# HOMOTOPY FIXED POINT SETS AND ACTIONS ON HOMOGENEOUS SPACES OF $p$–COMPACT GROUPS

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ABSTRACT. We generalize a result of Dror Farjoun and Zabrodsky on the relationship between fixed point sets and homotopy fixed point sets, which is related to the generalized Sullivan Conjecture. As an application, we discuss extension problems considering actions on homogeneous spaces of $p$–compact groups.

## Introduction.

For a group $\pi$, if $X$ is a $\pi$–space, the homotopy fixed point set $X^{h\pi}$ is the set of $\pi$–maps $map_\pi(E\pi, X)$. The fixed point set $X^\pi$ embeds in $X^{h\pi}$ as the subspace of constant maps. In [6], Dror Farjoun and Zabrodsky consider the relationship between $X^\pi$ and $X^{h\pi}$, when $\pi$ is a finite group and $X$ is a finite $\pi$–simplicial complex. Our work has been motivated by a result of theirs. Namely, if $\pi$ is a finite $p$–group, they show that $X^\pi$ is an empty set if and only if $X^{h\pi}$ is empty. We observe that, when the finiteness condition on $X$ is replaced by a $p$–local one, the corresponding result still holds. Recall [23, p557] that the mod $p$ cohomological dimension of $X$, denoted by $cd_p(X)$, means the supremum of the integer $m$ such that there exists a sheaf $F$ of $\mathbb{Z}/p\mathbb{Z}$–modules with $H^m(X;F) \neq 0$. If $X$ is the $p$–completion of a finite complex, then $cd_p(X) < \infty$.

**Theorem 0.** *For a finite $p$–group $\pi$, suppose a $\pi$–space $X$ is $\mathbb{F}_p$-complete and $cd_p(X)$ is finite. Then $X^\pi$ is an empty set if and only if the homotopy fixed point set $X^{h\pi}$ is empty.*

Our proof is analogous to the one given by Dror Farjoun and Zabrodsky [6]. Their argument uses Miller's theorem (Sullivan Conjecture) [19], and we use its $p$–compact group version [9]. Our result is related to the generalized Sullivan Conjecture. Some results on this matter can be found in [4] and [5].

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A $p$–compact group, [8], is a loop space $X$ such that $X$ is $\mathbb{F}_p$–finite and that its classifying space $BX$ is $\mathbb{F}_p$–complete. The $p$–completion of a compact Lie group $G$ is a $p$–compact group if $\pi_0(G)$ is a $p$–group. Suppose a map $i : X \longrightarrow Y$ is a monomorphism of $p$–compact groups so that the homotopy fiber $Y/X$ of the delooped map $BX \longrightarrow BY$ is $\mathbb{F}_p$–complete and $\mathbb{F}_p$–finite. For a map $f : B\pi \longrightarrow BY$, there is an extension map $\widetilde{f} : B\pi \longrightarrow BX$ if and only if $(Y/X)^{h\pi} \neq \emptyset$, [8, 3.4 Actions on homogeneous spaces].

$$\begin{array}{ccc} & & BX \\ & \overset{\tilde{f}}{\nearrow} & \downarrow{\scriptstyle Bi} \\ B\pi & \xrightarrow[f]{} & BY \end{array}$$

We consider what follows if $(Y/X)^{h\pi} \neq \emptyset$, and what conditions guarantee the existence of an extension map respectively.

**Theorem 1.** *Let $X \longrightarrow Y$ be a monomorphism of $p$–compact groups and let $\pi$ be a finite $p$–group. Suppose the $p$–compact group $X$ is abelian and $f : B\pi \longrightarrow BY$ is induced from a monomorphism. If there is an extension map $B\pi \longrightarrow BX$, the group $\pi$ is abelian.*

**Theorem 2.** *Suppose $V$ is an elementary abelian $p$–group and $T \longrightarrow Y$ is a maximal torus of a connected $p$–compact group $Y$ with Weyl group $W(Y)$. If the map $BT \longrightarrow BY$ induces $H^*(BY;\mathbb{F}_p) = H^*(BT;\mathbb{F}_p)^{W(Y)}$, any map $f : BV \longrightarrow BY$ has an extension map $BV \longrightarrow BT$.*

Let $G$ be a compact Lie group and let $H$ be a subgroup of $G$ with inclusion $i : H \longrightarrow G$. In [6, Example D1], it is shown that, for a homomorphism $\rho : \pi \longrightarrow G$, the group $\rho(\pi)$ is conjugate in $G$ to a subgroup of $H$ if and only if $B\rho : B\pi \longrightarrow BG$ lifts (up to homotopy) to $B\pi \longrightarrow BH$. Of course the condition is equivalent to $(G/H)^{\rho(\pi)} \neq \emptyset$. We consider a generalization of this result.

We recall that the $G$–action on $G/H$ is based on the following two actions. A left action $G \times X \longrightarrow X$ and a right action $X \times K \longrightarrow X$ give us the $G$–space $X/K$. In the case of $G/H$, take $X = G$ and $K = H$. In this particular case, the composite map $G \longrightarrow G \times X \longrightarrow X$ deloops to $BG \longrightarrow BX$, and similarly $BK \longrightarrow BX$, since the maps are induced by homomorphisms of groups. We consider the case when $X$ is a sphere $S^{2n-1}$. The unitary group $U(n)$ acts on the sphere $S^{2n-1}$ from both left and right in the usual way so that $S^{2n-1} = U(n)/U(n-1)$. Let a finite $p$–group $\pi$ is a subgroup of $U(n)$ and $K = S^1 = U(1) \hookrightarrow U(n)$. We note that $S^1$ is the maximal torus of the mod $p$ finite loop space $S^{2n-1}$ when $n$ divides $p-1$. If the map $\pi \longrightarrow (S^{2n-1})^\wedge_p$ is a homomorphism of $p$–compact groups, Theorem 0 implies that the delooped map lifts to $B\pi \longrightarrow B(S^1)^\wedge_p$ if and only if $(S^{2n-1}/S^1)^\pi \neq \emptyset$.

**Theorem 3.** *If a finite $p$–group $\pi$ which is a subgroup of $U(n)$ acting on $S^{2n-1}$ as above is abelian, the fixed point set $(S^{2n-1}/S^1)^\pi$ is non–empty.*

If $X \longrightarrow Y$ is a monomorphism of $p$–compact groups, and a finite $p$-group $\pi$ acts on both $BX$ and $BY$, then $(Y/X)^{h\pi}$ is $\mathbb{F}_p$–complete. A result of [8] implies that $(Y/X)^{h\pi}$ is $\mathbb{F}_p$–finite. We consider the space $(Y/X)^{h\pi}$ which is obtained by the induced fibration over $B\pi$ with fiber $Y/X$ from a map $B\pi \longrightarrow BY$.

$$\begin{array}{ccc} Y/X & = & Y/X \\ \downarrow & & \downarrow \\ E\pi \times_\pi (Y/X) & \longrightarrow & BX \\ \downarrow & & \downarrow \\ B\pi & \longrightarrow & BY \end{array}$$

**Theorem 4.** *Let $(Y/X)^{h\pi}$ be the homotopy fixed point set obtained as above for a monomorphism $X \longrightarrow Y$ of $p$–compact groups. If $(Y/X)^{h\kappa}$ is $\mathbb{F}_p$–good for any subgroup $\kappa$ of the finite $p$–group $\pi$, then $(Y/X)^{h\pi}$ is $\mathbb{F}_p$–finite.*

Any nilpotent space is $\mathbb{F}_p$–good, [3]. Assuming a nilpotency condition, the second author [18] shows the $\mathbb{F}_p$–finiteness of a finite complex with an action of a $p$–compact toral group.

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## 0. A $p$–local generalization of the result of Dror Farjoun and Zabrodsky.

The argument in this section is very similar to the one used in [6]. It can be considered as a $p$–local version.

A compact topological group $G$ is said to have the *extended homotopy fixed point property* (EHFPP) if for every $\mathbb{F}_p$–complete $G$–space $X$ with $cd_p(X) < \infty$, one has $X^G = \emptyset$ if and only if $X^{hG} = \emptyset$. If $G$ has no EHFPP, that is, there exist an $\mathbb{F}_p$–complete $G$–space $X$ with $cd_p(X) < \infty$ and $X^G = \emptyset$, and an equivariant map $EG \longrightarrow X$, we say that $G$ is *extended compressible.*

The following is an extended version of [6, Theorem B], and to show that (iii) implies (i), the result of [13, p45 Cor. 1] can be used as mentioned in [6].

**Proposition 0.1.** *For an elementary abelian $p$–group $V$, suppose a $V$–space $X$ is $\mathbb{F}_p$–complete and $cd_p(X)$ is finite. Then the following are equivalent.*

(i) $X^V \neq \emptyset$
(ii) $X^{hV} \neq \emptyset$
(iii) *The classifying map $\chi : EV \times_V X \longrightarrow BV$ induces a monomorphism on mod $p$ cohomology.*

The following is a $p$–local version of [6, Lemma 2.1]

**Lemma 0.2.** *Suppose the kernel $H$ of an epimorphism $G \longrightarrow G_0$ is a compact Lie group such that the loop space $\Omega(BH)^{\wedge}_p$ is a $p$–compact group. Let $X$ be an $\mathbb{F}_p$–complete and $\mathbb{F}_p$–finite $G_0$–complex. Then the natural composition $map_{G_0}(EG_0, X) \longrightarrow map_G(EG, X)$ is a weak equivalence of spaces.*

*Proof.* The argument is very similar to the one used in [6]. Notice that $map_G(EG, X) = map_G(EG/H, X) = map_G(BH, X)$, where $H$ has a trivial action on $X$ and $EG$ is a free contractible $H$–space. We consider the diagram

$$\begin{array}{ccc} map_G(EG, X) & \xrightarrow{=} & map_G(EG/H, X) \\ \phi \uparrow & & \downarrow = \\ map_{G_0}(EG_0, X) & \xrightarrow{\psi} & map_G(BH, X) \end{array}$$

To show $\phi$ is a weak equivalence, it suffices to show that $\psi$ is a weak equivalence. Taking the full function space from $\psi$, we get $\bar{\psi} : map(EG_0, X) \longrightarrow map(BH, X)$. According to the result of [9, Theorem 9.3] we can see that

$map(BH, X) \simeq X$, since $\Omega(BH)^\wedge_p$ is a $p$–compact group and $X$ is $\mathbb{F}_p$–complete and $\mathbb{F}_p$–finite. Now $\bar{\psi}$ is a $G$–map, and a weak homotopy equivalence when the $G$–actions are ignored. Therefore $\bar{\psi}$ induces a homotopy fixed point equivalence. Since $EG_0$ and $BH = EG/H$ are free $G_0$-space, we see that $map_{G_0}(BH, X) \simeq map_{G_0}(EG_0, map(BH, X)) \simeq map_{G_0}(EG_0, X)$ by [6, Lemma 2.2]. Consequently $\bar{\psi}$ must be a weak $G_0$-equivalence. Therefore $\psi$ is weakly equivalent. □

It is well-known that, for a compact Lie group $K$, the loop space $\Omega(BK)^\wedge_p$ is $p$–compact if $\pi_0(K)$ is a $p$–group. The problem on the conditions of a compact Lie group that its loop space of the $p$–completed classifying space be a $p$–compact group is considered in [15] and [16]. A result says that if $\Omega(BK)^\wedge_p$ is a $p$–compact group, then $\pi_0 K$ must be $p$–nilpotent.

Let $G$ be a group. Recall [12] that the Frattini subgroup of $G$ is the intersection of maximal subgroups of $G$, denoted by $\Phi(G)$. Let $G_0 = G/\Phi G$ be Frattini factor group of $G$. Using Lemma 0.2, the argument of the proof of [6, Theorem C] is applicable for the following result.

**Proposition 0.3.** *A finite $p$–group $G$ is extended compressible if and only if its Frattini factor group $G_0$ is extended compressible.*

*Proof of Theorem 0.* Since the Frattini factor of finite $p$–group is an elementary abelian $p$–group, the desired result is immediate from Proposition 0.1 and Proposition 0.3. □

Let $G$ be a $p$–compact toral group. It is known [8, Proposition 6.9] that any $p$–compact toral group $G$ has a discrete approximation $f : G_\infty \to G$. For the discrete toral group $G_\infty$, there exists an increasing chain $G_n \subset G_{n+1} \subset \cdots$ of finite subgroups of $G_\infty$ such that $G_\infty = \cup_{i \geq n} G_i$. Then we have the following result.

**Corollary 0.4.** *Suppose $X$ is an $\mathbb{F}_p$-complete space with the proxy action of a $p$-compact toral group $G$ and $cd_p(X) < \infty$. If $G_\infty$ acts on $X$ and $X^{G_i} = \emptyset$ for some finite $p$-subgroup $G_i$ of $G_\infty$, then $X^{hG}$ is empty.*

*Proof.* Since $X^{hG_\infty}$ is equivalent to the homotopy inverse limit of the tower $\{X^{hG_i}\}_{i \geq n}$, if $X^{G_i} = \emptyset$ for some $G_i$ then $X^{hG_i} = \emptyset$ by Theorem 0. This implies $X^{hG_\infty} = \emptyset$. According to [8, Proposition 6.8], the discrete approximation $f$ induces a homotopy equivalence $X^{hG} \to X^{hG_\infty}$. Therefore $X^{hG}$ is empty. □

**Corollary 0.5.** *Let $G_\infty$ be a $p$–discrete toral group. Suppose $G_\infty$–space $X$ is $\mathbb{F}_p$–complete and $cd_p(X) < \infty$. Then $X^{hG_\infty} = \emptyset$ if and only if $X^{G_\infty} = \emptyset$.*

*Proof.* It suffices to show that $X^{hG_\infty} \neq \emptyset$ implies $X^{G_\infty} \neq \emptyset$. So let $X^{hG_\infty} \neq \emptyset$. Then there is $m$ such that $X^{hG_i} \neq \emptyset$ for all $i \geq m$. Theorem 0 says $X^{G_i} \neq \emptyset$ for all $i \geq m$. Therefore $X^{G_\infty} \neq \emptyset$. □

Let $X_{h\pi}$ denote the Borel construction so that $X_{h\pi} = E\pi \times_\pi X$. Assume $X$ satisfies the condition of Theorem 0. According to [8, Theorem 7.4] together with Theorem 0, we immediately conclude that $X_{h\pi}$ is $\mathbb{F}_p$–finite if and only if the fixed point set $X^\kappa$ is empty for any subgroup $\kappa$ of the finite $p$–group $\pi$ of order $p$.

## 1. Extension problems and the mod $p$ cohomology.

In this section we consider extension problems, and prove Theorem 1 and Theorem 2. Some results of mod $p$ cohomology of classifying spaces will be used.

*Proof of Theorem 1.* Recall that any abelian $p$–compact group is equivalent to the product of a $p$–compact torus and a finite abelian $p$–group, so that $BX = (BG)^\wedge_p$ for a compact abelian Lie group $G$, [9] and [21]. Thus the extension map $\tilde{f} : B\pi \longrightarrow BX$ is induced by a group homomorphism $\rho : \pi \longrightarrow G$, since $\pi$ is a finite $p$–group, [11]. It is enough to show that this group homomorphism $\rho$ is injective. Since $f : B\pi \longrightarrow BY$ is a monomorphism, the induced homomorphism $f^* : H^*(BY;\mathbb{F}_p) \longrightarrow H^*(B\pi;\mathbb{F}_p)$ is finite, [8, Proposition 9.11]. This means that $H^*(B\pi;\mathbb{F}_p)$ is a finitely generated module over $f^*(H^*(BY;\mathbb{F}_p))$. Consider the following commutative diagram

$$\begin{array}{ccc} & & H^*(BX;\mathbb{F}_p) \\ & \overset{\tilde{f}^*}{\swarrow} & \uparrow \\ H^*(B\pi;\mathbb{F}_p) & \underset{f^*}{\longleftarrow} & H^*(BY;\mathbb{F}_p) \end{array}$$

Since $\tilde{f}^*(H^*(BX;\mathbb{F}_p))$ contains $f^*(H^*(BY;\mathbb{F}_p))$, we see that $H^*(B\pi;\mathbb{F}_p)$ is a finitely generated module over $\tilde{f}^*(H^*(BX;\mathbb{F}_p))$, and therefore $\tilde{f}^* = ((B\rho)^\wedge_p)^*$ is finite. So a result of Quillen [23] implies that the kernel of $\rho$ is trivial. $\square$

An argument analogous to the one used here shows the following result:

**Theorem 1.1.** *Let $X \longrightarrow Y$ be a monomorphism of $p$–compact groups and let $\pi$ be a finite $p$–group. Suppose $BX = (BG)^\wedge_p$ for a compact Lie group $G$ and $f : B\pi \longrightarrow BY$ is induced from a monomorphism. Assume there is an extension map $B\pi \longrightarrow (BG)^\wedge_p$ so that this map is induced from a group homomorphism $\rho : \pi \longrightarrow G$. Then $\rho$ is injective.*

*Proof of Theorem 2.* Since $H^*(BY;\mathbb{F}_p) = H^*(BT;\mathbb{F}_p)^{W(Y)}$, using a result of [2] one can show that there is a homomorphism $\phi : H^*(BT;\mathbb{F}_p) \longrightarrow H^*(BV;\mathbb{F}_p)$ which makes the following diagram commutative over the Steenrod algebra:

$$\begin{array}{ccc} & & H^*(BT;\mathbb{F}_p) \\ & \swarrow^{\phi} & \uparrow \\ H^*(BV;\mathbb{F}_p) & \xleftarrow[f^*]{} & H^*(BY;\mathbb{F}_p) \end{array}$$

Here $\phi$ factors through the polynomial part of $H^*(BV;\mathbb{F}_p)$. We note that $H^*(B(\mathbb{Z}/p)^n;\mathbb{F}_p) = \mathbb{F}_p[x_1,\cdots,x_n]\otimes\Lambda(y_1,\cdots,y_n)$ for odd prime $p$ where each $y_i$ has dimension 1 and each $x_i = \beta y_i$ has dimension 2. Since $V$ is an elementary abelian $p$–group, a result of [17] implies that there is a homomorphism $\rho : V \longrightarrow T$ such that $\phi = (B\rho)^*$, and the following diagram is commutative:

$$\begin{array}{ccc} & & BT \\ & \nearrow^{B\rho} & \downarrow \\ BV & \xrightarrow[f]{} & BY \end{array}$$

This completes the proof. $\square$

For a connected compact Lie group $G$, we note that, for instance, if $p$ is odd and $G$ is $p$–torsion free, then $H^*(BG;\mathbb{F}_p)$ is isomorphic to the invariant ring $H^*(BT_G;\mathbb{F}_p)^{W(G)}$, where $T_G$ is a maximal torus, and $W(G)$ denotes the Weyl group. Analogous results hold for connected $p$–compact groups $X$ when $H^*(X;\mathbb{Z}_p^\wedge)$ is torsion free, [10] and [22]. Next we recall that $SO(3)$ contains $\mathbb{Z}/2\oplus\mathbb{Z}/2$ as a subgroup. Considering Theorem 2 when $p=2$ and $Y = SO(3)_2^\wedge$, we notice that $H^*(BSO(3);\mathbb{F}_2) \not\cong H^*(BS^1;\mathbb{F}_2)^{\mathbb{Z}/2}$, and that there is no extension for the monomorphism $B(\mathbb{Z}/2\oplus\mathbb{Z}/2) \longrightarrow BSO(3)_2^\wedge$, since $rank(SO(3)) = 1$. Generally, we see that if $rank(G) < 2\text{–}rank(G)$, then $H^*(BG;\mathbb{F}_2)$ is not isomorphic to the invariant ring $H^*(BT_G;\mathbb{F}_2)^{W(G)}$.

## 2. Actions on homogeneous spaces and fixed point sets.

We recall that a left action $G\times X \longrightarrow X$ and a right action $X\times K \longrightarrow X$ give the $G$–space $X/K$. Let $[x] = xK \in X/K$. For $g\in G$, the action is given by $g\cdot[x] = [gx]$. If $[x_0]\in (X/K)^G$, for any $g$ we can find $k\in K$ such that $gx_0 = x_0k$. In the case $G = U(n)$, $X = S^{2n-1}$ and $K = S^1 = U(1) \hookrightarrow U(n)$ as mentioned in the introduction, the equation of $gx_0 = x_0k$ is expressed in the matrix form. For $n=2$, for instance, the expression is given by the following:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left[\begin{pmatrix} x_1 & x_2 \end{pmatrix}\cdot\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\right]^T$$

where $A^T$ denotes the transpose of a matrix $A$.

*Proof of Theorem 3.* Since $\pi$ is abelian, all the irreducible representations of $\pi$ have degree 1. Consequently there is $\sigma \in U(n)$ such that $\sigma^{-1}\pi\sigma$ is a subgroup of $T^n$, where $T^n$ is the maximal torus of $U(n)$ consisting of diagonal matrices. Let $g \in \pi$ and let $g' = \sigma^{-1}g\sigma$. If $g'x = xk$ for some $x \in S^{2n-1}$, then $gx_0 = x_0k$ where $x_0 = \sigma x$. It remains to find such $x \in S^{2n-1}$. Since $g'$ is a diagonal matrix, it is easy to see that if $x = (1, 0, \cdots, 0) \in S^{2n-1}$, then, as seen below,

$$\begin{pmatrix} \zeta_1 & & & 0 \\ & \zeta_2 & & \\ & & \ddots & \\ 0 & & & \zeta_n \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \left[ \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} \zeta_1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \right]^T$$

for any $g'$ there is $k \in S^1$ such that $g'x = xk$. $\square$

In Theorem 2, taking $Y = (S^{2n-1})^\wedge_p$ when $n$ divides $p-1$, we obtain $((S^{2n-1}/S^1)^\wedge_p)^{hV} \neq \emptyset$, and Theorem 0 says $((S^{2n-1}/S^1)^\wedge_p)^V \neq \emptyset$. Note that, in general, if $\pi$ is a finite $p$–group and $X$ is a finite $\pi$–complex, then $X^\pi \neq \emptyset$ if and only if $(X^\wedge_p)^\pi \neq \emptyset$. This result follows from the following diagram

$$\begin{array}{ccc} X^\pi & \longrightarrow & (X^\wedge_p)^\pi \\ \downarrow & & \downarrow \\ (X^\pi)^\wedge_p & \longrightarrow & (X^\wedge_p)^{h\pi} \end{array}$$

and the result $(X^\pi)^\wedge_p \simeq (X^\wedge_p)^{h\pi}$, [20, Theorem 2]. Consequently it follows that $(S^{2n-1}/S^1)^V \neq \emptyset$, which is a special case of Theorem 3 assuming the map $V \longrightarrow (S^{2n-1})^\wedge_p$ is a homomorphism of $p$–compact groups.

Next we consider the non–abelian case. Suppose $Q_8$ denotes the quaternion group in $SU(2)$;

$$Q_8 = < a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} >$$

where

$$a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Taking $x = (1,0)$ as the base point of $S^3$, the composite map $Q_8 \longrightarrow U(2) \longrightarrow U(2) \times S^3 \longrightarrow S^3$ is a homomorphism of groups. A direct calculation shows $(S^3/S^1)^{Q_8} = \emptyset$. This result can be obtained from Theorem 1, since $Q_8$ is non–abelian. We have the following generalization.

**Proposition 2.1.** *For $n \geq 2$, let $\rho : \pi \longrightarrow U(n)$ be an irreducible representation for a non–abelian finite $p$–group $\pi$. Then $(S^{2n-1}/S^1)^\pi = \emptyset$.*

*Proof.* The center of a nontrivial finite $p$–group contains more than one element. Since the representation $\rho$ is irreducible, Schur's lemma tells us that we can find $a \in \pi$ such that $\rho(a)$ is a diagonal matrix

$$\begin{pmatrix} \zeta & & & 0 \\ & \zeta & & \\ & & \ddots & \\ 0 & & & \zeta \end{pmatrix}$$

where $\zeta$ is a $p$–th primitive root of unity. If $(S^{2n-1}/S^1)^\pi \neq \emptyset$, then an argument analogous to the one used in our proof of Theorem 3 shows that the following equation should be satisfied:

$$\begin{pmatrix} \zeta & & & 0 \\ & \zeta & & \\ & & \ddots & \\ 0 & & & \zeta \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \left[ \begin{pmatrix} z_1 & z_2 & \cdots & z_n \end{pmatrix} \cdot \begin{pmatrix} \zeta & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \right]^T$$

for suitable $(z_1, z_2, \cdots, z_n) \in S^{2n-1}$. Consequently it follows that $z_i = 0$ for $i = 2, \cdots, n$. Now let $x_0 = (z_1, 0, \cdots, 0)$. Using the equations $gx_0 = x_0 k$ for all $g \in \rho(\pi)$, we see that all entries of the first column except the $(1,1)$–entry of each matrix $g$ are zero. This means that there would be a 1–dimensional invariant subspace. This is a contradiction, since the representation is irreducible and $n \geq 2$. Therefore $(S^{2n-1}/S^1)^\pi = \emptyset$. □

We have seen that the $G$–action on $G/H$ is based on the following two actions: $G \times X \longrightarrow X$ and $X \times K \longrightarrow X$. In the case $X = G$, the composite of the based maps $G \longrightarrow G \times X \longrightarrow X$ deloops to $BG \longrightarrow BX$. Here we consider the deloopability problem for $G = U(n)$ and $X = S^{2n-1}$ or $SU(n)$. Let $\psi : U(n) \times S^{2n-1} \longrightarrow S^{2n-1}$ be the $U(n)$–action on $S^{2n-1}$. For this action, we will show that the $p$–completed map $(U(n))^\wedge_p \longrightarrow (S^{2n-1})^\wedge_p$ is not deloopable for any prime $p$.

**Proposition 2.2.** *The composite map* $U(n) \longrightarrow U(n) \times S^{2n-1} \xrightarrow{\psi} S^{2n-1}$ *is not deloopable at any prime* $p$ *when* $n \geq 2$.

Our proof for the case $n \geq 3$ will use admissible maps, [1]. The case $n = 2$ is, however, treated separately. This is a special case of the following $U(n)$–action on $SU(n)$. The action $\mu : U(n) \times SU(n) \longrightarrow SU(n)$ is given by

$$\mu(A, B) = A \cdot B \cdot \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \det A^{-1} \end{pmatrix}$$

for $A \in U(n)$ and $B \in SU(n)$. This action is transitive, and the isotropy subgroup at the identity is isomorphic to $U(1)$.

**Proposition 2.3.** *The composite map* $U(n) \longrightarrow U(n) \times SU(n) \xrightarrow{\mu} SU(n)$ *is not deloopable at any prime* $p$.

*Proof.* There is a finite covering $\mathbb{Z}/n \longrightarrow SU(n) \times S^1 \xrightarrow{q} U(n)$, where $q|_{SU(n)}$ is the inclusion $SU(n) \hookrightarrow U(n)$ and $q(S^1)$ is the center of $U(n)$. If the map $U(n) \longrightarrow SU(n)$ is deloopable at $p$, we obtain a map induced from the composition

$$(BSU(n))^\wedge_p \times (BS^1)^\wedge_p \longrightarrow (BSU(n))^\wedge_p$$

The axis $(BSU(n))^\wedge_p \longrightarrow (BSU(n))^\wedge_p$ is the identity map. According to [14, Theorem 1], the other axis $(BS^1)^\wedge_p \longrightarrow (BSU(n))^\wedge_p$ should factor through $(B\mathbb{Z}/n)^\wedge_p$, where $\mathbb{Z}/n$ is the center of $SU(n)$. This means that the map $(BS^1)^\wedge_p \longrightarrow (BSU(n))^\wedge_p$ would be a zero map. This is a contradiction, since the map $S^1 \longrightarrow SU(n)$ is a monomorphism. Thus we obtain the desired result. □

**Lemma 2.4.** *The two* $U(2)$*–spaces* $SU(2)$ *and* $S^3$ *are* $U(2)$*–homeomorphic.*

*Proof.* A homeomorphism $\tau : SU(2) \longrightarrow S^3$ is given by a map sending $\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ where $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$. The desired result follows from the following commutative diagram:

$$\begin{array}{ccc} U(2) \times SU(2) & \xrightarrow{\mu} & SU(2) \\ {\scriptstyle 1\times\tau}\downarrow & & \downarrow{\scriptstyle \tau} \\ U(2) \times S^3 & \xrightarrow{\psi} & S^3 \end{array}$$

This completes the proof. □

*Proof of Proposition 2.2.* The case $n = 2$ is proved by Proposition 2.3 and Lemma 2.4. So we assume $n \geq 3$. If the map $U(n) \longrightarrow S^{2n-1}$ was deloopable at $p$, we would have a map $BU(n)^\wedge_p \longrightarrow B(S^{2n-1})^\wedge_p$. Notice that the Lie group $SU(n)$ is simple, and $rank(SU(n)) \geq 2$. According to [1, Proposition 2.12], the restriction of the delooped map on $BSU(n)^\wedge_p$ is null homotopic.

$$\begin{array}{ccc} BU(n)^\wedge_p & \longrightarrow & B(S^{2n-1})^\wedge_p \\ \uparrow & \nearrow & \\ BSU(n)^\wedge_p & & \end{array}$$

On the other hand, the restriction of $U(n) \longrightarrow S^{2n-1}$ on $U(1)$ is not null homotopic.

$$\begin{array}{ccc} BU(n)^\wedge_p & \longrightarrow & B(S^{2n-1})^\wedge_p \\ \uparrow & & \uparrow \\ BU(1)^\wedge_p & \xrightarrow{\text{id}} & B(S^1)^\wedge_p \end{array}$$

Consequently the map $BSU(n)^\wedge_p \longrightarrow B(S^{2n-1})^\wedge_p$ should be essential, since $U(1) \hookrightarrow SU(2) \hookrightarrow SU(n)$. This contradiction completes the proof. $\square$

## 3. Some properties of homotopy fixed point sets $(Y/X)^{h\pi}$.

Suppose that $X \longrightarrow Y$ is a homomorphism of $p$–compact groups, that a finite $p$–group $\pi$ acts on classifying spaces $BX$ and $BY$, and that $BX \longrightarrow BY$ is a $\pi$–map. According to [8, Lemma 10.6 and Proposition 5.8], if $BY^{h\pi} \neq \emptyset$, then $(Y/X)^{h\pi} \neq \emptyset$ and the space is $\mathbb{F}_p$–complete. If the map $X \longrightarrow Y$ is a monomorphism, then $Y/X$ is $\mathbb{F}_p$–finite. We see, by [8, Theorem 4.6], that $(Y/X)^{h\pi}$ is $\mathbb{F}_p$–finite.

As mentioned in the introduction, next we consider the space $(Y/X)^{h\pi}$ which is obtained by the induced fibration over $B\pi$ with fiber $Y/X$ from a map $B\pi \longrightarrow BY$, [8, Lemma 10.4].

$$\begin{array}{ccc} Y/X & = & Y/X \\ \downarrow & & \downarrow \\ E\pi \times_\pi (Y/X) & \longrightarrow & BX \\ \downarrow & & \downarrow \\ B\pi & \longrightarrow & BY \end{array}$$

A space $X$ is said to be $\mathbb{F}_p$–good, [3], if $H_*(X;\mathbb{F}_p) \longrightarrow H_*(X^\wedge_p;\mathbb{F}_p)$ induced from the $\mathbb{F}_p$–completion map $X \longrightarrow X^\wedge_p$ is an isomorphism. For instance, it is known [3, Ch VII Proposition 5.1] that if the fundamental group $\pi_1 X$ is finite, then $X$ is $\mathbb{F}_p$–good for any prime $p$.

*Proof of Theorem 4.* Recall [8, Remark 11.13] that a space is $\mathbb{F}_p$–complete if and only if $X$ is both $\mathbb{F}_p$–local and $\mathbb{F}_p$–good. Since $Y/X$ is $\mathbb{F}_p$–local, so is $(Y/X)^{h\kappa}$ for any subgroup $\kappa$ of the finite $p$–group $\pi$. From our assumption, we see that each $(Y/X)^{h\kappa}$ is $\mathbb{F}_p$–complete. Since the map $X \longrightarrow Y$ is a monomorphism of $p$–compact groups, the space $Y/X$ is $\mathbb{F}_p$–finite. Consequently [8, Theorem 4.6] implies that $(Y/X)^{h\pi}$ is $\mathbb{F}_p$–finite. □

As a special case, we notice [11, Lemma 2.3] that if the homotopy fiber $Y/X$ is nilpotent and mod $p$–acyclic, then $(Y/X)^{h\kappa}$ is also nilpotent and mod $p$–acyclic for any finite $p$–subgroup $\kappa$ of $\pi$. Since any nilpotent space is $\mathbb{F}_p$–good, each $(Y/X)^{h\kappa}$ is $\mathbb{F}_p$–good. This implies that $(Y/X)^{h\pi}$ is $\mathbb{F}_p$–finite.

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