ETALE REALIZATION ON THE $\mathbb{A}^1$-HOMOTOPY THEORY OF SCHEMES

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ABSTRACT. We compare Friedlander's definition of étale homotopy for simplicial schemes to another definition involving homotopy colimits of pro-simplicial sets. This can be expressed as a notion of hypercover descent for étale homotopy. We use this result to construct a homotopy invariant functor from the category of simplicial presheaves on the étale site of schemes over S to the category of pro-spaces. After completing away from the characteristics of the residue fields of S, we get a functor from the Morel-Voevodsky $\mathbb{A}^1$-homotopy category of schemes to the homotopy category of pro-spaces.

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1. Introduction

In the recent proof of the Milnor conjecture [31], a certain realization functor from the $\mathbb{A}^1$-homotopy category of schemes over $\mathbb{C}$ [25] to the ordinary homotopy category of spaces plays a useful role. The basic idea is to detect that a certain map in the stable $\mathbb{A}^1$-homotopy category is not homotopy trivial by checking that its image in the ordinary stable homotopy category is not homotopy trivial.

This realization functor is defined by extending the notion of the underlying analytic space of a complex variety, so we call it analytic realization. The analytic realization functor as defined in [25, § 3.3] has two shortcomings. First, it is defined directly on the homotopy categories. It would be much preferable to have a functor on the point-set level that is homotopy invariant and therefore induces a functor on the homotopy categories.

The second shortcoming is that the analytic realization does not work over fields with positive characteristic. Varieties over abstract fields have no underlying analytic topology. However, the étale topological type [2] [9] is a substitute. In characteristic zero, the étale topological type $\mathrm{Et}X$ of a variety X is the pro-finite completion of the underlying analytic space of X. In any characteristic, $\mathrm{Et}X$ carries information about the étale cohomology of X and the algebraic fundamental group of X.

The main goal of this paper to fix the second of these two shortcomings. Because the category of pro-spaces is complicated, it is essential to use model structures to establish the functor on the homotopy categories. Using a model structure for $\mathbb{A}^1$-homotopy theory slightly different than the one in [25], the étale topological type provides a functor from the category of simplicial presheaves on the Nisnevich site of smooth schemes over S to the category of pro-spaces. This functor is a left Quillen functor, which means that it automatically gives a functor on the homotopy categories.

The étale realization functor provides a calculational tool for $\mathbb{A}^1$-homotopy theory of schemes over fields of positive characteristic. In future work, we hope to take Galois group actions into account to obtain a realization functor into a homotopy category of equivariant pro-spaces. However, the foundations for a suitable equivariant homotopy theory of pro-spaces have not yet been established. We also hope to stabilize our techniques to obtain a functor on stable $\mathbb{A}^1$-homotopy theory. Although some progress on the foundations of the homotopy theory of pro-spectra has been made [5] [18], it is not yet clear whether these theories are suitable for the current application.

In this paper, the first shortcoming of the analytic realization described above remains unfixed. We plan to fill this gap in future work with Dan Dugger. The general program of this paper will work provided that for any open hypercover U of a topological space X, the geometric realization of U is weakly equivalent to X. This generalizes a result of [28, § 4] about Cech complexes of open covers.

A summary of the contents of the paper follows. Section 2 begins with a review of simplicial presheaves and their homotopy theory. We assume familiarity with closed model structures. General references on this topic include [13], [14], or [26]. We conform to the conventions of [13] as closely as possible. See also [6] for more details on model structures as applied to simplicial presheaves. Next comes the definition of the étale realization functor, and the first major result is that it is homotopy invariant on the local projective model structure for simplicial presheaves on the étale site. Specializing to the Nisnevich site of smooth schemes, étale realization is also homotopy invariant with respect to $\mathbb{A}^1$-weak equivalences but only after completing away from the characteristics of the residue fields of the base scheme S. The reason for this completion is that $\mathrm{Et}\mathbb{A}^1$ is non-trivial in positive characteristic. Completion takes care of this problem.

Section 2 closes with a corollary concerning the behavior of the étale topological type on elementary distinguished squares. This result can be interpreted as excision for étale topological types.

This finishes the main thrust of the paper. The remaining sections are dedicated to developing language and machinery suitable for proving Theorem 5.4, which is a key ingredient in the earlier sections. This theorem states that if U is a hypercover of a scheme X, then the natural map

$$\operatorname*{hocolim}_n \mathrm{Et}U_n \to \mathrm{Et}X$$

of pro-spaces is a weak equivalence.

Section 3 is dedicated to the study of rigid hypercovers. First come some technical results about finite limits of schemes and an introduction to the language of simplicial schemes. Next is the definition hypercovers and rigid hypercovers and then some redefinitions and clarifications of the constructions concerning the étale topological type that first appeared in [9].

Section 4 concerns pro-spaces. We review only the bare essentials of pro-spaces and their homotopy theory. See [15] for details. Some results from [16] on calculating colimits of pro-spaces are also necessary. A k-truncated realization functor is a necessary tool because the infinite colimits that are used to construct ordinary realizations are hard to handle in the category of pro-spaces.

Finally, Section 5 gives the hypercover descent theorem for étale topological types.

We make a few final remarks on terminology. We always mean simplicial sets [21] whenever we refer to spaces. We write *sSet* for the category of simplicial sets.

An étale map $U \to X$ is any map such that U is a (possibly infinite) disjoint union of schemes U^i and each map $U^i \to X$ is étale. This allows us to discuss étale covers in terms of single maps rather than collections.

Throughout, we assume that the base scheme S is locally Noetherian. Since all of our schemes are locally of finite type over S, every scheme that we consider is locally Noetherian. This is a technical requirement for the machinery of étale topological types [9, Ch. 4].

2. Etale Realizations

2.1. **Simplicial Presheaves.** Let S be a locally Noetherian scheme. Consider the big étale site $\mathbf{Sch}/\boldsymbol{S}$ of S [23, § II.1]. The objects of this category are schemes locally of finite type over S. Morphisms in Sch/S are just morphisms of schemes over S. Covers in this category are collections of étale maps that have surjective images. The site Sch/S is suitable for studying étale cohomology in the sense that for every X in Sch/S, the étale cohomology functors $H^*_{\mathrm{et}}(X;\cdot)$ are the derived functors of the functor taking a presheaf F to its group $F(X)$ of sections over X.

Let $\boldsymbol{sPre}(\mathbf{Sch}/\boldsymbol{S})$ be the category of simplicial presheaves on Sch/S. Objects of $sPre(\mathrm{Sch}/S)$ are contravariant functors from Sch/S to simplicial sets; equivalently, they are simplicial objects in the category of set-valued presheaves on Sch/S. Morphisms of $sPre(\mathrm{Sch}/S)$ are natural transformations of functors.

2.1.1. *Objectwise Model Structures on* $sPre(\mathrm{Sch}/S)$. Let us recall several model structures for $sPre(\mathrm{Sch}/S)$. In the first two, the weak equivalences are maps of presheaves $F \to G$ such that $F(X) \to G(X)$ is a weak equivalence for every X in Sch/S; we call such maps **objectwise weak equivalences**. An **injective cofibration** is an objectwise cofibration. An **injective fibration** is a map of presheaves having the right lifting property with respect to all objectwise acyclic injective cofibrations.

On the other hand, a **projective fibration** is an objectwise fibration, and a **projective cofibration** is a map of presheaves having the left lifting property with respect to all objectwise acyclic projective fibrations.

Theorem 2.1. [4, Prop. XI.8.1] [12, § II.4] *The definitions of objectwise weak equivalences, injective cofibrations, and injective fibrations satisfy the axioms for a simplicial proper model structure. The definitions of objectwise weak equivalences, projective cofibrations, and projective fibrations satisfy the axioms for a simplicial proper model structure.*

These are the **objectwise injective** and **objectwise projective** model structures respectively. The simplicial structure comes from objectwise tensoring and cotensoring.

Both model structures for represent the same homotopy category. This means we can use whichever model structure is most convenient for a particular purpose. For example, the injective cofibrations are simple to describe. In particular, every object is injective cofibrant. This is sometimes a convenient property. However, the price for this convenience is that there is no explicit description of the fibrations.

On the other hand, the projective fibrations are simple to describe, but the projective cofibrations are more complicated. Not every object is projective cofibrant. However, there is still a partially explicit description of the projective cofibrations. For every X in Sch/S, let the **representable presheaf** $\boldsymbol{rX}$ be the presheaf given by the formula

$$rX(Y) = \mathrm{Hom}_S(Y, X).$$

Note that rX is discrete in the sense that each space $rX(Y)$ is discrete. More generally, if X is a simplicial scheme over S, then rX is the (not necessarily discrete) presheaf given by the formula

$$rX(Y)_n = \mathrm{Hom}_S(Y, X_n).$$

Proposition 2.2. *Every map of the form* $rX \otimes \partial\Delta[k] \to rX \otimes \Delta[k]$ *is a projective cofibration. The maps of this form are a set of generating projective cofibrations.*

Proof. By the Yoneda lemma, a map $F \to G$ of simplicial presheaves has the right lifting property with respect to the map $rX \otimes \partial\Delta[k] \to rX \otimes \Delta[k]$ if and only if the map $F(X) \to G(X)$ of simplicial sets has the right lifting property with respect to the map $\partial\Delta[k] \to \Delta[k]$. A map of simplicial sets has the right lifting property with respect to the maps $\partial\Delta[k] \to \Delta[k]$ if and only if it is an acyclic fibration. □

2.1.2. *Local Model Structures on* $sPre(\mathrm{Sch}/S)$. The two objectwise structures of the previous section are intermediate stages to constructing the two model structures of chief interest. Usually, the weak equivalences of these interesting structures are defined using sheaves of homotopy groups [19, § 2] or weak equivalences of stalks.

Our approach is to define the local weak equivalences by a left Bousfield localization [13, Defn. 3.2.1] because this serves our particular purposes best. We follow [6] in this viewpoint.

Start with the objectwise projective model structure and define a set T of maps that are to become weak equivalences. First, let T contain the map $rU \to rX$ for every hypercover U of X. See Definition 3.12 for the definition of hypercovers. Second, let T contain the map $\coprod rV_i \to r(\coprod V_i)$ for every collection of objects in Sch/S.

Definition 2.3. The local projective model structure on $sPre(\mathrm{Sch}/S)$ is the T-localization of the objectwise projective model structure in the sense of [6, Defn. 5.2]. The **local weak equivalences** are the weak equivalences in this model structure.

The following proposition tells us that our definition of local weak equivalences is the same as the usual one.

Proposition 2.4. [7] *The local weak equivalences are the same as the topological weak equivalences of* [19, § 2].

Remark 2.5. Note that the same kind of T-localization can be applied to the objectwise injective model structure. We do not use this theory.

The local projective cofibrations are the same as the projective cofibrations. A **local projective fibration** is a map of presheaves that has the right lifting property with respect to all local acyclic projective cofibrations.

2.2. **Etale Realization on $sPre(\mathbf{Sch}/S)$.** The étale topological type functor from Sch/S to pro-spaces gives a method for constructing a homotopy invariant functor from $sPre(\mathrm{Sch}/S)$ to pro-spaces. See Section 5 or [9] for the definition and properties of this functor.

Recall the following constructions from [6, Prop. 2.1]. Suppose given any functor $F : \mathrm{Sch}/S \to \mathcal{C}$, where $\mathcal{C}$ is a simplicial cocomplete category. Define the **singular complex** functor $S^F\colon \mathcal{C} \to sPre(\mathrm{Sch}/S)$ by the formula

$$S^F Z(X) = \mathrm{Map}_{\mathcal{C}}(FX, Z).$$

The **realization** functor Re^F is the left adjoint of S^F. Moreover, the functors Re^F and S^F are simplicial adjoints in the sense that

$$\mathrm{Map}(Re^F G, Z) \cong \mathrm{Map}(G, S^F Z)$$

for every simplicial presheaf G and every object Z of $\mathcal{C}$.

For any X in Sch/S, note that $Re^F(rX) = FX$. In fact, Re^F is the unique colimit-preserving functor with this property. This follows from the fact that every presheaf is a colimit of representable presheaves.

The above paragraphs apply in particular to the functor $\mathrm{Et} : \mathrm{Sch}/S \to \text{pro-}sSet$ that takes a scheme X to the étale topological type [9, Defn. 4.4] of the constant simplicial scheme cX. Since pro-$sSet$ is a simplicial cocomplete category [15], we can apply the constructions of the previous paragraphs to obtain adjoint functors

$$Re^{\mathrm{et}} : sPre(\mathrm{Sch}/S) \to \text{pro-}sSet$$

and

$$S^{\mathrm{et}} : \text{pro-}sSet \to sPre(\mathrm{Sch}/S).$$

For every scheme X, the pro-space $Re^{\mathrm{et}}(rX)$ is equal to $\mathrm{Et}X$.

The next theorem is one of the main results of this paper.

Theorem 2.6. *With respect to the local projective model structure on* $sPre(\mathrm{Sch}/S)$ *and the model structure on* pro-$sSet$ *given in* [15], *the functors* Re^{et} *and* S^{et} *form a Quillen pair.*

Remark 2.7. The theorem is not true if we consider the local injective model structure on $sPre(\mathrm{Sch}/S)$. There are too many injective cofibrations.

Proof. By the universal property of T-localizations [6, Defn. 5.2], we need only show that Re^{et} takes maps in T to weak equivalences. Cofibrant replacements are no problem because the targets and sources of every map in T are already projective cofibrant. To show that rU is projective cofibrant for every hypercover U, use Lemma 3.13 to conclude that U is a split simplicial scheme.

First consider a collection $\{V_i\}$ of objects in Sch/S. Since Re^{et} commutes with coproducts of simplicial presheaves and Et commutes with coproducts of schemes [9, Prop. 5.2], it follows that $Re^{\mathrm{et}}(\coprod rV_i)$ and $Re^{\mathrm{et}}r(\coprod V_i)$ are both isomorphic to the pro-space $\coprod \mathrm{Et}V_i$.

Next consider a hypercover U of X. The simplicial presheaf rU is isomorphic to the realization $|n \mapsto rU_n|$. Since Re^{et} is a simplicial left adjoint, it commutes with realizations; therefore, $Re^{\mathrm{et}}rU$ is equal to the realization $|n \mapsto \mathrm{Et}U_n|$ in the category of pro-spaces. By Theorem 5.4, the map $Re^{\mathrm{et}}rU \to Re^{\mathrm{et}}rX$ is a weak equivalence of pro-spaces. □

The point of the previous theorem is that Re^{et} induces a homotopy invariant derived functor LRe^{et}. In order to evaluate LRe^{et} on a simplicial presheaf, one first takes a projective cofibrant replacement for the presheaf and then apply Re^{et}.

Corollary 2.8. *The functor* LRe^{et} *induces a functor from the local homotopy category of simplicial presheaves to the homotopy category of pro-spaces. On the level of homotopy categories, it has a right adjoint* RS^{et}*. Moreover,* $LRe^{et}(rX) = EtX$ *for every scheme* X *in* Sch/S.

Proof. The first two claims follow from the formal machinery of Quillen adjoint functors [13, § 8.6]. The last claim follows from the construction of LRe^{et} and the fact that every representable presheaf is projective cofibrant. □

The following corollary implies that the original definition of the étale topological type is recovered by the functor LRe^{et}.

Corollary 2.9. *Let* X *be a simplicial scheme in* Sch/S*. Then* $LRe^{et}rX$ *is weakly equivalent to* EtX*.*

Proof. The simplicial presheaf rX is equal to the realization $|n \mapsto rX_n|$. This realization is weakly equivalent to $\mathrm{hocolim}_n\, rX_n$ only if $n \to rX_n$ is a Reedy cofibrant diagram [13, Defn. 16.3.3] of simplicial presheaves. This diagram is *not* Reedy cofibrant with respect to the local projective model structure because X is not necessarily a split simplicial scheme. However, the diagram is Reedy cofibrant with respect to the local injective model structure. Since the local injective model structure and the local projective model structure have the same homotopy categories, we are entitled to use either model structure to construct homotopy colimits. Therefore, rX is weakly equivalent to the simplicial presheaf $\mathrm{hocolim}_n\, rX_n$. Since homotopy colimits commute with left derived functors, it follows that $LRe^{\mathrm{et}}rX$ is weakly equivalent to $\mathrm{hocolim}_n\, \mathrm{Et}X_n$. This homotopy colimit is weakly equivalent to $\mathrm{Et}X$ by Theorem 5.3. □

2.3. **Etale Realization on the $\mathbb{A}^1$-Homotopy Category of Schemes.** The étale realization functor Re^{et} also behaves well with respect to $\mathbb{A}^1$-local model structures [25]. Begin with the projective objectwise model structure on the site Sm/S of smooth schemes over S equipped with Nisnevich covers and define a set T' of maps that are to become weak equivalences. First, let T' contain the map $rU \to rX$ for every Nisnevich hypercover U of X. Nisnevich hypercovers are defined similarly to hypercovers except that we use Nisnevich covers, not étale covers. Since Nisnevich covers are a special kind of étale cover, Theorem 5.4 applies to them. Second, let T' contain the map $\coprod rV_i \to r(\coprod V_i)$ for every collection of objects in Sm/S. Third, let T' contain the map $r(X \times \mathbb{A}^1) \to rX$ for every scheme X in Sm/S.

Definition 2.10. The $\mathbb{A}^1$-local projective model structure on $sPre(\text{Sm}/S)$ is the T'-localization of the objectwise projective model structure in the sense of [6, Defn. 5.2]. The **$\mathbb{A}^1$-local weak equivalences** are the weak equivalences in this model structure.

These $\mathbb{A}^1$-local weak equivalences agree with the definition of [25, Defn. 3.2.1]. This follows from Proposition 2.4 and the fact that the $\mathbb{A}^1$-local model structure of [25] is defined as a localization.

Remark 2.11. The same process yields the T'-localization of the objectwise injective model structure; this leads to an $\mathbb{A}^1$-local injective model structure, which is exactly the model structure presented in [25]. In this paper we use the $\mathbb{A}^1$-local projective model structure on Sm/S; although the model structures are different, the homotopy categories are the same.

Remark 2.12. The **Nisnevich-local** projective model structure is formed by starting with the objectwise projective model structure on $sPre(\text{Sm}/S)$ and inverting all the maps in T' except for the projections $X \times \mathbb{A}^1 \to X$. This is the Nisnevich version of the local projective model structure considered in Section 2.1.2. Of course, the $\mathbb{A}^1$-local projective model structure is the localization of the Nisnevich-local projective structure at the projections $X \times \mathbb{A}^1 \to X$.

The importance of Nisnevich hypercovers here suggests that a "Nisnevich topological type" may give an interesting realization functor on $sPre(\text{Sm}/S)$. However, it turns out that the Nisnevich topological type of any scheme is always homotopy discrete. The problem is that the Nisnevich topological type only captures Nisnevich cohomology with locally constant coefficients, but these always vanish. Therefore, we continue to use the étale topological type.

Now we discuss the $\mathbb{A}^1$-homotopy invariance of the étale realization functor. In order to have an $\mathbb{A}^1$-homotopy invariant functor, it is necessary to complete away from the characteristics of the residues fields of S. We use here a functorial model for $\mathbb{Z}/p$-completion of pro-spaces as described in [17], where p is a prime not occurring as a characteristic of a residue field of S. Below is a summary of the details of this construction.

Let X be a pointed space. Then $\hat{X}$ is a tower

$$\cdots \to (\mathbb{Z}/p)_2 X \to (\mathbb{Z}/p)_1 X \to (\mathbb{Z}/p)_0 X \to X$$

of fibrations [4, I.4.3]. The inverse limit of this tower is one of the usual notions of the $\mathbb{Z}/p$-completion of X. However, we always consider $\hat{X}$ as a pro-space, not as an ordinary space. This definition extends to pro-spaces.

Definition 2.13. Let X be a pointed pro-space (*i.e.*, a pro-object in the category of pointed spaces). The **$\mathbb{Z}/p$-completion $\hat{X}$** of X is the pro-space given by the functor

$$(n,s) \mapsto (\mathbb{Z}/p)_n X_s.$$

The following result from [17] reminds us of the most important properties of $\mathbb{Z}/p$-completion.

Theorem 2.14. *Let $f : X \to Y$ be a map of pointed pro-spaces. Then the map $\hat{f} : \hat{X} \to \hat{Y}$ is a weak equivalence of pro-spaces in the sense of Section 4.2 if and only if for every $q \geq 0$ and every $\mathbb{Z}/p$-vector space V, the map*

$$H^q(Y;V) \to H^q(X;V)$$

is an isomorphism.

The above theorem uses the traditional formula of cohomology of pro-objects (see Section 4.2).

Now let p be a fixed prime that does not occur as the characteristic of any residue field of S, and let $\hat{\mathbf{Et}}$ be the functor from schemes to pro-spaces that takes X to the $\mathbb{Z}/p$-completion of $\mathrm{Et}X$. As in Section 2.2, let $\hat{\boldsymbol{Re}}^{\mathbf{et}}$ and $\hat{\boldsymbol{S}}^{\mathbf{et}}$ be the realization and singular complex functors corresponding to $\hat{\mathrm{Et}}$.

Theorem 2.15. *With respect to the $\mathbb{A}^1$-local projective model structure on sPre(Sm/S) and the model structure on* pro-*sSet described in Section 4.2, the functors $\hat{Re}^{et}$ and $\hat{S}^{et}$ form a Quillen pair.*

Remark 2.16. The theorem is not true when considering the $\mathbb{A}^1$-local injective model structure on *sPre*(Sm/S). There are too many injective cofibrations.

Proof. The argument is basically the same as in the proof of Theorem 2.6. The only significantly different part is in showing that

$$\hat{Re}^{\mathrm{et}} rX \to \hat{Re}^{\mathrm{et}} r(X \times \mathbb{A}^1)$$

is a weak equivalence for every scheme X in Sm/S. In other terms, we must show that $\hat{\mathrm{Et}}(X \times \mathbb{A}^1) \to \hat{\mathrm{Et}}X$ is a weak equivalence of pro-spaces. Since the functor Et commutes with coproducts and $X \times \mathbb{A}^1 \to X$ induces an isomorphism of connected components, it suffices to assume that X is connected. Moreover, we may choose an arbitrary basepoint for X so that we have a map of pointed connected pro-spaces. Invoking Theorem 2.14, it is necessary only to show that this map induces an isomorphism in cohomology with coefficients in $\mathbb{Z}/p$-vector spaces. In order to understand these cohomology maps, [9, Prop. 5.9] allows us to consider the map on étale cohomology induced by the projection

$$X \times \mathbb{A}^1 \to X.$$

The projection induces an isomorphism in étale cohomology by [23, Cor. VI.4.20]. □

Remark 2.17. It is also possible to use the completion of [24] in order to define a slightly different $\mathbb{A}^1$-homotopy invariant étale realization functor. See [17] for more details.

The next corollary follows from Theorem 2.15 in the same way that Corollary 2.8 follows from Theorem 2.6.

Corollary 2.18. *The functor $L\hat{Re}^{et}$ induces a functor from the homotopy category of schemes to the homotopy category of pro-spaces. On the level of homotopy categories, it has a right adjoint $R\hat{S}^{et}$.*

2.4. **Excision for the Etale Topological Type.** This section gives an interesting corollary about étale topological types and elementary distinguished squares. Recall that an **elementary distinguished square** [25, Defn. 3.1.3] is a diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[i]{} & X \end{array}$$

of smooth schemes over S in which i is an open inclusion and $p : p^{-1}(X-U) \to X-U$ is an isomorphism (where the schemes $p^{-1}(X-U)$ and $X-U$ are given the reduced structure). The relevance of such squares is that the maps i and p form a Nisnevich cover of X. One way of interpreting the next theorem is that these special Nisnevich covers generate all Nisnevich covers in a certain sense.

Theorem 2.19. [3, Lem. 4.1] *A simplicial presheaf F in $sPre(\mathrm{Sm}/S)$ is Nisnevich-local projective fibrant (see Remark 2.12) if and only if F is objectwise fibrant and takes elementary distinguished squares to homotopy pullback squares.*

This leads immediately to the following excision theorem for étale topological types.

Theorem 2.20. *Let*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

be an elementary distinguished square of smooth schemes over S. Then the square

$$\begin{array}{ccc} Et(U \times_X V) & \longrightarrow & EtV \\ \downarrow & & \downarrow \\ EtU & \longrightarrow & EtX \end{array}$$

is a homotopy pushout square of pro-spaces.

Proof. The argument of the proof of Theorem 2.6 shows that Re^{et} and S^{et} are a Quillen pair of adjoint functors between the category $sPre(\mathrm{Sm}/S)$ (equipped with the Nisnevich-local projective model structure) and the category of pro-spaces. Therefore, $S^{\mathrm{et}}Z$ is Nisnevich-local projective fibrant for every fibrant pro-space Z. By Theorem 2.19, the square

$$\begin{array}{ccc} \mathrm{Map}(\mathrm{Et}(U \times_X V), Z) & \longleftarrow & \mathrm{Map}(\mathrm{Et}V, Z) \\ \uparrow & & \uparrow \\ \mathrm{Map}(\mathrm{Et}U, Z) & \longleftarrow & \mathrm{Map}(\mathrm{Et}X, Z) \end{array}$$

is a homotopy pullback square for every fibrant Z, so

$$\begin{array}{ccc} \mathrm{Et}(U \times_X V) & \longrightarrow & \mathrm{Et}V \\ \downarrow & & \downarrow \\ \mathrm{Et}U & \longrightarrow & \mathrm{Et}X \end{array}$$

is a homotopy pushout square. □

Remark 2.21. The previous theorem can also be viewed in terms of the cohomological excision theorem of [23, III.1.27], at least with locally constant coefficients, because the étale cohomology of a scheme is isomorphic to the singular cohomology of its étale topological type.

3. Hypercovers of Simplicial Schemes

The point of this section is to study and define rigid hypercovers of simplicial schemes and to make some useful constructions concerning them.

3.1. **Finite Limits of Schemes.** We first study how finite limits interact with étale maps and separated maps. The results here are not particularly striking, but they do not appear in the standard literature [10] [11] [23] [30].

A technical result about fiber products comes first. The more general claim about arbitrary finite limits follows relatively easily.

Lemma 3.1. *Consider a diagram of schemes*

$$\begin{array}{ccccc} U & \longrightarrow & V & \longleftarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longleftarrow & Z \end{array}$$

*such that the three vertical maps are étale (*resp.*, separated). Then the induced map*

$$U \times_V W \to X \times_Y Z$$

*is also étale (*resp.*, separated).*

Proof. We prove the lemma for étale maps. The proof for separated maps is identical. See [10, Prop. I.5.3.1] for the necessary properties of separated maps.

Recall that base changes preserve étale maps [23, Prop I.3.3(c)]. Let f be the map in question. Factor f as

$$U \times_V W \longrightarrow U \times_Y W \longrightarrow X \times_Y W \longrightarrow X \times_Y Z.$$

The second map and third maps are étale because of the pullback squares

$$\begin{array}{ccc} U \times_Y W & \longrightarrow & U \\ \downarrow & & \downarrow \\ X \times_Y W & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} X \times_Y W & \longrightarrow & W \\ \downarrow & & \downarrow \\ X \times_Y Z & \longrightarrow & Z. \end{array}$$

It remains to show that the first map is also étale. The diagram

$$\begin{array}{ccc} U \times_V W & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ U \times_Y W & \longrightarrow & V \times_Y V \end{array}$$

is a pullback square, where Δ is the diagonal map. It suffices to observe that Δ is étale [23, Prop. I.3.5]. □

Proposition 3.2. *Let $f : U \to X$ be a map of finite diagrams of schemes such that the map $f^a : U^a \to X^a$ is étale (*resp.*, separated) for every a. Then the map $\lim f : \lim U \to \lim X$ is étale (*resp.*, separated).*

Proof. This is a formal consequence of Lemma 3.1 because every finite limit can be expressed in terms of finite products and fiber products. □

3.2. **Simplicial Schemes.** This section makes more precise many of the constructions and notations of [9]. We work in the category of schemes or more generally in the category of schemes over a fixed base scheme S; these two cases are actually the same since the category of schemes has a terminal object Spec$\mathbb{Z}$.

Let $\boldsymbol{\Delta}$ be the category whose objects are the non-empty ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$ and whose morphisms are the weakly monotonic maps. This is the usual indexing category for simplicial objects.

Let $\boldsymbol{\Delta[n]}$ be the simplicial set represented by $[n]$. Thus, $\Delta[n]_m = \mathrm{Hom}_\Delta([m], [n])$. Note that $\Delta[\mathbf{.}]$ is a cosimplicial space.

Let $\boldsymbol{\Delta_+}$ be the category Δ with an initial object $[-1]$ adjoined. The opposite of Δ_+ is the usual indexing category for augmented simplicial objects.

Definition 3.3. A **simplicial scheme** is a functor from Δ^{op} to schemes. An **augmented simplicial scheme** is a functor from Δ_+^{op} to schemes. A **bisimplicial scheme** is a functor from $(\Delta \times \Delta)^{\mathrm{op}}$ to schemes. An **augmented bisimplicial scheme** is a functor from $(\Delta \times \Delta_+)^{\mathrm{op}}$ to schemes.

Note that augmented bisimplicial schemes are augmented in only one direction. Augmented bisimplicial schemes are perhaps more correctly but awkwardly called simplicial augmented simplicial schemes.

For every scheme X, let $\boldsymbol{cX}$ be the constant simplicial scheme with value X.

Recall the nth latching object $L_n X$ of a simplicial object X [13, Defn. 16.2.5]. It is a finite colimit indexed by the subcategory of Δ consisting of the degeneracy maps $[m] \to [m-1]$ for $1 \leq m \leq n-1$. Beware that $L_n X$ does not necessarily exist for every simplicial scheme X because the category of schemes is not cocomplete.

Definition 3.4. A simplicial scheme X is **split** if $L_n X$ exists for every $n \geq 0$ and the canonical map $L_n X \to X_n$ is the inclusion of a direct summand. If X is split, let $\boldsymbol{NX_n}$ be the subscheme of X_n such that $X_n = L_n X \amalg NX_n$.

A simplicial scheme is split up to dimension n if the map $L_m X \to X_m$ is the inclusion of a direct summand for $m \leq n$.

The idea is that NX_n is the non-degenerate part of X_n and that X_n splits into a direct sum of its degenerate part and its non-degenerate part. Note that NX_n is well-defined because the category of schemes is locally connected [2, § 9].

3.3. **Skeleta and Coskeleta.** Let $\boldsymbol{\Delta^{(n)}}$ be the full subcategory of Δ on the objects $[m]$ for $m \leq n$. Note that $\Delta^{(n)}$ is a finite category.

Definition 3.5. An ***n*-truncated simplicial scheme** is a functor from $\left(\Delta^{(n)}\right)^{\mathrm{op}}$ to schemes.

Definition 3.6. If X is a simplicial scheme, then the **truncated *n*-skeleton $\mathbf{sk}_n X$** is the n-truncated simplicial scheme given by restriction of X along the inclusion $\left(\Delta^{(n)}\right)^{\mathrm{op}} \to \Delta^{\mathrm{op}}$. The ***n*-skeleton $\mathbf{Sk}_n X$** is the simplicial scheme given in dimension m by

$$(\mathrm{Sk}_n X)_m = \operatorname*{colim}_{\substack{\phi:[m]\to[k] \\ k \leq n}} X_k,$$

provided that it exists.

In general, $\mathrm{Sk}_n X$ does not exist because the necessary colimits may not exist in the category of schemes. However, $\mathrm{Sk}_n X$ does exist when X is split up to dimension n. In this case, $(\mathrm{Sk}_n X)_m$ is a disjoint union of one copy of NX_k for each surjective map $[m] \to [k]$ with $k \leq n$.

Remark 3.7. It is not usually necessary to distinguish between the functors sk_n and Sk_n in the category of simplicial sets. This is because Sk_n exists for every simplicial set since every simplicial set is split. For technical precision, the distinction is important in the category of simplicial schemes.

Definition 3.8. The ***n*th coskeleton** functor $\mathbf{cosk}_{\boldsymbol{n}}$ from n-truncated simplicial schemes to simplicial schemes is right adjoint to the functor sk_n.

We abuse notation and write $\mathrm{cosk}_n X$ instead of $\mathrm{cosk}_n(\mathrm{sk}_n X)$ for a simplicial scheme X. To avoid confusion, write cosk_n^S for the nth coskeleton functor in the category of schemes over S. By convention, define $\mathrm{cosk}_{-1} X$ to be the constant simplicial scheme $c\mathrm{Spec}\mathbb{Z}$. More generally, $\mathrm{cosk}_{-1}^S X$ is the constant simplicial scheme cS. In particular, $(\mathrm{cosk}_{-1}^S X)_0$ is equal to S. This convention makes our definition of hypercovers in Section 3.4.2 more concise.

In order to make the notation consistent, we should define another functor Cosk_n from simplicial schemes to simplicial schemes that is right adjoint to Sk_n. Because Sk_n is not defined on the full category of simplicial schemes, it is somewhat awkward to precisely state this adjointness. If X is a simplicial scheme such that $\mathrm{Sk}_n X$ exists, then for every simplicial scheme Y, maps $\mathrm{Sk}_n X \to Y$ are in one-to-one correspondence with maps $X \to \mathrm{cosk}_n Y$. Hence our abuse of notation in the previous paragraph makes it unnecessary to introduce another functor Cosk_n.

The functor cosk_n plays a critical role, so we recall some of its properties. Very importantly for us, each object $(\mathrm{cosk}_n X)_m$ is a finite limit of the objects X_k for $k \leq n$. Also important is that $(\mathrm{cosk}_n X)_m$ is isomorphic to X_m when $m \leq n$. In other words, $\mathrm{cosk}_n X$ and X agree up to dimension n.

For every simplicial scheme X, the unit map $X \to \mathrm{cosk}_n(\mathrm{sk}_n X)$ induces a natural map

$$X_m \to (\mathrm{cosk}_k X)_m.$$

These maps will appear again and again.

Note that $(\mathrm{cosk}_n X)_{n+1}$ is the nth matching object $M_n X$ of X [13, Defn. 16.2.5].

Proposition 3.9. *Let $f : U \to X$ be a map of n-truncated simplicial schemes such that for every $m \leq n$, the map f_m is étale. Then*

$$(\mathrm{cosk}_n f)_m : (\mathrm{cosk}_n U)_m \to (\mathrm{cosk}_n X)_m$$

is étale for every m.

Proof. This is just a special case of Proposition 3.2. □

Remark 3.10. For any finite simplicial set K and any simplicial scheme X, define $X \otimes K$ to be the simplicial scheme isomorphic to $\coprod_{K_n} X_n$ in dimension n. Then define the cotensor X^K such that the functors $(\cdot) \otimes K$ and $(\cdot)^K$ are adjoints. In these terms, the scheme $(\mathrm{cosk}_n X)_m$ is isomorphic to $\left(X^{\mathrm{Sk}_n \Delta[m]}\right)_0$.

The simplicial structure on simplicial schemes can be a very useful language. However, we will not need it here.

3.4. **Hypercovers.** Much of the material in this section can be found in [9]. We review the basic notions of hypercovers and rigid hypercovers. In the next few sections we formalize some useful constructions concerning them.

For any point x_0 of a scheme X, a **geometric point** of X over x_0 is a map $x : \mathrm{Spec}\overline{k} \to X$ with image x_0, where $\overline{k}$ is the separable closure of the residue field $k(x_0)$. For any scheme X, let $\boldsymbol{X(\overline{k})}$ be the set of geometric points. If $f : X \to Y$ is a map of schemes and $y : \overline{k} \to Y$ is a geometric point of Y, then a **lift** of y is a geometric point $x : \overline{k} \to X$ such that $y = f \circ x$. Equivalently, x goes to y under the set map $f(\overline{k}) : X(\overline{k}) \to Y(\overline{k})$. In this situation, we abuse notation and write $f(x) = y$.

3.4.1. *Rigid Covers.*

Definition 3.11. A **rigid cover** of a scheme X is a map of schemes $f : U \to X$ such that f is a separated étale surjective map satisfying the following properties. The connected components of U are in one-to-one correspondence with the geometric points of X. Write U_x for the component of U corresponding to the geometric point x, and it has a basepoint u_x, which is a geometric point of U_x such that $f(u_x) = x$.

If $f : U \to X$ and $f' : U' \to X'$ are rigid covers of X and X', then a rigid cover map $g : f \to f'$ over a scheme map $h : X \to X'$ is a commuting square

$$\begin{array}{ccc} U & \xrightarrow{g} & U' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X & \xrightarrow[h]{} & X' \end{array}$$

such that $g(u_x) = u'_{h(x)}$ for every geometric point x of X. The idea is that g preserves the basepoints of each component.

The importance of rigid covers is that there exists at most one rigid cover map between any two rigid covers of a scheme [9, Prop. 4.1].

3.4.2. *Hypercovers and Rigid Hypercovers.*

Definition 3.12. A **hypercover** of a scheme X is an augmented simplicial scheme U such that $U_{-1} = X$ and the map

$$U_n \to (\mathrm{cosk}_{n-1}^X U)_n$$

is étale surjective for all $n \geq 0$. A **hypercover** of a simplicial scheme X is an augmented bisimplicial scheme U such that $U_{\bullet,-1} = X$ and $U_{n,\bullet}$ is a hypercover of X_n for each n.

By convention, the map

$$U_n \to (\mathrm{cosk}_{n-1}^X U)_n$$

is equal to the map $U_0 \to X$ when $n = 0$. It is important to remember that $U_0 \to X$ must be étale surjective.

If U and U' are hypercovers of schemes (*resp.*, simplicial schemes) X and X', then a hypercover map $g : U \to U'$ over a map $h : X \to X'$ is an augmented simplicial scheme map (*resp.*, augmented bisimplicial scheme map) such that g_{-1} (*resp.*, $g_{\bullet,-1}$) is equal to h.

The following lemma is a key property of hypercovers. It provides a technical ingredient in the construction of rigid pullbacks and rigid limits of rigid hypercovers in Sections 3.5 and 3.6.3.

Lemma 3.13. *Every hypercover of a scheme is split.*

Proof. Let U be a hypercover of X. By induction and Proposition 3.9, U is a simplicial object in the category of étale schemes over X. The remark after [2, Defn. 8.1] finishes the argument. □

Definition 3.14. A **rigid hypercover** of a scheme X is a hypercover of X such that the map

$$U_n \to (\mathrm{cosk}^X_{n-1} U)_n$$

is a rigid cover for all $n \geq 0$.

If U and U' are rigid hypercovers of schemes X and X', then a rigid hypercover map $U \to U'$ over a map $X \to X'$ is a hypercover map such that the square

$$\begin{array}{ccc} U_n & \longrightarrow & U'_n \\ \downarrow & & \downarrow \\ (\mathrm{cosk}^X_{n-1} U)_n & \longrightarrow & (\mathrm{cosk}^{X'}_{n-1} U')_n \end{array}$$

is a rigid cover map for every $n \geq 0$.

Definition 3.15. A **rigid hypercover** of a simplicial scheme X is a hypercover of X such that $U_{n,\cdot}$ is a rigid hypercover of X_n for each n and $U_{n,\cdot} \to U_{m,\cdot}$ is a rigid hypercover map over $X_n \to X_m$ for every $[m] \to [n]$.

If U and U' are rigid hypercovers of simplicial schemes X and X', then a rigid hypercover map $g : U \to U'$ over a map $h : X \to X'$ is a hypercover map such that g_n is a rigid hypercover map over h_n for each n.

Similarly to rigid covers, there exists at most one map between two rigid hypercovers of a scheme (or simplicial scheme) [9, Prop. 4.3]. On the other hand, maps between hypercovers are unique only in a certain homotopical sense [2, Cor. 8.13].

Definition 3.16. For a scheme (or simplicial scheme) X, let $\mathbf{HRR}(\boldsymbol{X})$ be the category of rigid hypercovers of X.

The notation comes from [9]. The critical property of this category is that it is cofiltered [9, Prop. 4.3]. Since there is at most one map between any two objects of $\mathrm{HRR}(X)$, this category is actually a directed set.

Lemma 3.17. *Let X be a scheme. The categories $\mathrm{HRR}(X)$ and $\mathrm{HRR}(cX)$ are equivalent.*

Proof. Consider the functor $\mathrm{HRR}(X) \to \mathrm{HRR}(cX)$ that takes a rigid hypercover U of X to the hypercover V of cX given by the formula $V_{m,n} = U_n$. This functor is full and faithful, so it suffices to show that every rigid hypercover of cX belongs to the image of this functor.

Let V be an arbitrary rigid hypercover of cX. Then V is a simplicial diagram in the category $\mathrm{HRR}(X)$. There is at most one rigid hypercover map between any two rigid hypercovers of X. It follows that the map $V_{n,\cdot} \to V_{m,\cdot}$ is the identity map for all $[m] \to [n]$. □

3.5. **Rigid Pullbacks.** Suppose that $f : X \to Y$ is a map of schemes and $U \to Y$ is étale surjective. Then the base change $f^*U \to X$ is the projection $X \times_Y U \to X$, which is again étale surjective [23, Prop. I.3.3(c)]. This idea generalizes to rigid covers.

Definition 3.18. Let $f : X \to Y$ be any map of schemes and let $U \to Y$ be a rigid cover. Then the **rigid pullback $f^*U \to X$** is the rigid cover defined by the following construction. For each geometric point x of X, let $(f^*U)_x$ be the component of $X \times_Y U$ containing $x \times u_{f(x)}$, and let $x \times u_{f(x)}$ be the basepoint of $(f^*U)_x$.

Note that $(f^*U)_x$ is a component of $X \times_Y U_x$, but f^*U is not a restriction of $X \times_Y U$ to certain components since some components of $X \times_Y U$ may occur more than once as components of f^*U. Also note that there is a canonical rigid cover map

$$\begin{array}{ccc} f^*U & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

Lemma 3.19. *Let $f : X \to Y$ be any map of schemes and let $U \to Y$ be a rigid cover. Then the rigid cover $f^*U \to X$ has the following universal property. Let $V \to Z$ be an arbitrary rigid cover. Rigid cover maps*

$$\begin{array}{ccc} V & \longrightarrow & f^*U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

correspond to rigid cover maps

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

together with a factorization of the map $Z \to Y$ through X.

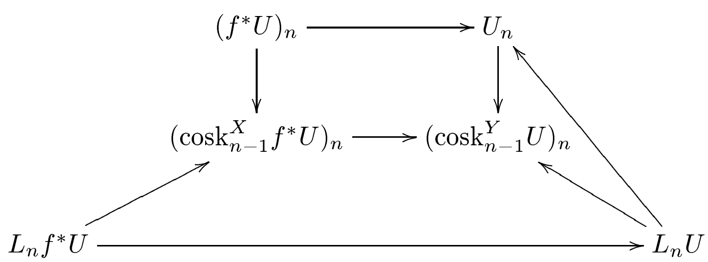
Proof. The category of connected pointed schemes has finite limits. To construct such limits, just take the basepoint component of the usual limit of schemes. The lemma now follows from this observation and the universal property of pullbacks of schemes. □

Rigid pullbacks of rigid covers generalizes to rigid hypercovers.

Definition 3.20. Suppose $f : X \to Y$ is a map of schemes and U is a rigid hypercover of Y. Then the **rigid pullback f^*U** is the hypercover of X constructed as follows. Let $(f^*U)_0$ be the rigid pullback along f of the rigid cover $U_0 \to Y$. Inductively define $(f^*U)_n$ to be the rigid pullback along $(\operatorname{cosk}_{n-1}^X f^*U)_n \to (\operatorname{cosk}_{n-1}^Y U)_n$ of the rigid cover $U_n \to (\operatorname{cosk}_{n-1}^Y U)_n$.

In order to describe the face and degeneracy maps of f^*U, it suffices to give inductively a factorization of $g : L_n f^*U \to (\operatorname{cosk}_{n-1}^X f^*U)_n$ through $(f^*U)_n$ for each n [13, Thm. 16.2.1]. Recall that $L_n f^*U$ is the nth latching object of f^*U and $(\operatorname{cosk}_{n-1}^X f^*U)_n$ is the nth matching object. Note that $L_n f^*U$ exists because (by induction) the $(n-1)$-truncated scheme f^*U is a truncated hypercover and is therefore split by Lemma 3.13. Also, $L_n U$ exists since U is a hypercover and is therefore split.

The diagram



induces a map

$$h' : L_n f^*U \to U_n \times_{(\mathrm{cosk}^Y_{n-1} U)_n} (\mathrm{cosk}^X_{n-1} f^*U)_n.$$

We can compose with the projection map to $(\mathrm{cosk}^X_{n-1} f^*U)_n$ to get a factorization of g, but this is not quite the desired factorization because $(f^*U)_n$ is not exactly equal to the fiber product

$$U_n \times_{(\mathrm{cosk}^Y_{n-1} U)_n} (\mathrm{cosk}^X_{n-1} f^*U)_n.$$

In order to produce the desired factorization $h : L_n f^*U \to (f^*U)_n$, we must specify which component of $(f^*U)_n$ is the target of each component of $L_n f^*U$.

Since $L_n f^*U$ is a disjoint union of copies of $(f^*U)_m$ for $m < n$, each component has a basepoint. Let C be a component of $L_n f^*U$ with basepoint c. Then h is defined to take C into the component $((f^*U)_n)_{g(c)}$ of $(f^*U)_n$.

Remark 3.21. Defining the face and degeneracy maps of f^*U is technically complex and unimportant in the big picture, but we see no way of avoiding it. This issue is not dealt with in [9, p. 37].

A careful inspection of the definitions yields the critically important property that rigid pullbacks of rigid hypercovers are functorial. This means that the definition of rigid pullbacks extends to rigid hypercovers of simplicial schemes.

Also note that there is a canonical rigid hypercover map $f^*U \to U$ over the map $f : X \to Y$.

Lemma 3.22. *Let U be a rigid hypercover and let $f : X \to U_{-1}$ be any map of schemes. The rigid hypercover f^*U of X has the following universal property. Let V be an arbitrary rigid hypercover. Rigid hypercover maps $V \to f^*U$ correspond to rigid hypercover maps $V \to U$ together with a factorization of the map $V_{-1} \to U_{-1}$ through X.*

Proof. This follows from Lemma 3.19 and induction. Because V, U, and f^*U are all split by Lemma 3.13, the degeneracy maps take care of themselves. □

3.6. **Rigid Limits.** In this section, Proposition 3.2 is generalized to rigid covers and rigid hypercovers.

3.6.1. *Limits of Covers and Hypercovers.*

Proposition 3.23. *Let $f : U \to X$ be a finite diagram of rigid cover maps. Then the map*

$$\lim_a f^a : \lim_a U^a \to \lim_a X^a$$

is étale surjective.

Proof. The map is étale by Proposition 3.2, so we need only show that every geometric point x of $\lim X$ lifts to $\lim U$. Let x^a be the composition of x with the obvious projection map $\lim X \to X^a$. Since each f^a is a rigid cover, there exist canonical lifts u^a of each x^a to U^a. They assemble to give a geometric point u of $\lim U$ because f is a diagram of rigid cover maps. □

The above proposition is not true if each f^a is only étale surjective. The problem is that a limit of surjective maps is not necessarily surjective. Note that $\lim f$ is only étale surjective; it is not a rigid cover. As the proof above indicates, there are canonical lifts for each geometric point of $\lim X$, but the components of $\lim U$ may not correspond one-to-one to the geometric points of $\lim X$.

Proposition 3.24. *Suppose that U is a finite diagram of rigid hypercovers. Then $\lim_a U^a$ is a hypercover.*

Proof. For convenience, write X for U_{-1}. We want to show that $\lim_a U^a$ is a hypercover of $\lim_a X^a$, so we must prove that

$$(\lim U)_{n-1} \to \operatorname{cosk}_n^{\lim X}(\lim U)_{n-1}$$

is étale surjective for all $n \geq 0$. This follows from Proposition 3.2 and the fact that

$$\operatorname{cosk}_n^{\lim X}(\lim U)_{n-1} \cong \lim_a (\operatorname{cosk}_n^{X^a} U^a)_{n-1}.$$

□

Again, the above proposition is not true if each U^a is only a hypercover. Also, $\lim U$ is only a hypercover, not a rigid hypercover.

3.6.2. *Rigid Limits of Rigid Covers.* As seen in the previous section, ordinary finite limits do not preserve rigid covers and rigid hypercovers. Thus, the notion of limit must be refined in order to get a rigid cover-preserving construction.

Definition 3.25. Let $f : U \to X$ be a finite diagram of rigid cover maps. Then the **rigid limit**

$$\operatorname{Rlim}_a f^a : \operatorname{Rlim}_a U^a \to \lim_a X^a$$

is the rigid cover defined as follows. For each geometric point $x = \lim_a x^a$ of $\lim_a X^a$, let $(\operatorname{Rlim}_a U^a)_x$ be the connected component of $\lim_a U^a$ containing $u_x = \lim_a u^a_{x^a}$, and let u_x be the basepoint of $(\operatorname{Rlim}_a U^a)_x$.

It is important to begin with a diagram of rigid cover maps, not just cover maps. Otherwise, the geometric points $u^a_{x^a}$ are not necessarily compatible and do not induce a geometric point u_x of $\lim_a U^a$.

The symbols $\overset{R}{\prod}$ and $\overset{R}{\times}$ denote rigid limits in the case of products or fiber products. Note that there is a natural map $\operatorname{Rlim} U \to \lim U$ over $\lim X$.

Lemma 3.26. *The rigid limit of a finite diagram of rigid covers is a rigid cover.*

Proof. The map $\mathrm{Rlim}_a U^a \to \lim_a X^a$ factors as a local isomorphism $\mathrm{Rlim}_a U^a \to \lim_a U^a$ followed by the map $\lim_a U^a \to \lim_a X^a$. The latter is étale and separated by Proposition 3.2, so the composition is also étale and separated. The other parts of the definition of a rigid cover are satisfied by construction. □

Lemma 3.27. *Let $f : U \to X$ be a finite diagram of rigid covers. Then* $\mathrm{Rlim}_a f^a$ *is universal in the following sense. Let $g : V \to Y$ be any rigid cover of a scheme Y. Rigid cover maps $g \to$* $\mathrm{Rlim} f$ *are in one-to-one correspondence with collections of rigid cover maps $g \to f^a$ such that for every map $f^a \to f^b$, the diagram*

$$\begin{array}{ccc} g & \longrightarrow & f^a \\ & \searrow & \downarrow \\ & & f^b \end{array}$$

of rigid cover maps commutes.

Proof. As in the proof of Lemma 3.19, it is important that the category of connected pointed schemes has finite limits. The lemma now follows from this observation and the universal property of limits. □

Remark 3.28. Rigid limits have the same kind of functoriality as ordinary limits. We make this more precise. Let $f : U \to X$ and $g : V \to Y$ be diagrams of rigid cover maps indexed by finite categories A and B respectively. Suppose given a functor $F : B \to A$, and let F^*f be the diagram of rigid cover maps indexed by B given by the formula $(F^*f)^b = f^{F(b)}$. Suppose given a natural transformation $\eta : F^*f \to g$. Then η induces a natural map $\mathrm{Rlim}_A f \to \mathrm{Rlim}_B g$. This is precisely what happens for ordinary limits.

Remark 3.29. Given a diagram U of rigid cover maps over a fixed scheme X, arbitrary finite rigid limits are not really necessary. Since the category of rigid covers of X is actually a directed set, the rigid limit of U is just the least upper bound of the rigid covers U^a. This is isomorphic to the rigid product $\prod^R U^a$.

More generally, suppose now that X is not a constant diagram. For every a, consider the rigid pullback $(\pi^a)^*U^a \to \lim_b X^b$ of $U^a \to X^a$ along the projection $\pi^a : \lim_b X^b \to X^a$. Then $\mathrm{Rlim}_a U^a$ is isomorphic to the rigid product $\prod^R (\pi^a)^*U^a$.

Hence arbitrary rigid limits can be constructed with rigid products and rigid pullbacks. This observation relies heavily on the fact that there is at most one rigid cover map between any two rigid covers.

Remark 3.30. Suppose that U and X are n-truncated schemes and $f : U \to X$ is a diagram of rigid cover maps. Write

$$(\mathrm{Rcosk}_n f)_k : (\mathrm{Rcosk}_n U)_k \to (\mathrm{cosk}_n X)_k$$

for the rigid limit of the finite diagram whose ordinary limit is $(\mathrm{cosk}_n f)_k$. Because of the functoriality expressed in Remark 3.28, these constructions assemble into a map

$$\mathrm{Rcosk}_n f : \mathrm{Rcosk}_n U \to \mathrm{cosk}_n X$$

of simplicial schemes that is a simplicial object in the category of rigid covers.

3.6.3. *Rigid Limits of Rigid Hypercovers.* Let U be a finite diagram of rigid hypercover maps, and let X equal U_{-1}. Each U^a is a rigid hypercover of X^a, and each $U^a \to U^b$ is a rigid hypercover map over $X^a \to X^b$.

Proposition 3.24 implies that $V = \lim_a U^a$ is a hypercover of $Y = \lim_a X^a$. We use rigid limits to define a canonical rigid hypercover $W = \mathrm{Rlim}_a U^a$ of Y and a natural hypercover map $W \to V$ over Y.

Begin by defining W_0 to be $\mathrm{Rlim}_a U_0^a$. There is a canonical map from W_0 to $V_0 = \lim_a U_0^a$.

Suppose for sake of induction that W_m and the map $W_m \to V_m$ have been defined for $m < n$. Thus there is a map $(\mathrm{cosk}_{n-1}^Y W)_n \to (\mathrm{cosk}_{n-1}^Y V)_n$. Let x be a geometric point of $(\mathrm{cosk}_{n-1}^Y W)_n$, and let y be its image in $(\mathrm{cosk}_{n-1}^Y V)_n$. Since $(\mathrm{cosk}_{n-1}^Y V)_n$ is isomorphic to $\lim_a (\mathrm{cosk}_{n-1}^{X^a} U^a)_n$, y gives compatible geometric points y^a in each of the schemes $(\mathrm{cosk}_{n-1}^{X^a} U^a)_n$. Each y^a has a canonical lift z^a in U_n^a since each U^a is a rigid hypercover. Moreover, these lifts are compatible since U is a diagram of rigid hypercover maps. This means that they assemble to give a geometric point z of $V_n = \lim_a U_n^a$, and z is a lift of y.

Now define $(W_n)_x$ to be the connected component of

$$V_n \times_{(\mathrm{cosk}_{n-1}^Y V)_n} (\mathrm{cosk}_{n-1}^Y W)_n$$

containing $z \times x$, and let $z \times x$ be the basepoint of $(W_n)_x$. This extends the definition of W to dimension n.

Remark 3.31. To describe the face and degeneracy maps of W, one must use an argument similar to that given in Section 3.5 for describing rigid pullbacks of rigid hypercovers. Although it is technically complex and unimportant in the big picture, we know of no way of avoiding it.

Lemma 3.32. *Rigid limits of rigid hypercovers have the following universal property. Suppose that U is a diagram of rigid hypercover maps, and let V be an arbitrary rigid hypercover of a scheme Y. Rigid hypercover maps from V to* $\mathrm{Rlim}\, U$ *are in one-to-one correspondence with collections of rigid hypercover maps $V \to U^a$ such that for every map $U^a \to U^b$, the diagram*

$$\begin{array}{ccc} V & \longrightarrow & U^a \\ & \searrow & \downarrow \\ & & U^b \end{array}$$

of rigid hypercover maps commutes.

Proof. This follows from Lemma 3.27 and induction. Because V, $\lim U$, and each U^a are all split by Lemma 3.13, the degeneracy maps take care of themselves. □

Remark 3.33. As for rigid limits of rigid covers, rigid limits of rigid hypercovers have the same kind of functoriality as ordinary limits. See Remark 3.28 for more details.

Arbitrary finite rigid limits of rigid hypercovers are not really necessary. In fact, rigid products and rigid pullbacks suffice to construct all rigid limits. See Remark 3.29 for more details.

We use the notation Rcosk_n for rigid hypercovers analogously to our use of this notation for rigid covers as in Remark 3.30.

3.6.4. *Cofinal Functors of Rigid Hypercovers.* The necessary constructions for rigid hypercovers have now been established. Our investment in the previous sections clarifies some of the technical complexities in the proofs of [9, Ch. 4].

For every simplicial scheme X and every $n \geq 0$, there is a forgetful functor $\mathrm{HRR}(X) \to \mathrm{HRR}(X_n)$ taking a rigid hypercover U of X to the rigid hypercover $U_{n,\cdot}$ of X_n. These functors assemble to give a functor

$$\mathrm{HRR}(X) \to \mathrm{HRR}(X_0) \times \mathrm{HRR}(X_1) \times \cdots \times \mathrm{HRR}(X_n).$$

The idea is that this functor forgets the face and degeneracy maps and only remembers the objects $U_{m,\cdot}$ for $m \leq n$.

Proposition 3.34. *Let X be a simplicial scheme. The functor*

$$\mathrm{HRR}(X) \to \mathrm{HRR}(X_0) \times \mathrm{HRR}(X_1) \times \cdots \times \mathrm{HRR}(X_n).$$

is cofinal.

This proposition is closely related to [9, Cor. 4.6], which show that the functor $\mathrm{HRR}(X) \to \mathrm{HRR}(X_n)$ is cofinal for every simplicial scheme X and every $n \geq 0$.

Proof. For convenience, let I be the category

$$\mathrm{HRR}(X_0) \times \mathrm{HRR}(X_1) \times \cdots \times \mathrm{HRR}(X_n).$$

Since each $\mathrm{HRR}(X_m)$ is actually a directed set, so is I. The category $\mathrm{HRR}(X)$ is also a directed set, so it suffices to show that for every object $U = (U_{0,\cdot}, U_{1,\cdot}, \ldots, U_{n,\cdot})$ of I, there is an object V of $\mathrm{HRR}(X)$ and a rigid hypercover map $V_{m,\cdot} \to U_{m,\cdot}$ over X_m for every $m \leq n$.

For each m, define $V_{m,\cdot}$ to be

$$\operatorname*{Rlim}_{\substack{\phi:[k]\to[m] \\ k\leq n}} U_{k,\cdot}.$$

The idea is that $V_{m,\cdot}$ is a "rigid right Kan extension". The rigid limit is finite because k is at most n.

The functoriality of rigid limits as expressed in Remark 3.33 assures us that V is in fact a rigid hypercover of X. The identity map $\mathrm{id} : [m] \to [m]$ gives a projection

$$V_{m,\cdot} \to U_{m,\cdot}.$$

These maps are the desired ones. □

4. Pro-Spaces

Having established the necessary background on hypercovers, we switch topics and proceed to develop some ideas about pro-spaces and realizations. Hypercovers reappear in Section 5.

4.1. **Preliminaries on Pro-Categories.**

We begin with a review of the necessary background on pro-categories. This section contains only standard material on pro-categories [1] [2] [8] [16].

Definition 4.1. For a category $\mathcal{C}$, the category **pro-$\mathcal{C}$** has objects all cofiltering diagrams in $\mathcal{C}$, and

$$\mathrm{Hom}_{\text{pro-}\mathcal{C}}(X,Y) = \lim_s \operatorname*{colim}_t \mathrm{Hom}_{\mathcal{C}}(X_t, Y_s).$$

Composition is defined in the natural way.

A category I is **cofiltering** if the following conditions hold: it is non-empty and small; for every pair of objects s and t in I, there exists an object u together with maps $u \to s$ and $u \to t$; and for every pair of morphisms f and g with the same source and target, there exists a morphism h such that fh equals gh. Recall that a category is **small** if it has only a set of objects and a set of morphisms. A diagram is said to be **cofiltering** if its indexing category is so.

Let $Y : I \to \mathcal{C}$ and $X : J \to \mathcal{C}$ be arbitrary pro-objects. Then X is **cofinal** in Y if there is a cofinal functor $F : J \to I$ such that X is equal to the composite YF. This means that for every s in I, the overcategory $F \downarrow s$ is cofiltered. In the case when F is an inclusion of directed sets, F is cofinal if and only if for every s in I there exists t in J such that $t \geq s$. The importance of this definition is that X is isomorphic to Y in pro-$\mathcal{C}$.

A **level representation** of a map $f : X \to Y$ is: a cofiltered index category I; pro-objects $\tilde{X}$ and $\tilde{Y}$ indexed by I and isomorphisms $X \to \tilde{X}$ and $Y \to \tilde{Y}$; and a collection of maps $f_s : \tilde{X}_s \to \tilde{Y}_s$ for all s in I such that for all $t \to s$ in I, there is a commutative diagram

$$\begin{array}{ccc} \tilde{X}_t & \longrightarrow & \tilde{Y}_t \\ \downarrow & & \downarrow \\ \tilde{X}_s & \longrightarrow & \tilde{Y}_s \end{array}$$

and such that the maps f_s represent a pro-map $\tilde{f} : \tilde{X} \to \tilde{Y}$ belonging to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow[\tilde{f}]{} & \tilde{Y} \end{array}$$

in pro-$\mathcal{C}$. That is, a level representation is just a natural transformation such that the maps f_s represent the element f of

$$\lim_s \operatorname*{colim}_t \mathrm{Hom}_{\mathcal{C}}(X_t, Y_s) \cong \lim_s \operatorname*{colim}_t \mathrm{Hom}_{\mathcal{C}}(\tilde{X}_t, \tilde{Y}_s).$$

Every map has a level representation [2, App. 3.2] [22].

More generally, suppose given any diagram $A \to \text{pro-}\mathcal{C} : a \mapsto X^a$. A **level representation** of X is: a cofiltered index category I; a functor $\tilde{X} : A \times I \to \mathcal{C} : (a, s) \mapsto \tilde{X}^a_s$; and isomorphisms $X^a \to \tilde{X}^a$ such that for every map $\phi : a \to b$ in A, $\tilde{X}^\phi$ is a level representation for X^ϕ. In other words, $\tilde{X}$ is a uniform level representation for all the maps in the diagram X. Not every diagram of pro-objects has a level representation, but every finite loopless diagram does have a level representation.

Level representations are important tools for calculating finite limits and colimits in pro-categories. In order to calculate the limit or colimit of a finite diagram of pro-objects, take a level representation for the diagram and then take the levelwise limit or colimit.

Suppose that $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are two pro-objects. A **strict representation** [9, p. 36] of a map $f : X \to Y$ is a functor $F : J \to I$ and a natural transformation $\eta : X \circ F \to Y$ such that the maps $\eta_s : X_{F(s)} \to Y_s$ represent the element f of

$$\lim_s \operatorname{colim}_t \operatorname{Hom}_{\mathcal{C}}(X_t, Y_s).$$

More generally, a **strict representation** of a diagram X in pro-$\mathcal{C}$ consists of strict representations (F^ϕ, η^ϕ) for every map ϕ in X such that for every pair of composable maps ϕ and ψ in X, the functor $F^{\psi\phi}$ equals $F^\phi F^\psi$ and the natural transformation $\eta^{\psi\phi}$ equals $\eta^\psi \circ \eta^\phi F^\psi$.

4.2. **Homotopy Theory of Pro-Spaces.** We now review from [15] the homotopy theory of pro-spaces suitable for studying the étale topological type.

Given a cofiltered diagram of pointed connected spaces X, application of the ith homotopy group functor yields a pro-group $\pi_i X$. A map $f : X \to Y$ of such pointed connected pro-spaces is a **weak equivalence** if $\pi_i f$ is an isomorphism of pro-groups for every $i \geq 0$.

In fact, one can define weak equivalences for pro-spaces that are not pointed and connected, but choices of basepoints become a complicated and messy issue. See [15] for more details.

These weak equivalences belong to a proper simplicial model structure for pro-spaces. The cofibrations are the maps that up to isomorphism have level representations that are levelwise cofibrations. It follows that every pro-space is cofibrant. The fibrations are defined by a lifting property, but an explicit description is possible [15, Prop. 6.6].

Given a pro-space X, the formula

$$H^*(X; M) = \operatorname{colim}_s H^*(X_s; M)$$

defines cohomology [2, 2.2] [29, § 2.2]. In fact, there is an isomorphism between $H^q(X; M)$ and the set $[X, cK(M, q)]_{\text{pro}}$ of maps in the homotopy category of pro-spaces [15, Lem. 8.1]. Here $cK(M, q)$ is the constant pro-space with value an Eilenberg-Mac Lane space.

4.3. n**-Truncated Realizations.** Let $\mathcal{C}$ be a simplicial category; this means that objects of $\mathcal{C}$ can be tensored and cotensored with simplicial sets, and these operations satisfy appropriate adjointness conditions. We assume that $\mathcal{C}$ is complete and cocomplete. Our application involves pro-spaces, which is a complete and cocomplete category [15, Prop. 11.1].

Recall the definition of the realization of a simplicial object in $\mathcal{C}$.

Definition 4.2. Given a simplicial object X in a simplicial category $\mathcal{C}$, its **realization *ReX*** is the coequalizer of the diagram

$$\coprod_{\phi:[m]\to[n]} X_n \otimes \Delta[m] \rightrightarrows \coprod_n X_n \otimes \Delta[n],$$

where the upper arrow is induced by maps $\operatorname{id} \otimes \phi_* : X_n \otimes \Delta[m] \to X_n \otimes \Delta[n]$ and the lower arrow is induced by maps $\phi^* \otimes \operatorname{id} : X_n \otimes \Delta[m] \to X_m \otimes \Delta[m]$.

The realization of X can be expressed as the coend over Δ of the simplicial object X with the cosimplicial object $\Delta[\cdot]$. The most important property of realization is that it is left adjoint to the functor sending an object Y of $\mathcal{C}$ to the simplicial object $Y^{\Delta[\cdot]}$.

Remark 4.3. Beware of the difference between the realizations of Definition 4.2 and of Section 2.2. Although we treat them as distinct concepts, they really are versions of the same idea. Since simplicial sets are contravariant functors on Δ, it is possible to think of simplicial sets as presheaves. Then the realization of Definition 4.2 coincides with the realization of Section 2.2 where F is the functor $[k] \mapsto \Delta[k]$.

Remark 4.4. Rather than think of ReX as a coequalizer, we prefer to think of it as the colimit of the following diagram. The diagram has one object $X_n \otimes \Delta[n]$ for each $n \geq 0$ and one object $X_n \otimes \Delta[m]$ for each $\phi : [m] \to [n]$. The maps of the diagram are of two types. The first type is of the form $\mathrm{id} \otimes \phi_* : X_n \otimes \Delta[m] \to X_n \otimes \Delta[n]$, and the second type is of the form $\phi^* \otimes \mathrm{id} : X_n \otimes \Delta[m] \to X_m \otimes \Delta[m]$. The colimit of this diagram is the realization ReX of X. Note that the diagram has no non-identity endomorphisms. This fact makes the analysis of realizations of pro-spaces simpler.

Realizations present some problems because they are colimits of infinite diagrams. Sometimes it will be necessary to use techniques applicable only to colimits of finite diagrams. Hence the following definitions are useful.

Definition 4.5. If X is a simplicial object in a simplicial category $\mathcal{C}$, then the ***n*-truncated realization $Re_n X$** of X is the coequalizer of the diagram

$$\coprod_{\substack{\phi:[m]\to[k] \\ m,k\leq n}} X_k \otimes \Delta[m] \rightrightarrows \coprod_{m\leq n} X_m \otimes \Delta[m].$$

This is essentially the same construction as ordinary realization except that only the objects X_m for $m \leq n$ are considered. It can be described as a coend over $\Delta^{(n)}$ of $\mathrm{sk}_n X$ with the n-truncated standard cosimplicial complex $\Delta^{(n)}[\bullet]$.

Remark 4.6. As for realizations, we prefer to think of n-truncated realizations not as coequalizers but as colimits of diagrams with no non-identity endomorphisms. See Remark 4.4 for more details.

Like ordinary realization, n-truncated realization is also a left adjoint. Namely, it is left adjoint to the functor sending an object Y of $\mathcal{C}$ to the simplicial object that is the nth coskeleton of the simplicial object $Y^{\Delta[\bullet]}$.

There is a canonical map $Re_n X \to ReX$ for every simplicial object X. Of course this map is not an isomorphism in general, but it is an isomorphism on low-dimensional simplices as stated in the next proposition.

Proposition 4.7. *Let X be a simplicial space. Then the natural map $\mathrm{Sk}_n Re_n X \to \mathrm{Sk}_n ReX$ is an isomorphism.*

Proof. We show that both functors $\mathrm{Sk}_n Re_n$ and $\mathrm{Sk}_n Re$ have the same right adjoint. The right adjoint of $\mathrm{Sk}_n Re$ is the functor taking a space Y to the simplicial space $(\mathrm{cosk}_n Y)^{\Delta[\bullet]}$. On the other hand, the right adjoint of $\mathrm{Sk}_n Re_n Y$ is the functor taking a space Y to the nth coskeleton of the simplicial space $(\mathrm{cosk}_n Y)^{\Delta[\bullet]}$. By direct computation, this right adjoint is isomorphic to the functor taking a space Y to the simplicial space $(\mathrm{cosk}_n Y)^{\mathrm{Sk}_n \Delta[\bullet]}$, which is is isomorphic to the simplicial space $(\mathrm{cosk}_n Y)^{\Delta[\bullet]}$. □

Corollary 4.8. *Let X be a simplicial space. Then for every $i < n$, the map $\pi_i Re_n X \to \pi_i ReX$ is an isomorphism.*

Proof. When $i < n$, the ith homotopy group of X only depends on $\mathrm{Sk}_n X$. Hence Proposition 4.7 gives the result. □

4.4. **Realizations of pro-spaces.** We are interested in studying realizations and k-truncated realizations of pro-spaces. One can define homotopy colimits in any simplicial model category [13, Ch. 19], so homotopy colimits of pro-spaces can be formed.

Proposition 4.9. *For any simplicial pro-space X, the realization ReX is weakly equivalent to* $\operatorname{hocolim} X$.

Proof. Realization agrees with homotopy colimit up to weak equivalence for Reedy cofibrant simplicial objects X [13, Thm. 19.6.4]. We show that every simplicial pro-space is Reedy cofibrant. It is necessary to prove that the map $L_n X \to X_n$ is a cofibration for every n. In order to calculate $L_n X \to X_n$, only the degeneracy maps $X_{m-1} \to X_m$ for $1 \leq m \leq n$. are relevant. Since these maps form a finite loopless diagram, we may take a level representation of the pro-spaces $X_0, X_1, \ldots, X_n$ and the degeneracy maps. Since finite colimits of pro-objects can be computed levelwise, it follows that $L_n X \to X_n$ can be computed levelwise. For a simplicial space S, $L_n S \to S_n$ is always a cofibration [13, Cor. 16.7.8]. Hence $L_n X \to X_n$ is a levelwise cofibration, so it is a cofibration of pro-spaces. □

Given any pro-space X, apply Sk_n to each X_s to obtain another pro-space $\mathrm{Sk}_n X$. Define $\mathrm{cosk}_n X$ similarly. A straightforward computation shows that Sk_n and cosk_n are adjoint functors from pro-spaces to pro-spaces.

Remark 4.10. Given a pro-space X, there are two ways to interpret the symbol $\mathrm{Sk}_n X$. First, we may think of the pro-space formed by taking the n-skeleton of each space X_s. From this viewpoint, Sk_n is left adjoint to cosk_n. This is what we always mean by the notation $\mathrm{Sk}_n X$ for a pro-space X.

On the other hand, we may think of X as a simplicial pro-set and then apply $\mathrm{Sk}_n X$ in the sense of Definition 3.6 to obtain another simplicial pro-set. Subtleties arise because the category of simplicial pro-sets is not equivalent to the category of pro-spaces [16, Rem. 3.5]. The problem is that the simplicial indexing category Δ^{op} is not cofinite. However, we have no need for this second construction, so this is no issue for us.

The following proposition is a direct analogue for pro-spaces of Proposition 4.7.

Proposition 4.11. *Let X be a simplicial object in the category of pro-spaces. Then the natural map $\mathrm{Sk}_n Re_n X \to \mathrm{Sk}_n ReX$ is an isomorphism.*

Proof. The proof is similar to the proof of Proposition 4.7. The right adjoint of $\mathrm{Sk}_n Re$ is the functor taking the pro-space Y to the simplicial pro-space $(\mathrm{cosk}_n Y)^{\Delta[\cdot]}$. The right adjoint of $\mathrm{Sk}_n Re_n$ is the functor taking the pro-space Y to the simplicial pro-space that is the nth coskeleton of the simplicial pro-space $(\mathrm{cosk}_n Y)^{\Delta[\cdot]}$. Each term of the nth coskeleton is a finite limit of pro-spaces, so they can be calculated levelwise. It follows by direct computation that the right adjoint takes Y to the simplicial pro-space $(\mathrm{cosk}_n Y)^{\mathrm{Sk}_n \Delta[\cdot]}$. Both functors cosk_n and $(\cdot)^{\mathrm{Sk}_n \Delta[n]}$ on pro-spaces are defined levelwise, so the same calculation as in the proof of Proposition 4.7 tells us that the right adjoint takes Y to the simplicial pro-space $(\mathrm{cosk}_n Y)^{\Delta[\cdot]}$. □

Corollary 4.12. *Let X be a simplicial object in the category of pointed pro-spaces. Then for every $i < n$, the map $\pi_i Re_n X \to \pi_i ReX$ is an isomorphism of pro-groups.*

Proof. When $i < n$, the ith homotopy pro-group of X only depends on $\mathrm{Sk}_n X$. Hence Proposition 4.11 gives the result. □

5. Hypercover Descent for the Etale Topological Type

We now review the etale topological type functor. We record some of its basic properties from [9].

For a simplicial scheme X, let $\mathbf{Et}X$ be the pro-space defined by the functor $Re \circ \pi$ from $\mathrm{HRR}(X)$ to spaces [9, Defn. 4.4]. Here π is the functor that takes a scheme to its set of connected components, and the category of bisimplicial sets is identified with the category of simplicial spaces in order to interpret the realization.

Recall the diagonal functor that takes a bisimplicial set T to its diagonal simplicial set $n \mapsto T_{n,n}$. This functor was used instead of realization in [9, Defn. 4.4]. However, the diagonal of a simplicial space is the same as its realization [27, p. 94], so our definition is the same.

Given a simplicial scheme map $f : X \to Y$, rigid pullback as described in Definition 3.20 gives a functor $\mathrm{HRR}(Y) \to \mathrm{HRR}(X)$. If U is a rigid hypercover of Y, then there is a canonical rigid hypercover map $f^*U \to U$. These maps induce a strict map $\mathrm{Et}X \to \mathrm{Et}Y$. The fact that this map of pro-spaces is strict is critical for the proof of Proposition 5.2.

When X is a scheme, define $\mathrm{Et}X$ to be $\mathrm{Et}(cX)$. In this case, there is a slightly simpler formula for $\mathrm{Et}X$. It is just the functor π from $\mathrm{HRR}(X)$ to spaces. This follows from Lemma 3.17.

If X is a pointed and connected scheme, then $\mathrm{Et}X$ is a pointed and connected pro-space [9, Prop. 5.2]. By [15, Cor. 7.5], the pro-groups $\pi_i \mathrm{Et}X$ determine the homotopy type of the pro-space $\mathrm{Et}X$. The étale topological type commutes with coproducts [9, Prop. 5.2], so the study of arbitrary schemes reduces easily to the study of pointed and connected schemes by considering one component at a time and choosing an arbitrary basepoint for each component.

The realization in the definition of the étale topological type is an infinite colimit. This creates problems when trying to analyze the associated pro-spaces. Therefore, n-truncated realizations enter into the picture.

Let $\mathbf{Et}_n X$ be the pro-space given by the functor $Re_n \circ \pi$ from $\mathrm{HRR}(X)$ to spaces. In general, $\mathrm{Et}_n X$ is not equivalent to $\mathrm{Et}X$, but the next proposition tells us that the pro-spaces $\mathrm{Et}_n X$ are close enough to $\mathrm{Et}X$ to determine its homotopy type.

Proposition 5.1. *Suppose that X is a pointed and connected scheme. The pro-map $\pi_i Et_n X \to \pi_i EtX$ is an isomorphism of pro-groups whenever $i < n$.*

Proof. This follows immediately from Corollary 4.8. □

If X is a simplicial scheme, then we can calculate $\mathrm{Et}X_n$ for each n separately. These constructions assemble into a simplicial pro-space. This diagram is actually a strict representation.

We would like to compare $\mathrm{Et}X$ with $\mathrm{hocolim}_n \mathrm{Et}X_n$, where the homotopy colimit is calculated in the category of pro-spaces. In general they are not isomorphic. The problem is that the realization in the definition of $\mathrm{Et}X$ is constructed levelwise. Since this is an infinite colimit, it is not equal to the realization of the simplicial pro-space $n \mapsto \mathrm{Et}X_n$. Nevertheless, we shall prove that the natural map $\mathrm{hocolim}_n \mathrm{Et}X_n \to \mathrm{Et}X$ is a weak equivalence of pro-spaces.

Proposition 5.2. *The pro-space $Et_n X$ is isomorphic to the pro-space $Re_n\,(m \mapsto EtX_m)$.*

Proof. For simplicity of notation, let Y be the pro-space $Re_n\,(m \mapsto EtX_m)$.

As described in Remarks 4.4 and 4.6, the diagram for calculating Y is a cofinite diagram of strict maps of pro-spaces. Moreover, each of the categories $\mathrm{HRR}(X_n)$ has finite limits because of the existence of rigid limits. According to [16, § 3.1], the index set K for Y is the product category

$$\mathrm{HRR}(X_0) \times \mathrm{HRR}(X_1) \times \cdots \times \mathrm{HRR}(X_n).$$

For each $V = (V_{0,\cdot}, V_{1,\cdot}, \ldots, V_{n,\cdot})$ in K, the space Y_V is the coequalizer of the diagram

$$\coprod_{\substack{\phi:[m]\to[k]\\ m,k\leq n}} \pi(V_{k,\cdot} \overset{R}{\times} \phi^* V_{m,\cdot}) \otimes \Delta[m] \rightrightarrows \coprod_{m\leq n} \pi(V_{m,\cdot}) \otimes \Delta[m].$$

In this diagram, the upper map is induced by the maps $\phi_* : \Delta[m] \to \Delta[k]$ and the projections $V_{k,\cdot} \overset{R}{\times} \phi^* V_{m,\cdot} \to V_{k,\cdot}$, while the lower map is induced by the maps

$$V_{k,\cdot} \overset{R}{\times} \phi^* V_{m,\cdot} \to \phi^* V_{m,\cdot} \to V_{m,\cdot}.$$

The functor $\mathrm{HRR}(X) \to K$ is cofinal by Proposition 3.34. Therefore, take $\mathrm{HRR}(X)$ to be the indexing category for Y. If V is a rigid hypercover of X, then Y_V is the coequalizer of the diagram

$$\coprod_{\substack{\phi:[m]\to[k]\\ m,k\leq n}} \pi(V_{k,\cdot} \overset{R}{\times} \phi^* V_{m,\cdot}) \otimes \Delta[m] \rightrightarrows \coprod_{m\leq n} \pi(V_{m,\cdot}) \otimes \Delta[m].$$

For every $\phi : [m] \to [k]$, the rigid hypercover map $V_{k,\cdot} \to V_{m,\cdot}$ gives us a map $V_{k,\cdot} \to \phi^* V_{m,\cdot}$. Since $\mathrm{HRR}(X_k)$ is actually a directed set, this means that $V_{k,\cdot} \overset{R}{\times} \phi^* V_{m,\cdot}$ is isomorphic to $V_{k,\cdot}$. It follows that Y_V is isomorphic to the coequalizer of the diagram

$$\coprod_{\substack{\phi:[m]\to[k]\\ m,k\leq n}} \pi(V_{k,\cdot}) \otimes \Delta[m] \rightrightarrows \coprod_{m\leq n} \pi(V_{m,\cdot}) \otimes \Delta[m].$$

In other words, Y_V is $Re_n(m \mapsto \pi V_{m,\cdot})$. This is precisely the definition of $\mathrm{Et}_n X$. □

The next theorem describes the étale topological type of a simplicial scheme X in terms of the étale topological types of each scheme X_n and homotopy colimits of pro-spaces.

Theorem 5.3. *For any simplicial scheme X, the natural map*

$$\operatorname*{hocolim}_n EtX_n \to EtX$$

is a weak equivalence in the category of pro-spaces.

Proof. By Proposition 4.9, it suffices to consider the realization of the simplicial pro-space $n \mapsto \mathrm{Et}X_n$.

First suppose that X is pointed and connected. By [15, Cor. 7.5], it suffices to show that the natural map $Re(n \mapsto \mathrm{Et}X_n) \to \mathrm{Et}X$ induces an isomorphism of pro-homotopy groups in all dimensions. By Corollary 4.12 and Proposition 5.1, we may as well consider the map $Re_m\,(n \mapsto \mathrm{Et}X_n) \to \mathrm{Et}_m X$ to study the homotopy groups in dimension less than m. This map induces an isomorphism on pro-homotopy groups by Proposition 5.2. Since m was arbitrary, the map $\pi_i Re(n \mapsto \mathrm{Et}X_n) \to \pi_i \mathrm{Et}X$ is a pro-isomorphism for all i.

Now suppose that X is not necessarily connected and pointed. Since Et commutes with coproducts, we reduce to the case when X is connected but not necessarily pointed. However, choosing an arbitrary basepoint for X makes $\mathrm{Et}X$ into a based and connected pro-space. □

Now we come to the key ingredient for the proof of Theorem 2.6. The following result is a kind of hypercover descent theorem for the étale topological type.

Theorem 5.4. *Let U be a hypercover of a scheme X. Then the natural map*

$$\operatorname*{hocolim}_n EtU_n \to EtX$$

is a weak equivalence of pro-spaces.

Proof. This follows immediately from Theorem 5.3 and [9, Prop. 8.1]. □

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