

COMPLETIONS OF PRO-SPACES

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ABSTRACT. For every ring R , we present a pair of model structures on the category of pro-spaces. In the first, the weak equivalences are detected by cohomology with coefficients in R . In the second, the weak equivalences are detected by cohomology with coefficients in all R -modules (or equivalently by pro-homology with coefficients in R). In the second model structure, fibrant replacement is essentially just the Bousfield-Kan R -tower. When $R = \mathbb{Z}/p$, the first homotopy category is equivalent to a homotopy theory defined by Morel but has some convenient categorical advantages.

1. INTRODUCTION

The notion of R -completion has been a valuable tool to homotopy theorists. The basic idea is to start with a space X and then construct another space X_R^\wedge whose homotopy type is entirely determined by the singular cohomology $H^*(X; R)$ of X with coefficients in R . In other words, X_R^\wedge remembers the R -cohomology of X but forgets all other information.

Bousfield and Kan [BK] constructed R -completions for a large class of spaces that they called R -good spaces. The basic construction goes as follows. Start with a space X . Then define a cosimplicial space \tilde{R}^*X . This cosimplicial space gives rise to a tower

$$\cdots \rightarrow R_2X \rightarrow R_1X \rightarrow R_0X$$

of fibrations. Finally, X_R^\wedge is the homotopy limit of this tower.

Unfortunately, this process only works for the R -good spaces. In fact, the *tower* described above is correct for all spaces, but the homotopy limit causes problems when X is not R -good.

Thus, one approach to generalizing the construction of Bousfield and Kan to arbitrary spaces is to consider X_R^\wedge not as a single space but rather as the whole tower. In fact, it is best to think of X_R^\wedge as a pro-space [D]. This paper is concerned with the homotopical foundations for pro-spaces suitable for this viewpoint on R -completion.

When $R = \mathbb{Z}/p$, Morel [Mo] constructed a homotopy theory of simplicial pro-finite sets that is suitable for studying \mathbb{Z}/p -completions of spaces. Unfortunately, there are a few problems with the approach in [Mo]. Namely, it is *not* true that the category of simplicial pro-finite sets is equal to the category of pro-simplicial finite sets. In fact, the former is only a retract of the latter. This means that we must be very careful with our intuitive ideas about simplicial pro-finite sets.

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We prove the following theorem in Section 6.

Theorem 1.1. *Let R be any ring. There is a model structure on the category of pro-simplicial sets in which the weak equivalences are maps $f : X \rightarrow Y$ such that $H^*(Y; R) \rightarrow H^*(X; R)$ is an isomorphism.*

One of our main results is that the homotopy theory of pro-simplicial sets from Theorem 1.1 is equivalent to Morel's homotopy theory of simplicial pro-finite sets when $R = \mathbb{Z}/p$. The advantage of our approach is that one does not have to worry about the unintuitive nature of simplicial pro-finite sets.

The proofs of [Mo] rely in an essential way on notions of finiteness and take great advantage of the fact that \mathbb{Z}/p is a finite ring. In fact, finiteness is not really such an important ingredient, as demonstrated by the proof of Theorem 1.1, which works for any ring R .

In some contexts [AM] [I4], one wants to take R -completions of pro-spaces. Although [Mo] handles \mathbb{Z}/p -completions of spaces perfectly well, it is not quite right for R -completions of arbitrary pro-spaces. Here is the underlying reason. If $X \rightarrow Y$ is a map of pro-spaces such that $H^*(Y; R) \rightarrow H^*(X; R)$ is an isomorphism, then $H^*(Y; M) \rightarrow H^*(X; M)$ is *not* necessarily an isomorphism for all R -modules M (see Example 5.3). These observations motivate the following theorem, which is also proved in Section 6.

Theorem 1.2. *Let R be any ring. There is a model structure on the category of pro-simplicial sets in which the weak equivalences are maps $f : X \rightarrow Y$ such that $H^*(Y; M) \rightarrow H^*(X; M)$ is an isomorphism for all R -modules M .*

We will show later that the weak equivalences in the model structure of Theorem 1.2 can also be described as the maps $f : X \rightarrow Y$ such that $H_n(X; R) \rightarrow H_n(Y; R)$ is an isomorphism of pro-groups for each $n \geq 0$.

For R -completions of pro-spaces, the homotopy theory constructed in Theorem 1.2 is better than the homotopy theory of Theorem 1.1. It retains more information, remembering not just the cohomology with coefficients in R but the cohomology with coefficients in all R -modules. Moreover, the homotopy theory of Theorem 1.2 has a close link with the Bousfield-Kan R -tower [BK]. In particular, the Bousfield-Kan R -tower of a pointed connected space X is basically the same thing as the fibrant replacement of X in the R -homological model structure. We explore this link in Section 7.

The main tool in the proof of Theorem 1.1 is a general localization result from [CI]. The proof of Theorem 1.2 is similar, but we have to be careful about the set-theoretical complications introduced by allowing cohomology with coefficients in arbitrarily large R -modules.

In both model structures, we give an explicit description of fibrant objects. This allows for computations, as demonstrated in Section 9, where we compute the R -completion of the classifying space of a finitely-generated free group. In order to describe the fibrant objects, we must introduce a notion of nilpotence for spaces that is related to but distinct from the usual notion of nilpotence. See Section 3 for more details.

At the end of the paper, we list several questions concerning this subject that remain unanswered. We hope that this will encourage future work on this topic.

1.1. Background. We assume familiarity with model structures. The original reference is [Q], but [Ho] and [Hi] are the modern thorough references. Also, [DS] is

a good introduction to the subject. We warn the reader about one convention concerning the model structure axioms. We will always start with a model category \mathcal{C} in which factorizations are functorial. However, when we produce model categories on the pro-category $\text{pro-}\mathcal{C}$, we will not necessarily obtain functorial factorizations. Recent work of Chorny [C] suggests that functoriality is obtainable, but we will ignore the question here.

We work exclusively with simplicial sets, rather than topological spaces. From now on, the word *space* always means *simplicial set*. It is possible to obtain all of the results of this paper for topological spaces instead of simplicial sets, but simplicial sets are somewhat easier to work with.

We also assume a certain amount of familiarity with pro-categories, although we give a brief review of the most important points in Section 2. Although there are many established and thorough references for pro-categories such as [AM], [SGA4], or [EH], the reader is encouraged to look also at [I1], [I2], and [I3] for aspects of pro-categories that are particularly relevant to this paper.

1.2. Organization. We begin in Section 2 with a brief review of pro-categories, touching only on the issues that are most important for present purposes. We also recall the strict model structure [EH] [I3] on the category of pro-spaces and a general localization result from [CI] that will allow us to produce new model structures for pro-spaces. In the following section, we study the fibrant objects in these localized model structures. For this purpose, we need a variation on the standard notion of nilpotence for spaces.

Section 4 contains some relatively straightforward material on homological algebra in pro-abelian categories; this is basically just transferring well-known results to a new setting.

The main part of the paper begins in Section 5 with the study of pro-maps that induce isomorphisms in various kinds of singular cohomology. Section 6 describes the model structures that have these cohomology isomorphisms as weak equivalences.

The rest of the paper is dedicated to applications and connections to other established theories. We begin in Section 7 with the link between our constructions and the Bousfield-Kan notion of R -completion. Then in Section 8 we compare our constructions to Morel's theory of \mathbb{Z}/p -completion. We warn the reader that Section 8 is a bit tricky because it involves a comparison of several categories that feel very similar but are definitely distinct. In Section 9, we compute the \mathbb{Z}/p -completion of the classifying space of a finitely-generated free group. Finally, we list some open questions in Section 10.

2. PRO-CATEGORIES AND HOMOTOPY THEORIES FOR PRO-SPACES

2.1. Pro-Categories. We begin with a brief overview of pro-categories and the homotopy theory of pro-spaces. Standard references on pro-categories include [SGA4], [AM], and [EH]. See also [I2] and [I3] for details specifically relevant to the homotopy theory of pro-categories.

Definition 2.1. For a category \mathcal{C} , the category **pro- \mathcal{C}** has objects all cofiltering diagrams in \mathcal{C} , and

$$\text{Hom}_{\text{pro-}\mathcal{C}}(X, Y) = \lim_s \text{colim}_t \text{Hom}_{\mathcal{C}}(X_t, Y_s).$$

Composition is defined in the natural way.

The word *pro-object* refers to objects of pro-categories. A **constant** pro-object is one indexed by the category with one object and one (identity) map.

A **level representation** of a map $f : X \rightarrow Y$ is: a cofiltered index category I ; cofiltered diagrams \tilde{X} and \tilde{Y} indexed by I that are pro-isomorphic to X and Y respectively; and a natural transformation $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ representing a pro-map that is isomorphic to f . Every map has a level representation [AM, App. 3.2] [Me].

A pro-object X satisfies a certain property **levelwise** if each X_s satisfies that property, and X satisfies this property **essentially levelwise** if it is isomorphic to another pro-object satisfying this property levelwise. Similarly, a level representation $X \rightarrow Y$ satisfies a certain property **levelwise** if each $X_s \rightarrow Y_s$ has this property. A map of pro-objects satisfies this property **essentially levelwise** if it has a level representation satisfying this property levelwise.

Let $c : \mathcal{C} \rightarrow \text{pro-}\mathcal{C}$ be the functor taking an object X to the constant pro-object with value X . Note that this functor makes \mathcal{C} a full subcategory of $\text{pro-}\mathcal{C}$. The limit functor $\lim : \text{pro-}\mathcal{C} \rightarrow \mathcal{C}$ is the right adjoint of c . To avoid confusion, we write \lim^{pro} for limits computed within the category $\text{pro-}\mathcal{C}$.

We recall the construction of cofiltered limits in $\text{pro-}\mathcal{C}$ (see, for example, [I1, § 4]). The specific details of this construction will be used in several places later. Start with a functor $X : A \rightarrow \text{pro-}\mathcal{C} : a \mapsto X^a$, where A is a cofiltered index category. The index category I for $\lim^{\text{pro}} X$ consists of all pairs (a, s) such that a belongs to A and s belongs to the indexing category of X^a . A morphism $(a, s) \rightarrow (b, t)$ consists of a morphism $a \rightarrow b$ in A together with a map $X_s^a \rightarrow X_t^b$ in \mathcal{C} that represents the pro-map $X^a \rightarrow X^b$. Finally, $\lim^{\text{pro}} X$ is defined to be the functor $I \rightarrow \mathcal{C}$ that takes (a, s) to X_s^a .

2.2. Strict Homotopy Theory of Pro-Spaces and Its Localizations. We now review from [I3] the strict homotopy theory of pro-spaces. The strict model structure was originally defined in [EH]. The **strict weak equivalences** (*resp.*, **cofibrations**) are the essentially levelwise weak equivalences (*resp.*, cofibrations), and the **strict fibrations** are defined by the right lifting property. In fact, a more explicit description of the fibrations in terms of matching maps is possible [I3, § 4]. The strict model structure is proper and simplicial.

Recall that the n th singular cohomology group $H^n(X; M)$ of a pro-space X with coefficients in an abelian group M is defined to be $\text{colim}_s H^n(X_s; M)$ [AM, 2.2] [S]. In fact, there is an isomorphism between $H^n(X; M)$ and the set $[X, cK(M, n)]_{\text{pro}}$ of weak homotopy classes of maps of pro-spaces. The pro-space $cK(M, n)$ is the constant pro-space with value an Eilenberg-Mac Lane space.

We recall the following localization result for pro-spaces. The full proof (in greater generality), which owes much to [Hi, Ch. 5], appears in [CI].

Theorem 2.2. *Let K be a set of fibrant spaces. There exists a left proper simplicial model structure on the category of pro-spaces such that the cofibrations are the essentially levelwise cofibrations and such that a map $f : X \rightarrow Y$ is a weak equivalence if and only if*

$$\text{Map}_{\text{pro}}(Y, cA) = \text{colim}_s \text{Map}(Y_s, A) \rightarrow \text{colim}_t \text{Map}(X_t, A) = \text{Map}_{\text{pro}}(X, cA)$$

is a weak equivalence for every object A in K .

The weak equivalences in the above theorem are called **K -colocal weak equivalences**.

3. NILPOTENT SPACES

We will need to understand the fibrant objects in the localized model categories of Theorem 2.2.

Definition 3.1. Let K be any collection of fibrant spaces. The class of **K -nilpotent spaces** is the smallest class of fibrant spaces such that:

- (1) the space $*$ is K -nilpotent;
- (2) the K -nilpotent spaces are closed under weak equivalences between fibrant spaces;
- (3) and if X is K -nilpotent, A belongs to K , and $X \rightarrow A^{\partial\Delta^k}$ is any map, then the fiber product $X \times_{A^{\partial\Delta^k}} A^{\Delta^k}$ is also K -nilpotent.

Because the fiber product in (3) above is actually a homotopy fiber product, it is only the weak homotopy types of the spaces in K that matter. Thus, the K -nilpotent spaces are more properly a collection of weak homotopy types rather than a collection of actual spaces. The consequence is that we are allowed to choose any (fibrant) models for the weak homotopy types in K that are most convenient for our purposes.

Lemma 3.2. *A space X is K -nilpotent if and only if it is fibrant and weakly equivalent to a space that can be built from $*$ by finitely many pullbacks of type (3) in Definition 3.1.*

In other words, when constructing a K -nilpotent space, it is not necessary to use any weak equivalences until the very last step.

Proof. Let C_n be the class of K -nilpotent spaces that can be built from $*$ with fewer than $n + 1$ pullbacks (and possibly also weak equivalences), and let D_n be the class of fibrant spaces that are weakly equivalent to a space that can be built from $*$ with fewer than $n + 1$ pullbacks (but without any weak equivalences). By definition, D_n is contained in C_n . We will show by induction that C_n and D_n are equal.

The classes C_0 and D_0 consist of the fibrant contractible spaces, so they are equal. Now suppose that C_{n-1} and D_{n-1} are equal. Let X belong to C_n . Then X is weakly equivalent to a space $X' \times_{A^{\partial\Delta^k}} A^{\Delta^k}$, where X' belongs to C_{n-1} . By the induction assumption, X' also belongs to D_{n-1} , so X' is weakly equivalent to a space X'' that can be built from $*$ by fewer than n pullbacks. Now $X' \times_{A^{\partial\Delta^k}} A^{\Delta^k}$ is weakly equivalent to $X'' \times_{A^{\partial\Delta^k}} A^{\Delta^k}$ because the fiber products are actually homotopy fiber products. Thus X is weakly equivalent to $X'' \times_{A^{\partial\Delta^k}} A^{\Delta^k}$, which is a space that can be built from $*$ using fewer than $n + 1$ pullbacks. Therefore, X belongs to D_n . \square

The next theorem demonstrates the relevance of K -nilpotent objects. It is proved in [CI, Prop. 4.9].

Theorem 3.3. *Let K be a set of fibrant spaces. In the model structure of Theorem 2.2, the fibrant objects are precisely the pro-spaces that are both strictly fibrant and essentially levelwise K -nilpotent.*

Recall that a fibration is **principal** if it is the base change of a fibration with contractible total space. We do not require that the base be connected. Therefore, some of the fibers of a principal fibration may be empty; however, the non-empty fibers are all weakly homotopic.

Lemma 3.4. *The fibration $p : K(A, n)^{\Delta^k} \rightarrow K(A, n)^{\partial\Delta^k}$ is principal for all $n \geq 0$, $k \geq 0$, and every abelian group A . Its non-empty fiber is weakly equivalent to $K(A, n - k)$.*

Proof. Throughout this proof, we use models for $K(A, n)$ that are simplicial abelian groups [Ma]. In particular, this means that $K(A, n)$ is fibrant. Also, $K(A, n)$ is based at 0.

Only one path-component of $K(A, n)^{\partial\Delta^k}$ is in the image of p ; it consists of the maps $\partial\Delta^k \rightarrow K(A, n)$ that are null-homotopic. If $n \neq k - 1$, then $K(A, n)^{\partial\Delta^k}$ only has one component, but this is irrelevant. Thus, we only have to compute the fiber of p over the zero map $0 : \partial\Delta^k \rightarrow K(A, n)$ (which is a point of $K(A, n)^{\partial\Delta^k}$), and this fiber is $\Omega^k K(A, n)$, as desired.

We still have to check that p is principal. Let v be the 0th vertex of Δ^k . Then we have a short exact sequence

$$0 \rightarrow X_{n,k} \rightarrow K(A, n)^{\Delta^k} \rightarrow K(A, n)^{\{v\}} = K(A, n) \rightarrow 0$$

of simplicial abelian groups, where $X_{n,k}$ is the subspace of $K(A, n)^{\Delta^k}$ consisting of all maps that take v to 0. The projection $\Delta^k \rightarrow \{v\}$ gives a splitting, so $K(A, n)^{\Delta^k}$ is isomorphic to $K(A, n) \times X_{n,k}$. Similarly, $K(A, n)^{\partial\Delta^k}$ is isomorphic to $K(A, n) \times Y_{n,k}$, where $Y_{n,k}$ is the subspace of $K(A, n)^{\partial\Delta^k}$ consisting of all maps that take v to 0. The fibration $X_{n,k} \rightarrow Y_{n,k}$ is principal because $X_{n,k}$ is contractible (which follows from the fact that Δ^k is contractible). The identity map on $K(A, n)$ is of course principal, and a product of two principal fibrations is again principal. This shows that p is principal. \square

Proposition 3.5. *Let C be a class of R -modules, and let K be the class of Eilenberg-Mac Lane spaces $K(M, n)$ such that M belongs to C . A space X is K -nilpotent if and only if it is fibrant and there exists a finite tower*

$$X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = *,$$

where X_n is weakly equivalent to X and each map $X_k \rightarrow X_{k-1}$ is a principal fibration whose non-empty fibers belong to K .

Proof. For one direction, Lemma 3.4 tells us that for every M in C , the map $p : K(M, n)^{\Delta^k} \rightarrow K(M, n)^{\partial\Delta^k}$ is a principal fibration whose non-empty fibers are weakly equivalent to $K(M, n - k)$. In view of Lemma 3.2, this shows that every K -nilpotent space X has a tower of the desired form.

Now we will show that if X has a tower of the desired form, then X is K -nilpotent. By induction on the length of the tower, we just need to show that if $p : E \rightarrow B$ is a principal fibration whose non-empty fibers are weakly equivalent to $K(M, n)$ and B is K -nilpotent, then E is K -nilpotent. We can rewrite p in the form $\phi \coprod E \rightarrow B_0 \coprod B_1$, where the fibers over B_0 are all empty and the fibers over B_1 are all weakly equivalent to $K(M, n)$. By the following lemma, we know that B_1 is K -nilpotent. Thus it suffices to replace B with B_1 and assume that p is surjective.

Let $p' : E' \rightarrow B'$ be a fibration with E' contractible such that p is a base change of p' . Since p is surjective, the image of the map $B \rightarrow B'$ lies in the same component as the image of p' . This means that we may assume that B' is connected. Since every fiber of p' is weakly equivalent to $K(M, n)$, we know that B' is weakly equivalent to $K(M, n + 1)$.

We now know that E is the homotopy pullback of the diagram

$$B \rightarrow K(M, n + 1) \leftarrow *$$

Recall the spaces $X_{n+1,1}$ and $Y_{n+1,1}$ from the proof of Lemma 3.4. Since $\partial\Delta^k$ is a model for S^{k-1} , the space $Y_{n,k}$ is weakly equivalent to $\Omega^{k-1}K(M, n)$, which is a model for $K(M, n - k + 1)$. Thus E is the homotopy pullback of the diagram

$$B \rightarrow Y_{n+1,1} \leftarrow X_{n+1,1},$$

so E is also the homotopy pullback of the diagram

$$B \rightarrow K(M, n + 1) \times Y_{n+1,1} \leftarrow K(M, n + 1) \times X_{n+1,1}.$$

This identifies E up to homotopy as a fiber product

$$B \times_{K(M, n+1)^{\partial\Delta^1}} K(M, n + 1)^{\Delta^1},$$

which implies that E is K -nilpotent. \square

Lemma 3.6. *Let C be a class of R -modules, and let K be the class of Eilenberg-Mac Lane spaces $K(M, n)$ such that M belongs to C . Let $X = Y \amalg Z$. If X is K -nilpotent, then Y and Z are K -nilpotent.*

We make no assumptions about whether Y and Z are connected.

Proof. We choose the model for $K(M, 0)$ that consists simply of the underlying set of M , viewed as a discrete simplicial set. The map $K(M, 0)^{\Delta^1} \rightarrow K(M, 0)^{\partial\Delta^1}$ then becomes the diagonal $M \rightarrow M \times M$. Take any map $X \rightarrow K(M, 0)^{\partial\Delta^1}$ such that Y maps to the diagonal and Z maps off the diagonal. Then the pullback $X \times_{K(M, 0)^{\partial\Delta^1}} K(M, 0)^{\Delta^1}$ is isomorphic to Y . This shows that Y is K -nilpotent. The same argument shows that Z is K -nilpotent. \square

When C is the collection of all R -modules, the notion of K -nilpotence defined here is related but not equivalent to the notion of R -nilpotence in [BK]. However, as Proposition 3.5 and [BK, Prop. III.5.3] show, every space that is nilpotent in the sense of Bousfield and Kan is also nilpotent in our sense. The converse is not true. For example, a K -nilpotent space (such as $K(R, 0)$) need not be connected. One way to see the difference is to note that the two notions of nilpotence are based upon different notions of principality. The question is whether the base spaces are required to be connected (or, equivalently, whether empty fibers are allowed).

It would be nice to have algebraic conditions on the homotopy groups of a disconnected space that guarantee that the space is K -nilpotent. However, such a condition has eluded us so far. See Section 10 for an elaboration of this problem.

4. PRO-HOMOLOGICAL ALGEBRA

In this section, we let \mathcal{A} be any abelian category. Recall that the category $\text{pro-}\mathcal{A}$ is again abelian [AM, App. 4.5]. The monomorphisms are the essentially levelwise monomorphisms, and the epimorphisms are the essentially levelwise epimorphisms. We also assume that \mathcal{A} has enough injectives. This implies that $\text{pro-}\mathcal{A}$ also has enough injectives [Z].

Lemma 4.1. *Let \mathcal{A} be an abelian category, and let I be an injective object of \mathcal{A} . Then cI is an injective object of $\text{pro-}\mathcal{A}$.*

Proof. Let $i : A \rightarrow B$ be a monomorphism in $\text{pro-}\mathcal{A}$; we may assume that i is a level monomorphism. Let $f : A \rightarrow cI$ be any map in $\text{pro-}\mathcal{A}$. This map is represented by a map $f_s : A_s \rightarrow I$ in \mathcal{A} for some s . Now f_s extends over i_s since I is injective and i_s is a monomorphism. This extension represents a map $g : B \rightarrow I$, and g extends f over i . \square

Lemma 4.2. *Let \mathcal{A} be an abelian category, and let A be any object of $\text{pro-}\mathcal{A}$. The groups $\text{Ext}_{\text{pro-}\mathcal{A}}^n(A, cB)$ and $\text{colim}_t \text{Ext}_{\mathcal{A}}^n(A_s, B)$ are isomorphic for every object B of \mathcal{A} and every $n \geq 0$.*

Proof. Let

$$B \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

be an injective resolution of B in \mathcal{A} . Then

$$cB \rightarrow cI^0 \rightarrow cI^1 \rightarrow \dots$$

is an injective resolution of cB in $\text{pro-}\mathcal{A}$ by Lemma 4.1 and the fact that c preserves exactness. Therefore, $\text{Ext}_{\text{pro-}\mathcal{A}}^*(A, cB)$ is the homology of the complex

$$\text{Hom}_{\text{pro-}\mathcal{A}}(A, cB) \rightarrow \text{Hom}_{\text{pro-}\mathcal{A}}(A, cI^0) \rightarrow \text{Hom}_{\text{pro-}\mathcal{A}}(A, cI^1) \rightarrow \dots,$$

which is equal to the complex

$$\text{colim}_s \text{Hom}_{\mathcal{A}}(A_s, B) \rightarrow \text{colim}_s \text{Hom}_{\mathcal{A}}(A_s, I^0) \rightarrow \text{colim}_s \text{Hom}_{\mathcal{A}}(A_s, I^1) \rightarrow \dots.$$

Since filtered colimits are exact, the homology of the last complex is equal to $\text{colim}_s \text{Ext}_{\mathcal{A}}^*(A_s, B)$. \square

Now we construct a universal coefficients spectral sequence for pro-spaces. Let X be an arbitrary pro-space, and let M be an R -module. Choose an injective resolution

$$0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots.$$

Consider the bicomplex $K^{*,*}$ given by the formula

$$K^{p,q} = C^q(X; I^p) = \text{colim}_s C^q(X_s; I^p) = \text{colim}_s \text{Hom}(C_q(X_s; R), I^p).$$

This is a first-quadrant bicomplex with cohomological grading. Now $C_q(X_s; R)$ is a free R -module, so the complex $\text{colim}_s \text{Hom}(C_q(X_s; R), I^*)$ is exact. Taking cohomology of $K^{*,*}$ with respect to the p -differential gives that

$$E_1^{0,q} = \text{colim}_s \text{Hom}(C_q(X_s; R), M) = C^q(X; M)$$

and $E_1^{p,q} = 0$ if $p > 0$. Therefore, $E_2^{0,q} = H^q(X; M)$ and $E_2^{p,q} = 0$ if $p > 0$. Thus, the spectral sequence collapses, and the cohomology of the total complex of $K^{*,*}$ is $H^*(X; M)$.

Now we compute in the other order. Taking cohomology of $K^{*,*}$ with respect to the q -differential gives

$$E_1^{p,q} = H^q(X; I^p) = \operatorname{colim}_s H^q(X_s; I^p).$$

Because I^p is injective, this equals $\operatorname{colim}_s \operatorname{Hom}(H_q(X_s; R), I^p)$. Therefore, taking cohomology with respect to the p -differential gives

$$E_2^{p,q} = \operatorname{colim}_s \operatorname{Ext}_R^p(H_q(X_s; R), M) = \operatorname{Ext}_{\operatorname{pro}}^p(H_q(X; R), M).$$

The second equality above relies on Lemma 4.2.

Hence, we have a convergent first-quadrant cohomological spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_{\operatorname{pro}}^p(H_q(X; R), cM) \implies H^{p+q}(X; M).$$

This spectral sequence is called the **pro-universal coefficients spectral sequence**.

5. COHOMOLOGICAL AND HOMOLOGICAL WEAK EQUIVALENCES

In this section we collect some results about the cohomology and pro-homology of pro-spaces, emphasizing the differences and similarities with the situation of ordinary spaces.

The first lemma looks quite strange at first glance, but it becomes plausible when one remembers that filtered limits are exact in pro-categories [I1] [AHJM].

Lemma 5.1. *If $a \mapsto X^a$ is a cofiltered diagram of pro-spaces, then the cohomology group $H^n(\lim_a^{\operatorname{pro}} X^a; M)$ is isomorphic to $\operatorname{colim}_a H^n(X^a; M)$, where $\lim_a^{\operatorname{pro}}$ is the limit internal to the category of pro-spaces.*

Proof. This follows by direct computation using the construction of cofiltered limits in pro-categories given in Section 2.1. \square

Definition 5.2. Let R be any ring. An **R -cohomology weak equivalence** is a map of pro-spaces $X \rightarrow Y$ inducing isomorphisms $H^n(Y; R) \rightarrow H^n(X; R)$ for every $n \geq 0$.

Suppose that f is a map of ordinary spaces inducing an R -cohomology isomorphism. Then f induces an isomorphism in cohomology with coefficients in an arbitrary product of copies of R . But every free R -module is a retract of some product of copies of R , so f induces an isomorphism in cohomology with coefficients in any free R -module. Now every R -module M belongs to a short exact sequence

$$0 \rightarrow F_1 \rightarrow F_2 \rightarrow M \rightarrow 0$$

in which F_1 and F_2 are free, so the long exact sequence of cohomology groups and the five lemma imply that f induces an isomorphism in cohomology with coefficients in any R -module.

In contrast to the above paragraph, if f is a map of pro-spaces that is an R -cohomology weak equivalence, then f does not necessarily induce a cohomology isomorphism with coefficients in all R -modules. The argument from the previous paragraph breaks down because cohomology of pro-spaces does not commute with arbitrary products of coefficients. Actually, cohomology only commutes with finite products. One explanation for this difference is that $K(\prod_t M_t, n)$ is weakly equivalent to $\prod_t K(M_t, n)$ for ordinary spaces, but $cK(\prod_t M_t, n)$ is not weakly equivalent to $\prod_t cK(M_t, n)$ as pro-spaces.

Example 5.3. Let V be an infinite-dimensional \mathbb{Z}/p -vector space, and let $n \geq 1$. We give an example of a pro-space X for which $H^n(X; \mathbb{Z}/p) = 0$ but $H^n(X; V)$ is non-zero. Consider the pro-vector space W consisting of all subspaces of V with finite codimension; the structure maps are the inclusions of subspaces. Let X be the pro-space $K(W, n)$ obtained by applying levelwise the functor $K(-, n)$ to W .

Now $H^n(X; \mathbb{Z}/p)$ equals $\text{colim}_s \text{Hom}_{\mathbb{Z}/p}(W_s, \mathbb{Z}/p)$. This colimit is zero because the kernel of any homomorphism $W_s \rightarrow \mathbb{Z}/p$ is equal to W_t for some t . On the other hand, $H^n(X; V)$ equals $\text{colim}_s \text{Hom}_{\mathbb{Z}/p}(W_s, V)$, which is non-zero because it contains the element represented by any of the inclusions $W_s \rightarrow V$.

Because of this phenomenon, we introduce the following definition, which is distinct from Definition 5.2.

Definition 5.4. Let R be any ring. An **R -homology weak equivalence** is a map of pro-spaces $X \rightarrow Y$ inducing isomorphisms $H_n(X; R) \rightarrow H_n(Y; R)$ of pro-groups for all $n \geq 0$.

This definition can be reformulated in terms of cohomology. The next result implies that every R -homology weak equivalence is an R -cohomology weak equivalence. As shown in Example 5.3, the converse is not true.

Proposition 5.5. *A map $f : X \rightarrow Y$ is an R -homology weak equivalence if and only if it induces an isomorphism $f^* : H^n(Y; M) \rightarrow H^n(X; M)$ for all $n \geq 0$ and all R -modules M .*

The following proof is inspired by [BK, Prop. III.6.7].

Proof. First suppose that f is an R -homology weak equivalence. For any R -module M , f induces an isomorphism of E_2 -terms of the pro-universal coefficients spectral sequence (see Section 4). Therefore, the map on abutments is also an isomorphism.

For the other implication, suppose that f is a cohomology isomorphism with coefficients in any R -module. If I is an injective R -module, then

$$H^n(X; I) = \text{colim}_s \text{Hom}(H_n(X_s; R), I) = \text{Hom}_{\text{pro}}(H_n(X; R), cI).$$

Similarly, $H^n(Y; I) = \text{Hom}_{\text{pro}}(H_n(Y; R), cI)$. It follows that

$$\text{Hom}_{\text{pro}}(H_n(Y; R), cI) \rightarrow \text{Hom}_{\text{pro}}(H_n(X; R), cI)$$

is an isomorphism for every injective R -module I .

Now let M be an arbitrary pro- R -module. Choose a monomorphism $M \rightarrow I$ such that I is an injective pro- R -module. Then there is a diagram

$$\begin{array}{ccc} \text{Hom}_{\text{pro}}(H_n(Y; R), M) & \longrightarrow & \text{Hom}_{\text{pro}}(H_n(Y; R), I) \\ \downarrow & & \downarrow \\ \text{Hom}_{\text{pro}}(H_n(X; R), M) & \longrightarrow & \text{Hom}_{\text{pro}}(H_n(X; R), I) \end{array}$$

in which the horizontal maps are monomorphisms. Since the right vertical map is an isomorphism, the left vertical map must be a monomorphism. We conclude that $H_n(X; R) \rightarrow H_n(Y; R)$ is an epimorphism of pro-abelian groups.

Now let K be the pro-group that is the kernel of the map $H_n(X; R) \rightarrow H_n(Y; R)$. Since $\text{Hom}_{\text{pro}}(-, I)$ is exact for all injective pro-groups I , the first paragraph tells us that $\text{Hom}_{\text{pro}}(K, I)$ is zero for all injectives I . But the category of pro-abelian

groups has enough injectives, so this can only happen if K equals zero. This means that the map $H_n(X; R) \rightarrow H_n(Y; R)$ is an isomorphism. \square

From now on, we will freely switch between the cohomological and homological descriptions of R -homology weak equivalences, as given in Definition 5.4 and Proposition 5.5.

We will need to rephrase cohomology isomorphisms in terms of weak equivalences of mapping spaces.

Proposition 5.6. *Let M be any R -module, and let $K(M, n)$ be a fibrant Eilenberg-Mac Lane space. If $f : X \rightarrow Y$ is any map of pro-spaces, then $H^n(f; M)$ is an isomorphism for all $n \geq 0$ if and only if the map $\text{Map}_{\text{pro}}(Y, cK(M, n)) \rightarrow \text{Map}_{\text{pro}}(X, cK(M, n))$ is a weak equivalence for all $n \geq 0$.*

Proof. First suppose that the maps between mapping spaces are weak equivalences. We get isomorphisms after taking π_0 , which gives us the desired cohomology isomorphisms.

Now suppose that the maps $H^n(f; M)$ are isomorphisms for all $n \geq 0$. Since all pro-spaces are cofibrant, the homotopy types of the mapping spaces do not change if we alter X or Y up to strict weak equivalence. This means that we may assume that f is a levelwise cofibration; let Z be its cofiber, which is computed levelwise. Recall that Z is pointed canonically. Using the long exact sequence in cohomology, note that the reduced cohomology group $\tilde{H}^n(Z; M)$ is zero for all $n \geq 0$.

Now we want to show that the fibration

$$\text{Map}_{\text{pro}}(Y, cK(M, n)) \rightarrow \text{Map}_{\text{pro}}(X, cK(M, n))$$

is actually an acyclic fibration of simplicial sets. We can do this by showing that it has the right lifting property with respect to all generating cofibrations $\partial\Delta^k \rightarrow \Delta^k$. After the usual adjointness arguments, we need to show that every diagram

$$\begin{array}{ccc} \partial\Delta^k \otimes Y \amalg_{\partial\Delta^k \otimes X} \Delta^k \otimes X & \longrightarrow & cK(M, n) \\ \downarrow & & \\ \Delta^k \otimes Y & & \end{array}$$

has a lift. Since $K(M, n)$ is fibrant, formal model category arguments show that we only need obtain a lift up to homotopy; then we can adjust this lift to get an actual lift. In other words, we need to show that the map

$$H^n(\Delta^k \otimes Y; M) \longrightarrow H^n(\partial\Delta^k \otimes Y \amalg_{\partial\Delta^k \otimes X} \Delta^k \otimes X; M)$$

is surjective. The cofiber of the vertical map above is $S^k \wedge Z$, where the smash product is constructed levelwise. Using the long exact sequence in cohomology, it suffices to show that the reduced cohomology group $\tilde{H}^{n+1}(S^k \wedge Z; M)$ is zero. But this group is equal to $\tilde{H}^{n+1-k}(Z; M)$, which we already computed to be zero. \square

From now on, we will frequently express cohomology isomorphisms in terms of weak equivalences of mapping spaces as in Proposition 5.6. For example, we have the following corollary.

Corollary 5.7.

- (1) A map f is an R -cohomology weak equivalence if and only if the map

$$\mathrm{Map}_{\mathrm{pro}}(f, cK(R, n))$$

is a weak equivalence for every $n \geq 0$, where $K(R, n)$ is a fibrant Eilenberg-Mac Lane space.

- (2) A map f is an R -homology weak equivalence if and only if the map

$$\mathrm{Map}_{\mathrm{pro}}(f, cK(M, n))$$

is a weak equivalence for every $n \geq 0$ and every R -module M , where $K(M, n)$ is a fibrant Eilenberg-Mac Lane space.

Proof. The first claim follows immediately from Definition 5.2 and Proposition 5.6. The second claim follows immediately from Propositions 5.5 and 5.6. \square

6. THE R -COHOMOLOGICAL AND R -HOMOLOGICAL MODEL STRUCTURES

We next establish model structures whose weak equivalences are the R -cohomology weak equivalences and the R -homology weak equivalences.

Definition 6.1. A **cofibration** of pro-spaces is an essentially levelwise cofibration.

This is the same notion of cofibration as in the strict model structure [I2] or the π_* -model structure [I3]. In fact, the notion of cofibration is the same for every model structure on pro-spaces in the entire paper.

Definition 6.2. An **R -cohomology fibration** is a map having the right lifting property with respect to all maps that are both cofibrations and R -cohomology weak equivalences.

Theorem 6.3. *The cofibrations, R -cohomology weak equivalences, and R -cohomology fibrations give a left proper simplicial model structure on the category of pro-spaces.*

Proof. The proof is an application of Theorem 2.2 with K equal to the set of Eilenberg-Mac Lane objects of the form $K(R, n)$. By Corollary 5.7(1), the K -colocal weak equivalences are the same as the R -cohomology weak equivalences. \square

If λ is any infinite cardinal, then an R -module M is **λ -generated** if M has a generating set of cardinality at most λ . Note that there is a set of isomorphism types of λ -generated R -modules. Also note that a λ -generated R -module might have more than λ elements because R might be large.

A pro-map is a **λ -generated R -cohomology weak equivalence** if it induces an isomorphism on cohomology with coefficients in all λ -generated R -modules. Similarly, a pro-map is a **λ -generated R -cohomology fibration** if it has the right lifting property with respect to all pro-maps that are both cofibrations and λ -generated R -cohomology weak equivalences.

Theorem 6.4. *For any infinite cardinal λ , the cofibrations, λ -generated R -cohomology weak equivalences, and λ -generated R -cohomology fibrations give a left proper simplicial model structure on the category of pro-spaces.*

Proof. The proof is an application of Theorem 2.2 with K equal to the set of Eilenberg-Mac Lane objects of the form $K(M, n)$ with M a λ -generated R -module. \square

Actually, the λ -generated model structures are not so interesting. What we really want is a model structure in which the weak equivalences are detected by R -homology, or equivalently, according to Proposition 5.5, by cohomology with coefficients in all R -modules. This is the main purpose of the rest of this section.

Definition 6.5. An **R -homology fibration** is any map that is a λ -generated R -cohomology fibration for some λ .

In other words, the class of R -homology fibrations is the union of the classes of λ -generated R -cohomology fibrations as λ ranges over all cardinals. By Proposition 5.5, the class of R -homology weak equivalences is the intersection of the classes of λ -generated R -cohomology weak equivalences as λ ranges over all cardinals.

Lemma 6.6. *For any λ , the acyclic λ -generated R -cohomology fibrations are the same as the acyclic R -homology fibrations.*

Proof. For every λ , the acyclic λ -generated R -cohomology fibrations are detected by the same class of cofibrations, so these classes of acyclic λ -generated R -cohomology fibrations are all equal. Therefore, if p is an acyclic λ -generated R -cohomology fibration, then it is a μ -generated R -cohomology weak equivalence for all μ . This implies that p is an R -homology weak equivalence by Proposition 5.5.

For the other direction, suppose that p is an acyclic R -homology fibration. Then it is a μ -generated R -cohomology weak equivalence for every μ and a λ -generated R -cohomology fibration for some λ . Thus, p is an acyclic λ -generated R -cohomology fibration for some λ . As in the first paragraph, this implies that p is an acyclic λ -generated R -cohomology fibration for all λ . \square

Theorem 6.7. *The cofibrations, R -homology weak equivalences, and R -homology fibrations are a left proper simplicial model structure on the category of pro-spaces.*

Proof. Most of the proof follows formally from the existence of the λ -generated R -cohomology model structures. Given any collection of classes that satisfy the two-out-of-three axiom, their intersection also satisfies the two-out-of-three axiom. Similarly, arbitrary unions and intersections of classes preserve the retract axiom.

Since Lemma 6.6 tells us what the acyclic R -homology fibrations are, one of the lifting axioms and one of the factoring axioms follows immediately from Theorem 6.4. For the other lifting axiom, suppose that i is an acyclic R -homology cofibration and p is an R -homology fibration. Then there exists some λ such that p is a λ -generated R -cohomology fibration, and i is also an acyclic λ -generated R -cohomology cofibration. Thus i has the left lifting property with respect to p because of Theorem 6.4.

Factorizations into acyclic R -homology cofibrations followed by R -homology fibrations are significantly more difficult. We construct these below in Proposition 6.10.

The simplicial structure also follows formally from the λ -generated model structures of Theorem 6.4. Namely, let $i : A \rightarrow B$ be a cofibration and let $p : X \rightarrow Y$ be an R -homology fibration. Then p is a λ -generated R -homology fibration for some λ , so the map

$$\text{Map}(i, p) : \text{Map}(B, X) \rightarrow \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y)$$

is a fibration of simplicial sets because the λ -generated model structure is simplicial. If i or p is acyclic, then it is a λ -generated R -cohomology weak equivalence, so the

above map $\text{Map}(i, p)$ is also a weak equivalence because of the λ -generated model structure of Theorem 6.4.

Left properness follows immediately from the fact that every object is cofibrant. \square

Functorially in M , choose a fibrant Eilenberg-Mac Lane space $K(M, n)$ for each finitely generated R -module M . Let $p^k(M, n)$ be the fibration $K(M, n)^{\Delta^k} \rightarrow K(M, n)^{\partial\Delta^k}$.

Now for any R -module M , define $K(M, n)$ to be $\text{colim} K(N, n)$, where the colimit ranges over all finitely generated submodules N of M . This construction is a specific fibrant model for the Eilenberg-Mac Lane space of type (M, n) . We also get maps $p^k(M, n)$, which are again fibrations because the colimit is filtered. Note also that $K(M, n)^{\Delta^k}$ is equal to $\text{colim}_{N \leq M} (K(N, n)^{\Delta^k})$ because Δ^k is a finite simplicial set (and similarly for $K(M, n)^{\partial\Delta^k}$).

The constant map $cp^k(M, n) : cK(M, n)^{\Delta^k} \rightarrow cK(M, n)^{\partial\Delta^k}$ is a λ -generated R -cohomology fibration if M is λ -generated. This (or rather its dual) is proved in [CI, Lem. 2.3(b)]; it can also be deduced from Proposition 5.6 using adjointness. Therefore, $cp^k(M, n)$ is an R -homology fibration for every M .

A pro-space X is **λ -bounded** if each space X_s has at most λ elements.

Lemma 6.8. *Let $f : X \rightarrow Y$ be any map between λ -bounded pro-spaces. Then f has the left lifting property with respect to all maps of the form $cp^k(M, n)$ with M any R -module, if and only if f has the left lifting property with respect to all maps of the form $cp^k(M, n)$ with M any λ -generated R -module.*

Proof. One direction is tautological. For the other direction, let f have the left lifting property with respect to all maps of the form $cp^k(M, n)$ with M any λ -generated R -module. Suppose there is a square

$$\begin{array}{ccc} X & \longrightarrow & cK(M, n)^{\Delta^k} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & cK(M, n)^{\partial\Delta^k} \end{array}$$

with M any R -module; we want to produce a lift. This square can be represented as a square

$$\begin{array}{ccc} X_s & \xrightarrow{g} & K(M, n)^{\Delta^k} \\ \downarrow & & \downarrow \\ Y_t & \xrightarrow{h} & K(M, n)^{\partial\Delta^k} \end{array}$$

of ordinary spaces. For each x in X_s , $f(x)$ belongs to $K(P_x, n)^{\Delta^k}$ for some finitely generated submodule P_x of M . Similarly, for each y in Y_t , $g(y)$ lies in $K(P_y, n)^{\partial\Delta^k}$ for some finitely generated submodule P_y of M . Since there are at most λ choices for x and y , we can choose a λ -generated submodule N containing each P_x and

each P_y . Now we have a diagram

$$\begin{array}{ccccc} X & \longrightarrow & cK(N, n)^{\Delta^k} & \longrightarrow & cK(M, n)^{\Delta^k} \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & cK(N, n)^{\partial\Delta^k} & \longrightarrow & cK(M, n)^{\partial\Delta^k} \end{array}$$

for some λ -generated submodule N of M . A lift exists in the left square by assumption, and this gives us the desired lift. \square

The importance of Lemma 6.8 is that the second condition involves only a set of maps of the form $cp^k(M, n)$, while the first condition does not.

Lemma 6.9. *A cofibration $i : A \rightarrow B$ induces an isomorphism in cohomology with coefficients in M if and only if it has the right lifting property with respect to the maps $cp^k(M, n)$.*

Proof. By Proposition 5.6, instead of considering whether $H^n(i; M)$ is an isomorphism for all n , we shall consider whether

$$\text{Map}_{\text{pro}}(i, cK(M, n)) : \text{Map}_{\text{pro}}(B, cK(M, n)) \rightarrow \text{Map}_{\text{pro}}(A, cK(M, n))$$

is a weak equivalence for all n . Since i is a cofibration, this map is a fibration of simplicial sets. It is an acyclic fibration if and only if it has the left lifting property with respect to the maps $\partial\Delta^k \rightarrow \Delta^k$. By adjointness, this lifting property is equivalent to the desired lifting property. \square

Proposition 6.10. *Any map $f : X \rightarrow Y$ of pro-spaces factors into $i : X \rightarrow Z$ followed by $q : Z \rightarrow Y$, where i is an acyclic R -homology cofibration and q is an R -homology fibration.*

Proof. Choose a cardinal λ_0 such that X and Y are both λ_0 -bounded, and let $f_0 = f$. Factor the map f_0 into an acyclic λ_0 -generated R -cohomology cofibration $f_1 : X \rightarrow Z_1$ followed by a λ_0 -generated R -cohomology fibration $Z_1 \rightarrow Y$. Here we are using Theorem 6.4.

Proceeding inductively, choose a cardinal λ_n such that X and Z_n are both λ_n -bounded. Factor the map f_n into an acyclic λ_n -generated cofibration $f_{n+1} : X \rightarrow Z_{n+1}$ followed by a λ_n -generated fibration $Z_{n+1} \rightarrow Z_n$. This process yields a diagram

$$X \rightarrow \cdots \rightarrow Z_2 \rightarrow Z_1 \rightarrow Y,$$

and we let $Z = \lim_n^{\text{pro}} Z_n$, where the limit is computed within the category of pro-spaces.

Choose a cardinal λ_∞ such that $\lambda_\infty \geq \lambda_n$ for each n . Note that X , Y , and each Z_n are λ_∞ -bounded. Using the explicit construction of cofiltered limits in pro-categories given in Section 2.1, it follows that Z is also λ_∞ -bounded.

If $\lambda \leq \mu$, then the μ -generated R -cohomology weak equivalences are contained in the λ -generated R -cohomology weak equivalences. Therefore, the acyclic μ -generated R -cohomology cofibrations are contained in the λ -generated R -cohomology cofibrations. From the nature of lifting properties, it follows that the λ -generated R -cohomology fibrations are contained in the μ -generated R -cohomology fibrations.

Since $\lambda_n \leq \lambda_\infty$, each map $Z_n \rightarrow Z_{n-1}$ (and also $Z_1 \rightarrow Y$) is a λ_∞ -generated R -cohomology fibration. Thus, the map $q : Z \rightarrow Y$ is a composition of a countable tower of λ_∞ -generated R -cohomology fibrations. Formal arguments with lifting properties imply that q is also a λ_∞ -generated R -cohomology fibration. This means that q is an R -homology fibration.

Each map $X \rightarrow Z_n$ is a cofibration. Since cofibrations are closed under cofiltered limits [I1, Cor. 5.3], we conclude that $i : X \rightarrow Z$ is a cofibration. By Lemma 6.9, it remains only to show that i has the left lifting property with respect to the maps $cp^k(M, n)$ for all R -modules M .

Suppose given a square

$$\begin{array}{ccc} X & \longrightarrow & cK(M, n)^{\Delta^k} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & cK(M, n)^{\partial\Delta^k} \end{array}$$

with M a λ_∞ -generated R -module. Using the explicit construction of filtered limits of pro-objects given in Section 2.1, this diagram factors as

$$\begin{array}{ccc} X & \longrightarrow & cK(M, n)^{\Delta^k} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z_j \longrightarrow cK(M, n)^{\partial\Delta^k} \end{array}$$

for some j . Because X and Z_j are both λ_j -bounded, this diagram further factors into

$$\begin{array}{ccccc} X & \longrightarrow & cK(N, n)^{\Delta^k} & \longrightarrow & cK(M, n)^{\Delta^k} \\ \downarrow & & \downarrow & & \downarrow \\ Z_j & \longrightarrow & cK(N, n)^{\partial\Delta^k} & \longrightarrow & cK(M, n)^{\partial\Delta^k} \end{array}$$

as in the proof of Lemma 6.8, where N is a λ_j -generated submodule of M .

Lemma 6.9 tells us that there is a lift in the diagram

$$\begin{array}{ccccc} X & \longrightarrow & cK(N, n)^{\Delta^k} & \longrightarrow & cK(M, n)^{\Delta^k} \\ \downarrow & \nearrow \text{dotted} & \downarrow & & \downarrow \\ Z_{j+1} & \longrightarrow & Z_j \longrightarrow cK(N, n)^{\partial\Delta^k} & \longrightarrow & cK(M, n)^{\partial\Delta^k} \end{array}$$

because $X \rightarrow Z_{j+1}$ is an acyclic λ_j -generated cofibration and N is a λ_j -generated R -module. This gives us the desired lift. \square

7. R -COMPLETIONS

The model structures of Theorems 6.3 and 6.7 allow us to define the R -completion of any pro-space.

Definition 7.1. Let X be a pro-space. The **cohomological R -completion** X_{R-c}^\wedge of X is a fibrant replacement for X in the cohomological model structure of Theorem 6.3. The **homological R -completion** X_{R-h}^\wedge of X is a fibrant replacement for X in the homological model structure of Theorem 6.7.

Philosophically, the cohomological R -completion of a pro-space X should preserve the cohomology of X with coefficients in R but forget all other information. Similarly, the homological R -completion of X should preserve the cohomology of X with coefficients in any R -module but forget all other information. Thus, homological R -completion contains more information than the cohomological R -completion. See Example 5.3 for a pro-space X such that X_{R-h}^\wedge and X_{R-c}^\wedge are distinct.

These ideas are made precise in the following theorem.

Theorem 7.2. *A map $f : X \rightarrow Y$ of pro-spaces (or of ordinary spaces) induces an isomorphism in cohomology with coefficients in R if and only if X_{R-c}^\wedge and Y_{R-c}^\wedge are simplicially homotopy equivalent pro-spaces. Similarly, the map f induces an isomorphism in cohomology with coefficients in all R -modules if and only if X_{R-h}^\wedge and Y_{R-h}^\wedge are simplicially homotopy equivalent pro-spaces.*

Of course, if two pro-spaces are simplicially homotopy equivalent, then they are strictly weakly equivalent or equivalent with respect to any reasonable notion of homotopy theory for pro-spaces.

Proof. The maps $X \rightarrow X_{R-c}^\wedge$ and $Y \rightarrow Y_{R-c}^\wedge$ are R -cohomological weak equivalences. Thus $H^*(f; R)$ is an isomorphism if and only if X_{R-c}^\wedge and Y_{R-c}^\wedge are R -cohomologically weakly equivalent. Since every pro-space is cofibrant, X_{R-c}^\wedge and Y_{R-c}^\wedge are both cofibrant and fibrant with respect to the R -cohomological model structure of Theorem 6.3. Thus, X_{R-c}^\wedge and Y_{R-c}^\wedge are R -cohomologically weakly equivalent if and only if they are simplicially homotopy equivalent.

The argument for homological R -completions is identical except that it uses the R -homological model structure of Theorem 6.7. \square

For any pro-space X such that each X_s is pointed and connected, we will now show how to construct X_{R-h}^\wedge in terms of the Bousfield-Kan R -towers [BK] of the spaces X_s . The moral is that at least for pointed connected spaces, the Bousfield-Kan R -tower (with a minor modification) is the same thing as fibrant replacement in the R -homological model structure.

Proposition 7.3. *Let X be a pro-object in the category of pointed connected spaces, and let I be the indexing category of X . Construct a new pro-space Y with indexing category $I \times \mathbb{N}$ by defining $Y_{s,n}$ to be the n th Postnikov section $P_n R_n X_s$ of the n th stage of the Bousfield-Kan R -tower for X_s . Then the strict fibrant replacement \hat{Y} of Y is a fibrant replacement for X in the R -homological model structure.*

Remark 7.4. In the previous proposition, it is also possible to define $Y_{s,n}$ to be just $R_n X_s$, not its n th Postnikov section. However, then \hat{Y} must be a π_* -fibrant replacement, i.e., a fibrant replacement in the model structure on pro-spaces in which weak equivalences are detected by pro-homotopy groups [I2].

Proof. Let K be the collection of all fibrant spaces of the form $K(M, n)$, where $n \geq 0$ and M is any R -module. From [BK, Cor. III.5.6] and [BK, Prop. III.5.3], we know that the Postnikov tower of $R_n X_s$ can be refined to a sequence of principal fibrations whose fibers belong to K . Therefore, the Postnikov tower of $P_n R_n X_s$ consists of a finite sequence of principal fibrations whose fibers belong to K . Proposition 3.5 tells us that each $P_n R_n X_s$ is K -nilpotent, so Y is essentially levelwise K -nilpotent. It follows from [CI, Prop. 3.7] that \hat{Y} is also essentially levelwise K -nilpotent, so Theorem 3.3 tells us that \hat{Y} is fibrant in the R -homological model structure.

It remains to show that the map $X \rightarrow \hat{Y}$ is an R -homology weak equivalence. Since $Y \rightarrow \hat{Y}$ is a strict weak equivalence, it suffices to show that $X \rightarrow Y$ is an R -homology weak equivalence. We know from [BK, Prop. III.6.5] (or [D]) that $cH_k(X_s; R) \rightarrow H_k(P_*R_*X_s; R)$ is a pro-isomorphism for each s ; here $P_*R_*X_s$ is the pro-space

$$\cdots \rightarrow P_2R_2X_s \rightarrow P_1R_1X_s \rightarrow P_0R_0X_s,$$

and $cH_k(X_s; R)$ is the constant pro-group with value $H_k(X_s; R)$. Now $H_k(X; R)$ is isomorphic to $\lim_s^{\text{pro}} cH_k(X_s; R)$, where the limit is computed within the category of pro-groups. Similarly, $H_k(Y; R)$ is isomorphic to $\lim_s^{\text{pro}} H_k(P_*R_*X_s; R)$ (see the construction of limits in pro-categories given in Section 2.1). Thus, the map $H_k(X; R) \rightarrow H_k(Y; R)$ is a pro-isomorphism. \square

8. \mathbb{Z}/p -COHOMOLOGY

In this section, fix a prime p , and let R be the finite ring \mathbb{Z}/p . We will compare the \mathbb{Z}/p -cohomological model structure of Theorem 6.3 to the \mathbb{Z}/p -cohomological model structure of [Mo] and show that they yield the same homotopy categories. Throughout this section, whenever we discuss the category of pro-simplicial sets, we are always thinking of it equipped with the \mathbb{Z}/p -cohomological model structure.

We first recall some ideas from [Mo]. Let \mathcal{F} be the category of finite sets. The category $\text{pro-}\mathcal{F}$ of pro-finite sets is equivalent to the category of totally disconnected compact Hausdorff topological spaces.

The main object of study in [Mo] is the category $\text{spro-}\mathcal{F}$ of simplicial pro-finite sets (or, equivalently, simplicial totally disconnected compact Hausdorff topological spaces). The weak equivalences in this category are the continuous cohomology isomorphisms with \mathbb{Z}/p -coefficients, the cofibrations are the degreewise monomorphisms, and the fibrations are defined by a lifting property.

The main purpose of this model structure on $\text{spro-}\mathcal{F}$ is to describe a \mathbb{Z}/p -completion functor. Given any set X , let \hat{X} be the pro-finite set of all finite quotients of X . Applying this construction degreewise gives a functor from simplicial sets to simplicial pro-finite sets. The \mathbb{Z}/p -completion of a simplicial set X is defined to be a fibrant replacement for \hat{X} .

In order to compare the category $\text{spro-}\mathcal{F}$ to the category of pro-simplicial sets, we need the intermediate category $\text{pro-}s\mathcal{F}$ of pro-simplicial finite sets. Despite claims in [Mo], [R], and elsewhere, this category is not equivalent to $\text{spro-}\mathcal{F}$. See [I1, Ex. 3.7] for a counterexample. Beware that a simplicial finite set is not the same as a finite simplicial set. A finite simplicial set can only have finitely many non-degenerate simplices, while a simplicial finite set need only be finite degreewise.

There is an inclusion functor $i : \text{pro-}s\mathcal{F} \rightarrow \text{pro-}s\text{Set}$ from pro-simplicial finite sets to pro-simplicial sets. We next define its adjoint.

Definition 8.1. If cX is any constant pro-space, then $\mathbf{Finc}X$ is the system of all simplicial finite quotients of X . If X is an arbitrary pro-space, then $\mathbf{Fin}X$ is $\lim_s^{\text{pro}} \mathbf{Finc}X_s$, where the limit is calculated within the category of pro-simplicial finite sets.

Lemma 8.2. *The functor \mathbf{Fin} is the left adjoint of the inclusion i .*

Proof. We begin by showing that $\mathbf{Finc}X$ is a cofiltered system of spaces. Let X_1 and X_2 be two simplicial finite quotients of X . We need to find another simplicial

finite quotient X_3 of X that refines both X_1 and X_2 . Note that $X_1 \times X_2$ is a simplicial finite set but not necessarily a quotient of X because the canonical map $X \rightarrow X_1 \times X_2$ is not surjective. Define X_3 to be the image of the map $X \rightarrow X_1 \times X_2$. Now X_3 is a simplicial finite set because it is a subobject of $X_1 \times X_2$. The map $X \rightarrow X_3$ is surjective by construction, so X_3 is a simplicial finite quotient of X . The two maps $X \rightarrow X_1$ and $X \rightarrow X_2$ factor through X_3 . This shows that $\text{Finc}X$ is a cofiltered system.

Next we will show that $\text{Finc}X$ has the correct adjoint property. We want to show that $\text{Hom}_{\text{pro}}(\text{Finc}X, Y)$ is isomorphic to $\text{Hom}_{\text{pro}}(cX, Y)$ for every pro-simplicial finite set Y . This follows from the fact that every map from X into a simplicial finite set factors through a simplicial finite quotient of X .

Now let X be an arbitrary pro-simplicial set. The construction of limits in pro-categories given in Section 2.1 implies that

$$\text{Hom}_{\text{pro}}(\lim_a^{\text{pro}} Z^a, cY) = \text{colim}_a \text{Hom}_{\text{pro}}(Z^a, cY)$$

for any cofiltered system $a \mapsto Z^a$ of pro-objects and any constant pro-object cY . The desired adjointness property for $\text{Fin}X$ now follows formally. \square

The adjoint functors Fin and i connect the categories of pro-simplicial sets and pro-simplicial finite sets. Now we have to connect the categories of pro-simplicial finite sets and simplicial pro-finite sets. As described explicitly in [I1, § 3], there are functors $F : \text{pro-}s\mathcal{F} \rightarrow \text{spro-}\mathcal{F}$ and $G : \text{spro-}\mathcal{F} \rightarrow \text{pro-}s\mathcal{F}$ such that G is the left adjoint of F and such that the composition FG is naturally isomorphic to the identity on $\text{spro-}\mathcal{F}$. This uses the fact that the category of simplicial finite sets is small and that the simplicial indexing category Δ^{op} has finite morphism sets. In other words, the category $\text{spro-}\mathcal{F}$ is a retract of the category $\text{pro-}s\mathcal{F}$. As observed above, F and G are not inverse equivalences of categories because GF is not naturally isomorphic to the identity functor.

The construction of G is complicated; fortunately we will not need the details here. For later reference, we describe the functor F . Let X be a pro-simplicial finite set. For each $n \geq 0$, X_n is a pro-finite set. Thus, $[n] \mapsto X_n$ is a simplicial pro-finite set, and this is FX .

In order to pass between pro-simplicial sets and simplicial pro-finite sets, we use the compositions $F \circ \text{Fin}$ and $i \circ G$. Unfortunately, these functors are the composition of a left adjoint and a right adjoint. Thus, they do not have nice adjointness properties. This means that we will not be able to produce a Quillen equivalence [Hi, Defn. 8.5.20] between $\text{pro-}s\text{Set}$ and $\text{spro-}\mathcal{F}$.

One might hope that there is a \mathbb{Z}/p -cohomology model structure on the intermediate category $\text{pro-}s\mathcal{F}$. Then there would be a zig-zag of Quillen equivalences

$$\text{pro-}s\text{Set} \rightleftarrows \text{pro-}s\mathcal{F} \rightleftarrows \text{spro-}\mathcal{F}.$$

However, since the category of simplicial finite sets is not a model category, the techniques used in this paper do not seem to apply. Possibly there is another approach altogether.

In the absence of a Quillen equivalence, we have to show directly that the homotopy categories $\text{Ho}(\text{pro-}s\text{Set})$ and $\text{Ho}(\text{spro-}\mathcal{F})$ are equivalent.

Lemma 8.3. *A map f of pro-simplicial finite sets is a \mathbb{Z}/p -cohomology isomorphism if and only if Ff is a \mathbb{Z}/p -cohomology isomorphism of simplicial pro-finite*

sets. A map g of simplicial pro-finite sets is a \mathbb{Z}/p -cohomology isomorphism if and only if Gg is a \mathbb{Z}/p -cohomology isomorphism of pro-simplicial finite sets.

Proof. Let X be a pro-simplicial finite set. The cochain complex C^*X used to compute $H^*(X; \mathbb{Z}/p)$ is given by $C^n X = \operatorname{colim}_s \operatorname{Hom}((X_s)_n, \mathbb{Z}/p)$. Using the description of the functor F above, we see that this is equal to the cochain complex used to compute $H^*(FX; \mathbb{Z}/p)$. This proves the first claim.

For the second claim, let Y be a simplicial pro-finite set. We want to show that Y and GY have naturally isomorphic \mathbb{Z}/p -cohomology. By the previous paragraph, it suffices to compare FY and FGY . Now FGY is isomorphic to Y , so we just need to use the previous paragraph again. \square

We have observed that the functor GF is not well-behaved categorically. Nevertheless, it does have good cohomological properties.

Corollary 8.4. *The counit natural transformation from the functor GF to the identity functor on $\operatorname{pro}\text{-}s\mathcal{F}$ is a natural \mathbb{Z}/p -cohomology isomorphism.*

Proof. If X is any pro-simplicial finite set, both parts of the proof of Lemma 8.3 imply that $H^*(X; \mathbb{Z}/p)$ is isomorphic to $H^*(GF X; \mathbb{Z}/p)$. \square

Lemma 8.5. *Let $f : X \rightarrow Y$ be a map between pro-simplicial sets. Then f is a \mathbb{Z}/p -cohomology isomorphism if and only if $\operatorname{Fin}(f)$ is a \mathbb{Z}/p -cohomology isomorphism.*

Proof. We will show that for every pro-simplicial set X , the natural map $X \rightarrow \operatorname{Fin} X$ is a \mathbb{Z}/p -cohomology isomorphism. Because of Lemma 5.1 and the definition of Fin , it suffices to assume that X is a simplicial set. We must show that the natural map

$$\operatorname{colim}_Y H^n(Y; \mathbb{Z}/p) \rightarrow H^n(X; \mathbb{Z}/p)$$

is an isomorphism, where Y ranges over all simplicial finite quotients of X . To do this, we consider reduced cochain complexes given in degree n by functions into \mathbb{Z}/p from the non-degenerate part NX_n of X in degree n .

To show that the map of reduced cochain complexes is surjective, consider an arbitrary cochain α , which is just a function $NX_n \rightarrow \mathbb{Z}/p$. We need to construct a simplicial finite quotient X' of X and a cochain α' on X' that pulls back to α . Begin by defining an n -dimensional simplicial set Y whose $(n-1)$ -skeleton is trivial and whose non-degenerate n -simplices correspond to the elements of \mathbb{Z}/p . There is an obvious map $\operatorname{sk}_n X \rightarrow Y$ induced by α . Adjointness gives a map $X \rightarrow \operatorname{cosk}_n Y$. Since Y is a simplicial finite set, so is $\operatorname{cosk}_n Y$. Finally, take X' to be the image in $\operatorname{cosk}_n Y$ of X .

To show that the map of reduced cochain complexes is injective, suppose that X' and X'' are two simplicial finite quotients of X , and let α' and α'' be reduced cochains on X' and X'' respectively that pull back to the same reduced cochain α on X . There exists a simplicial finite quotient Y of X refining both X' and X''

(see the proof of Lemma 8.2). We now have the diagram

$$\begin{array}{ccccc}
 X & & & & \\
 \searrow & & & & \\
 & Y & \longrightarrow & X' & \\
 & \downarrow & & \downarrow \alpha' & \\
 & X'' & \xrightarrow{\alpha''} & \mathbb{Z}/p & \\
 & & & &
 \end{array}$$

in which the outer quadrilateral and the two triangles are commutative. We want to show that the square is also commutative. This follows from the fact that the map $X \rightarrow Y$ is surjective. \square

Proposition 8.6. *The functor $F \circ \text{Fin}$ induces a functor*

$$\text{Ho}(\text{pro-}s\text{Set}) \rightarrow \text{Ho}(\text{spro-}\mathcal{F})$$

on homotopy categories, and the functor $i \circ G$ induces a functor

$$\text{Ho}(\text{spro-}\mathcal{F}) \rightarrow \text{Ho}(\text{pro-}s\text{Set})$$

on homotopy categories.

Proof. By the universal property of localizations of categories, it suffices to show that the two functors take weak equivalences to weak equivalences. Let f be any \mathbb{Z}/p -cohomology isomorphism of pro-simplicial sets. Lemmas 8.3 and 8.5 imply that $F \circ \text{Fin}f$ is a \mathbb{Z}/p -cohomology isomorphism. Hence, $F \circ \text{Fin}$ preserves weak equivalences. For $i \circ G$, this is the second part of Lemma 8.3. \square

Theorem 8.7. *The functors $F \circ \text{Fin}$ and $i \circ G$ induce inverse equivalences between the homotopy categories $\text{Ho}(\text{pro-}s\text{Set})$ and $\text{Ho}(\text{spro-}\mathcal{F})$.*

Proof. The composition $(F \circ \text{Fin}) \circ (i \circ G)$ is isomorphic to the identity because $\text{Fin} \circ i$ is the identity by construction of Fin and because $F \circ G$ is isomorphic to the identity. On the other hand, Corollary 8.4 and Lemma 8.5 tell us that for every pro-simplicial set X , there are natural weak equivalences

$$X \xrightarrow{\sim} \text{Fin}X \xleftarrow{\sim} GF \circ \text{Fin}X.$$

Thus X and $(i \circ G) \circ (F \circ \text{Fin})X$ are naturally isomorphic in $\text{Ho}(\text{pro-}s\text{Set})$. \square

9. FREE GROUPS

The point of this section is to describe both the cohomological and homological \mathbb{Z}/p -completions of the Eilenberg-Mac Lane space $K(F_n, 1)$, where F_n is the free group on n generators. This means that we need to find a fibrant replacement for $cK(F_n, 1)$ in the \mathbb{Z}/p -cohomological model structure and the \mathbb{Z}/p -homological model structure. As we will see below, these two completions turn out to be the same.

Part of the definition of these fibrant replacements requires that the pro-space be strictly fibrant. For the rest of this section, we will drop this requirement. This change preserves the homotopy type of each space in the cofiltered system because of the nature of strict weak equivalences. Thus, for calculational purposes we do not really need the strict fibrancy.

Theorem 9.1. *Consider the system $\{K(F_n/H, 1)\}$ as H ranges over all normal subgroups of F_n such that F_n/H is a finite p -group; the structure maps are induced by the canonical quotient maps. This pro-space is the fibrant replacement for $K(F_n, 1)$ in either the \mathbb{Z}/p -cohomological or \mathbb{Z}/p -homological model structure.*

To be precise, we really should also produce a map $cK(F_n, 1) \rightarrow X$ that induces an isomorphism in \mathbb{Z}/p -cohomology. We will not worry about this because the map will be obvious and natural in everything that we do.

Proof. For notational convenience, write X_H for the space $K(F_n/H, 1)$. We need to show that $H^*(X; M)$ is isomorphic to $H^*(F_n; M)$ for every \mathbb{Z}/p -module M . This will show that X and $K(F_n, 1)$ are weakly equivalent in both the \mathbb{Z}/p -cohomological and \mathbb{Z}/p -homological model structures. We also have to show that each space X_H is nilpotent in the sense of Definition 3.1 with respect to the class of Eilenberg-Mac Lane spaces of the form $K(\mathbb{Z}/p, n)$. This will show that X is fibrant in both model structures.

First of all, the diagram X is a cofiltered system because $F_n/(H \cap K)$ is a finite p -group whenever F_n/H and F_n/K are finite p -groups.

Since $K(F_n, 1)$ is just a wedge of n circles, $H^q(F_n; \mathbb{Z}/p)$ is isomorphic to \mathbb{Z}/p in dimension 0; to $(\mathbb{Z}/p)^n$ in dimension 1; and to the zero group otherwise. This tells us exactly what the cohomology of X should be.

Let Z be any connected space whose homotopy groups are finite p -groups. Then Z is \mathbb{Z}/p -nilpotent in the sense of Bousfield and Kan. This can be proved by showing that if G is a finite p -group acting on another finite p -group A , then G acts nilpotently. If in addition Z has only finitely many non-zero homotopy groups, then Z must be K -nilpotent in the sense of in the sense of Definition 3.1 because of [BK, Prop. III.5.3] and Proposition 3.5.

Since each X_H satisfies the hypotheses in the previous paragraph, we conclude that each X_H is K -nilpotent. It remains to calculate the \mathbb{Z}/p -cohomology of X . This is done below in Lemma 9.4. \square

Lemma 9.2. *There exists a normal subgroup H of F_n such that F_n/H is a finite p -group and such that the map $H^1(F_n; M) \rightarrow H^1(H; M)$ is the zero map for all \mathbb{Z}/p -modules M .*

Proof. By an argument similar to the one given after Definition 5.2, it suffices to consider the case $M = \mathbb{Z}/p$. Let H be the kernel of the homomorphism $F_n \rightarrow (\mathbb{Z}/p)^n$ that is the composition of abelianization with reduction modulo p . Now $K(H, 1) \rightarrow K(F_n, 1)$ is a covering map of degree p^n . More concretely, $K(H, 1)$ is the Cayley graph of the group $(\mathbb{Z}/p)^n$ relative to the standard basis. Thus, $K(H, 1)$ has one vertex for each element of $(\mathbb{Z}/p)^n$. The edges of $K(H, 1)$ are of the form v to $v + e_i$, where v is any element of $(\mathbb{Z}/p)^n$ and e_i is any element of the standard basis.

We use a wedge of n circles as our model for $K(F_n, 1)$. Let α be a 1-cocycle on $K(F_n, 1)$ whose value on the i th circle of $K(F_n, 1)$ is the element α_i of \mathbb{Z}/p . Let β be the 1-cocycle on $K(H, 1)$ induced by α . The value of β on the edge from v to $v + e_i$ is α_i .

We construct a 0-cocycle γ whose coboundary is β . Let the value of γ on the vertex (v_1, \dots, v_n) of $K(H, 1)$ equal $\alpha_1 v_1 + \dots + \alpha_n v_n$. Thus, β is zero in cohomology, which means that the desired map is zero. \square

For each normal subgroup H of F_n such that F_n/H is a finite p -group, we have a short exact sequence

$$H \rightarrow F_n \rightarrow F_n/H$$

which gives rise to a fiber sequence

$$K(H, 1) \rightarrow K(F_n, 1) \rightarrow K(F_n/H, 1).$$

Each such sequence has an associated cohomological Serre spectral sequence

$$E_2^{st} = H^s(F_n/H; \mathcal{H}^t(H; M)) \Rightarrow H^{s+t}(F_n; M),$$

where $\mathcal{H}^t(H; M)$ is a local system on $K(F_n/H, 1)$. Since the Serre spectral sequence is natural and since filtered colimits respect filtrations, we can take colimits everywhere and get another spectral sequence

$$E_2^{st} = \operatorname{colim}_H H^s(F_n/H; \mathcal{H}^t(H; M)) \Rightarrow H^{s+t}(F_n; M).$$

Recall how the structure maps of the colimit in the above formula are constructed. If K is a subgroup of H , let π be the projection $F_n/K \rightarrow F_n/H$. The map

$$H^s(F_n/H; \mathcal{H}^t(H; M)) \rightarrow H^s(F_n/K; \mathcal{H}^t(K; M))$$

is the composition

$$H^s(F_n/H; \mathcal{H}^t(H; M)) \rightarrow H^s(F_n/K; \pi^* \mathcal{H}^t(H; M)) \rightarrow H^s(F_n/K; \mathcal{H}^t(K; M)),$$

where $\pi^* \mathcal{H}^t(H; M)$ is a pullback of local systems.

Lemma 9.3. *The group $\operatorname{colim}_H H^s(F_n/H; \mathcal{H}^t(H; M))$ is zero unless $t = 0$.*

Proof. For $t \geq 2$, each local system $\mathcal{H}^t(H; M)$ is zero because H is a free group. It only remains to consider the case $t = 1$. Because H is free, Lemma 9.2 implies that there exists a subgroup K such that the map

$$\pi^* \mathcal{H}^1(H; M) \rightarrow \mathcal{H}^1(K; M)$$

of local systems is zero. This gives the desired result for $t = 1$. \square

Lemma 9.4. *The map $\operatorname{colim}_H H^q(F_n/H; M) \rightarrow H^q(F_n; M)$ is an isomorphism, where the colimit ranges over all normal subgroups of F_n such that F_n/H is a finite p -group.*

Proof. By the previous lemma, the E_2 -term of the Serre spectral sequence described above is concentrated on the line $t = 0$. This gives the desired isomorphism. \square

10. QUESTIONS

The work in this paper leaves some obvious further questions unanswered. We mention a few of these here in the interest of encouraging future work on the subject.

Question 10.1. *Are the model structures of Theorems 1.1 and 1.2 are right proper?*

The general machinery of localizations does not automatically produce right proper model structures. Presumably the Serre spectral sequence is the way to approach this problem, but one has to deal with twisted coefficients.

Question 10.2. *If X is a space considered as a constant pro-space, how do its two fibrant replacements compare?*

We know that the model structures of Theorems 1.1 and 1.2 are distinct. However, in Section 9 we showed that the two fibrant replacements of $K(F_n, 1)$ are the same. It is easy to imagine that this would generalize to any space X with some kind of finiteness hypothesis on the cohomology of X .

Question 10.3. *If the ground ring R is \mathbb{Z}/p , what is the difference between the two fibrant replacements of a pro-space that is an étale topological type?*

Certain kinds of pro-spaces are more relevant to applications than others. The pro-spaces that arise as étale topological types of well-behaved schemes [AM] [F] are particularly interesting. Perhaps theorems in algebraic geometry about étale cohomology with finite coefficients can be used to conclude that the two fibrant replacements are the same.

Question 10.4. *Let R be an infinite ring. If X is a space such that each component is nilpotent in the sense of Bousfield and Kan and such that the size of $\pi_0 X$ is no bigger than the size of R , can we conclude that X is also nilpotent in the sense of Section 3?*

Nilpotence in the sense of Section 3 is definitely distinct from Bousfield-Kan nilpotence. If X is nilpotent in the sense of Section 3, then each component of X is nilpotent in the sense of Bousfield and Kan, and $\pi_0 X$ cannot be bigger than R . The question is whether the implication works in reverse.

Question 10.5. *Let R be a finite ring. If X is a space such that each component is nilpotent in the sense of Bousfield and Kan and such that $\pi_0 X$ is finite, can we conclude that X is also nilpotent in the sense of Section 3?*

This question is just a minor variation on Question 10.4. Again, we know that nilpotence implies the given conditions.

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