

# THE WEAK CONJECTURE ON SPHERICAL CLASSES

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ABSTRACT. Let  $\mathcal{A}$  be the mod 2 Steenrod algebra. We construct a chain-level representation of the dual of Singer's algebraic transfer,  $Tr_k^* : Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2) \rightarrow \mathbf{F}_2 \otimes \mathbf{F}_2[x_1, \dots, x_k]$ , which maps Singer's invariant-theoretic model of the dual of the Lambda algebra,  $\Gamma_k^{\wedge}$ , to  $\mathbf{F}_2[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$  and is the inclusion of the Dickson algebra,  $D_k \subset \Gamma_k^{\wedge}$ , into  $\mathbf{F}_2[x_1, \dots, x_k]$ . This chain-level representation allows us to confirm the weak conjecture on spherical classes (see [9]), assuming the truth of (1) either the conjecture that *the Dickson invariants of at least  $k = 3$  variables are homologically zero in  $Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$* , (2) or a conjecture on  *$\mathcal{A}$ -decomposability of the Dickson algebra in  $\Gamma_k^{\wedge}$* . We prove the conjecture in item (1) for  $k = 3$  and also show a weak form of the conjecture in item (2).

## 1. INTRODUCTION AND STATEMENT OF RESULTS

This paper continues our study of spherical classes that started in [9]. To make the paper self-contained, we first recall certain background of the classical conjecture on spherical classes, which has been given in the introduction of our paper [9].

We are interested in the following conjecture on spherical classes in  $Q_0S^0$ , i.e. elements belonging to the image of the Hurewicz homomorphism

$$H : \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \rightarrow H_*(Q_0S^0).$$

Here and throughout the paper, the coefficient ring for homology and cohomology is always  $\mathbf{F}_2$ , the field of two elements.

**Conjecture 1.1.** (conjecture on spherical classes). *There are no spherical classes in  $Q_0S^0$ , except the Hopf invariant one and the Kervaire invariant one elements.*

Some topologists believe that the conjecture is due to I. Madsen, while some others say it is due to E. Curtis. (See Curtis [6] and Wellington [24] for a discussion.)

Let  $E^k$  be an elementary abelian 2-group of rank  $k$ . It is also viewed as a  $k$ -dimensional vector space over  $\mathbf{F}_2$ . So, the general linear group  $GL_k = GL(k, \mathbf{F}_2)$  acts on  $E^k$  and therefore on  $H^*(BE^k)$  in the usual way. Let  $D_k$  be the Dickson algebra of  $k$  variables, i.e. the algebra of invariants

$$D_k := H^*(BE^k)^{GL_k} \cong \mathbf{F}_2[x_1, \dots, x_k]^{GL_k},$$

where  $P_k = \mathbf{F}_2[x_1, \dots, x_k]$  is the polynomial algebra on  $k$  generators  $x_1, \dots, x_k$ , each of dimension 1. As the action of the (mod 2) Steenrod algebra,  $\mathcal{A}$ , and that of  $GL_k$  on  $P_k$  commute with each other,  $D_k$  is an algebra over  $\mathcal{A}$ .

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One way to attack Conjecture 1.1 is to study the Lannes–Zarati homomorphism

$$\varphi_k : Ext_{\mathcal{A}}^{k, k+i}(\mathbf{F}_2, \mathbf{F}_2) \rightarrow (\mathbf{F}_2 \otimes_{\mathcal{A}} D_k)_i^*,$$

which is compatible with the Hurewicz homomorphism (see [13], [14, p. 46]). The domain of  $\varphi_k$  is the  $E_2$ -term of the Adams spectral sequence converging to  $\pi_*^s(S^0) \cong \pi_*(Q_0S^0)$ . According to Madsen’s theorem [16], which asserts that  $D_k$  is dual to the coalgebra of Dyer–Lashof operations of length  $k$ , the range of  $\varphi_k$  is a submodule of  $H_*(Q_0S^0)$ . By compatibility of  $\varphi_k$  and the Hurewicz homomorphism we mean  $\varphi_k$  is a “lifting” of the latter from the “ $E_\infty$ -level” to the “ $E_2$ -level”.

The Hopf invariant one and the Kervaire invariant one elements are respectively represented by certain permanent cycles in  $Ext_{\mathcal{A}}^{1,*}(\mathbf{F}_2, \mathbf{F}_2)$  and  $Ext_{\mathcal{A}}^{2,*}(\mathbf{F}_2, \mathbf{F}_2)$ , on which  $\varphi_1$  and  $\varphi_2$  are non-zero (see Adams [1], Browder [5], Lannes–Zarati [14]).

Therefore, Conjecture 1.1 is a consequence of the following one.

**Conjecture 1.2.**  $\varphi_k = 0$  in any positive stem  $i$  for  $k > 2$ .

It is well known that the Ext group has intensively been studied, but remains very mysterious. In order to avoid the shortage of our knowledge of the Ext group, we want to restrict  $\varphi_k$  to a certain subgroup of the Ext group which (1) is large enough and worthwhile to pursue and (2) could be handled more easily than the Ext group itself. To this end, we combine the above data with Singer’s algebraic transfer.

Singer defined in [22] the algebraic transfer

$$Tr_k : \mathbf{F}_2 \otimes_{GL_k} PH_i(BE^k) \rightarrow Ext_{\mathcal{A}}^{k, k+i}(\mathbf{F}_2, \mathbf{F}_2),$$

where  $PH_*(BE^k)$  denotes the submodule consisting of all  $\mathcal{A}$ -annihilated elements in  $H_*(BE^k)$ . It is shown to be an isomorphism for  $k \leq 2$  by Singer [22] and for  $k = 3$  by Boardman [3]. Singer also proved in [22] that it is *not* an isomorphism for  $k = 5$ , and conjectured that  $Tr_k$  is a *monomorphism* for any  $k$ .

Restricting  $\varphi_k$  to the image of  $Tr_k$ , we stated in [9] the following conjecture.

**Conjecture 1.3.** (weak conjecture on spherical classes).

$$\varphi_k \cdot Tr_k : \mathbf{F}_2 \otimes_{GL_k} PH_*(BE^k) \rightarrow P(\mathbf{F}_2 \otimes_{GL_k} H_*(BE^k)) := (\mathbf{F}_2 \otimes_{\mathcal{A}} D_k)^*$$

is zero in positive dimensions for  $k > 2$ .

In other words, there are no spherical classes in  $Q_0S^0$ , which can be detected by the algebraic transfer, except the Hopf invariant one and the Kervaire invariant one elements.

In [9], we have proved that *the inclusion of  $D_k$  into  $P_k$  is a chain-level representation of  $Tr_k^* \cdot \varphi_k^* : \mathbf{F}_2 \otimes_{\mathcal{A}} D_k \rightarrow (\mathbf{F}_2 \otimes_{\mathcal{A}} P_k)^{GL_k}$* . So, we get the following result.

**Theorem 1.4.** *The weak conjecture on spherical classes is equivalent to the conjecture that the homomorphism*

$$j_k : \mathbf{F}_2 \otimes_{\mathcal{A}} (P_k^{GL_k}) \rightarrow (\mathbf{F}_2 \otimes_{\mathcal{A}} P_k)^{GL_k}$$

*induced by the identity map on  $P_k$  is zero in positive dimensions for  $k > 2$ .*

Let  $D_k^+, \mathcal{A}^+$  be respectively the submodules of  $D_k$  and  $\mathcal{A}$  consisting of all elements of positive dimensions. Then the above conjecture on  $j_k$  can equivalently be stated as follows.

**Conjecture 1.5.** *If  $k > 2$ , then*

$$D_k^+ \subset \mathcal{A}^+ \cdot P_k.$$

In [9], we have got a proof of this conjecture for  $k = 3$ .

To introduce a new approach, we need to summarize Singer's invariant-theoretic description of the lambda algebra [21]. According to Dickson [7], one has

$$D_k \cong \mathbf{F}_2[Q_{k,k-1}, \dots, Q_{k,0}],$$

where  $Q_{k,i}$  denotes the Dickson invariant of dimension  $2^k - 2^i$ . Singer set  $\Gamma_k = D_k[Q_{k,0}^{-1}]$ , the localization of  $D_k$  given by inverting  $Q_{k,0}$ , and defined  $\Gamma_k^\wedge$  to be a certain "not too large" submodule of  $\Gamma_k$ . He also equipped  $\Gamma^\wedge = \bigoplus_k \Gamma_k^\wedge$  with a differential  $\partial : \Gamma_k^\wedge \rightarrow \Gamma_{k-1}^\wedge$  and a coproduct. Then, he showed that the differential coalgebra  $\Gamma^\wedge$  is dual to the lambda algebra of the six authors of [4]. Thus,  $H_k(\Gamma^\wedge) \cong \text{Tor}_k^A(\mathbf{F}_2, \mathbf{F}_2)$ . (Originally, Singer used the notation  $\Gamma_k^+$  to denote  $\Gamma_k^\wedge$ . However, by  $D_k^+$ ,  $\mathcal{A}^+$  we always mean the submodules of  $D_k$  and  $\mathcal{A}$  respectively consisting of all elements of positive dimensions, so Singer's notation  $\Gamma_k^+$  would make a confusion in this paper. Therefore, we prefer the notation  $\Gamma_k^\wedge$  to  $\Gamma_k^+$ .)

One of the main results of this paper is to construct a homomorphism  $\mathcal{T}_k : \Gamma_k^\wedge \rightarrow \mathbf{F}_2[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$  with the following properties.

**Theorem 3.2** *The homomorphism  $\mathcal{T}_k : \Gamma_k^\wedge \rightarrow \mathbf{F}_2[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$  maps the submodule of all cycles in  $\Gamma_k^\wedge$  to  $P_k$  and is a chain-level representation of  $\text{Tr}_k^* : \text{Tor}_k^A(\mathbf{F}_2, \mathbf{F}_2) \rightarrow \mathbf{F}_2 \otimes_A P_k$ . Moreover, its restriction to  $D_k \subset \Gamma_k^\wedge$  is the inclusion of  $D_k$  into  $P_k$ .*

Note that every  $q \in D_k^+$  is a cycle in the differential module  $\Gamma_k^\wedge$ . By means of this theorem, the weak conjecture on spherical classes is an immediate consequence of the following conjecture.

**Conjecture 1.6.** *If  $q \in D_k^+$ , then  $[q] = 0$  in  $\text{Tor}_k^A(\mathbf{F}_2, \mathbf{F}_2)$  for  $k > 2$ .*

This is a corollary of the following conjecture. (See Proposition 4.2 for a proof.)

**Conjecture 4.1** *Let  $\text{Ker} \partial_k$  be the submodule of all cycles in  $\Gamma_k^\wedge$ . Then, for  $k > 2$ ,*

$$D_k^+ \subset \mathcal{A}^+ \cdot \text{Ker} \partial_k.$$

We get the following result, which is weaker than the above conjecture.

**Theorem 4.3** *For  $k > 2$ ,*

$$D_k^+ \subset \mathcal{A}^+ \cdot \Gamma_k^\wedge.$$

We also have the following theorem.

**Theorem 4.8** *Conjecture 1.6 is true for  $k = 3$ .*

Conjecture 1.5 is related to the difficult problem of determination of  $\mathbf{F}_2 \otimes_A P_k$ . This problem has first been studied by F. Peterson [18], R. Wood [26], W. Singer [22], S. Priddy [19]... who show its relationships to several classical problems in Homotopy Theory.  $\mathbf{F}_2 \otimes_A P_k$  has explicitly been computed for  $k \leq 3$ . The cases  $k = 1$  and

2 are not difficult, while the case  $k = 3$  is very complicated and was solved by M. Kameko [12]. There is also another approach, the qualitative one, to the problem. By this we mean giving conditions on elements of  $P_k$  to show that they go to zero in  $\mathbf{F}_2 \otimes P_k$ , i. e. belong to  $\mathcal{A}^+ \cdot P_k$ . Peterson's conjecture, which was established by Wood [26], claims that  $\mathbf{F}_2 \otimes P_k = 0$  in dimension  $d$  such that  $\alpha(d+k) > k$ . Here  $\alpha(n)$  denotes the number of ones in the dyadic expansion of  $n$ . Recently, W. Singer, K. Monks, J. Silverman... have refined the method of R. Wood to show that many more monomials in  $P_k$  are in  $\mathcal{A}^+ \cdot P_k$ . (See Silverman [20] and its references.) *Conjecture 1.5 presents a large family, whose elements are predicted to be in  $\mathcal{A}^+ \cdot P_k$ .*

The paper is organized as follows. Section 2 is a preliminary on invariant theory and Singer's algebraic transfer. Sections 3 and 4 are respectively devoted to prove Theorems 3.2, 4.3 and 4.8. Finally, in Section 5, we propose a way to prove the classical conjecture on spherical classes.

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## 2. PRELIMINARY

This section begins with a few words on invariant theory and ends with a brief sketch of Singer's definition of the algebraic transfer.

Let  $T_k$  be the Sylow 2-subgroup of  $GL_k$  consisting of all upper triangular  $k \times k$ -matrices with 1 on the main diagonal. The  $T_k$ -invariant ring,  $M_k = P_k^{T_k}$ , is called the Mui algebra. In [17], Mui showed that

$$P_k^{T_k} = \mathbf{F}_2[V_1, \dots, V_k],$$

where

$$V_i = \prod_{\lambda_j \in \mathbf{F}_2} (\lambda_1 x_1 + \dots + \lambda_{i-1} x_{i-1} + x_i).$$

Then, the Dickson invariant  $Q_{k,i}$  can inductively be defined by

$$Q_{k,i} = Q_{k-1,i-1}^2 + V_k \cdot Q_{k-1,i},$$

where, by convention,  $Q_{k,k} = 1$  and  $Q_{k,i} = 0$  for  $i < 0$ .

Let  $S(k) \subset P_k$  be the multiplicative subset generated by all the non-zero linear forms in  $P_k$ . Let  $\Phi_k$  be the localization:  $\Phi_k = (P_k)_{S(k)}$ . Using the results of Dickson [7] and Mui [17], Singer noted in [21] that

$$\Delta_k := (\Phi_k)^{T_k} = \mathbf{F}_2[V_1^{\pm 1}, \dots, V_k^{\pm 1}],$$

$$\Gamma_k := (\Phi_k)^{GL_k} = \mathbf{F}_2[Q_{k,k-1}, \dots, Q_{k,1}, Q_{k,0}^{\pm 1}].$$

Further, he set

$$v_1 = V_1, \quad v_k = V_k / V_1 \cdots V_{k-1} \quad (k \geq 2),$$

so that

$$V_k = v_1^{2^{k-2}} v_2^{2^{k-3}} \cdots v_{k-1} v_k \quad (k \geq 2).$$

Then, he obtained

$$\Delta_k = \mathbf{F}_2[v_1^{\pm 1}, \dots, v_k^{\pm 1}],$$

with  $\dim v_i = 1$  for every  $i$ .

Singer defined  $\Gamma_k^\wedge$  to be the submodule of  $\Gamma_k = D_k[Q_{k,0}^{-1}]$  spanned by all monomials  $\gamma = Q_{k,k-1}^{i_{k-1}} \cdots Q_{k,0}^{i_0}$  with  $i_{k-1}, \dots, i_1 \geq 0$ ,  $i_0 \in \mathbf{Z}$ , and  $i_0 + \dim \gamma \geq 0$ .

In the remaining part of this section, we briefly sketch Singer's definition of  $Tr_k^*$ .

Let  $P_1 = \mathbf{F}_2[x]$  with  $|x| = 1$ . Let  $\hat{P} \subset \mathbf{F}_2[x, x^{-1}]$  be the submodule spanned by all powers  $x^i$  with  $i \geq -1$ . The canonical  $\mathcal{A}$ -action on  $P_1$  is extended to an  $\mathcal{A}$ -action on  $\mathbf{F}_2[x, x^{-1}]$  (see Adams [2], Wilkerson [25]).  $\hat{P}$  is an  $\mathcal{A}$ -submodule of  $\mathbf{F}_2[x, x^{-1}]$ . We have a short-exact sequence of  $\mathcal{A}$ -modules

$$\mathbf{2.1.} \quad 0 \rightarrow P_1 \xrightarrow{\iota} \hat{P} \xrightarrow{\pi} \mathbf{F}_2 \rightarrow 0,$$

where  $\iota$  is the inclusion and  $\pi$  is given by  $\pi(x^i) = 0$  if  $i \neq -1$  and  $\pi(x^{-1}) = 1$ . Let  $e_1$  be the corresponding element in  $Ext_{\mathcal{A}}^1(\mathbf{F}_2, P_1)$ . Then, Singer defined

$$\mathbf{2.2.} \quad e_k := e_1 \otimes \cdots \otimes e_1 \in Ext_{\mathcal{A}}^k(\mathbf{F}_2, P_k) \quad (k \text{ times}).$$

Now,  $Tr_k^* : Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2) \rightarrow Tor_0^{\mathcal{A}}(\mathbf{F}_2, P_k) \cong \mathbf{F}_2 \otimes_{\mathcal{A}} P_k$  is defined by

$$\mathbf{2.3.} \quad Tr_k^*(z) := e_k \cap z.$$

Note that, since  $\pi$  raises internal degree by one,  $Tr_k^*$  lowers it by  $k$ .

We need to relate  $Tr_k^*$  with connecting homomorphisms. Let  $N, P, Q, R$  be (left)  $\mathcal{A}$ -modules. From MacLane [15, p. 229], one has

$$f \otimes g = (f \otimes R) \circ (N \otimes g)$$

for  $f \in Ext_{\mathcal{A}}^*(N, P), g \in Ext_{\mathcal{A}}^*(Q, R)$ . Hence

$$\mathbf{2.4.} \quad e_k = (e_1 \otimes P_{k-1}) \circ \cdots \circ (e_1 \otimes P_1) \circ e_1.$$

Cap and Yoneda products are related by the formula

$$(h \circ f) \cap z = h \cap (f \cap z)$$

for  $z \in Tor_*^{\mathcal{A}}(M, N), f \in Ext_{\mathcal{A}}^*(N, P), h \in Ext_{\mathcal{A}}^*(P, Q)$ .

Suppose  $f \in Ext_{\mathcal{A}}^1(N_3, N_1)$  is represented by the short-exact sequence of left  $\mathcal{A}$ -modules  $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$ . Let  $\Delta(f) : Tor_s^{\mathcal{A}}(M, N_3) \rightarrow Tor_{s-1}^{\mathcal{A}}(M, N_1)$  be the connecting homomorphism associated with this short-exact sequence, for any right  $\mathcal{A}$ -module  $M$ . Then one verifies easily

$$\Delta(f)(z) = f \cap z$$

for any  $z \in Tor_s^{\mathcal{A}}(M, N_3)$ .

Consequently, we get

$$\mathbf{2.5.} \quad Tr_k^* = \Delta(e_1 \otimes P_{k-1}) \circ \cdots \circ \Delta(e_1 \otimes P_1) \circ \Delta e_1.$$

(See Singer [22, p. 498].)

Now we prepare to construct a chain-level representation of  $Tr_k^*$ .

For every left  $\mathcal{A}$ -module  $N$ , Singer defined in [21] a chain complex  $\Gamma^\wedge N$ : It is given in homological dimension  $k$  by  $(\Gamma^\wedge N)_k = \Gamma_k^\wedge \otimes N$  as an  $\mathbf{F}_2$ -vector space and its differential  $\partial : (\Gamma^\wedge N)_k \rightarrow (\Gamma^\wedge N)_{k-1}$  is defined by

$$\partial(v_1^{a_1} \cdots v_k^{a_k} \otimes y) := (v_1^{a_1} \cdots v_{k-1}^{a_{k-1}} \otimes Sq^{a_k+1}(y)),$$

for  $y \in N$ . Then he proved that

$$H_k(\Gamma^\wedge N) \cong Tor_k^{\mathcal{A}}(\mathbf{F}_2, N).$$

This isomorphism is natural in  $N$ . Further, if  $(f) : 0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$  is any short-exact sequence of left  $\mathcal{A}$ -modules, then  $0 \rightarrow \Gamma^\wedge N_1 \rightarrow \Gamma^\wedge N_2 \rightarrow \Gamma^\wedge N_3 \rightarrow 0$  is a short-exact sequence of chain complexes, from which a chain-level representation of the connecting homomorphism  $\Delta(f) : Tor_k^{\mathcal{A}}(\mathbf{F}_2, N_3) \rightarrow Tor_{k-1}^{\mathcal{A}}(\mathbf{F}_2, N_1)$  can be computed.

### 3. A CHAIN-LEVEL REPRESENTATION OF THE DUAL OF THE ALGEBRAIC TRANSFER

Following Singer [21], the homomorphism

$$\begin{aligned} \partial_k : \mathbf{F}_2[v_1^{\pm 1}, \dots, v_k^{\pm 1}] &\rightarrow \mathbf{F}_2[v_1^{\pm 1}, \dots, v_{k-1}^{\pm 1}] \\ \partial_k(v_1^{a_1} \cdots v_k^{a_k}) &:= \begin{cases} v_1^{a_1} \cdots v_{k-1}^{a_{k-1}} & \text{if } a_k = -1, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

maps  $\Gamma_k^\wedge$  to  $\Gamma_{k-1}^\wedge$ . Moreover, it is a differential on  $\Gamma^\wedge = \bigoplus_k \Gamma_k^\wedge$  with  $H_*(\Gamma^\wedge) \cong \text{Tor}_*^{\mathbf{A}}(\mathbf{F}_2, \mathbf{F}_2)$ . This is an isomorphism of bigraded modules. Here the bidegree of  $\Gamma^\wedge$  is given by

$$\text{bideg}(v_1^{a_1} \cdots v_k^{a_k}) := (k, k + \sum a_i).$$

Note that, for every  $q \in D_k$ , its expansion

$$q = \sum_{(a_1, \dots, a_k)} v_1^{a_1} \cdots v_k^{a_k}$$

always has  $a_k \geq 0$  in any term of the sum. Therefore,  $\partial_k(q) = 0$ . This means that every Dickson invariant is a cycle in the complex  $\Gamma^\wedge$ , whose homology is  $\text{Tor}_*^{\mathbf{A}}(\mathbf{F}_2, \mathbf{F}_2)$ .

Now we construct a homomorphism, denoted  $\mathcal{T}_k$ , as follows.

**Definition 3.1.** *The homomorphism  $\mathcal{T}_k : \mathbf{F}_2[v_1^{\pm 1}, \dots, v_k^{\pm 1}] \rightarrow \mathbf{F}_2[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$  is defined by*

$$\mathcal{T}_k(v_1^{a_1} \cdots v_k^{a_k}) := Sq^{a_1+1}(x_1^{-1} \cdots Sq^{a_{k-1}+1}(x_{k-1}^{-1} Sq^{a_k+1}(x_k^{-1})) \cdots),$$

where  $a_1, \dots, a_k$  are arbitrary integers. Here, we mean  $Sq^i = 0$  for  $i < 0$ .

Let  $\text{Ker} \partial_k$  be the submodule of all cycles in  $\Gamma_k^\wedge$ . Then, the goal of this section is to prove the following theorem.

**Theorem 3.2.** *The homomorphism  $\mathcal{T}_k$  satisfies the following two properties:*

- (i)  $\mathcal{T}_k(\text{Ker} \partial_k) \subset P_k$ . Furthermore,  $\mathcal{T}_k|_{\Gamma_k^\wedge} : \Gamma_k^\wedge \rightarrow \mathbf{F}_2[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$  is a chain-level representation of

$$\text{Tr}_k^* : H_k(\Gamma^\wedge) \cong \text{Tor}_k^{\mathbf{A}}(\mathbf{F}_2, \mathbf{F}_2) \rightarrow \mathbf{F}_2 \otimes_{\mathbf{A}} P_k.$$

- (ii)  $\mathcal{T}_k|_{D_k}$  is the inclusion of  $D_k$  into  $P_k$ .

The first part of Theorem 3.2 is shown by the following lemma.

**Lemma 3.3.**  *$\mathcal{T}_k$  maps  $\text{Ker} \partial_k$  to  $P_k$  and is a chain-level representation of  $\text{Tr}_k^*$ .*

*Proof.* In order to construct a chain-level representation for  $\text{Tr}_k^*$ , following 2.5, we first construct a chain-level representation for

$$\Delta(e_1) : \text{Tor}_k^{\mathbf{A}}(\mathbf{F}_2, \mathbf{F}_2) \rightarrow \text{Tor}_{k-1}^{\mathbf{A}}(\mathbf{F}_2, P_1),$$

the connecting homomorphism associated with the short-exact sequence 2.1. Let us consider the induced short-exact sequence

$$0 \rightarrow \Gamma^\wedge P_1 \xrightarrow{\iota} \Gamma^\wedge \hat{P} \xrightarrow{\pi} \Gamma^\wedge \mathbf{F}_2 \rightarrow 0.$$

A lifting of a cycle  $z \in \Gamma^\wedge = \Gamma^\wedge \mathbf{F}_2$  over the chain map  $\pi$  is given by the chain  $z \otimes x_k^{-1} \in \Gamma^\wedge \hat{P}$ , where we are writing  $P_1 = \mathbf{F}_2[x_k]$ ,  $\hat{P} = \text{Span}\{x_k^i | i \geq -1\}$ . The

boundary  $\partial(z \otimes x_k^{-1})$  pulls back  $z \otimes x_k^{-1}$  under  $\iota$  to a cycle in  $\Gamma^\wedge P_1$ , which represents  $\Delta(e_1)(z)$  in  $Tor_{k-1}^A(\mathbf{F}_2, P_1)$ . If  $z = \sum v_1^{a_1} \cdots v_k^{a_k}$ , then by (4.1) of [21],

$$\partial(z \otimes x_k^{-1}) = \partial\left(\sum v_1^{a_1} \cdots v_k^{a_k} \otimes x_k^{-1}\right) = \sum v_1^{a_1} \cdots v_{k-1}^{a_{k-1}} \otimes Sq^{a_k+1}(x_k^{-1}).$$

This means that the map given by

$$v_1^{a_1} \cdots v_k^{a_k} \mapsto v_1^{a_1} \cdots v_{k-1}^{a_{k-1}} \otimes Sq^{a_k+1}(x_k^{-1})$$

is a chain-level representation of  $\Delta(e_1)$ .

We similarly construct a chain-level representation for the homomorphism  $\Delta(e_1 \otimes P_1) : Tor_{k-1}^A(\mathbf{F}_2, P_1) \rightarrow Tor_{k-2}^A(\mathbf{F}_2, P_2)$ , which is the connecting homomorphism associated with the short-exact sequence of chain complexes

$$0 \rightarrow \Gamma^\wedge P_2 \xrightarrow{\iota \otimes P_1} \Gamma^\wedge(\hat{P} \otimes P_1) \xrightarrow{\pi \otimes P_1} \Gamma^\wedge P_1 \rightarrow 0.$$

Here we are writing  $P_1 = \mathbf{F}_2[x_k]$ ,  $P_2 = \mathbf{F}_2[x_{k-1}, x_k]$  and  $\hat{P} = \text{Span}\{x_{k-1}^i \mid i \geq -1\}$ . By the argument similar to the one given above, the map

$$v_1^{a_1} \cdots v_{k-1}^{a_{k-1}} \otimes x_k^i \mapsto \partial(v_1^{a_1} \cdots v_{k-1}^{a_{k-1}} \otimes x_{k-1}^{-1} x_k^i) = v_1^{a_1} \cdots v_{k-2}^{a_{k-2}} \otimes Sq^{a_{k-1}+1}(x_{k-1}^{-1} x_k^i)$$

is a chain-level representation of  $\Delta(e_1 \otimes P_1)$ . So the composite map

$$v_1^{a_1} \cdots v_k^{a_k} \mapsto v_1^{a_1} \cdots v_{k-2}^{a_{k-2}} \otimes Sq^{a_{k-1}+1}\left(x_{k-1}^{-1} Sq^{a_k+1}(x_k^{-1})\right)$$

sends a cycle in  $\Gamma^\wedge P_1$  to a cycle in  $\Gamma^\wedge P_2$  and it is a chain-level representation of the composite homomorphism  $\Delta(e_1 \otimes P_1) \circ \Delta(e_1)$ .

By repeating the above argument, it is easy to see that the map  $\mathcal{T}_k$  given by Definition 3.1 is a chain-level representation of  $Tr_k^* = \Delta(e_1 \otimes P_{k-1}) \circ \cdots \circ \Delta(e_1 \otimes P_1) \circ \Delta(e_1)$ . In particular, we have

$$\mathcal{T}_k(Ker \partial_k) \subset \Gamma_0^\wedge P_k = P_k.$$

For the convenience of the latter use,  $\mathcal{T}_k$  is thought of as a morphism from  $\mathbf{F}_2[v_1^{\pm 1}, \dots, v_k^{\pm 1}]$  to  $\mathbf{F}_2[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$ .

The lemma follows.

**Remark 3.4.** Since the chain-level representation  $\mathcal{T}_k$  has been extended to a homomorphism, whose domain is  $\mathbf{F}_2[v_1^{\pm 1}, \dots, v_k^{\pm 1}]$ , the computation of  $\mathcal{T}_k(V_i)$  makes sense as one can see in Lemmas 3.7 and 3.8 below.

The second part of Theorem 3.2 is proved by a number of lemmata.

Let  $M$  be a graded  $\mathcal{A}$ -module, which is concentrated in non-negative dimensions. We are concerned with the Steenrod homomorphism  $d^*P : M \rightarrow \mathbf{F}_2[x^{\pm 1}] \otimes M$  given by

$$d^*P(u) = d^*P_x(u) = \sum_{i=0}^{|u|} x^{|u|-i} \otimes Sq^i(u),$$

where  $u \in M$ ,  $x$  is of dimension 1 and  $\mathbf{F}_2[x^{\pm 1}]$  supports the canonical  $\mathcal{A}$ -action as discussed in Section 2. Actually, Steenrod defined his  $Sq^i$  by using  $d^*P$ , which is constructed by means of cohomology of the weath products (see [23, Ch. VII]).

Suppose additionally that  $M$  is an  $\mathcal{A}$ -algebra. Then, by the Cartan formula,  $d^*P$  is a homomorphism of algebras.

**Lemma 3.5.** *Let  $M$  be a unstable  $\mathcal{A}$ -algebra. Suppose  $\alpha, \beta \in M$ , and  $a \geq |\alpha|, b \geq |\beta|$ . Then*

$$Sq^{a+b+1}(x^{-1} \otimes \alpha\beta) = Sq^{a+1}(x^{-1} \otimes \alpha)Sq^{b+1}(x^{-1} \otimes \beta).$$

*Proof.* Recall that  $Sq^i(x^{-1}) = x^{i-1}$  for every integer  $i$ . From the unstability of  $M$  and  $a \geq |\alpha|$ , one gets

$$Sq^{a+1}(x^{-1} \otimes \alpha) = \sum_{i=0}^{|\alpha|} x^{a-i} \otimes Sq^i(\alpha) = x^{a-|\alpha|} d^* P(\alpha).$$

As  $d^*P$  is an algebra homomorphism, the lemma follows.

Applying repeatedly this lemma to  $M = P_k$ , we obtain

**Lemma 3.6.** *Suppose  $v = v_1^{a_1} \cdots v_k^{a_k}$  and  $v' = v_1^{b_1} \cdots v_k^{b_k}$  satisfy the conditions:*

$$a_i \geq a_{i+1} + \cdots + a_k \geq 0, \quad b_i \geq b_{i+1} + \cdots + b_k \geq 0,$$

for  $1 \leq i \leq k$ . Then

$$\mathcal{T}_k(v \cdot v') = \mathcal{T}_k(v) \cdot \mathcal{T}_k(v').$$

Furthermore, the both sides are elements of  $P_k$ .

*Proof.* Set  $\tau_i(v) = Sq^{a_i+1}(x_i^{-1} \cdots Sq^{a_k+1}(x_k^{-1}))$ . It is easy to see that  $\tau_k(vv') = \tau_k(v)\tau_k(v')$  in  $P_k$ . Suppose inductively that  $\tau_{i+1}(vv') = \tau_{i+1}(v)\tau_{i+1}(v')$  in  $P_k$ . Then, applying Lemma 3.5 with  $a = a_i, b = b_i, \alpha = \tau_{i+1}(v), \beta = \tau_{i+1}(v')$ , we get  $\tau_i(vv') = \tau_i(v)\tau_i(v')$  in  $P_k$ .

Thus, the lemma follows as  $\mathcal{T}_k = \tau_1$ .

**Lemma 3.7.** *The restriction of  $\mathcal{T}_k$  to the Mùì algebra  $M_k = \mathbf{F}_2[V_1, \dots, V_k]$  is a homomorphism of algebras.*

*Proof.* Recall that  $V_i = v_1^{2^{i-2}} v_2^{2^{i-3}} \cdots v_{i-1} v_i$ . So, one easily verifies that every element  $v \in M_k$  is a sum of monomials  $v_1^{a_1} \cdots v_k^{a_k}$ , which satisfy the conditions of Lemma 3.6. The lemma follows from the previous one.

**Lemma 3.8.**

$$\mathcal{T}_k(V_i) = V_i \quad (1 \leq i \leq k).$$

Therefore,  $\mathcal{T}_k|_{M_k}$  is the inclusion of  $M_k$  into  $P_k$ .

*Proof.* This is proved by induction on  $k$ . It is easy for  $k = 1$ . Suppose inductively that it has been shown for  $k - 1$ . Then, using the expansion  $V_i = v_1^{2^{i-2}} v_2^{2^{i-3}} \cdots v_{i-1} v_i$ , one has

$$\begin{aligned} \mathcal{T}_k(V_i) &= \partial \left( v_1^{2^{i-2}} \otimes x_1^{-1} \mathcal{T}_{k-1}(V_{i-1}(x_2, \dots, x_i)) \right) \\ &= Sq^{2^{i-2}+1} \left( x_1^{-1} V_{i-1}(x_2, \dots, x_i) \right) \quad (\text{by inductive hypothesis}) \\ &= d^* P_{x_1} \left( V_{i-1}(x_2, \dots, x_i) \right) \quad (\text{since } |V_{i-1}| = 2^{i-2}) \\ &= V_i(x_1, x_2, \dots, x_i). \end{aligned}$$

The last equality is showed in Mùì [17, Lemma 5.3], (see also [8]).

The lemma is proved.

Since  $D_k \subset M_k$ , Lemmas 3.7 and 3.8 show that  $\mathcal{T}_k|_{D_k}$  is the inclusion of  $D_k$  into  $P_k$ .

Theorem 3.2 is completely proved.

#### 4. HOMOLOGICAL CLASSES INDUCED BY DICKSON INVARIANTS

According to Singer [21],  $\Gamma_k^\wedge$  supports a canonical  $\mathcal{A}$ -action as follows. The usual action of  $\mathcal{A}$  on  $P_k$  commutes with the action of  $GL_k$ . So, it induces an action of  $\mathcal{A}$  on  $D_k = P_k^{GL_k} = \mathbf{F}_2[Q_{k,k-1}, \dots, Q_{k,0}]$ . This is extended to an action of  $\mathcal{A}$  on  $\Gamma_k = D_k[Q_{k,0}^{-1}]$ . By definition [21],  $\Gamma_k^\wedge$  is the submodule of  $\Gamma_k$  spanned by all monomials  $\gamma = Q_{k,k-1}^{i_{k-1}} \cdots Q_{k,0}^{i_0}$  with  $-\infty < i_0 < +\infty, 0 \leq i_1, \dots, i_{k-1}$  and  $0 \leq i_0 + \dim \gamma$ . By means of the Hai–Hung formula for the  $\mathcal{A}$ -action on  $D_k$  (see [8] or 4.4 below), it is easy to verify that  $\Gamma_k^\wedge$  is an  $\mathcal{A}$ -submodule of  $\Gamma_k$ .

From Singer [21], the  $\mathcal{A}$ -action on  $\Gamma^\wedge = \bigoplus_k \Gamma_k^\wedge$  commutes with the differential of  $\Gamma^\wedge$ .

Now we show that the weak conjecture on spherical classes follows from the following one.

**Conjecture 4.1.** *Let  $Ker \partial_k$  be the submodule of all cycles in  $\Gamma_k^\wedge$ . Then,*

$$D_k^+ \subset \mathcal{A}^+ \cdot Ker \partial_k,$$

for  $k > 2$ .

**Proposition 4.2.** *Conjecture 4.1 implies Conjecture 1.6, which in turn implies the weak conjecture on spherical classes.*

*Proof.* By Conjecture 4.1, for a given  $q \in D_k^+$ , we have  $q = \sum_i Sq^i(\gamma_i)$  with some  $i > 0$  and  $\gamma_i \in Ker \partial_k$ .

On the other hand, from Singer [21, Th. 1.3], the induced  $\mathcal{A}$ -action on  $H_k(\Gamma^\wedge) = Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$  is trivial. So  $[q] = \sum_i Sq^i[\gamma_i] = 0$  in  $Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$ . By Theorem 3.2,  $[q] = Tr^*[q] = 0$  in  $\mathbf{F}_2 \otimes P_k$  for every  $q \in D_k^+$ , or equivalently,  $D_k^+ \subset \mathcal{A}^+ \cdot P_k$ , for  $k > 2$ . The proof is complete.

Here is an alternative proof for the fact that Conjecture 4.1 implies the weak conjecture on spherical classes.

By Theorem 3.2,  $\mathcal{T}_k(Ker \partial_k) \subset P_k$ ,  $\mathcal{T}_k(D_k) = D_k$ .

Since  $\mathcal{T}_k$  is an  $\mathcal{A}$ -homomorphism, Conjecture 4.1 implies

$$D_k^+ \subset \mathcal{A}^+ \cdot P_k,$$

for  $k > 2$ . That means Conjecture 1.5 holds. Hence, the weak conjecture on spherical classes is proved.

The following theorem is a weak form of Conjecture 4.1.

**Theorem 4.3.** *If  $k > 2$ , then*

$$D_k^+ \subset \mathcal{A}^+ \cdot \Gamma_k^\wedge.$$

In order to prove the theorem, we need the Hai–Hung formula for the action of  $\mathcal{A}$  on  $D_k$  (see [8]):

$$4.4. \quad Sq^i(Q_{k,j}) = \begin{cases} Q_{k,r} & i = 2^j - 2^r, r \leq j, \\ Q_{k,t} Q_{k,r} & i = 2^k - 2^t + 2^j - 2^r, r \leq j < t, \\ Q_{k,j}^2 & i = 2^k - 2^j, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$Sq^{2^{s+j-1}}(Q_{k,j}^{2^s}) = Q_{k,j-1}^{2^s},$$

for  $0 < j < k$ .

The following remark is an immediate consequence of 4.4 and the Cartan formula.

**Remark 4.5.** (on jump steps). If  $Sq^i(Q_{k,j}^{2^s}) \neq 0$ , then either  $i = 0$  or  $i \geq 2^{s+j-1}$ . Furthermore, if  $Sq^i X = Q_{k,j-1}^{2^s}$  for some  $X \in D_k$ , then either  $i = 0$  or  $i \geq 2^{s+j-1}$ .

*Proof of Theorem 4.3.* We use techniques of the proof in [11] for the fact that

$$\mathbf{F}_2 \otimes_{\mathcal{A}} \Gamma_k = 0,$$

for  $k > 1$ .

We follow the notations of [10]. For a given non-negative integer  $i$ , we denote by  $s(i)$  the number with  $2^{s(i)}$  being the first missing 2-power in the dyadic expansion of  $i$ . In other words,  $i \equiv 2^{s(i)} - 1 \pmod{2^{s(i)+1}}$ .

We prove the theorem by a downward induction.

Let  $Q^I = Q_{k,k-1}^{i_{k-1}} \cdots Q_{k,0}^{i_0}$ ,  $I = (i_{k-1}, \dots, i_0)$  with  $i_j \geq 0$  if  $j \geq 1$  and  $i_0 \in \mathbf{Z}$ , be a monomial in  $\Gamma_k$ . Set  $f_j(I) = s(i_j) + j - 1$  for  $j \geq 1$  and  $f(I) = \min\{f_j(I) \mid j \geq 1\}$ .

Suppose  $f_j(I) = f(I)$  with  $j \geq 2$ , then  $s(i_{j-1}) + j - 2 = f_{j-1}(I) \geq f_j(I) = s(i_j) + j - 1$ , so  $s(i_{j-1}) \geq s(i_j) + 1$ . In particular,  $i_{j-1} \geq 2^{s(i_j)}$ . Hence, applying 4.4 we get

$$4.6. \quad Sq^{2^{f(I)}} Q^{(i_{k-1}, \dots, i_j + 2^{s(i_j)}, i_{j-1} - 2^{s(i_j)}, \dots, i_0)} = Q^I + \Sigma Q^L.$$

We show that  $s(\ell_{j-1}) \leq s(i_j)$  and thus  $f_{j-1}(L) = s(\ell_{j-1}) + j - 2 < s(i_j) + j - 1 = f_j(I) = f(I)$ . Hence  $f(L) < f(I)$  for any  $L$  in the sum. Let  $\alpha_m(a)$  denote the coefficient of  $2^m$  in the dyadic expansion of  $a$ , and  $s = s(i_j)$ . There are two cases. In the first case,  $\alpha_s(\ell_{j-1}) = \alpha_s(i_{j-1}) = 1$ . Then, using the remark on jump steps, we easily verify that  $Sq^{2^{f(I)}}$  acts only to send  $Q_{k,j}^{2^s}$  to  $Q_{k,j-1}^{2^s}$ . So  $Q^L = Q^I$ , but not an extra term. In the second case,  $\alpha_s(\ell_{j-1}) = \alpha_s(i_{j-1} - 2^s) = 0$ . Then, by definition,  $s(\ell_{j-1}) \leq s = s(i_j)$ .

It should be noted that if  $Q^I \in D_k$ , then the ‘‘killer’’

$$Q^J = Q^{(i_{k-1}, \dots, i_j + 2^{s(i_j)}, i_{j-1} - 2^{s(i_j)}, \dots, i_0)}$$

and thus any extra term  $Q^L$  is in  $D_k$ .

Next we consider the case  $f_1(I) = f(I)$  and  $f_j(I) > f(I)$  if  $1 < j < k$ . Similarly as in 4.6, we have

$$4.7. \quad Sq^{2^{f(I)}} Q^{(i_{k-1}, \dots, i_1 + 2^{s(i_1)}, i_0 - 2^{s(i_1)})} = Q^I + \Sigma Q^L.$$

We show that, for any  $L$  in the sum, there exists  $j$  ( $1 \leq j < k$ ) such that  $f_j(L) < f_1(I)$ , therefore  $f(L) < f(I)$ . Suppose the contrary that  $f_j(L) \geq f_1(I)$  or equivalently  $s(\ell_j) + j - 1 \geq s(i_1)$  for  $1 \leq j < k$ . As  $s(\ell_1) \geq s(i_1)$ , then  $\ell_1 = i_1 + 2^{s(i_1)}$ . Otherwise, by the remark on jump steps,  $Sq^{2^{f(I)}} = Sq^{2^{s(i_1)}}$  acts only to send  $Q_{k,1}^{2^{s(i_1)}}$  to  $Q_{k,0}^{2^{s(i_1)}}$ , thus  $Q^L = Q^I$ , but not an extra term. Suppose inductively that  $\ell_1 = i_1 + 2^{s(i_1)}$ ,  $\ell_2 = i_2, \dots, \ell_{j-1} = i_{j-1}$ . We show  $\ell_j = i_j$  ( $1 \leq j < k$ ). Indeed, if  $\ell_j \neq i_j$ , then combining  $s(\ell_j) + j - 1 \geq s(i_1)$ ,  $s(i_j) + j - 1 > s(i_1)$  with the remark on jump steps, we easily verify that actually  $s(\ell_j) + j - 1 = s(i_1)$  and  $Sq^{2^{f(I)}} = Sq^{2^{s(i_1)}}$  acts only to send  $Q_{k,j}^{2^{s(i_j)}}$  to  $Q_{k,j-1}^{2^{s(i_j)}}$ . This contradicts the

hypothesis  $\ell_{j-1} = i_{j-1}$  (or the hypothesis  $\ell_1 = i_1 + 2^{s(i_1)}$  if  $j = 2$ ). Consequently, we get  $\ell_1 = i_1 + 2^{s(i_1)}, \ell_2 = i_2, \dots, \ell_{k-1} = i_{k-1}$ . So  $Sq^{2^{f(I)}} = Sq^{2^{s(i_1)}}$  acts only to increase the power of  $Q_{k,0}$ . However,  $\dim Q_{k,0} = 2^k - 1 \not\equiv 2^{s(i_1)}$  if  $k > 1$ . This is a contradiction.

Now we must show that if  $Q^I \in D_k$ , then the “killer”

$$Q^J = Q^{(i_{k-1}, \dots, i_1 + 2^{s(i_1)}, i_0 - 2^{s(i_1)})}$$

and every “killer” needed in the procedure of using 4.6 and 4.7 to kill extra terms  $Q^L$  or to kill extra terms of extra terms ... are all in  $\Gamma_k^\wedge$ . Suppose  $Q^K$  is such a “killer”. Let  $s = s(i_1)$ . As  $f = f(I)$  decreases in the procedure, then

$$k_0 \geq i_0 - 2^s - 2^{s-1} - \dots - 1.$$

On the other hand, combining  $f(I) = s$  with the two facts that the dimension of any extra term equals to  $\dim Q^I$  and  $f(\text{extra term}) < f(I)$ , we have

$$\dim Q^K \geq \dim Q^I - 2^{f(I)} = \dim Q^I - 2^s.$$

Thus

$$\begin{aligned} k_0 + \dim Q^K &\geq (i_0 - 2^s - 2^{s-1} - \dots - 1) + (\dim Q^I - 2^s) \\ &= (i_0 - 2^{s+1} + 1) + (\dim Q^I - 2^s) \\ &\geq \dim Q^I - 3 \cdot 2^s + 1. \end{aligned}$$

From  $s(i_1) = s$ , it implies  $i_1 \geq 2^s - 1$ . Then we get

$$\dim Q^I \geq \dim(Q_{k,1}^{i_1}) \geq (2^s - 1)(2^k - 2).$$

Hence

$$\begin{aligned} k_0 + \dim Q^K &\geq (2^s - 1)(2^k - 2) - 3 \cdot 2^s + 1 \\ &= (2^s - 1)(2^k - 2) - 3 \cdot (2^s - 1) - 3 + 1 \\ &= (2^s - 1)(2^k - 5) - 2 \\ &\geq 2^k - 7 > 0 \quad (\text{as } k > 2), \end{aligned}$$

except for  $s = 0$ . (This case is handled by the next step.) Consequently,  $Q^K \in \Gamma_k^\wedge$ .

To start the induction, assume  $Q^I \in D_k$  with  $f(I) = 0$ . Then  $f_1(I) = f(I) = 0$  as, by definition,  $f_j(I) > 0$  for  $1 < j < k$ . Then  $s = s(i_1) = 0$ , and  $i_1 \equiv 0 \pmod{2}$ . Thus

$$Sq^1 Q^{(i_{k-1}, \dots, i_1+1, i_0-1)} = Q^I.$$

The “killer”  $Q^{(i_{k-1}, \dots, i_1+1, i_0-1)}$  is in  $\Gamma_k^\wedge$ , because

$$(i_0 - 1) + \dim Q^{(i_{k-1}, \dots, i_1+1, i_0-1)} = i_0 + \dim Q^I - 2 \geq 2^{k-1} - 2 > 0,$$

for  $k > 2$ . The first inequality holds as there exists at least one  $i_j \neq 0$  and  $\dim(Q_{k,j}) \geq 2^{k-1}$  for  $0 \leq j < k$ .

The theorem is proved.

The following theorem establishes Conjecture 1.6 for  $k = 3$ .

**Theorem 4.8.** *Let  $k = 3$ . Then, for every  $q \in D_3^+$ ,  $[q] = 0$  in  $H_3(\Gamma^\wedge) \cong \text{Tor}_3^A(\mathbf{F}_2, \mathbf{F}_2)$ .*

*Proof.* Let  $\langle \cdot, \cdot \rangle$  denote the dual pairing between  $Tor_3^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$  and  $Ext_{\mathcal{A}}^3(\mathbf{F}_2, \mathbf{F}_2)$ , and also the one between  $(\mathbf{F}_2 \otimes P_3)^{GL_3}$  and  $\mathbf{F}_2 \otimes_{GL_3} PH_*(BE^3)$ . Here, by  $PH_*(BE^3)$  we mean the submodule consisting of all  $\mathcal{A}$ -annihilated elements of  $H_*(BE^3)$ .

By Boardman [3],  $Tr_3 : \mathbf{F}_2 \otimes_{GL_3} PH_*(BE^3) \rightarrow Ext_{\mathcal{A}}^3(\mathbf{F}_2, \mathbf{F}_2)$  is an isomorphism.

In particular, we have

$$\begin{aligned} \langle [q], Ext_{\mathcal{A}}^3(\mathbf{F}_2, \mathbf{F}_2) \rangle &= \langle [q], Tr_3(\mathbf{F}_2 \otimes_{GL_3} PH_*(BE^3)) \rangle \\ &= \langle Tr_3^*[q], \mathbf{F}_2 \otimes_{GL_3} PH_*(BE^3) \rangle \\ &= \langle [q], \mathbf{F}_2 \otimes_{GL_3} PH_*(BE^3) \rangle \quad (\text{by Theorem 3.2}) \\ &= 0, \end{aligned}$$

because  $[q] = 0$  in  $(\mathbf{F}_2 \otimes P_3)^{GL_3}$ , by Theorem 3.2 of our paper [9].

Hence  $[q] = 0$  in  $Tor_3^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$ , for every  $q \in D_3^+$ . The theorem is proved.

## 5. FINAL REMARKS

**Remark 5.1.** The weak conjecture on spherical classes is actually equivalent to the fact that for every  $q \in D_k^+$  and any  $k > 2$ ,  $[q] \in Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$  vanishes on the image of Singer's  $k$ -th transfer,  $Tr_k(\mathbf{F}_2 \otimes_{GL_k} PH_*(BE^k)) \subset Ext_{\mathcal{A}}^k(\mathbf{F}_2, \mathbf{F}_2)$ . This observation can be read off from the proof of Theorem 4.8.

**Remark 5.2.** Conjecture 1.6, and therefore Conjecture 4.1, is false when  $k = 1$  or 2. Indeed,  $[Q_{1,0}^{2^i-1}]$  and  $[Q_{2,1}^{2^i-1}]$  are non-zero in  $Tor_*^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$ . They are respectively dual to the Adams element  $h_i \in Ext_{\mathcal{A}}^1(\mathbf{F}_2, \mathbf{F}_2)$  and its square  $h_i^2 \in Ext_{\mathcal{A}}^2(\mathbf{F}_2, \mathbf{F}_2)$ . One easily verifies this assertion by combining Theorem 3.2 with Theorem 2.1 of [9] and Proposition 5.3 of [14]. The only Hopf invariant one elements are represented by  $h_i$  for  $i = 1, 2, 3$  (see Adams [1]). Furthermore, the only Kervaire invariant one elements are represented by  $h_i^2$ , wherever  $h_i^2$  is a permanent cycle in the Adams spectral sequence for spheres (see Browder [5]).

Let  $\varphi_k^* : \mathbf{F}_2 \otimes D_k \rightarrow Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$  be the dual of the Lannes–Zarati homomorphism, which is compatible with the Hurewicz one  $H : \pi_*(Q_0S^0) \rightarrow H_*(Q_0S^0)$ . In [9] we have proved that the inclusion  $D_k \subset P_k$  is a chain-level representation of  $Tr_k^* \cdot \varphi_k^* : \mathbf{F}_2 \otimes D_k \rightarrow \mathbf{F}_2 \otimes P_k$ . This together with Theorem 3.2 lead us to the following conjecture.

**Conjecture 5.3.** *The inclusion  $D_k \subset \Gamma_k^\wedge$  is a chain-level representation of  $\varphi_k^*$ .*

This conjecture and Conjecture 1.6 imply the classical conjecture on spherical classes.

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**Note added in proof:** Recently, we have established Conjecture 5.3.