective if and only if  $\pi_*(IY)$  is flat. Once  $\pi_*(IY)$  is flat, then the map

$$\pi_*(IY) \otimes_{\pi_*S} \pi_*X \rightarrow \pi_*(IY \wedge X)$$

is a natural transformation of homology theories that is an isomorphism when  $X = S$ , so is always an isomorphism.

Now Lemma 2 of [Jen66] implies that, over a countable ring like  $\pi_*S$ , any flat module has projective dimension  $\leq 1$ . Since  $\pi_*S$  is a local ring, projectives are free [Kap58]. Thus, if  $\pi_*(IY)$  had projective dimension 0, it would be free. From that it is easy to conclude that  $IY$  is a coproduct of spheres. But, since  $IY = DY \wedge I$  is  $BP$ -acyclic (see [HPS97, Lemma B.11]),  $IY$  would have to be trivial. Since this is false, the projective dimension of  $IY$  is 1.

Thus there is a short exact sequence

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow \pi_*IY \rightarrow 0$$

of  $\pi_*S$ -modules, where  $F_1$  and  $F_0$  are free. In fact, by tensoring over  $\mathbb{Z}_{(p)}$  with  $\mathbb{Q}$ , we see that  $F_1$  and  $F_0$  are isomorphic. By choosing generators, we can find a coproduct of spheres  $W$  with  $\pi_*W \cong F_1 \cong F_0$ . By looking at the image of the generators in homotopy, we can find maps

$$W \xrightarrow{f} W \xrightarrow{g} IY$$

such that the induced maps on homotopy give our original free resolution of  $\pi_*IY$ . In fact, this sequence is a cofiber sequence. Indeed, the composite  $gf$  is null, so there is an induced map from the cofiber of  $f$  to  $IY$ , which one can easily see induces an isomorphism on homotopy.

Now, given Freyd's generating hypothesis with target  $S$ , we claim that the map  $\Sigma^{-1}IY \xrightarrow{\delta} W$  that is the fiber of  $f$  is phantom. Indeed, if  $F$  is finite, and  $h: F \rightarrow \Sigma^{-1}IY$  is a map, then  $\delta h$  must factor through some a map  $h': F \rightarrow W'$  for some finite subcoproduct of spheres  $W'$ . If  $\delta h$  is nonzero, then  $h'$  is nonzero, and so, by Freyd's generating hypothesis with target  $S$ , must induce a nontrivial map on homotopy. Then  $\delta h$  also induces a nontrivial map on homotopy, as does the trivial map  $f\delta h$ . This contradiction implies  $\delta$  is phantom.  $\square$

Corollary C has some interesting consequences. Suppose that Freyd's generating hypothesis with target  $S$  holds, so that we have the cofiber sequence

$$\Sigma^{-1}I \xrightarrow{\delta} W \xrightarrow{f} W \rightarrow I$$

where  $W$  is a coproduct of spheres and  $\delta$  is a phantom map. Then  $E_*f$  is a monomorphism for all  $E$ , and is an isomorphism for the many  $E$  for which  $E_*I = 0$ ,

such as all  $BP$ -module spectra and  $I$  itself. In fact,  $E^*f$  is an isomorphism for all  $BP$ -module spectra and all harmonic spectra  $E$ , since  $I$  is  $BP$ -acyclic and dissonant.

On the other hand, suppose  $E$  is one of the many spectra for which  $[E, S]_* = 0$ , such as  $I, H\mathbb{F}_p, K(n)$  for  $n > 0$ , or any dissonant spectrum. Then any map  $E \rightarrow W$  goes to 0 in the corresponding product  $P$  of spheres, and hence factors through the fiber of  $W \rightarrow P$ . Since this is a phantom map,  $[E, W]_*$  consists entirely of phantom maps, which necessarily go to 0 in  $[E, I]_*$ . Hence  $[E, f]_*$  is in fact surjective in this case. One might think that this happens because  $[E, W]_* = 0$ , but in fact  $[E, W]_* = 0$  if and only if  $E = 0$ , since if  $[E, W]_* = 0$  then  $[E, I]_* = 0$ , and so  $E = 0$ .

Another corollary is the following.

**Corollary 2.1.** *Suppose Freyd's generating hypothesis with target  $S$  holds. Then there is a product  $J$  of suspensions of  $I$  such that  $S_p \vee J \cong J$ .*

*Proof.* Apply the functor  $F(-, I)$  to the cofiber sequence

$$\Sigma^{-1}I \xrightarrow{\delta} W \rightarrow W \rightarrow I$$

to get a cofiber sequence

$$F(I, I) = S_p \rightarrow J = F(W, I) \rightarrow J = F(W, I) \rightarrow F(\Sigma^{-1}I, I) = \Sigma S_p.$$

On homotopy, the last map takes a map  $\alpha: \Sigma^i W \rightarrow I$  into the composite  $\delta \circ \alpha$ , which is necessarily 0 since there are no phantom maps to  $I$ . On the other hand, because  $\pi_* S_p$  is an injective  $\pi_* S$ -module, any map into  $S_p$  that is trivial on homotopy is in fact trivial. Hence the cofiber sequence above splits, giving the corollary.  $\square$

### 3. OTHER CONSEQUENCES OF THE GENERATING HYPOTHESIS

In this section, we use Theorem B to draw some further consequences of the generating hypothesis.

First, we give a more precise version of Freyd's "faithful implies full" result [Fre66, Proposition 9.7].

**Corollary 3.1.** *Suppose Freyd's generating hypothesis with target  $Y$  holds. Then the natural map*

$$[X, Y] \rightarrow \text{Hom}_{\pi_* S}(\pi_* X, \pi_* Y)$$

*is an isomorphism for all finite  $X$ .*

*Proof.* We again let  $C$  denote the cofiber of  $Y \rightarrow Y_p$ , so that we get the same commutative diagram used in the proof of Proposition 1.4.

$$\begin{array}{ccccc} [X, Y] & \longrightarrow & [X, Y_p] & \longrightarrow & [X, C] \\ \downarrow & & \downarrow & & \downarrow \\ \text{Hom}_{\pi_* S}(\pi_* X, \pi_* Y) & \longrightarrow & \text{Hom}_{\pi_* S}(\pi_* X, \pi_* Y_p) & \longrightarrow & \text{Hom}_{\pi_* S}(\pi_* X, \pi_* C) \end{array}$$

The top row of this diagram is exact for all finite  $X$  since  $\Sigma^{-1}C \rightarrow Y$  is phantom. The bottom row is also exact, as pointed out in Proposition 1.5. The right-hand vertical map is an isomorphism by Proposition 1.3, and the middle vertical map is an isomorphism by Theorem B. A simple diagram chase shows the left-hand vertical map is an isomorphism for all finite  $X$ .  $\square$

Proposition 1.5 immediately gives us the following corollary of Theorem B.

**Corollary 3.2.** *If Freyd's generating hypothesis with target  $Y$  holds, then*

$$\mathrm{Ext}_{\pi_*S}^n(\pi_*X, \pi_*Y) = 0$$

for all finite  $X$ .

Using the well-known fact that  $\mathbb{Z}_p/\mathbb{Z}_{(p)}$  is a rational vector space, we get the following consequence of the generating hypothesis.

**Corollary 3.3.** *If Freyd's generating hypothesis with target  $Y$  holds, then*

$$0 \rightarrow \pi_*Y \rightarrow \pi_*Y_p \rightarrow \pi_*Y_p/\pi_*Y \rightarrow 0$$

is an injective resolution of  $\pi_*Y$  in the category of  $\pi_*S$ -modules. In particular, if  $Y$  is a finite spectrum of type 0, then  $\pi_*Y$  has injective dimension 1.

#### 4. INJECTIVE $\pi_*S$ -MODULES

Theorem B focusses attention on injective  $\pi_*S$ -modules; in this section we prove a few facts about them. Without assuming Freyd's generating hypothesis, we still know that  $\pi_*Y$  has some injective hull  $J_Y$ . We cannot say very much about  $J_Y$ , but we can say a little.

**Proposition 4.1.** *The map  $\pi_*S \rightarrow \pi_*S_p$  is an essential extension of  $\pi_*S$ -modules.*

Hence, whatever  $J_S$  is, at least it contains  $\pi_*S_p$ .

*Proof.* The only elements in  $\pi_*S_p$  not in  $\pi_*S$  are elements in  $\pi_0S_p \cong \mathbb{Z}_p$ . Choose a nonzero  $x \in \mathbb{Z}_p$ , and suppose  $p^n$  divides  $x$  but  $p^{n+1}$  does not, so that  $x$  is congruent to an integer of the form  $kp^n \in \pi_0S$  modulo  $p^{n+1}$ , where  $k$  is a unit. Now choose an element  $\alpha \in \pi_*S$  of order  $p^{n+1}$ , which can be done in the image of the  $J$  homomorphism. Then  $\alpha x = kp^n x$ , which is a nontrivial element of  $\pi_*S$ . Therefore,  $(x) \cap \pi_*S$  is nonzero, completing the proof. is a  $y \in \pi_0S$  such that  $x - y$  is a multiple of  $p$  in  $\mathbb{Z}_p$ .  $\square$

In fact, we know a little more about  $J$ .

**Proposition 4.2.** *Let  $J$  denote the injective hull of  $\pi_*S_p$  as a  $\pi_*S_p$ -module. The inclusion  $\mathbb{Z}_p \rightarrow J_0$  is a split monomorphism of abelian groups.*

*Proof.* We will prove that  $\mathbb{Z}_p \rightarrow J_0$  is a pure monomorphism. That is, we will show that if we have an equation  $x = p^n y$  for  $x \in \mathbb{Z}_p$  and  $y \in J_0$ , then in fact we have an equation  $x = p^n z$  for some  $z \in \mathbb{Z}_p$ . Indeed, we may as well assume  $x = p^k$ , so that we have  $p^k \alpha = p^n \alpha y$  for all  $\alpha \in \pi_*S_p$ . But then, if  $n > k$ , we may take  $\alpha$  to be an element of exact order  $p^n$  and conclude that  $p^k \alpha = 0$ . This contradiction shows that  $\mathbb{Z}_p \rightarrow J_0$  is pure.

Since  $\mathbb{Z}_p$  is a pure injective abelian group, the proposition follows.  $\square$

Since  $\pi_*S_p \rightarrow J$  is an essential extension, for every element  $y \in J$ , there is an element  $\alpha_y \in \pi_*S_p$  with  $\alpha_y y \in \pi_*S_p$ . We can thus look for an element  $x$  in  $\pi_*S_p$  of lowest possible degree such that  $x = \gamma y$  for some  $y \in J \setminus \pi_*S_p$ . Proposition 4.2 tells us the degree of  $x$  must be positive. Our knowledge of  $\pi_*S$  is sufficient to rule out some possibilities for the pair  $(x, \gamma)$ , but insufficient, as far as the author knows, to say anything systematic.

We point that there is one more injective  $\pi_*S$ -module known, besides the rational ones and, conjecturally,  $\pi_*S_p$ .

**Proposition 4.3.** *Let  $I$  denote the Brown-Comenetz dual of  $S$ . Then  $\pi_*I$  is the injective hull of  $\mathbb{F}_p$  as a  $\pi_*S$ -module.*

The same argument as in the proof below shows that  $\pi_*IL_nS$  is an injective  $\pi_*L_nS$ -module for any  $n$ .

*Proof.* We have  $\pi_*I = \text{Hom}_{\mathbb{Z}_{(p)}}(\pi_*S, \mathbb{Q}/\mathbb{Z}_{(p)})$ . Since  $\mathbb{Q}/\mathbb{Z}_{(p)}$  is an injective abelian group,  $\pi_*I$  is injective by a standard result about injective modules [Lam99, Corollary 3.6B]. The obvious inclusion  $\mathbb{F}_p \rightarrow \pi_0I \rightarrow \pi_*I$  is obviously a map of  $\pi_*S$ -modules. The action of  $\pi_*S$  on  $\pi_*I$  is given by

$$\mu: \pi_kS \otimes \text{Hom}_{\mathbb{Z}_{(p)}}(\pi_nS, \mathbb{Q}/\mathbb{Z}_{(p)}) \rightarrow \text{Hom}(\pi_{n-k}S, \mathbb{Q}/\mathbb{Z}_{(p)})$$

where  $\mu(x \otimes f)(y) = f(xy)$ . In particular, if  $f$  is a nontrivial element of  $\pi_{-n}I = \text{Hom}(\pi_nS, \mathbb{Q}/\mathbb{Z}_{(p)})$ , then there is an  $x \in \pi_nS$  such that  $f(x)$  is a nonzero element of  $\mathbb{F}_p \subseteq \mathbb{Q}/\mathbb{Z}_{(p)}$ . It follows that  $\mu(x \otimes f)$  is a nonzero element of  $\mathbb{F}_p$ , and therefore that  $\mathbb{F}_p \rightarrow \pi_*I$  is an essential extension of  $\pi_*S$ -modules.  $\square$

Note that it is tempting to conclude from Proposition 4.3 that  $\mathbb{F}_p$  has injective dimension 1 as a  $\pi_*S$ -module. This is wrong, however. The cokernel of  $\mathbb{F}_p \rightarrow \pi_*I$  is isomorphic as a graded abelian group to  $\pi_*I$ , but not as a  $\pi_*S$ -module.

## 5. INFINITELY GENERATED HOMOTOPY

It was G. Whitehead who realized that the generating hypothesis implies that the homotopy of a finite complex  $Y$  is not finitely generated over  $\pi_*S$  unless  $Y$  is a finite coproduct of spheres [Fre66, Proposition 9.5]. The proof of this fact is so easy we recall it here. Suppose  $Y$  has finitely generated homotopy, so that we have a cofiber sequence

$$F \xrightarrow{f} W \xrightarrow{g} Y \xrightarrow{h} \Sigma F$$

where  $W$  is a finite wedge of spheres and  $\pi_*(g)$  is onto. Then  $\pi_*h = 0$ , so, by the generating hypothesis,  $h = 0$ . Hence  $Y$  is a retract of  $W$ , so  $\pi_*Y$  is projective, and hence free. Thus  $Y$  is itself a wedge of spheres.

Don Kahn [Kah69] has shown that, for any finite spectrum  $Y$ , it is possible to attach two cells (one if  $Y$  is not torsion) to  $Y$  to get a new complex  $Z$  with  $\pi_*Z$  not finitely generated. Thus there are many finite spectra whose homotopy is not finitely generated.

We can use the existence of  $v_n$  self-maps to prove Theorem D, which, we recall, says that the homotopy of a finite torsion spectrum is never finitely generated.

*Proof of Theorem D.* By the nilpotence theorem [HS98],  $X$  has a non-nilpotent self-map  $v$  of positive degree. This map can be taken to have Adams-Novikov filtration 0; see, for example, [HS99, Theorem 4.6]. The spectrum  $v^{-1}X$  is nonzero, since  $v$  is not nilpotent, and  $\pi_*(v^{-1}X) = v^{-1}\pi_*X$ . Hence we can choose a  $\beta \in \pi_*X$  such that  $v^k \circ \beta$  is not zero for all  $k \geq 0$ ; in fact, we choose such a  $\beta$  with minimal Adams-Novikov filtration.

Now suppose  $\{x_1, \dots, x_n\}$  is a generating set for  $\pi_*X$  as a  $\pi_*S$ -module, arranged so that their Adams-Novikov filtrations are non-decreasing. Suppose that the  $x_j$  have Adams-Novikov filtration less than that of  $\beta$  for  $j < i$ , and write  $\beta = x_1 \circ \alpha_1 + \dots + x_n \circ \alpha_n$  for some  $\alpha_j \in \pi_*S$ . Then for large  $k$ , we have

$$v^k \circ \beta = v^k \circ x_i \circ \alpha_i + \dots + v^k \circ x_n \circ \alpha_n.$$

By comparing Adams-Novikov filtrations, we see that there must be a  $j$  such that the Adams-Novikov filtration of  $x_j$  is the same as that of  $\beta$ , and also that  $\alpha_j$  must have Adams-Novikov filtration 0. But this means that the degree of  $x_j$  is the same as the degree of  $\beta$ .

Hence, there must be a generator of  $\pi_*X$  in the degree of  $v^k\beta$  for all  $k \geq 0$ . Thus  $\pi_*X$  is not finitely generated.  $\square$

## 6. THE GENERATING HYPOTHESIS AND THICK SUBCATEGORIES

One difficulty that the generating hypothesis has always posed is that knowing the generating hypothesis with target  $Y$  does not seem to say very much about the generating hypothesis with other targets. Freyd's work does imply, however, that if the generating hypothesis with target  $Y$  is true for all finite torsion spectra  $Y$ , then it is true for all finite  $Y$  (this can be obtained from the proof of Theorem 9.9 of [Fre66]). In this section we extend Freyd's result to finite spectra of type at least  $n$ .

**Proposition 6.1.** *Suppose  $X$  is a type  $n$  finite spectrum for some  $n$ , with  $v_n$  self-map  $v$ . Let  $X/v^k$  denote the cofiber of  $v^k$ , and consider the cofiber sequence*

$$Z \xrightarrow{\delta} X \rightarrow \prod_{k \geq 1} X/v^k.$$

*Then  $\delta$  is a phantom map.*

*Proof.* Suppose first that  $n = 0$ , so that  $v = p$ . If  $F$  is a finite spectrum, the group  $[F, X]$  is finitely generated abelian, and therefore any  $f: F \rightarrow X$  is not divisible by  $p^k$  for large enough  $k$ . Hence the image of  $f$  in  $[F, X/p^k]$  is nonzero for large enough  $k$ . Thus

$$[F, X] \rightarrow [F, \prod X/p^k]$$

is a monomorphism, and so  $\delta$  is phantom.

Now suppose  $n \geq 1$ , so that the map  $v$  has some positive degree  $d$  (see [HS98]). Let  $F$  be a finite spectrum, and suppose  $f: F \rightarrow X$  is a nontrivial map. We claim that the composite  $F \rightarrow X \rightarrow X/v^k$  is nontrivial for some  $k$ . Indeed, if not, then  $f$  factors through  $\Sigma^{dk}X$  for all  $k$ . For  $k$  large enough, every cell of  $F$  will be in lower degree than all the cells of  $\Sigma^{dk}X$ , and so  $[F, \Sigma^{dk}X] = 0$  and  $f = 0$ . Thus

$$[F, X] \rightarrow [F, \prod X/v^k]$$

is a monomorphism, and so  $\delta$  is phantom.  $\square$

**Corollary 6.2.** *Suppose  $X$  is a type  $n$  finite spectrum for some  $n$ , with  $v_n$  self-map  $v$ . Then  $X_p$  is a retract of  $\prod_{k \geq 1} X/v^k$ .*

*Proof.* Recall that completion is really Bousfield localization  $L_{M(p)}$ . The space  $\prod X/v^k$  is already  $L_{M(p)}$ -local, since each  $X/v^k$  is so. Hence we have a cofiber sequence

$$L_{M(p)}Z \xrightarrow{L_{M(p)}\delta} L_{M(p)}X \rightarrow \prod_{k \geq 1} X/v^k.$$

The map  $L_{M(p)}\delta$  is determined by its restriction to  $Z$ , which is phantom by Proposition 6.1. Since there are no phantom maps to  $X_p$ , we conclude that  $L_{M(p)}\delta = 0$ , giving us the desired splitting.  $\square$

**Corollary 6.3.** *Fix  $n \geq 0$ . The generating hypothesis with target  $Y$  is true for all finite spectra  $Y$  if and only if it is true for all finite  $Y$  of type at least  $n$ .*

Corollary 6.3 is the closest we can come to showing that the collection of all  $Y$  for which the generating hypothesis with target  $Y$  is true is a thick subcategory.

*Proof.* It is enough to show that, if the generating hypothesis with target  $Y$  is true for all finite  $Y$  of type at least  $k$ , then the generating hypothesis with target  $Y$  is true for all finite  $Y$  of type at least  $k - 1$ . Suppose  $X$  has type  $k - 1$ . Choose a  $v_{k-1}$  self-map  $v$  of  $X$ . By Corollary 6.2,  $X_p$  is a summand in  $\coprod X/v^k$ . Each  $X/v^k$  has type  $k$ , and so  $\pi_* X/v^k$  is an injective  $\pi_* S$ -module, by Theorem B. It follows that  $\pi_* X_p$  is an injective  $\pi_* S$ -module, and so the generating hypothesis with target  $X_p$  is true. But then Proposition 1.4 implies that the generating hypothesis with target  $X$  is true.  $\square$

Another interesting corollary of Proposition 6.1 is the following. Let  $\mathcal{C}_n$  denote the thick subcategory of finite spectra whose type is at least  $n$ .

**Corollary 6.4.** *The subcategory  $\mathcal{C}_n$  generates and cogenerates the category of finite spectra.*

This means that, given a nonzero map  $f: X \rightarrow Y$  of finite spectra, there are maps  $g: Z \rightarrow X$  and  $h: Y \rightarrow W$  with  $Z, W \in \mathcal{C}_n$  and  $f \circ g$  and  $h \circ f$  both nonzero. This corollary was proved by Freyd [Fre66, Proposition 6.8] in the case  $n = 1$ .

*Proof.* Let  $f: X \rightarrow Y$  be a nonzero map. Suppose  $Y$  is of type  $k$ . Then it follows from Proposition 6.1 that there is a  $Z$  of type  $k + 1$ , namely  $Y/v^r$  for large  $r$ , and a map  $h: Y \rightarrow Z$  such that  $hf$  is nonzero. We can then proceed by induction to see that  $\mathcal{C}_n$  cogenerates the category of finite spectra.

Given this, consider the Spanier-Whitehead dual  $Df$  of  $f$ . There is a spectrum  $V$  of type at least  $n$  and a map  $k: DX \rightarrow V$  such that  $k \circ Df$  is nonzero. Dualizing, we see that  $f \circ Dk$  is nonzero, and  $DV$  also has type at least  $n$ .  $\square$

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