

Implications of the Ganea Condition

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Abstract

Suppose the spaces X and $X \times A$ have the same Lusternik-Schnirelmann category: $\text{cat}(X \times A) = \text{cat}(X)$. Then there is a *strict* inequality $\text{cat}(X \times (A \times B)) < \text{cat}(X) + \text{cat}(A \times B)$ for every space B , provided the connectivity of A is large enough (depending only on X). This is applied to give a partial verification of a conjecture of Iwase on the category of products of spaces with spheres.

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Introduction

The product formula $\text{cat}(X \times Y) \leq \text{cat}(X) + \text{cat}(Y)$ [1] is one of the most basic relations of Lusternik-Schnirelmann category. Taking $Y = S^r$, it implies that $\text{cat}(X \times S^r) \leq \text{cat}(X) + 1$ for any $r > 0$. In [5], Ganea asked whether the inequality can ever be strict in this special case. The study of the ‘Ganea condition’ $\text{cat}(X \times S^r) = \text{cat}(X) + 1$ has been, and remains, a formidable challenge to all techniques for the calculation of Lusternik-Schnirelmann category. In fact, it was only recently that techniques were developed which were powerful enough to identify a space which does not satisfy the Ganea condition [8] (see also [9, 12]). It is still not well understood exactly which spaces X do not satisfy the Ganea condition, although it has been conjectured that they are precisely the spaces for which $\text{cat}(X)$ is not equal to the related invariant $\text{Qcat}(X)$ (see [14, 17]).

Since the failure of the Ganea condition appears to be a strange property for a space to have, it is reasonable to expect that such failure would have useful and interesting implications. In this paper we explore some of the implications of the equation $\text{cat}(X \times A) = \text{cat}(X)$ for general spaces A , and for $A = S^r$ in particular.

A brief look at the method of the paper [8] will help to put our results into proper perspective. The new techniques begin with the following question: if $Y = X \cup_f e^{t+1}$, the cone on $f : S^t \rightarrow X$, then how can we tell if $\text{cat}(Y) > \text{cat}(X)$? It is shown (see [9, Thm. 5.2] and [12, Thm. 3.6]) that, if $t \geq \dim(X)$, then $\text{cat}(Y) = \text{cat}(X) + 1$ if and only if a certain Hopf invariant $\mathcal{H}_s(f)$ (which is a set of homotopy classes) does not contain the trivial map $*$. It is also shown [9, Thm. 3.8] that if $* \in \Sigma^r \mathcal{H}_s(f)$, then $\text{cat}(Y \times S^r) \leq \text{cat}(X) + 1$. Thus Y does not satisfy Ganea's condition if $* \notin \mathcal{H}_s(f)$, but there is at least one $h \in \mathcal{H}_s(f)$ such that $\Sigma^r h \simeq *$.

Of course, if $\Sigma^r h \simeq *$, then $\Sigma^{r+1} h \simeq *$ as well, and this suggests the following conjecture (formulated in [8, Conj. 1.4]):

CONJECTURE *If $\text{cat}(X \times S^r) = \text{cat}(X)$, then $\text{cat}(X \times S^{r+1}) = \text{cat}(X)$.*

In this paper we prove that this conjecture is true, provided r is large enough.

Theorem 1 *Suppose X is a $(c - 1)$ -connected space and let $r > \dim(X) - c \cdot \text{cat}(X) + 2$. If $\text{cat}(X \times S^r) = \text{cat}(X)$, then*

$$\text{cat}(X \times S^t) = \text{cat}(X)$$

for all $t \geq r$.

The conjecture remains open for small values of r .

Our main result is much more general: it shows how the equation $\text{cat}(X \times A) = \text{cat}(X)$ governs the Lusternik-Schnirelmann category of products of X with a vast collection of other spaces.

Theorem 2 *Let X be a $(c - 1)$ -connected space and let A be $(r - 1)$ -connected with $r > \dim(X) - c \cdot \text{cat}(X) + 2$. If $\text{cat}(X \times A) = \text{cat}(X)$ then*

$$\text{cat}(X \times (A \times B)) < \text{cat}(X) + \text{cat}(A \times B)$$

for every space B .

When A is a suspension, the half-smash product decomposes as $A \rtimes B \simeq A \vee (A \wedge B)$ (see, for example [12, Lem. 5.9]), so we obtain the following.

Corollary *Under the conditions of Theorem 2, if A is a suspension, then*

$$\text{cat}(X \times (A \wedge B)) = \text{cat}(X)$$

for every space B .

Our partial verification of the conjecture is an immediate consequence of this corollary: it the special case $A = S^r$ and $B = S^{t-r}$.

Organization of the paper. In Section 1 we recall the necessary background information on homotopy pushouts, cone length and Lusternik-Schnirelmann category. We introduce an auxiliary space and establish its important properties in Section 2. The proof of Theorem 2 is presented in Section 3.

1 Preliminaries

In this paper all spaces are based and have the pointed homotopy type of CW complexes; maps and homotopies are also pointed. We denote by $*$ the one point space and any nullhomotopic map. Much of our exposition uses the language of homotopy pushouts; we refer to [11] for the definitions and basic properties.

1.1 Homotopy Pushouts

We begin by recalling some basic facts about homotopy pushout squares. We call a sequence $A \rightarrow B \rightarrow C$ a **cofiber sequence** if the associated square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ * & \longrightarrow & C \end{array}$$

is a homotopy pushout square. The space C is called the **cofiber** of the map f . One special case that we use frequently is the **half-smash product** $A \rtimes B$, which is the cofiber of the inclusion $B \rightarrow A \times B$.

Finally, we recall the following result on products and homotopy pushouts.

Proposition 3 *Let X be any space. Consider the squares*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \quad \text{and} \quad \begin{array}{ccc} X \times A & \longrightarrow & X \times B \\ \downarrow & & \downarrow \\ X \times C & \longrightarrow & X \times D. \end{array}$$

If the first square is a homotopy pushout, then so is the second.

Proof This follows from Theorem 6.2 in [15]. □

1.2 Cone Length and Category

A **cone decomposition** of a space Y is a diagram of the form

$$\begin{array}{ccccccc} L_0 & & L_1 & & & & L_{k-1} \\ \downarrow & & \downarrow & & & & \downarrow \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & \cdots & \longrightarrow & Y_{k-1} & \longrightarrow & Y_k \end{array}$$

in which $Y_0 = *$, each sequence $L_i \rightarrow Y_i \rightarrow Y_{i+1}$ is a cofiber sequence, and $Y_k \simeq Y$; the displayed cone decomposition has **length** k . The **cone length** of Y , denoted $\text{cl}(Y)$, is defined by

$$\text{cl}(Y) = \begin{cases} 0 & \text{if } Y \simeq * \\ \infty & \text{if } Y \text{ has no cone decomposition, and} \\ k & \text{if the shortest cone decomposition of } Y \text{ has length } k. \end{cases}$$

The Lusternik-Schnirelmann category of X may be defined in terms of the cone length of X by the formula

$$\text{cat}(X) = \inf\{\text{cl}(Y) \mid X \text{ is a homotopy retract of } Y\}.$$

Berstein and Ganea proved this formula in [3, Prop. 1.7] with $\text{cl}(Y)$ replaced by the strong category of Y ; the formula above follows from another result of Ganea — strong category is equal to cone length [7]. It follows directly from this definition that if X is a homotopy retract of Y , then $\text{cat}(X) \leq \text{cat}(Y)$. The reader may refer to [10] for a survey of Lusternik-Schnirelmann category.

The category of X can be defined in another way that is essential to our work. Let $F_0(X) \longrightarrow G_0(X) \xrightarrow{p_0} X$ be the familiar path-loop fibration sequence $\Omega(X) \longrightarrow \mathcal{P}(X) \longrightarrow X$. Given the n^{th}

Ganea fibration sequence $F_n(X) \longrightarrow G_n(X) \xrightarrow{p_n} X$, let $\overline{G}_{n+1}(X) = G_n(X) \cup CF_n(X)$ be the cofiber of p_n and define $\overline{p}_{n+1} : \overline{G}_{n+1}(X) \rightarrow X$ by sending the cone to the base point of X . The $(n+1)^{\text{st}}$ Ganea fibration $p_{n+1} : G_{n+1}(X) \rightarrow X$ results from converting the map \overline{p}_{n+1} to a fibration. The following result is due to Ganea (cf. Svarc).

Theorem 4 *For any space X ,*

- (a) $\text{cl}(G_n(X)) \leq n$,
- (b) *the map $p_n : G_n(X) \rightarrow X$ has a section if and only if $\text{cat}(X) \leq n$, and*
- (c) $F_n(X) \simeq (\Omega(X))^{*(n+1)}$, *the $(n+1)$ -fold join of ΩX with itself.*

Proof Assertion (a) follows immediately from the construction. For parts (b) and (c), see [6]; these results also appear, from a different point of view, in [16]. \square

2 An Auxilliary Space

Let \tilde{G}_n denote the homotopy pushout in the square

$$\begin{array}{ccc} G_{n-1}(X) & \xrightarrow{i_1} & G_{n-1}(X) \times A \\ \downarrow & & \downarrow \\ G_n(X) & \longrightarrow & \tilde{G}_n. \end{array}$$

The maps $p_n : G_n(X) \rightarrow X$ and $1_A : A \rightarrow A$ piece together to give a map $\tilde{p}_n : \tilde{G}_n \rightarrow X \times A$. The space \tilde{G}_n and the map \tilde{p}_n play key roles in the forthcoming constructions; this section is devoted to establishing some of their properties.

2.1 Category Properties of \tilde{G}_n

We begin by estimating the category of \tilde{G}_n .

Proposition 5 *For any noncontractible A and $n > 0$,*

$$\text{cat}(\tilde{G}_n) < n + \text{cat}(A).$$

Proof Let $\text{cat}(A) = k$. The space A is a retract of a space A' which has $\text{cl}(A') = k$. Let $\tilde{G}'_n = G_n \cup G_{n-1} \times A'$; clearly \tilde{G}'_n is a homotopy retract of \tilde{G}'_n and so it suffices to show that $\text{cl}(\tilde{G}'_n) < n + k$. Let

$$\begin{array}{ccccccc} & L_0 & & L_1 & & & L_{k-1} \\ & \downarrow & & \downarrow & & & \downarrow \\ A'_0 & \longrightarrow & A'_1 & \longrightarrow & \cdots & \longrightarrow & A'_{k-1} \longrightarrow A_k \end{array}$$

be a cone decomposition of A' . According to a result of Baues [2] (see also [13, Prop. 2.9]), there are cofiber sequences

$$F_{i-1} * A'_{j-1} \longrightarrow G_i \times A'_{j-1} \cup G_{i-1} \times A'_j \longrightarrow G_i \times A'_j.$$

Now let $W_s = G_{s+1}(X) \cup \bigcup_{i+j=s, i < n} G_i \times A'_j \subseteq \tilde{G}'_n$ and observe that there are cofiber sequences

$$F_s \vee \bigvee F_{i-1} * A'_{j-1} \longrightarrow W_s \longrightarrow W_{s+1}$$

and since $\tilde{G}'_n = W_{s+k-1}$, we have the result. \square

Next, we show that the map $\tilde{p}_n : \tilde{G}_n \rightarrow X \times A$ has one of the category-detecting properties of $p_n : G_n(X \times A) \rightarrow X \times A$.

Proposition 6 *If $\text{cat}(X \times A) = \text{cat}(X) = n$, then \tilde{p}_n has a homotopy section.*

Proof We follow [4] (see also [8, Thm. 2.7]) and define

$$\widehat{G}'_n(X \times A) = \bigcup_{i+j=n} G_i(X) \times G_j(A).$$

There is a natural map $h : \widehat{G}'_n(X \times A) \rightarrow X \times A$ induced by the Ganea fibrations over X and A . According to [4, Thm. 2.3], $\text{cat}(X \times A) = n$ if and only if h has a homotopy section.

Each map $G_i(X) \times G_j(A) \rightarrow X \times A$ (with $j > 0$) factors through $G_i(X) \times A$ and these factorizations are compatible because p_{i+1} extends p_i . So h factors as $\widehat{G}'_n(X \times A) \rightarrow \tilde{G}'_n \rightarrow X \times A$. Therefore, if $\text{cat}(X \times A) = n$, then h , and hence \tilde{p}_n , has a section. \square

2.2 Comparison of \tilde{G}_n with $G_n(X) \times A$

Let $j : \tilde{G}_n \rightarrow G_n(X) \times A$ denote the natural inclusion map.

Proposition 7 *Assume that X is $(c-1)$ -connected and that A is $(r-1)$ -connected. Then the homotopy fiber F of the map j is $(nc+r-2)$ -connected.*

Proof There is a cofiber sequence

$$\tilde{G}_n \xrightarrow{j} G_n(X) \times A \longrightarrow \Sigma F_{n-1}(X) \wedge A.$$

Therefore the homotopy fiber of j and the space $\Omega(\Sigma F_{n-1}(X) \wedge A) \simeq \Omega(\Omega(X)^{*n} * A)$ have the same connectivity, namely $nc+r-2$. \square

Corollary 8 *Assume $\dim(Z) < nc+r-2$ and let $f, g : Z \rightarrow \tilde{G}_n$. Then $f \simeq g$ if and only if $jf \simeq jg$.*

The proof is standard, and we omit it.

2.3 New Sections from Old Ones

Suppose that $\text{cat}(X) = \text{cat}(X \times A) = n$. By Proposition 6 there is a section $\sigma : X \times A \rightarrow \tilde{G}_n$ of the map $\tilde{p}_n : \tilde{G}_n \rightarrow X \times A$. Define a new map $\sigma' : X \rightarrow G_n(X)$ by the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma'} & G_n(X) \\ i_1 \downarrow & & \uparrow \text{pr}_1 \\ X \times A & \xrightarrow{\sigma} & \tilde{G}_n \xrightarrow{j} G_n(X) \times A \end{array}$$

We need the following basic properties of σ' .

Proposition 9 *If $\text{cat}(X \times A) = \text{cat}(X) = n$, then*

- (a) σ' is a homotopy section of the projection $p_n : G_n(X) \rightarrow X$, and
- (b) if X is $(c-1)$ -connected and A is $(r-1)$ -connected with $r > \dim(X) - nc + 2$, then the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma'} & G_n(X) \\ i_1 \downarrow & & \downarrow k \\ X \times A & \xrightarrow{\sigma} & \tilde{G}_n \end{array}$$

commutes up to homotopy.

Proof First consider the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\sigma'} & G_n(X) & \xlongequal{\quad} & G_n(X) & \xrightarrow{p_n} & X \\
\downarrow i_1 & & \downarrow k & & \uparrow \text{pr}_1 & & \uparrow p_n \\
X \times A & \xrightarrow{\sigma} & \tilde{G}_n & \xrightarrow{j} & G_n(X) \times A & \xrightarrow{\text{pr}_1} & G_n(X) \\
& \searrow 1_{X \times A} & & & \downarrow p_n \times 1_A & & \downarrow p_n \\
& & & & X \times A & \xrightarrow{\text{pr}_1} & X.
\end{array}$$

The diagram of solid arrows is evidently commutative. Therefore, we have $p_n \circ \sigma' \simeq \text{pr}_1 \circ 1_{X \times A} \circ i_1 \simeq 1_X$, proving (a).

To prove (b) we have to show that two maps $X \rightarrow \tilde{G}_n$ are homotopic. Since $\dim(X) < nc + r - 2$, it suffices by Corollary 8 to show that $j \circ (\sigma \circ i_1) \simeq j \circ (k \circ \sigma')$. Since $\text{pr}_2 \circ j \circ (\sigma \circ i_1) \simeq * \simeq \text{pr}_2 \circ j \circ (k \circ \sigma')$, it remains to show that $\text{pr}_1 \circ j \circ (\sigma \circ i_1) \simeq \text{pr}_1 \circ j \circ (k \circ \sigma')$. But both of these maps are homotopic to σ' . \square

3 Proof of the Main Theorem

Proof of Theorem 2 We have $n = \text{cat}(X) = \text{cat}(X \times A)$ by hypothesis. It follows from Proposition 6 that there is a section $\sigma : X \times A \rightarrow \tilde{G}_n$ of the map $\tilde{p}_n : \tilde{G}_n \rightarrow X \times A$. We then get the section $\sigma' : X \rightarrow G_n(X)$ that was constructed and studied in Section 2.3.

Consider the following diagram and the induced sequence of maps on the homotopy pushouts of the rows

$$\begin{array}{ccccc}
(X \times A) \times B & \xleftarrow{i_1 \times 1_B} & X \times B & \xrightarrow{\text{pr}_1} & X \\
\sigma \times 1_B \downarrow \simeq s & & \downarrow \sigma' \times 1_B & & \downarrow \sigma' \\
\tilde{G}_n \times B & \xleftarrow{k \times 1_B} & G_n(X) \times B & \xrightarrow{\text{pr}_1} & G_n(X) \\
\tilde{p}_n \times 1_B \downarrow & & \downarrow p_n \times 1_B & & \downarrow p_n \\
(X \times A) \times B & \xleftarrow{i_1 \times 1_B} & X \times B & \xrightarrow{\text{pr}_1} & X
\end{array}
\quad \begin{array}{c}
\text{homotopy} \\
\text{pushout} \\
\text{---} \\
\text{---} \\
\text{---}
\end{array}
\quad \begin{array}{c}
Y \\
\downarrow \\
P \\
\downarrow \\
Y.
\end{array}$$

Proposition 9 implies that the upper left square commutes up to homotopy. Since $i_1 \times 1_B$ is a cofibration, we can apply homotopy extension and replace the map $\sigma \times 1_B : (X \times A) \times B \rightarrow \tilde{G}_n \times B$ with a homotopic

map s which makes that square strictly commute. All other squares are strictly commutative as they stand.

Since the composites $(\tilde{p}_n \times 1_B) \circ (\sigma' \times 1_B)$ and $p_n \circ \sigma'$ are the identity maps and $(\tilde{p}_n \times 1_B) \circ s$ is a homotopy equivalence, each vertical composite in the modified diagram is a homotopy equivalence. Thus Y is a homotopy retract of P , and consequently $\text{cat}(Y) \leq \text{cat}(P)$.

The space Y is the homotopy pushout of the top row in the diagram, which is the product of the homotopy pushout diagram

$$\begin{array}{ccc} B & \longrightarrow & * \\ \downarrow & & \downarrow \\ A \times B & \longrightarrow & A \rtimes B \end{array}$$

with the space X . Therefore $Y \simeq X \times (A \rtimes B)$ by Proposition 3. Since Y is a homotopy retract of P , it follows that

$$\text{cat}(X \times (A \rtimes B)) \leq \text{cat}(P),$$

the proof will be complete once we establish that $\text{cat}(P) < \text{cat}(X) + \text{cat}(A \rtimes B)$. This is accomplished in Lemma 10, which is proved below.

□

Lemma 10 *The space P constructed in the proof of Theorem 2 satisfies $\text{cat}(P) \leq \text{cl}(P) \leq n$.*

Proof The space \tilde{G}_n is defined by the homotopy pushout square

$$\begin{array}{ccc} G_{n-1}(X) & \longrightarrow & G_n(X) \\ \downarrow & & \downarrow \\ G_{n-1}(X) \times A & \longrightarrow & \tilde{G}_n. \end{array}$$

Take the product of this square with the space B and adjoin the homotopy pushout square that defines P to obtain the diagram

$$\begin{array}{ccccc} G_{n-1}(X) \times B & \longrightarrow & G_n(X) \times B & \longrightarrow & G_n(X) \\ \downarrow & & \downarrow & & \downarrow \\ G_{n-1}(X) \times A \times B & \longrightarrow & \tilde{G}_n \times B & \longrightarrow & P. \end{array}$$

By [11, Lem. 13], the outer square

$$\begin{array}{ccc} G_{n-1}(X) \times B & \longrightarrow & G_n(X) \\ \downarrow & & \downarrow \\ G_{n-1}(X) \times A \times B & \longrightarrow & P \end{array}$$

is also a homotopy pushout square. The top map is the composite

$$G_{n-1}(X) \times B \xrightarrow{\text{pr}_1} G_{n-1}(X) \longrightarrow G_n(X),$$

and so we have a new factorization into homotopy pushout squares:

$$\begin{array}{ccccc} G_{n-1}(X) \times B & \xrightarrow{\text{pr}_1} & G_{n-1}(X) & \longrightarrow & G_n(X) \\ \downarrow & & \downarrow & & \downarrow \\ G_{n-1}(X) \times A \times B & \longrightarrow & L & \longrightarrow & P. \end{array}$$

To identify the space L , observe that the left square is simply the product of the space $G_{n-1}(X)$ with the homotopy pushout square

$$\begin{array}{ccc} B & \longrightarrow & * \\ \downarrow & & \downarrow \\ A \times B & \longrightarrow & A \rtimes B. \end{array}$$

By Proposition 3, $L \simeq G_{n-1}(X) \times (A \rtimes B)$. Hence the right-hand square is the homotopy pushout square

$$\begin{array}{ccc} G_{n-1}(X) & \longrightarrow & G_n(X) \\ \downarrow & & \downarrow \\ G_{n-1}(X) \times (A \rtimes B) & \longrightarrow & P. \end{array}$$

Therefore $\text{cl}(P) \leq \text{cat}(A) + \text{cat}(A \rtimes B)$ by Proposition 5. \square

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