

ON TRIVIALITY OF DICKSON INVARIANTS IN THE HOMOLOGY OF THE STEENROD ALGEBRA

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Dedicated to Professor Nguyễn Duy Tiến on the occasion of his sixtieth birthday

ABSTRACT. Let \mathcal{A} be the mod 2 Steenrod algebra and D_k the Dickson algebra of k variables. We study the Lannes-Zarati homomorphisms

$$\varphi_k : Ext_{\mathcal{A}}^{k, k+i}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} D_k)_i^*$$

which correspond to an associated graded of the Hurewicz map $H : \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \rightarrow H_*(Q_0S^0)$. An algebraic version of the long-standing conjecture on spherical classes predicts that $\varphi_k = 0$ in positive stems, for $k > 2$. That the conjecture is no longer valid for $k = 1$ and 2 is respectively an exposition of the existence of Hopf invariant one classes and Kervaire invariant one classes.

This conjecture has been proved for $k = 3$ in [9]. It has been shown that φ_k vanishes on decomposable elements for $k > 2$ in [14] and on the image of Singer's algebraic transfer for $k > 2$ in [9] and [12]. In this paper, we establish the conjecture for $k = 4$. To this end, our main tools include (1) an explicit chain-level representation of φ_k and (2) a squaring operation Sq^0 on $(\mathbb{F}_2 \otimes_{\mathcal{A}} D_k)^*$, which commutes with the classical Sq^0 on $Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ through the Lannes-Zarati homomorphism.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $H : \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \rightarrow H_*(Q_0S^0)$ be the Hurewicz homomorphism of the basepoint component Q_0S^0 in the infinite loop space $QS^0 = \lim_n \Omega^n S^n$. Here and throughout the paper, homology and cohomology are taken with coefficients in \mathbb{F}_2 , the field of two elements. The long-standing conjecture on spherical classes states as follows: *Only the classes of Hopf invariant one and those of Kervaire invariant one are detected by the Hurewicz homomorphism.* (See Curtis [6], Snaithe and Tornehave [26] and Wellington [27] for a discussion.)

An algebraic version of this problem, which we are interested in, goes as follows. Let $P_k = \mathbb{F}_2[x_1, \dots, x_k]$ be the polynomial algebra on k generators x_1, \dots, x_k , each of degree 1. Let the general linear group $GL_k = GL(k, \mathbb{F}_2)$ and the mod 2 Steenrod algebra \mathcal{A} both act on P_k in the usual way. The Dickson algebra of k variables, D_k , is the algebra of invariants

$$D_k := \mathbb{F}_2[x_1, \dots, x_k]^{GL_k}.$$

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Since the action of \mathcal{A} and that of GL_k on P_k commute with each other, D_k is an algebra over \mathcal{A} . In [17], Lannes and Zarati construct homomorphisms

$$\varphi_k : Ext_{\mathcal{A}}^{k,k+i}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} D_k)_i^*,$$

which correspond to an associated graded of the Hurewicz map. The proof of this assertion is unpublished, but it is sketched by Lannes [16] and by Goerss [8]. The Hopf invariant one and the Kervaire invariant one classes are respectively represented by certain permanent cycles in $Ext_{\mathcal{A}}^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$ and $Ext_{\mathcal{A}}^{2,*}(\mathbb{F}_2, \mathbb{F}_2)$, on which φ_1 and φ_2 are non-zero (see Adams [1], Browder [5], Lannes-Zarati [17]). Therefore, we are led to the following conjecture.

Conjecture 1.1. $\varphi_k = 0$ in any positive stem i for $k > 2$.

The conjecture has been proved for $k = 3$ in [9] and for $k = 4$ in a range of stems in [14]. It has been shown that φ_k vanishes on decomposable elements for $k > 2$ in [14] and on the image of Singer's algebraic transfer $Tr_k : ((\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)^{GL_k})^* \rightarrow Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ for $k > 2$ in [9] and [12].

The following is the main result of the present paper.

Theorem 1.2. $\varphi_4 = 0$ in positive stems.

An ingredient in our proof of this theorem is the squaring operation Sq^0 on $(\mathbb{F}_2 \otimes_{\mathcal{A}} D_k)^*$, which is defined in our paper [9]. The key step in the proof is to show the following theorem.

Theorem 1.3. *The squaring operation Sq^0 on $(\mathbb{F}_2 \otimes_{\mathcal{A}} D_k)^*$ commutes with the classical squaring operation Sq^0 on $Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ through the Lannes-Zarati homomorphism φ_k , for any k .*

Applying this theorem, we get a proof of Theorem 1.2 by combining the computation of $Ext_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$ by W. H. Lin [18] and that of $\mathbb{F}_2 \otimes_{\mathcal{A}} D_4$ by the author and Peterson [13].

In order to prove Theorem 1.3, we need to exploit Singer's invariant-theoretic description of the lambda algebra [24]. According to Dickson [7], one has

$$D_k \cong \mathbb{F}_2[Q_{k,k-1}, \dots, Q_{k,0}],$$

where $Q_{k,i}$ denotes the Dickson invariant of degree $2^k - 2^i$. Singer sets $\Gamma_k = D_k[Q_{k,0}^{-1}]$, the localization of D_k given by inverting $Q_{k,0}$, and defines Γ_k^\wedge to be a certain "not too large" submodule of Γ_k . He also equips $\Gamma^\wedge = \bigoplus_k \Gamma_k^\wedge$ with a differential $\partial : \Gamma_k^\wedge \rightarrow \Gamma_{k-1}^\wedge$ and a coproduct. Then, he shows that the differential coalgebra Γ^\wedge is dual to the (opposite) lambda algebra of the six authors of [4]. Thus, $H_k(\Gamma^\wedge) \cong Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. (Originally, Singer uses the notation Γ_k^+ to denote Γ_k^\wedge . However, by D_k^+ , \mathcal{A}^+ we always mean the submodules of D_k and \mathcal{A} respectively consisting of all elements of positive degrees, so Singer's notation Γ_k^+ would make a confusion in this paper. Therefore, we prefer the notation Γ_k^\wedge .)

The following result plays a key role in our proof of Theorem 1.3.

Theorem 1.4. ([11]) *The inclusion $D_k \subset \Gamma_k^\wedge$ is a chain-level representation of the Lannes-Zarati dual homomorphism*

$$\varphi_k^* : (\mathbb{F}_2 \otimes_{\mathcal{A}} D_k)_i \rightarrow Tor_{k,k+i}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

By this theorem, Conjecture 1.1 is equivalent to our conjecture on the triviality of Dickson invariants in the homology of the Steenrod algebra:

Conjecture 1.5. ([10]) Let D_k^+ denote the submodule of all positive degree elements in D_k . If $q \in D_k^+$, then $[q] = 0$ in $H_k(\Gamma^\wedge) \cong \text{Tor}_k^A(\mathbb{F}_2, \mathbb{F}_2)$ for $k > 2$.

Therefore, Theorem 1.2 can be restated as follows.

Theorem 1.6. *Every positive-degree Dickson invariant of four variables represents the 0 class in the homology, $\text{Tor}_*^A(\mathbb{F}_2, \mathbb{F}_2)$, of the Steenrod algebra.*

Also, the theorem that φ_k vanishes on the image of the (Singer) algebraic transfer $\text{Tr}_k : ((\mathbb{F}_2 \otimes P_k)^{GL_k})^* \rightarrow \text{Ext}_A^k(\mathbb{F}_2, \mathbb{F}_2)$ for $k > 2$ is restated as follows: *Every positive-degree Dickson invariant of k variables represents a class in the kernel of the algebraic transfer's dual $\text{Tr}_k^* : \text{Tor}_k^A(\mathbb{F}_2, \mathbb{F}_2) \rightarrow (\mathbb{F}_2 \otimes P_k)^{GL_k}$ for $k > 2$ (see [10], [12]).* It should be noted that the algebraic transfer is computationally showed to be highly nontrivial by Singer [25] and by Boardman [3].

The paper contains four sections. Section 2 is a recollection on modular invariant theory. Its goal is to make the paper self-contained by recalling Singer's invariant-theoretic description of the lambda algebra and our chain-level representation of the Lannes-Zarati dual map. Section 3 and Section 4 are respectively devoted to the proofs of Theorem 1.3 and Theorem 1.2.

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2. RECOLLECTION ON MODULAR INVARIANT THEORY

The purpose of this section is to make the paper self-contained. First, we summarize Singer's invariant-theoretic description of the lambda algebra.

Let T_k be the Sylow 2-subgroup of GL_k consisting of all upper triangular $k \times k$ -matrices with 1 on the main diagonal. The T_k -invariant ring, $M_k = P_k^{T_k}$, is called the Mui algebra. In [22], Mui shows that

$$P_k^{T_k} = \mathbb{F}_2[V_1, \dots, V_k],$$

where

$$V_i = \prod_{c_j \in \mathbb{F}_2} (c_1 x_1 + \dots + c_{i-1} x_{i-1} + x_i).$$

Then, the Dickson invariant $Q_{k,i}$ can inductively be defined by

$$Q_{k,i} = Q_{k-1,i-1}^2 + V_k \cdot Q_{k-1,i},$$

where, by convention, $Q_{k,k} = 1$ and $Q_{k,i} = 0$ for $i < 0$.

Let $S(k) \subset P_k$ be the multiplicative subset generated by all the non-zero linear forms in P_k . Let Φ_k be the localization: $\Phi_k = (P_k)_{S(k)}$. Using the results of Dickson [7] and Mui [22], Singer notes in [24] that

$$\Delta_k := (\Phi_k)^{T_k} = \mathbb{F}_2[V_1^{\pm 1}, \dots, V_k^{\pm 1}],$$

$$\Gamma_k := (\Phi_k)^{GL_k} = \mathbb{F}_2[Q_{k,k-1}, \dots, Q_{k,1}, Q_{k,0}^{\pm 1}].$$

Further, he sets

$$v_1 = V_1, \quad v_k = V_k/V_1 \cdots V_{k-1} \quad (k \geq 2),$$

so that

$$V_k = v_1^{2^{k-2}} v_2^{2^{k-3}} \cdots v_{k-1} v_k \quad (k \geq 2).$$

Then, he obtains

$$\Delta_k = \mathbb{F}_2[v_1^{\pm 1}, \dots, v_k^{\pm 1}],$$

with $\deg v_i = 1$ for every i .

Singer defines Γ_k^\wedge to be the submodule of $\Gamma_k = D_k[Q_{k,0}^{-1}]$ spanned by all monomials $\gamma = Q_{k,k-1}^{i_{k-1}} \cdots Q_{k,0}^{i_0}$ with $i_{k-1}, \dots, i_1 \geq 0, i_0 \in \mathbb{Z}$, and $i_0 + \deg \gamma \geq 0$. He also shows in [24] that the homomorphism

$$\begin{aligned} \partial_k : \mathbb{F}_2[v_1^{\pm 1}, \dots, v_k^{\pm 1}] &\rightarrow \mathbb{F}_2[v_1^{\pm 1}, \dots, v_{k-1}^{\pm 1}], \\ \partial_k(v_1^{j_1} \cdots v_k^{j_k}) &:= \begin{cases} v_1^{j_1} \cdots v_{k-1}^{j_{k-1}}, & \text{if } j_k = -1, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

maps Γ_k^\wedge to Γ_{k-1}^\wedge . Moreover, it is a differential on $\Gamma^\wedge = \bigoplus_k \Gamma_k^\wedge$. This module is bigraded by putting $\text{bideg}(v_1^{j_1} \cdots v_k^{j_k}) = (k, k + \sum j_i)$.

Let Λ be the (opposite) lambda algebra, in which the product in lambda symbols is written in the order opposite to that used in [4]. It is also bigraded by putting (as in [23, p. 90]) $\text{bideg}(\lambda_i) = (1, 1 + i)$. Singer proves in [24] that Γ^\wedge is a differential bigraded coalgebra, which is dual to the differential bigraded lambda algebra Λ via the isomorphisms

$$\begin{aligned} \text{2.1.} \quad \Gamma_k^\wedge &\rightarrow \Lambda_k^*, \\ v_1^{j_1} \cdots v_k^{j_k} &\mapsto (\lambda_{j_1} \cdots \lambda_{j_k})^*. \end{aligned}$$

Here the duality $*$ is taken with respect to the basis of admissible monomials of Λ . As a consequence, one gets an isomorphism of bigraded coalgebras

$$\text{2.2.} \quad H_*(\Gamma^\wedge) \cong \text{Tor}_*^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

As stated in Theorem 1.4, we prove in [11] that *the inclusion $D_k \subset \Gamma_k^\wedge$ is a chain-level representation of the Lannes–Zarati dual homomorphism*

$$\varphi_k^* : (\mathbb{F}_2 \otimes D_k)_i \rightarrow \text{Tor}_{k,k+i}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

In the remaining part of this section, we recall definition of the classical squaring operation on $\text{Ext}_{\mathcal{A}}^*(\mathbb{F}_2, \mathbb{F}_2)$.

Liulevicius was perhaps the first person who noted in [20] that there are squaring operations $Sq^i : \text{Ext}_{\mathcal{A}}^{k,t}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow \text{Ext}_{\mathcal{A}}^{k+i,2t}(\mathbb{F}_2, \mathbb{F}_2)$, which share most of the properties with Sq^i on the cohomology of spaces. In particular, $Sq^i(\alpha) = 0$ if $i > k$, $Sq^k(\alpha) = \alpha^2$ for $\alpha \in \text{Ext}_{\mathcal{A}}^{k,t}(\mathbb{F}_2, \mathbb{F}_2)$, and the Cartan formula holds for the Sq^i 's. However, Sq^0 is not the identity. In fact, Sq^0 can be defined in terms of the lambda algebra as follows:

$$\begin{aligned} \text{2.3.} \quad Sq^0 : \Lambda_k &\rightarrow \Lambda_k, \\ Sq^0(\lambda_{i_1} \cdots \lambda_{i_k}) &= \lambda_{2i_1+1} \cdots \lambda_{2i_k+1}. \end{aligned}$$

So, by dualizing, the following map

$$\begin{aligned} \text{2.4.} \quad Sq_v^0 : \Gamma_k^\wedge &\rightarrow \Gamma_k^\wedge, \\ Sq_v^0(v_1^{j_1} \cdots v_k^{j_k}) &= \begin{cases} v_1^{\frac{j_1-1}{2}} \cdots v_k^{\frac{j_k-1}{2}}, & j_1, \dots, j_k \text{ odd,} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

is a chain-level representation of the dual squaring operation

$$Sq_*^0 : Tor_k^A(\mathbb{F}_2, \mathbb{F}_2) \rightarrow Tor_k^A(\mathbb{F}_2, \mathbb{F}_2).$$

3. THE SQUARING OPERATIONS

Given a module M over the dual of the Steenrod algebra \mathcal{A}_* , let $P(M)$ denote the submodule of M spanned by all elements annihilated by any operations of positive degrees in \mathcal{A}_* .

Let \mathbb{V}_k be an \mathbb{F}_2 -vector space of dimension k . As is well known, $H^*(B\mathbb{V}_k) \cong P_k$. Then, it is easily seen that $P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k))$ and $\mathbb{F}_2 \otimes_{GL_k} PH_*(B\mathbb{V}_k)$ are respectively dual to $\mathbb{F}_2 \otimes_{\mathcal{A}} (P_k)^{GL_k}$ and $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)^{GL_k}$.

In [9], we have defined a squaring operation

$$Sq^0 : P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k)) \rightarrow P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k)),$$

which is derived from Kameko's squaring operation Sq^0 on $\mathbb{F}_2 \otimes_{GL_k} PH_*(B\mathbb{V}_k)$ (see [15], [3]). We also prove in [9, Proposition 4.2] that these two squaring operations commute with each other through the canonical homomorphism

$$j_k^* : \mathbb{F}_2 \otimes_{GL_k} PH_*(B\mathbb{V}_k) \rightarrow P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k))$$

induced by the identity map on \mathbb{V}_k .

The goal of this section is to show that the Sq^0 on $P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k))$ commutes with the classical squaring operation Sq^0 on $Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ through the Lannes-Zarati map φ_k .

Now we recall the definitions of the above mentioned squaring operations.

As is well known, $H_*(B\mathbb{V}_k)$ is a divided power algebra

$$H_*(B\mathbb{V}_k) = \Gamma(a_1, \dots, a_k)$$

generated by a_1, \dots, a_k , each of degree 1, where a_i is dual to $x_i \in H^1(B\mathbb{V}_k)$. Here, the duality is taken with respect to the basis of $H^*(B\mathbb{V}_k)$ consisting of all monomials in x_1, \dots, x_k .

In [15] Kameko defines a GL_k -homomorphism

$$\begin{aligned} Sq^0 : H_*(B\mathbb{V}_k) &\rightarrow H_*(B\mathbb{V}_k), \\ a_1^{(i_1)} \dots a_k^{(i_k)} &\mapsto a_1^{(2i_1+1)} \dots a_k^{(2i_k+1)}, \end{aligned}$$

where $a_1^{(i_1)} \dots a_k^{(i_k)}$ is dual to $x_1^{i_1} \dots x_k^{i_k}$. He shows that Sq^0 maps $PH_*(B\mathbb{V}_k)$ to itself. (See also [2].) The induced homomorphism, which is also denoted by Sq^0 ,

$$Sq^0 : \mathbb{F}_2 \otimes_{GL_k} PH_*(B\mathbb{V}_k) \rightarrow \mathbb{F}_2 \otimes_{GL_k} PH_*(B\mathbb{V}_k)$$

is called Kameko's squaring operation.

In [9], we consider the homomorphism

$$Sq_D^0 = 1 \otimes_{GL_k} Sq^0 : \mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k) \rightarrow \mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k)$$

and show that it sends the primitive part $P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k))$ to itself. The resulting homomorphism will be redenoted by Sq^0 for short:

$$Sq^0 : P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k)) \rightarrow P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k)).$$

The following theorem, which is a re-statement of Theorem 1.3, is the main result of this section.

Theorem 3.1. *For an arbitrary positive integer k , the squaring operation Sq^0 on $P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k))$ commutes with the classical Sq^0 on $Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ through the Lannes-Zarati homomorphism φ_k . In other words, the following diagram commutes:*

$$\begin{array}{ccc} Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2) & \xrightarrow{\varphi_k} & P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k)) \\ \downarrow Sq^0 & & \downarrow Sq^0 \\ Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2) & \xrightarrow{\varphi_k} & P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k)). \end{array}$$

We will prove this theorem by showing its dual version. To this end, let us consider the dual homomorphism of Kameko's one:

$$Sq_x^0 = Sq_x^0 : \mathbb{F}_2[x_1, \dots, x_k] \rightarrow \mathbb{F}_2[x_1, \dots, x_k],$$

$$Sq_x^0(x_1^{j_1} \cdots x_k^{j_k}) = \begin{cases} x_1^{\frac{j_1-1}{2}} \cdots x_k^{\frac{j_k-1}{2}}, & j_1, \dots, j_k \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

In order to explain the behavior of this homomorphism on modular invariants, we present a homomorphism:

$$Sq_v^0 : \mathbb{F}_2[V_1, \dots, V_k] \rightarrow \mathbb{F}_2[V_1, \dots, V_k],$$

$$Sq_v^0(v_1^{j_1} \cdots v_k^{j_k}) = \begin{cases} v_1^{\frac{j_1-1}{2}} \cdots v_k^{\frac{j_k-1}{2}}, & j_1, \dots, j_k \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, this map coincides with the map in 2.4 on the intersection of their domains.

The two homomorphisms Sq_x^0 and Sq_v^0 depend on k and, when necessary, will respectively be denoted by $Sq_{x,k}^0$ and $Sq_{v,k}^0$.

Technically, the following proposition is the key point in our proof of Theorem 3.1.

Proposition 3.2. *Sq_x^0 coincides with Sq_v^0 on $\mathbb{F}_2[V_1, \dots, V_k]$, for any k .*

This proposition will be shown by means of the following two lemmata, which directly come from the definitions of Sq_x^0 and Sq_v^0 given above.

Lemma 3.3. (i) $Sq_{x,k}^0(ab^2) = Sq_{x,k}^0(a)b$, for any $a, b \in \mathbb{F}_2[x_1, \dots, x_k]$.

(ii) $Sq_{v,k}^0(AB^2) = Sq_{v,k}^0(A)B$, for any $A, B \in \mathbb{F}_2[V_1, \dots, V_k]$.

Lemma 3.4. (i) $Sq_{x,k}^0(ax_k) = Sq_{x,k-1}^0(a)$, for any $a \in \mathbb{F}_2[x_1, \dots, x_{k-1}]$.

(ii) $Sq_{v,k}^0(Av_k) = Sq_{v,k-1}^0(A)$, for any $A \in \mathbb{F}_2[V_1, \dots, V_{k-1}]$.

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2. The proof proceeds by induction on k .

For $k = 1$, since $x_1 = v_1$, we get obviously $Sq_{x,1}^0 = Sq_{v,1}^0$.

Let $k > 1$ and suppose inductively that $Sq_{x,k-1}^0 = Sq_{v,k-1}^0$. We need to show $Sq_{x,k}^0 = Sq_{v,k}^0$. Let $V = V_1^{i_1} \cdots V_k^{i_k}$ be an arbitrary monomial in $M_k = \mathbb{F}_2[V_1, \dots, V_k]$. We consider the following two cases.

Case 1: i_k is even. Recall that

$$V_k = Q_{k-1,0}x_k + Q_{k-1,1}x_k^2 + \cdots + Q_{k-1,k-1}x_k^{2^{k-1}}$$

(see Mùì [22, Appendix]). Since $Q_{k-1,0}, \dots, Q_{k-1,k-1}, V_1, \dots, V_{k-1}$ all do not depend on x_k , we have

$$V = V_1^{i_1} \cdots V_k^{i_k} = \sum_{j_k \text{ even}} x_1^{j_1} \cdots x_k^{j_k},$$

where j_k is even in every monomial of the sum. Therefore, by definition of $Sq_{x,k}^0$,

$$Sq_{x,k}^0(V) = \sum_{j_k \text{ even}} Sq_{x,k}^0(x_1^{j_1} \cdots x_k^{j_k}) = 0.$$

On the other hand, from the expansions of V_i 's in terms of v_j 's, we get

$$V = V_1^{i_1} \cdots V_k^{i_k} = v_1^{\ell_1} \cdots v_k^{\ell_k},$$

where $\ell_k = i_k$ is even. Hence, by definition of $Sq_{v,k}^0$,

$$Sq_{v,k}^0(V) = Sq_{v,k}^0(v_1^{\ell_1} \cdots v_k^{\ell_k}) = 0.$$

Case 2: $i_k = 2n + 1$. We have

$$\begin{aligned} V &= V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} V_k^{i_k} \\ &= V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} (Q_{k-1,0}x_k + Q_{k-1,1}x_k^2 + \cdots + Q_{k-1,k-1}x_k^{2^{k-1}}) V_k^{2n}. \end{aligned}$$

Since $V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0}x_k V_k^{2n}$ is the only term in the above expansion of V with power of x_k odd, we get

$$Sq_{x,k}^0(V) = Sq_{x,k}^0(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0}x_k V_k^{2n}).$$

Note that $V_1, \dots, V_{k-1}, Q_{k-1,0}$ all do not depend on x_k . Combining Lemma 3.3, Lemma 3.4 and the inductive hypothesis, we obtain

$$\begin{aligned} Sq_{x,k}^0(V) &= Sq_{x,k}^0(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0}x_k) V_k^{2n} && \text{(by Lemma 3.3)} \\ &= Sq_{x,k-1}^0(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0}) V_k^{2n} && \text{(by Lemma 3.4)} \\ &= Sq_{v,k-1}^0(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0}) V_k^{2n} && \text{(by the inductive hypothesis)} \\ &= Sq_{v,k}^0(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0}v_k) V_k^{2n} && \text{(by Lemma 3.4)} \\ &= Sq_{v,k}^0(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0}v_k V_k^{2n}) && \text{(by Lemma 3.3)} \\ &= Sq_{v,k}^0(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} V_k^{2n+1}). \end{aligned}$$

The last equality comes from the expansions

$$Q_{k-1,0}v_k = V_1 \cdots V_{k-1}v_k = V_k.$$

The proposition is completely proved.

Now we come back to Theorem 3.1.

Proof of Theorem 3.1. We will show the commutativity of the dual diagram:

$$\begin{array}{ccc} \mathbb{F}_2 \otimes_{\mathcal{A}} D_k & \xrightarrow{\varphi_k^*} & \text{Tor}_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \\ \downarrow Sq_*^0 & & \downarrow Sq_*^0 \\ \mathbb{F}_2 \otimes_{\mathcal{A}} D_k & \xrightarrow{\varphi_k^*} & \text{Tor}_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) . \end{array}$$

This will be obtained from a commutative diagram of appropriate chain-level representations of the homomorphisms in questions.

Indeed, by definition of Sq^0 on $(\mathbb{F}_2 \otimes_{\mathcal{A}} D_k)^* = P(\mathbb{F}_2 \otimes_{GL_k} H_*(B\mathbb{V}_k))$, the restriction of Sq_x^0 on D_k is a chain-level representation of $Sq_*^0 : \mathbb{F}_2 \otimes_{\mathcal{A}} D_k \rightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} D_k$. On the other hand, from 2.4, the map

$$Sq_v^0 : \Gamma_k^\wedge \rightarrow \Gamma_k^\wedge, \\ Sq_v^0(v_1^{j_1} \cdots v_k^{j_k}) = \begin{cases} \Gamma_k^\wedge, & \\ v_1^{\frac{j_1-1}{2}} \cdots v_k^{\frac{j_k-1}{2}}, & j_1, \dots, j_k \text{ odd,} \\ 0, & \text{otherwise.} \end{cases}$$

is a chain-level representation of $Sq_*^0 : \text{Tor}_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \rightarrow \text{Tor}_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. Now, since $D_k \subset M_k = \mathbb{F}_2[V_1, \dots, V_k]$, Proposition 3.2 implies the commutativity of the diagram:

$$\begin{array}{ccc} D_k & \xrightarrow{\subset} & \Gamma_k^\wedge \\ \downarrow Sq_x^0 & & \downarrow Sq_v^0 \\ D_k & \xrightarrow{\subset} & \Gamma_k^\wedge . \end{array}$$

By Theorem 1.4, the inclusion $D_k \subset \Gamma_k^\wedge$ is a chain-level representation of the Lannes-Zarati's dual map φ_k^* . Therefore, the last commutative diagram shows the commutativity of the previous one.

Theorem 3.1 is proved.

4. THE TRIVIALITY OF φ_4

The goal of this section is to prove Theorem 1.2, the main result of this paper.

To this end, we need to recall the computation of $\text{Ext}_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$ by W. H. Lin and that of $\mathbb{F}_2 \otimes_{\mathcal{A}} D_4$ by F. P. Peterson and the author.

Theorem 4.1. (W. H. Lin [18], see also [19, Theorem2.2]). *The following classes form an \mathbb{F}_2 -basis for the vector space of indecomposable elements in $\text{Ext}_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$:*

- (1) $d_i = [(Sq^0)^i(\lambda_6 \lambda_2 \lambda_3^2 + \lambda_4^2 \lambda_3^2 + \lambda_2 \lambda_4 \lambda_5 \lambda_3 + \lambda_1 \lambda_5 \lambda_1 \lambda_7)]$
 $\in \text{Ext}_{\mathcal{A}}^{4, 2^{i+4} + 2^{i+1}}, i \geq 0,$
- (2) $e_i = [(Sq^0)^i(\lambda_8 \lambda_3^3 + \lambda_4(\lambda_5^2 \lambda_3 + \lambda_7 \lambda_3^2) + \lambda_2(\lambda_3 \lambda_5 \lambda_7 + \lambda_1 \lambda_{11} \lambda_3))]$
 $\in \text{Ext}_{\mathcal{A}}^{4, 2^{i+4} + 2^{i+2} + 2^i}, i \geq 0,$

- (3) $f_i = [(Sq^0)^i(\lambda_4\lambda_0\lambda_7^2 + \lambda_3(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_2^2\lambda_7^2)]$
 $\in Ext_{\mathcal{A}}^{4,2^{i+4}+2^{i+2}+2^{i+1}}, i \geq 0,$
- (4) $g_{i+1} = [(Sq^0)^i(\lambda_6\lambda_0\lambda_7^2 + \lambda_5(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_3(\lambda_5\lambda_9\lambda_3 + \lambda_{11}\lambda_3^2))]$
 $\in Ext_{\mathcal{A}}^{4,2^{i+4}+2^{i+3}}, i \geq 0,$
- (5) $p_i = [(Sq^0)^i(\lambda_{14}\lambda_5\lambda_7^2 + \lambda_{10}\lambda_9\lambda_7^2 + \lambda_6\lambda_9\lambda_{11}\lambda_7)]$
 $\in Ext_{\mathcal{A}}^{4,2^{i+5}+2^{i+2}+2^i}, i \geq 0,$
- (6) $D_3(i) = [(Sq^0)^i(\lambda_{22}\lambda_1\lambda_7\lambda_{31} + \lambda_{16}\lambda_7^2\lambda_{31} + \lambda_{14}\lambda_9\lambda_7\lambda_{31} + \lambda_{12}\lambda_{11}\lambda_7\lambda_{31})]$
 $\in Ext_{\mathcal{A}}^{4,2^{i+6}+2^i}, i \geq 0,$
- (7) $p'_i = [(Sq^0)^i(\lambda_0\lambda_{39}\lambda_{15}^2 + \lambda_0\lambda_{15}\lambda_{23}\lambda_{31})]$
 $\in Ext_{\mathcal{A}}^{4,2^{i+6}+2^{i+3}+2^i}, i \geq 0.$

To simplify notation, we will denote $Q_{4,3}^a Q_{4,2}^b Q_{4,1}^c Q_{4,0}^d$ by $Q(a, b, c, d)$ in the following theorem.

Theorem 4.2. (Hirng-Peterson [13]). *The following elements form an \mathbb{F}_2 -basis for the vector space $\mathbb{F}_2 \otimes_{\mathcal{A}} D_4$:*

- (1) $Q(2^s - 1, 0, 0, 0), \quad s \geq 0,$
- (2) $Q(2^r - 2^s - 1, 2^s - 1, 1, 0), \quad r > s > 0,$
- (3) $Q(2^t - 2^r - 1, 2^r - 2^s - 1, 2^s - 1, 2), \quad t > r > s > 1,$
- (4) $Q(2^r - 2^{s+1} - 2^s - 1, 2^s - 1, 2^s - 1, 2), \quad r > s + 1 > 2.$

They are of degrees $2^{s+3} - 8, 2^{r+3} + 2^{s+2} - 6, 2^{t+3} + 2^{r+2} + 2^{s+1} - 4$ and $2^{r+3} + 2^{s+1} - 4$, respectively.

Now we come back to prove Theorem 1.2.

Proof of Theorem 1.2.

In [14], F. Peterson and the author have proved that φ_k vanishes on any decomposable elements for $k > 2$ by showing that $\varphi_* = \bigoplus_k \varphi_k$ is a homomorphism of algebras and, more importantly, that the product of the algebra $\bigoplus_k (\mathbb{F}_2 \otimes_{\mathcal{A}} D_k)^*$ is trivial, except for the case

$$(\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \otimes (\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \rightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} D_2)^*.$$

Therefore, we need only to show φ_4 vanishing on any indecomposable elements.

Let a_0 denote one of the seven generators

$$d_0, e_0, f_0, g_1, p_0, D_3(0), p'_0,$$

each of which is the element of lowest stem in its own family. Furthermore, set $a_i = (Sq^0)^i(a_0)$, for $i \geq 0$. From Theorem 3.1, we have

$$\varphi_4(a_i) = \varphi_4(Sq^0)^i(a_0) = (Sq^0)^i \varphi_4(a_0).$$

So, in order to prove that $\varphi_4(a_i) = 0$ for any i , it suffices to show $\varphi_4(a_0) = 0$. We will do this by checking that the stem of a_0 is different from degrees of all the generators of $\mathbb{F}_2 \otimes_{\mathcal{A}} D_4$ given in Theorem 4.2.

Now let us check it case by case.

Case 1: For d_0 of stem $2^4 + 2^1 - 4 = 14$,
 $2^{s+3} = 8 + 14 = 22$, no solution;
 $2^{r+3} + 2^{s+2} = 6 + 14 = 16 + 4$, $r = 1, s = 0$, it does not satisfy $s > 0$;
 $2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 14 = 16 + 2$, no solution;
 $2^{r+3} + 2^{s+1} = 4 + 14 = 16 + 2$, $r = 1, s = 0$ it does not satisfy $r > s + 1 > 2$.

Case 2: For e_0 of stem $2^4 + 2^2 + 2^0 - 4 = 17$,
 $2^{s+3} = 8 + 17 = 25$, no solution;
 $2^{r+3} + 2^{s+2} = 6 + 17 = 16 + 4 + 2 + 1$, no solution;
 $2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 17 = 16 + 4 + 1$, no solution;
 $2^{r+3} + 2^{s+1} = 4 + 17 = 16 + 4 + 1$, no solution.

Case 3: For f_0 of stem $2^4 + 2^2 + 2^1 - 4 = 18$,
 $2^{s+3} = 8 + 18 = 26$, no solution;
 $2^{r+3} + 2^{s+2} = 6 + 18 = 16 + 8$, $r = 1, s = 1$ it does not satisfy $r > s$;
 $2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 18 = 16 + 4 + 2$, $t = 1, r = s = 0$ it does not satisfy
 $r > s > 1$;
 $2^{r+3} + 2^{s+1} = 4 + 18 = 16 + 4 + 2$, no solution.

Case 4: For g_1 of stem $2^4 + 2^3 - 4 = 20$,
 $2^{s+3} = 8 + 20 = 28$, no solution;
 $2^{r+3} + 2^{s+2} = 6 + 20 = 16 + 8 + 2$, no solution;
 $2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 20 = 16 + 8$, no solution;
 $2^{r+3} + 2^{s+1} = 4 + 20 = 16 + 8$, $r = 1, s = 2$, it does not satisfy $r > s + 1 > 2$.

Case 5: For p_0 of stem $2^5 + 2^2 + 2^0 - 4 = 33$,
 $2^{s+3} = 8 + 33 = 41$, no solution;
 $2^{r+3} + 2^{s+2} = 6 + 33 = 32 + 4 + 2 + 1$, no solution;
 $2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 33 = 32 + 4 + 1$, no solution;
 $2^{r+3} + 2^{s+1} = 4 + 33 = 32 + 4 + 1$, no solution.

Case 6: For $D_3(0)$ of stem $2^6 + 2^0 - 4 = 61$,
 $2^{s+3} = 8 + 61 = 69$, no solution;
 $2^{r+3} + 2^{s+2} = 6 + 61 = 64 + 2 + 1$, no solution;
 $2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 61 = 64 + 1$, no solution;
 $2^{r+3} + 2^{s+1} = 4 + 61 = 64 + 1$, no solution.

Case 7: For p'_0 of stem $2^6 + 2^3 + 2^0 - 4 = 69$,
 $2^{s+3} = 8 + 69 = 77$, no solution;
 $2^{r+3} + 2^{s+2} = 6 + 69 = 64 + 8 + 2 + 1$, no solution;
 $2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 69 = 64 + 8 + 1$, no solution;
 $2^{r+3} + 2^{s+1} = 4 + 69 = 64 + 8 + 1$, no solution.

Therefore, φ_4 vanishes on any indecomposable elements.

In summary, $\varphi_4 = 0$ in positive stems. Theorem 1.2 is completely proved.

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