Higher order principal bundles

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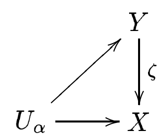
Introduction

Suppose that G is a sheaf of groups on a topological space X, and that $\zeta : Y \to X$ is a principal G-bundle for X. The bundle is locally trivial, so there is an open covering $\{U_\alpha\}$ of X such that the restriction of the bundle to each member of the covering admits a section



On intersections $U_\alpha \cap U_\beta$ of the covering family, the restrictions may be different but they are related by multiplication by a unique element $g_{\alpha\beta}$ of the sections $G(U_\alpha \cap U_\beta)$ of the group G. The collection of all elements $g_{\alpha\beta}$ define a cocycle for the covering with coefficients in the sheaf of groups G, and cohomologous cocycles correspond to isomorphic bundles. In this way, the set of isomorphism classes of G bundles ζ which trivialize over the covering $\{U_\alpha\}$ is isomorphic to a set of naive homotopy classes of maps $\pi(U_\bullet, BG)$ of simplicial sheaves from the Čech resolution $U_\bullet$ corresponding to the covering to the classifying simplicial sheaf BG. This line of argument is classical and well known and, subject to placing oneself in the context of simplicial sheaves, is most of the proof of the theorem that says that there is a natural bijection

$$H^1(X, G) \cong [*, BG]$$

relating the non-abelian H^1 invariant associated to G (aka. isomorphism classes of principal G-bundles) with morphisms in the homotopy category of simplicial sheaves on X from the terminal object $*$ to the classifying simplicial sheaf BG. The link between $H^1(X, G)$ and the homotopy theory object $[*, BG]$, or rather the desciption of it arising from the generalized Verdier hypercovering theorem, amounts to the observation that the fundamental groupoid of a hypercover is a Čech resolution.

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The argument is also universal, in that it gives a bijection

$$H^1(\mathcal{E}, G) \cong [*, BG] \tag{1}$$

relating isomorphism classes of G-torsors with morphisms in the simplicial sheaf homotopy category, which holds in any Grothendieck topos $\mathcal{E}$ and for all groups G in the topos. This result has been known for some time now [6].

The purpose of this note is to give a description of the corresponding homotopy theoretic invariant

$$[*, \overline{W}G] \cong [*, dBG]$$

when G is either a presheaf of simplicial groups or G is a presheaf of groupoids enriched in simplicial sets. Of course, the last case is the most general — a simplicial group is a simplicial groupoid with one object — but a separate development is given for presheaves of simplicial groups in the first two sections of this paper.

The objects $\overline{W}G$ and dBG are models for a classifying space construction for G: $\overline{W}G$ is the universal cocycles construction, and as such is a generalization of a classicial construction of Eilenberg-Mac Lane [5], while dBG is the diagonal of the bisimplicial object arising from standard nerve functor applied to a simplicial groupoid G. These constructions are shown to be weakly equivalent in the final section of this paper. They have been used to create model structures for various flavours of sheaves and presheaves of simplicial groupoids [3], [9], [11], all of which give models for the homotopy category of simplicial sheaves and presheaves.

The following observation is a central idea of this paper. If G is an ordinary sheaf of groups on a Grothendieck site $\mathcal{C}$, then a G-torsor is a sheaf X admitting a free (or principal) G-action such that the coinvariant sheaf X/G is a copy of the terminal sheaf $*$ up to isomorphism. Backing up a bit, one knows that, in the presence of a principal G-action on a sheaf Y, the corresponding Borel construction $EG \times_G Y$ is a simplicial sheaf which is weakly equivalent to the discrete object Y/G. In fact, the converse is true: if $EG \times_G Y \to Y/G$ is a local weak equivalence then the G-action on Y is principal. Thus a G-torsor is a sheaf X with a G-action such that the Borel construction $EG \times_G X$ is locally weakly equivalent to the terminal sheaf $*$ in the sense of simplicial sheaf homotopy theory.

The analogue of a group action for a presheaf of groupoids G enriched in simplicial sets is a simplicial functor X defined on G and taking values in simplicial presheaves. Each such simplicial functor X has a homotopy colimit $\underrightarrow{\text{holim}}_G X$, and one says, by direct analogy with the Borel construction (aka. homotopy colimit) description of ordinary torsors, that X is a G-torsor if this homotopy colimit is locally weakly equivalent to the point $*$. There is a category of G-torsors $G - \mathbf{Tors}$ which has a class $\pi_0(G - \mathbf{Tors})$ of path components, and the main result of this paper (Theorem 17) asserts that there is a natural bijection

$$[*, dBG] \cong \pi_0(G - \mathbf{Tors}).$$

The special case of Theorem 17 corresponding to the case where G is a presheaf of simplicial groups is given its own proof in the second section of this paper, and appears as Corollary 10.

This is done to display a quick application of our main new technical device, which is an expanded notion of cocyle which appears in Lemma 8. Cocycles have previously been interpreted (in the most general formulation) as maps defined on hypercovers. Hypercovers are most precisely defined as locally fibrant presheaves which are weakly equivalent to the terminal object $*$. As such, they are examples of simplicial presheaves U which are weakly equivalent to a point, and the magic thing here is that when one looks at the path components of the category $\mathbf{Triv}/Y$ of all morphisms $U \to Y$ (no homotopy classes), one gets a class $\pi_0(\mathbf{Triv}/Y)$ which is isomorphic to $[*, Y]$. Furthermore this result holds in great generality: Y can be any member of an arbitrary right proper model category having a cofibrant terminal object.

We also give a new demonstration of the bijection (1) relating isomorphism classes of G-torsors with homotopy classes of maps $[*, BG]$ for sheaves of groups G, in Remark 11.

With the new approach to cocycle theory in hand, Theorem 17 is a rather easy consequence of a result of Joyal and Tierney [9] which asserts that their homotopy category of sheaves of simplicial groupoids is equivalent to the homotopy category of simplicial sheaves via the classifying space functor dB. The point is that when G is a simplicial groupoid with discrete objects and X is a G-torsor, the homotopy colimit $\underrightarrow{\text{holim}}_G\, X$ can be taken apart and put back together again with well known results of Quillen. There is no appeal to amenable objects [9] in the proof of Theorem 17. Also, our torsors do not coincide with the pseudo-torsors of [8] (see Remark 19).

One can define G-torsors, and one has an analog of Theorem 17 for all presheaves of simplicial groupoids G. In the case where G has discrete objects, a result of Moerdijk [13] can be used to show that a G-torsor X is locally a copy of the loop space object ΩdBG of the classifying space. In particular, in this case, any morphism of G-torsors is a local weak equivalence. This is the analogue of the well known observation that any morphism of ordinary torsors for a sheaf of groups is an isomorphism.

There is another antecedent for our theory in the description of torsors for a sheaves of groupoids which appears in [7], and Theorem 17 is a generalization of [7, Th.14]. The reader should be aware that the proof of the older result contains an error which is fixed in the proof of Theorem 17 — see the explanation in Remark 18 at the end of the third section.

The final section of this paper contains a first application: Theorem 23 identifies the class of path components of the category of G-gerbes for a sheaf of groups G with set of isomorphism classes of right torsors over the automorphism 2-groupoid object $\mathbf{Aut}(G)$. In the world of ordinary groups, the automorphism 2-groupoid of a group H has one 0-cell, 1-cells given by the automorphisms of H and 2-cells given by their homotopies. A gerbe is a locally connected stack, and a G-gerbe is a gerbe which is locally equivalent to either G or its associated stack of G-torsors. Our classification theorem can be inferred from a result of Breen [2, Prop.7.3] and its proof employs similar constructions, but we avoid a discussion of bitorsors and do not encounter homotopy coherence issues. Theorem 23 is a direct homotopy theoretic classification of G-gerbes up to local equivalence.

The results of Sections 1–4 of this paper appeared in preliminary form in the thesis of Zhiming Luo (the second author), [12].

Contents

1 Torsor categories for presheaves of simplicial groups

Suppose that G is a presheaf of simplicial groups on a (small) site $\mathcal{C}$, and write $G - s\,\mathrm{Pre}(\mathcal{C})$ for the category of all simplicial presheaves X admitting a G-action $G \times X \to X$. We shall also call these objects simplicial G-presheaves.

Lemma 1. *There is a cofibrantly generated closed model structure on the category $G - s\,\mathrm{Pre}(\mathcal{C})$ of simplicial G-presheaves, where a map $f : X \to Y$ is a fibration (respectively weak equivalence) if the underlying map of simplicial presheaves is a global fibration (respectively local weak equivalence).*

Proof. The product $G \times X$ is the free simplicial G-presheaf on a simplicial presheaf X. It follows that the free simplicial G-presheaf functor preserves cofibrations and weak equivalences. Colimits in simplicial G-presheaves are formed as in simplicial presheaves. Thus, if $i : A \to B$ is a trivial cofibration of simplicial presheaves and the diagram

$$\begin{array}{ccc} G \times A & \longrightarrow & X \\ {\scriptstyle 1\times i}\downarrow & & \downarrow{\scriptstyle i_*} \\ G \times B & \longrightarrow & Y \end{array}$$

is a pushout in the category of simplicial G-presheaves, then the map i_* is a weak equivalence. It follows that the generating set $A \to B$ for the class of trivial cofibrations of simplicial presheaves determines a generating family

$G \times A \to G \times B$ for the trivial cofibrations of simplicial G-presheaves. The set of morphisms $G \times K \to G \times L_U(\Delta^n)$ of simplicial G-presheaf morphisms induced by the simplicial presheaf inclusions $K \subset L_U(\Delta^n)$ is a generating set for the class of all cofibrations of simplicial G-presheaves. $\square$

As usual, $L_U(\Delta^n)$ denotes the simplicial presheaf which is freely generated by an n-simplex in U-sections.

Say that G acts freely on X or that X is G-free if the simplicial group $G(U)$ acts freely on the simplicial set $X(U)$ for all $U \in \mathcal{C}$.

Suppose that G acts freely on X. Let X/G be the quotient simplicial presheaf and let $\pi : X \to X/G$ be the canonical map. Then the lifting ϕ exists in the diagram

$$\begin{array}{ccc} & & X \\ & \overset{\phi}{\nearrow} & \downarrow \pi \\ L_U(\Delta^n) & \xrightarrow[x]{} & X/G \end{array}$$

for each simplex $x \in X/G(U)$ for all $U \in \mathcal{C}$. Multiplying the map ϕ by the action of G determines a simplicial presheaf map $G \times L_U(\Delta^n) \to X$, and it is easy to see that the freeness of the action implies this map factors through an isomorphism

$$\phi_* : G \times L_U(\Delta^n) \xrightarrow{\cong} L_U(\Delta^n) \times_{X/G} X.$$

The cofibrant simplicial G-presheaves X in the model structure of Lemma 1 are (sectionwise) principal G-bundles $X \to X/G$, on account of the following:

Lemma 2. *Suppose that X is a cofibrant simplicial G-presheaf. Then G acts freely on X in all sections.*

Proof. The maps $G \times K \to G \times L_U(\Delta^n)$ generate the cofibrations of the category of simplicial G-presheaves, and any pushout

$$\begin{array}{ccc} G \times K & \longrightarrow & Z \\ \downarrow & & \downarrow \\ G \times L_U(\Delta^n) & \longrightarrow & W \end{array}$$

has the effect of adding some freely generated $G(U)$-space to $Z(U)$ for each $U \in \mathcal{C}$. The cofibration $\emptyset \to X$ has a factorization

$$\begin{array}{ccc} \emptyset & \longrightarrow & V \\ & \searrow & \downarrow \pi \\ & & X \end{array}$$

where π is a trivial fibration and the map $\emptyset \to V$ is a transfinite colimit of pushouts of the above form. It follows that G acts freely on V. But then, by a standard argument, X is a retract of V (since π is a trivial fibration) so that G acts freely on X. $\square$

Remark 3. The model structure of Lemma 1 specializes to the standard model structure for G-spaces in the case where the site $\mathcal{C}$ is a point, and where G is a simplicial group. In the case of G-spaces, Lemma 2 has a converse [5, V.2.10]. It follows that a simplicial G-presheaf X having free G-action is a diagram of $G(U)$-spaces $X(U)$, each of which is cofibrant. It is not clear that X itself is cofibrant.

Write $G-$**Tors** for the category of cofibrant simplicial G-presheaves X such that the canonical map $X/G \to *$ is a hypercover (ie. a local trivial fibration). A morphism $f : X \to Y$ of $G-$**Tors** is just a G-equivariant map of simplicial presheaves. Write $G-$**Tors** for the corresponding category.

Choose a factorization

$$\begin{array}{ccc} \emptyset & \xrightarrow{i} & EG \\ & \searrow & \downarrow{\scriptstyle \pi} \\ & & * \end{array}$$

where i is a cofibration and π is a trivial fibration in the category of simplicial G-presheaves. Write $BG = EG/G$. Observe that BG is a presheaf of Kan complexes on account of Lemma 2, [5, V.2.7], and [5, V.3.7].

The homotopy type of BG is independent of the choice of the object EG. If $E'G$ is a second such choice with quotient $B'G = E'G/G$ then there is a G-equivariant homotopy equivalence $EG \to E'G$ since both objects are fibrant and cofibrant. This homotopy equivalence induces a homotopy equivalence $BG \to B'G$ of the quotients.

Remark 4. The Eilenberg-Mac Lane object WG has a free G-action, and the map $WG \to *$ is a sectionwise trivial fibration. There are maps of simplicial G-presheaves

$$WG \xleftarrow{p} \tilde{W}G \xrightarrow{j} EG$$

such that p is a trivial fibration and $\tilde{W}G$ is cofibrant, and such that j is a trivial cofibration and EG is fibrant. The map p is in particular a sectionwise weak equivalence of sectionwise cofibrant G-spaces, and therefore induces a sectionwise weak equivalence $\tilde{W}G/G \to \overline{W}G$. The map j is a trivial cofibration of cofibrant simplicial G-presheaves, and therefore induces a weak equivalence $\tilde{W}G/G \to EG/G = BG$.

A similar argument works for the diagonal map $d(EG) \to d(BG)$ induced by the standard bisimplicial sheaf map $EG \to BG$, because the induced map of diagonal simplicial objects is a sectionwise principal G-fibration and the object $d(EG)$ is weakly equivalent to a point. It follows that $d(BG) \simeq BG$ for the two different senses of BG.

Remark 5. Every trivial cofibration $i : A \to B$ of simplicial G-presheaves induces a trivial cofibration $i_* : A/G \to B/G$. In effect, i has the left lifting property with respect to all global fibrations $p : X \to Y$ of simplicial presheaves with trivial G-action.

Suppose that X is a cofibrant simplicial G-presheaf such that the induced map $X/G \to *$ is a local weak equivalence. Find a trivial cofibration $j : X \to \tilde{X}$ in the category of simplicial G-presheaves such that $\tilde{X}$ is fibrant. Then the induced map $j_* : X/G \to \tilde{X}/G$ is a trivial cofibration of simplicial presheaves, and $\tilde{X}/G$ is a presheaf of Kan complexes so the map $\tilde{X}/G \to *$ is a hypercover. Write $G - \mathbf{Tors}_0$ for the category of cofibrant simplicial G-presheaves X such that $X/G \to *$ is a local weak equivalence. Then the inclusion

$$G - \mathbf{Tors} \subset G - \mathbf{Tors}_0$$

induces an isomorphism

$$\pi_0(G - \mathbf{Tors}) \cong \pi_0(G - \mathbf{Tors}_0).$$

Remark 6. Write $G - \mathbf{Tors}_1$ for the category of simplicial G-presheaves Y such that the canonical map $d(EG \times_G Y) \to *$ is a local weak equivalence, where $d(X)$ denotes the diagonal of a bisimplicial object X. Then there is an inclusion

$$G - \mathbf{Tors}_0 \subset G - \mathbf{Tors}_1$$

since the canonical map $d(EG \times_G Z) \to Z/G$ is a sectionwise weak equivalence if Z is cofibrant. On the other hand, if X is a simplicial G-sheaf such that $d(EG \times_G X) \to *$ is a weak equivalence, there is a trivial fibration $Z \to X$ of simplicial G-presheaves such that Z is cofibrant. The induced map $d(EG \times_G Z) \to d(EG \times_G X)$ is a local weak equivalence, so that Z is an object of $G - \mathbf{Tors_0}$. It follows that there is an isomorphism

$$\pi_0(G - \mathbf{Tors}_0) \cong \pi_0(G - \mathbf{Tors}_1).$$

2 Cocycles

Suppose that $\mathbf{M}$ is a model category with a terminal object $*$, and let X be an object of $\mathbf{M}$. Write $\mathbf{Triv}/X$ for the category whose objects are all morphisms $W \to X$ of $\mathbf{M}$ such that the map $W \to *$ is a weak equivalence. Observe that there is a function

$$\psi_X : \pi_0(\mathbf{Triv}/X) \to [*, X]$$

which is defined by associating to an object $W \to X$ the composite

$$* \xleftarrow{\simeq} W \to X$$

in the homotopy category.

Lemma 7. *Suppose that* $\mathbf{M}$ *is a right proper model category with terminal object* $*$. *Suppose that the map* $g : X \to Y$ *is a weak equivalence. Then the induced function*

$$g_* : \pi_0(\mathbf{Triv}/X) \to \pi_0(\mathbf{Triv}/Y)$$

is a bijection.

Proof. The function g_* is induced by a functor which is defined by associating to the object $W \to X$ the composite

$$W \to X \xrightarrow{g} Y.$$

Suppose that $v : U \to Y$ is an object of $\mathbf{Triv}/Y$. Choose a factorization

$$\begin{array}{ccc} U & \xrightarrow{j} & V \\ & \searrow^{v} & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

of v, where j is a trivial cofibration and p is a fibration. Form the pullback

$$\begin{array}{ccc} X \times_Y V & \xrightarrow{g_*} & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[g]{} & Y \end{array}$$

Then the map g_* is a weak equivalence by the right properness assumption, so that the projection $X \times_Y V \to X$ is an object of $\mathbf{Triv}/X$. The path component of this object is independent of the choices made, and is independent of the choice of representative for the path component of $U \to Y$.

In effect, if

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & U' \\ & {\scriptstyle v}\searrow \quad \swarrow{\scriptstyle v'} & \\ & Y & \end{array}$$

is a morphism of $\mathbf{Triv}/Y$ and $v' = p' \cdot j'$ is a factorization of v' with j' a trivial cofibration and p' a fibration, then there is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{j'\alpha} & V' \\ {\scriptstyle j}\downarrow & \nearrow^{\omega} & \downarrow{\scriptstyle p'} \\ V & \xrightarrow[p]{} & Y \end{array}$$

and so there is a commutative diagram

$$\begin{array}{ccc} X \times_Y V & \xrightarrow{\omega_*} & X \times_Y V' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

It follows that there is a well-defined function

$$g' : \pi_0(\mathbf{Triv}/Y) \to \pi_0(\mathbf{Triv}/X).$$

The composite functions $g' \cdot g_*$ and $g_* \cdot g'$ are both identities. $\square$

Lemma 8. *Suppose that Y is an object of a right proper model category* $\mathbf{M}$ *in which the terminal object $*$ is cofibrant. Then the function*

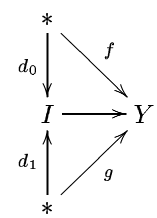
$$\psi_Y : \pi_0(\mathbf{Triv}/Y) \to [*, Y]$$

is a bijection.

Proof. By Lemma 7, it is enough to suppose that Y is fibrant. Then the function

$$\pi(*, Y) \to [*, Y]$$

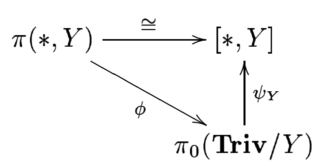
is a bijection since $*$ is cofibrant. Here, $\pi(*, Y)$ denotes homotopy classes of maps with respect to a fixed cylinder object I of $*$. If two maps $f, g : * \to Y$ are homotopic, then there is a diagram



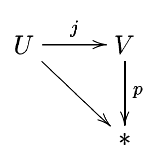
Then the morphisms d_0 and d_1 are weak equivalences, so that f and g are in the same path component of $\mathbf{Triv}/Y$. It follows that there is a well defined function

$$\phi : \pi(*, Y) \to \pi_0(\mathbf{Triv}/Y)$$

and that the diagram



commutes. Finally, if $U \to Y$ is an object of $\mathbf{Triv}/Y$, there is a factorization



where j is a trivial cofibration and p is a trivial fibration. The fibration p has a section $s : * \to V$ since $*$ is cofibrant, and the map $U \to Y$ extends to a map $V \to Y$ since j is a trivial cofibration and Y is fibrant. It follows that the function ϕ is surjective, and is therefore a bijection.

The map ψ_Y is therefore a bijection if Y is fibrant. □

Lemma 9. *Suppose that G is a presheaf of simplicial groups. Then there is a bijection*

$$[*, BG] \cong \pi_0(G - \mathbf{Tors}_0).$$

Recall that $G - \mathbf{Tors}_0$ is the category of cofibrant simplicial G-presheaves X such that the map $X/G \to *$ is a local weak equivalence.

Proof. We establish the existence of a bijection

$$\pi_0(\mathbf{Triv}/BG) \cong \pi_0(G - \mathbf{Tors}_0).$$

Then the desired result follows from Lemma 8.

First of all, there is a function

$$\pi_0(\mathbf{Triv}/BG) \to \pi_0(G - \mathbf{Tors}_0)$$

which is defined by associating a cofibrant model $Z(X)$ of the simplicial G-presheaf $X \times_{BG} EG$ to the object $X \to BG$ of $\mathbf{Triv}/BG$.

Here, one means that a choice of trivial fibration $Z(X) \to X \times_{BG} EG$ is made in the category of simplicial G-presheaves such that $Z(X)$ is cofibrant. This can be done functorially since the model structure on the category of simplicial G-presheaves is cofibrantly generated. Observe that the induced map $Z(X)/G \to X$ is a sectionwise weak equivalence since $X \times_{BG} EG$ is G-free.

Suppose that Z is a cofibrant simplicial G-presheaf such that $Z/G \to *$ is a local weak equivalence. Then there is a G-equivariant map $Z \to EG$ and an induced map $Z/G \to BG$. The class of the object $Z/G \to BG$ in $\pi_0(\mathbf{Triv}/BG)$ is independent of the choices that have been made: any two G-equivariant maps $Z \to EG$ are naively homotopic and so the induced maps $Z/G \to BG$ are naively homotopic and hence represent the same element of $\pi_0(\mathbf{Triv}/BG)$. It follows that there is a well defined function

$$\pi_0(G - \mathbf{Tors}_0) \to \pi_0(\mathbf{Triv}/BG)$$

and this function is the inverse of the function in π_0 which is induced by the functor of the previous paragraph. □

Corollary 10. *There is a bijection*

$$[*, BG] \cong \pi_0(G - \mathbf{Tors}_1).$$

Recall that the objects of the category $G - \mathbf{Tors}_1$ are simplicial G-presheaves Z such that $d(EG \times_G Z) \to *$ is a local weak equivalence. Lemma 9 is equivalent to Corollary 10, by Remark 6.

Remark 11. If G is a sheaf of groups, then a G-torsor X is naturally a member of $G - \mathbf{Tors}_1$ after identification of X with a constant simplicial G-sheaf, and in this way the category $G - \mathbf{tors}$ of ordinary G-torsors imbeds in $G - \mathbf{Tors}_1$. We claim that the induced function

$$\pi_0(G - \mathbf{tors}) \to \pi_0(G - \mathbf{Tors}_1) \quad (2)$$

is a bijection.

Suppose that X is a simplicial G-presheaf such that the map $d(EG \times_G X) \to *$ is a local weak equivalence, or that X is a member of $G - \mathbf{Tors}_1$. Then the canonical map $d(EG \times_G X) \to BG$ is a local fibration with fibre X according to Lemma 12 below. The total space object $d(EG \times_G X)$ is locally weakly equivalent to a point by assumption, so that X is non-equivariantly locally equivalent to $\Omega BG \simeq G$, where the sheaf of groups G is identified with a constant simplicial sheaf. It follows in particular that the G-equivariant map $X \to \pi_0 X$ is a local weak equivalence. The sheaf of groups G acts on the associated sheaf $\tilde{\pi}_0 X$, and the composite

$$X \to \pi_0 X \to \tilde{\pi}_0 X$$

is a G-equivariant local weak equivalence. The induced map

$$d(EG \times_G X) \to EG \times_G \tilde{\pi}_0 X$$

is also a local weak equivalence, so that $EG \times_G \tilde{\pi}_0 X$ is locally equivalent to a point. This last statement means precisely that the sheaf $\tilde{\pi}_0 X$ is a G-torsor: the freeness of the G-action is the vanishing of the sheaf $\tilde{\pi}_1(EG \times_G \tilde{\pi}_0 X)$, and $\tilde{\pi}_0(EG \times_G \tilde{\pi}_0 X) \cong (\tilde{\pi}_0 X)/G \cong *$ as a sheaf. All constructions are natural, so the function

$$\pi_0(G - \mathbf{tors}) \to \pi_0(G - \mathbf{Tors}_1)$$

is a bijection with inverse specified by $X \mapsto \tilde{\pi}_0 X$.

3 Torsors for presheaves of simplicial groupoids

Write $s\mathbf{Gpd}_0$ to denote the category of presheaves of groupoids enriched in simplicial sets, and write $s\mathbf{Gpd}$ for the full category of presheaves of simplicial groupoids. A groupoid enriched in simplicial sets is a simplicial groupoid with discrete objects, and the two ways of describing such an object will be used interchangeably. All sheaves or presheaves in this section are defined on a fixed small Grothendieck site $\mathcal{C}$.

The purpose of this section is to analyze the set of morphisms $[*, \overline{W}G]$ for a presheaf of simplicial groupoids with discrete objects. Here, $\overline{W}G$ is the universal cocycle construction of [5] and [11] — see also Section 4. It is also shown in Section 4 that there is a natural weak equivalence $j : dBG \to \overline{W}G$, where dBG denotes the diagonal of the usual bisimplicial nerve BG. The homotopy type of dBG is also insensitive to whether or not G is a sheaf, and we shall therefore focus attention on computing $[*, dBG]$ when G is a sheaf of groupoids with discrete objects.

Joyal and Tierney have a model structure for sheaves of simplicial groupoids [9] for which a map $G \to H$ is a weak equivalence if and only if the induced map $dBG \to dBH$ is a local weak equivalence of simplicial sheaves. The Joyal-Tierney model structure is proper [9, Th.9]. They also show [9, Th.12] that the functor dB determines a functor

$$dB : s\mathbf{Gpd}/G \to s\mathbf{Shv}/dBG$$

which induces an equivalence of homotopy categories. It follows from Lemma 8 that the functor dB induces an isomorphism

$$dB : \pi_0(\mathbf{Triv}/G) \cong \pi_0(\mathbf{Triv}/dBG)$$

for all sheaves of simplicial groupoids G. If one says that a map $f : H \to H'$ of presheaves of simplicial groupoids is a weak equivalence if the induced map $dBH \to dBH'$ is a local weak equivalence of simplicial presheaves, then it's clear that the functor dB and the associated sheaf functor $H \mapsto \tilde{H}$ together induce a commutative diagram

$$\begin{array}{ccc} \pi_0(\mathbf{Triv}/H) & \xrightarrow{dB} & \pi_0(\mathbf{Triv}/dBH) \\ \cong \downarrow & & \downarrow \cong \\ \pi_0(\mathbf{Triv}/\tilde{H}) & \xrightarrow[dB]{\cong} & \pi_0(\mathbf{Triv}/dB\tilde{H}) \end{array}$$

The function dB in the diagram is therefore a bijection for all presheaves of simplicial groupoids H.

The following result is a restatement of a theorem of Moerdijk, specifically Theorem 2.1 of [13]. It can also be proved with the techniques used to prove the group completion theorem in [5]. As Moerdijk observes in [13], the group completion theorem is a consequence of this result.

Lemma 12. *Suppose that C is a category enriched in simplicial sets and that $X : C \to \mathbf{S}$ is a simplicial functor taking values in simplicial sets. Suppose that all arrows $a \to b$ of C_0 induce weak equivalences $X(a) \to X(b)$. Then the map $X(a) \to F_a$ taking values in the homotopy fibre over a of the simplicial set map $d(\underrightarrow{\mathrm{holim}}_C X) \to d(BC)$ is a weak equivalence.*

The object $\underrightarrow{\mathrm{holim}}_C X$ is the bisimplicial set with simplicial set

$$\bigsqcup_{(a_0,a_1,\dots,a_n)} X(a_0) \times G(a_0,a_1) \times \cdots \times G(a_{n-1},a_n)$$

in horizontal degree n. In vertical degree m, it is the simplicial set $\underrightarrow{\mathrm{holim}}_{G_m} X_m$.

Corollary 13. *Suppose that G is a groupoid enriched in simplicial sets, and that $X : G \to \mathbf{S}$ is a simplicial functor taking values in simplicial sets. Then the map $X(a) \to F_a$ taking values in the homotopy fibre over a of the simplicial set map $d(\underrightarrow{\mathrm{holim}}_G X) \to d(BG)$ is a weak equivalence.*

A simplicial functor $X : G \to \mathbf{S}$ can alternatively be described as simplicial set $X = \bigsqcup_{a \in \mathrm{Ob}(G)} X_a$ fibred over the object set $\mathrm{Ob}(G)$ in the sense that there is a simplicial set map $f : X \to \mathrm{Ob}(G)$ which collapses summands to points. Suppose that the simplicial set $X \times_s \mathrm{Mor}(G)$ is defined by the pullback diagram

$$\begin{array}{ccc} X \times_s \mathrm{Mor}(G) & \longrightarrow & \mathrm{Mor}(G) \\ \downarrow & & \downarrow s \\ X & \xrightarrow[f]{} & \mathrm{Ob}(G) \end{array}$$

where s is the source map. Then the other piece of data required for the simplicial functor X is a simplicial set map $m : X \times_s \mathrm{Mor}(G) \to X$ which fits into a commutative diagram

$$\begin{array}{ccc} X \times_s \mathrm{Mor}(G) & \xrightarrow{m} & X \\ \downarrow & & \downarrow f \\ \mathrm{Mor}(G) & \xrightarrow[t]{} & \mathrm{Ob}(G) \end{array}$$

where t is the target map. The map m must respect identities of G is an obvious way.

There is a canonical diagram

$$\begin{array}{ccc} X & \longrightarrow & d(\underrightarrow{\mathrm{holim}}_G X) \\ \downarrow & & \downarrow \\ \mathrm{Ob}(G) & \longrightarrow & d(BG) \end{array} \tag{3}$$

and then Corollary 13 has the following equivalent formulation

Corollary 14. *Suppose that G is a groupoid enriched in simplicial sets, and that $X : G \to \mathbf{S}$ is a simplicial functor taking values in simplicial sets. Then the diagram (3) is homotopy cartesian.*

Lemma 12 can be expressed in terms of a similar homotopy cartesian diagram.

Example 15. Suppose that H is a simplicial groupoid with discrete objects and let $f : U \to H$ be a morphism of simplicial groupoids (U does not necessarily have discrete objects). Take $a \in \mathrm{Ob}(H)$ and write $f \downarrow a$ for the simplicial category given in degree n by the comma category $f_n \downarrow a$ arising from the functor $f_n : U_n \to H_n$. Then the functors $H_n \to \mathbf{cat}$ given by $a \mapsto f_n \downarrow a$ define a simplicial functor $dB(f \downarrow\ ) : H \to \mathbf{S}$. The forgetful functors $f_n \downarrow a \to U_n$ also assemble to define a weak equivalence

$$\underrightarrow{\mathrm{holim}}_H\, B(f \downarrow\ ) \xrightarrow{\alpha} BU$$

The simplicial sets $dB(f \downarrow a)$ therefore become identified with the homotopy fibres of the diagonal simplicial set map associated to the canonical bisimplicial set map

$$\underrightarrow{\mathrm{holim}}_H\, dB(f \downarrow\ ) \xrightarrow{\beta} BH$$

In "horizontal degree" n, this map can be identified with the projection

$$dB(f \downarrow a_0) \times H(a_0, a_1) \times \cdots \times H(a_{n-1}, a_n) \to H(a_0, a_1) \times \cdots \times H(a_{n-1}, a_n)$$

Remark 16. Suppose that C is a small category, and consider the simplicial set maps

$$BC \xleftarrow{\alpha} d(\bigsqcup_{x_0\to\cdots\to x_n} B(C\downarrow x_0) \xrightarrow{\beta} d(\bigsqcup_{x_0\to\cdots\to x_n} *) = BC$$

arising from the simplicial set construction underlying Example 15. In other words the map α is induced by the forgetful functors $C\downarrow a \to C$, while β is the canonical map induced by the simplicial set maps $B(C\downarrow x_0) \to *$. Both maps are weak equivalences.

The object

$$X = d(\bigsqcup_{x_0\to\cdots\to x_n} B(C\downarrow x_0))$$

is the simplicial set consisting of strings (y,x) of arrows

$$y_0 \to \cdots \to y_n \to x_0 \to \cdots \to x_n$$

of length $2n+1$ in C, and the map α takes this string to the string $y_0 \to \cdots \to y_n$ while β maps this element to the string $x_0 \to \cdots \to x_n$. The n-simplices of the simplicial object X can therefore be identified with functors $\mathbf{n} * \mathbf{n} \to C$ defined on the poset join $\mathbf{n} * \mathbf{n}$, and with simplicial structure maps induced by precomposition with maps $\theta * \theta : \mathbf{m} * \mathbf{m} \to \mathbf{n} * \mathbf{n}$. The maps α and β are induced by the inclusions $\mathbf{n} \to \mathbf{n} * \mathbf{n}$ of the left and right substrings of length n respectively.

There is a poset map $h_n : \mathbf{n} \times \mathbf{1} \to \mathbf{n} * \mathbf{n}$ which is defined by

$$(i,\epsilon) \mapsto \begin{cases} i & \text{if } \epsilon = 0, \text{ and} \\ n+i & \text{if } \epsilon = 1. \end{cases}$$

As a picture, h_n is the diagram

$$\begin{array}{ccccccc} y_0 & \longrightarrow & y_1 & \longrightarrow & \cdots & \longrightarrow & y_n \\ \downarrow & & \downarrow & & & & \downarrow \\ x_0 & \longrightarrow & x_1 & \longrightarrow & \cdots & \longrightarrow & x_n \end{array}$$

The maps h_n are natural in ordinal numbers n. It follows that the composites

$$\Delta^n \times \Delta^1 \xrightarrow{h_n} B(\mathbf{n} * \mathbf{n}) \xrightarrow{(y,x)} BC$$

together define a simplicial set map $X \times \Delta^1 \to BC$ from α to β. This construction is natural in all small categories.

Suppose that G is a presheaf of simplicial groupoids with discrete objects. A *torsor* for G is a simplicial functor $X : G \to s\operatorname{Pre}(\mathcal{C})$ taking values in simplicial presheaves such that the associated simplicial presheaf $d(\underrightarrow{\operatorname{holim}}_G X)$ is weakly equivalent to a point. A morphism $f : X \to Y$ of G-torsors is a natural transformation of simplicial functors; it may also be described as a simplicial presheaf morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & \mathrm{Ob}(G) & \end{array}$$

which respects the G-structure. It is an immediate consequence of Corollary 14 that any such map f must be a local weak equivalence. This map also induces weak equivalences of all local choices of fibres.

Write $\mathbf{G} - \mathbf{Tors}$ for the category of G-torsors, and let $\pi_0(\mathbf{G} - \mathbf{Tors})$ denote its set of path components. There is a well-defined function

$$\phi : \pi_0(G - \mathbf{Tors}) \to \pi_0(\mathbf{Triv}/dBG)$$

which is induced by associating to a G-torsor X the element represented by the map $d(\underrightarrow{\mathrm{holim}}_G X) \to dBG$.

Note that the map ϕ is morally induced by the *elts* map of Joyal and Tierney [9, p.288], although it is defined on enriched diagrams of presheaves rather than sheaves.

There is a function

$$\psi : \pi_0(\mathbf{Triv}/G) \cong \pi_0(\mathbf{Triv}/dBG) \to \pi_0(G - \mathbf{Tors})$$

which is defined as follows. Let $f : U \to G$ be an object of $\mathbf{Triv}/G$ and perform the construction of Example 15 sectionwise to construct the diagram

$$\begin{array}{ccc} dBU & \xleftarrow{\simeq} & d(\underrightarrow{\mathrm{holim}}_G\ dB(f \downarrow\ )) \\ \downarrow & & \downarrow f_* \\ dBG & \underset{\simeq}{\xleftarrow{\alpha}} & d(\underrightarrow{\mathrm{holim}}_G\ dB(G \downarrow\ )) \\ & & \simeq \downarrow \beta \\ & & dBG \end{array}$$

Then the simplicial G-functor $a \mapsto dB(f \downarrow\ )$ is a G-torsor. This construction is functorial and defines the function ψ.

The composites $\beta \cdot f_*$ and $\alpha \cdot f_*$ are homotopic by the construction of Remark 16. It follows that the canonical map $\beta \cdot f_*$ and the original map $dBU \to dBG$ represent the same element of $\pi_0(\mathbf{Triv}/dBG)$, and so the composite $\phi \cdot \psi$ is the identity function.

If X is a G-torsor, then the canonical map $d(\underrightarrow{\mathrm{holim}}_G X) \to dBG$ is induced by a morphism $f : E_G X \to G$ of presheaves of simplicial groupoids (where $E_G X$ is the translation category for the functor $X_n : G_n \to \mathbf{Set}$ in each degree. There is a G-natural functor $f \downarrow a \to X_n(a)$ which induces a map (also a weak equivalence)

$$dB(f \downarrow a) \to X_n(a)$$

for all n and a, and hence determines a map of G-torsors

$$dB(f\downarrow) \to X$$

It follows that the composite $\psi \cdot \phi$ is the identity function. We have therefore proved the following

Theorem 17. *Suppose that G is a presheaf of simplicial groupoids with discrete objects. Then the natural function*

$$\phi : \pi_0(G - \mathbf{Tors}) \to \pi_0(\mathbf{Triv}/dBG) \cong [*, dBG]$$

is a bijection.

Proof. The displayed isomorphism is a consequence of Lemma 8. The proof that ϕ is a bijection is given above. □

Remark 18. Theorem 17 generalizes Theorem 14 of [7]. The proof of Theorem 17 also implicitly fixes an error in the proof of that result, which does not properly take into account the phenomenon discussed in Remark 16.

Remark 19. A simplicial sheaf of groupoids G for which the coequalizer $c(G)$ of the source and target maps $s,t : \mathrm{Mor}(G) \to \mathrm{Ob}(G)$ $\mathrm{Ob}(G) \to \tilde{\pi}_0(\mathrm{Ob}(G))$ is simplicially discrete is said to be locally transitive in [8]. All sheaves of simplicial groupoids with discrete objects are locally transitive in this sense.Joyal and Tierney define a G-pseudo torsor for a locally transitive object G to be a simplicial sheaf X on which G acts freely (in each simplicial degree), and such that the colimit X/G is locally weakly equivalent to a point. All G-pseudo torsors are G-torsors in the sense of this paper for G with discrete objects, but the class of G-torsors is larger. Theorem 24 of [8] gives a homotopy classification of pseudo-torsors, and implies that the categories of G-pseudo torsors and G-torsors have isomorphic presheaves of path components in the case where G is a simplicial sheaf of groupoids with discrete objects.

4 Universal cocycles for simplicial categories

The simplicial set $\overline{W}G$ for a groupoid enriched in simplicial sets (aka. simplicial groupoid with discrete objects), is defined as a space of universal cocycles in [5, V.7]. We show here how to extend the definition of this construction to all simplicial categories C, and we construct a comparison map $j : dBC \to \overline{W}C$. We show in Lemma 20 that the map $j : dBG \to \overline{W}G$ is a weak equivalence for groupoids G enriched in simplicial sets.

Suppose that C is a simplicial object in the category of small categories. Write E_C for the following variant of the Grothendieck construction: the set of objects of E_C consists of all pairs (x, n) with $x \in C_n$, and a morphism $(f, \theta) : (x, m) \to (y, n)$ is a pair consisting of an ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ and a morphism $f : x \to \theta^* y$ of C_m. There is an obvious forgetful functor $\pi : E_C \to \mathbf{\Delta}$ which takes values in the ordinal number category $\mathbf{\Delta}$.

The segment category $Seg(\mathbf{n})$ of subintervals $[j,n]$ of $\mathbf{n}=[0,n]$ can be identified with the opposite $\mathbf{n}^{op}$ via the functor $[j,n]\mapsto j$. There is a functor $c_n:\mathbf{n}^{op}\to\mathbf{\Delta}$ which is defined by $j\mapsto \mathbf{n}-\mathbf{j}$.

An n-cocyle taking values in the simplicial category C is a functor $X:\mathbf{n}^{op}\to E_C$ which is a lifting of c_n in the sense that the diagram of functors

$$\begin{array}{ccc} & & E_C \\ & \overset{X}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \mathbf{n}^{op} & \xrightarrow[c_n]{} & \mathbf{\Delta} \end{array}$$

commutes. This is a generalization of the definition of an n-cocycle taking values in a groupoid enriched in simplicial sets, in view of the identification of the categories $Seg(\mathbf{n})$ and $\mathbf{n}^{op}$.

The n-cocycle $X:\mathbf{n}^{op}\to E_C$ is otherwise described as a string of arrows

$$(x_0,n)\leftarrow(x_1,n-1)\leftarrow\cdots\leftarrow(x_n,0)$$

each of which has the form (α_i,d^0), with $\alpha_i:x_{n-i}\to d_0(x_{n-i-1})$. This means that the string consists of objects $x_i\in C_{n-i}$ and morphisms $x_i\to d_0(x_{i-1})$.

Every ordinal number map $\theta:\mathbf{m}\to\mathbf{n}$ induces a commutative diagram

$$\begin{array}{ccccc} \mathbf{m}-\mathbf{i} & \xrightarrow{\cong} & [i,m] & \longrightarrow & \mathbf{m} \\ {\scriptstyle \theta_i}\downarrow & & {\scriptstyle \theta_i}\downarrow & & \downarrow{\scriptstyle \theta} \\ \mathbf{n}-\theta(\mathbf{i}) & \xrightarrow[\cong]{} & [\theta(i),n] & \longrightarrow & \mathbf{n} \end{array}$$

and there is a corresponding diagram

$$\begin{array}{ccccccc} (\theta_0^*x_{\theta(0)},m) & \longleftarrow & (\theta_1^*x_{\theta(1)},m-1) & \longleftarrow & \cdots & \longleftarrow & (\theta_m^*x_{\theta(m)},0) \\ {\scriptstyle (1,\theta_0)}\downarrow & & {\scriptstyle (1,\theta_1)}\downarrow & & & & \downarrow{\scriptstyle (1,\theta_m)} \\ (x_{\theta(0)},n-\theta(0)) & \longleftarrow & (x_{\theta(1)},n-\theta(1)) & \longleftarrow & \cdots & \longleftarrow & (x_{\theta(m)},n-\theta(m)) \end{array}$$

The string on top is denoted by θ^*X.

In this way, a simplicial set $\overline{W}C$ is defined, with $\overline{W}C_n$ given by the set of n-cocycles in C. The functoriality follows from the relations

$$\theta_{\tau(i)}\tau_i=(\theta\tau)_i$$

associated to composable ordinal number maps

$$\mathbf{k}\xrightarrow{\tau}\mathbf{m}\xrightarrow{\theta}\mathbf{n}.$$

There is a function

$$j:dBC_n=(BC_n)_n\to\overline{W}C_n$$

which sends a string

$$x_0 \leftarrow x_1 \leftarrow \cdots \leftarrow x_n$$

in C_n (note the Bousfield-Kan indexing [1, p.328]) to the cocycle consisting of the objects $d_0^j x_{n-j} \in C_{n-j}$ and the induced morphisms

$$d_0^j \alpha_{n-j} : d_0^j x_{n-j} \to d_0^j x_{n-j-1} = d_0 d_0^{n-j-1} x_{n-j-1},$$

or rather to the string

$$(x_0, \mathbf{n}) \leftarrow (d_0 x_1, \mathbf{n-1}) \leftarrow \cdots \leftarrow (d_0^n x_n, \mathbf{0})$$

in the Grothendieck construction E_C.

Suppose that $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal number map. One checks that the composite

$$dBC_n \xrightarrow{j} \overline{W}C_n \xrightarrow{\theta^*} \overline{W}C_m$$

sends the string of arrows $x_0 \leftarrow x_1 \leftarrow \cdots \leftarrow x_n$ in C_n to the string

$$(\theta_0^* d_0^{\theta(0)} x_{\theta(0)}, \mathbf{m}) \leftarrow (\theta_1^* d_0^{\theta(1)} x_{\theta(1)}, \mathbf{m-1}) \leftarrow \cdots \leftarrow (\theta_m^* d_0^{\theta(m)} x_{\theta(m)}, \mathbf{0})$$

while the composite

$$dBC_n \xrightarrow{\theta^*} dBC_m \xrightarrow{j} \overline{W}C_m$$

sends that same string in C_n to the string

$$(\theta^* x_{\theta(0)}, \mathbf{m}) \leftarrow (d_0 \theta^* x_{\theta(1)}, \mathbf{m-1}) \leftarrow \cdots \leftarrow (d_0^m \theta^* x_{\theta(m)}, \mathbf{0}).$$

Then $\theta_i^* d_0^{\theta(i)} x_{\theta(i)} = d_0^i \theta^* x_{\theta(i)}$, and it follows that the maps j respect the simplicial structure.

Lemma 20. *Suppose that G is a groupoid enriched in simplicial sets. Then the map $j : dBG \to \overline{W}G$ is a weak equivalence.*

Proof. The functors dB and $\overline{W}$ preserve homotopy equivalences and disjoint unions. If H is a simplicial group, the map $j : dBH \to \overline{W}H$ classifies the H-bundle $dEH \to dBH$, and so j is a weak equivalence for simplicial groups. Every simplicial groupoid G is homotopy equivalent to a disjoint union of simplicial groups. □

5 Gerbes

In homotopy theoretic terms, but according to the standard definition [2], [4], [10], a gerbe is a locally connected stack on a (small) Grothendieck site $\mathcal{C}$.

If G is a sheaf of groups on $\mathcal{C}$, then a G-gerbe (following [2]) is a stack $\mathcal{D}$ such that there is a covering family $U \to *$ of the terminal sheaf such that there are equivalences

$$\mathcal{D}|_U \to \mathrm{St}(G|_U)$$

for each U in the covering family, where $\mathrm{St}(G|_U)$ is the stack completion (stack of $G|_U$-torsors) of the restricted sheaf of groups $G|_U$. In this case, the stack $\mathcal{D}$ is automatically locally connected.

We can alternatively say that a stack $\mathcal{D}$ is a G-gerbe if there is a covering family $V \to *$ such that there are local equivalences

$$G|_V \to \mathcal{D}|_V$$

for each V in the cover. In effect, locally, there is an object x of $(G|_U) - \mathbf{tors}$ which lifts to $\mathcal{D}|_U$ up to isomorphism, any equivalence of groupoids induces isomorphisms of automorphism groups, and the sheaf of automorphisms of any G-torsor is isomorphic to G.

A presheaf of groupoids E is said to be a *G-gerbe* if there is a covering $W \to *$ of the terminal object by objects of $\mathcal{C}$ such that there are local weak equivalences

$$G|_W \to E|_W$$

for each W in the covering. Write $G - \mathbf{gerbe}$ for the corresponding category of G-gerbes and morphisms $E \to E'$ of presheaves of groupoids which are local weak equivalences. We shall only be interested in local weak equivalence classes of G-gerbes, so it will be irrelevant whether our gerbes are sheaves or presheaves of groupoids. Note that the definition of G-gerbe works equally well when G is a presheaf of groups, and that the following is easily proved:

Lemma 21. *Suppose that G is a presheaf of groups, and let $\tilde{G}$ denote its associated sheaf. Then the natural functor $\tilde{G} - \mathbf{gerbe} \to G - \mathbf{gerbe}$ defined by restriction of structure induces a bijection*

$$\pi_0(\tilde{G} - \mathbf{gerbe}) \cong \pi_0(G - \mathbf{gerbe}).$$

If H is a group, write $\mathrm{Aut}(H)$ for the 2-groupoid with one object, a 1-cell for each automorphism of H and a 2-cell for each homotopy (conjugation by an element of H) between automorphisms. One can check that the object $\mathrm{Aut}(H)$ is a group object in groupoids, so the simplicial groupoid (automorphisms and all their strings of homotopies) corresponding to $\mathrm{Aut}(H)$ is a simplicial group, and there is a natural inclusion

$$\mathrm{Aut}(H) \subset \mathbf{hom}(BH, BH).$$

We shall identify the 2-groupoid $\mathrm{Aut}(H)$ with this simplicial group.

In fact, $\mathrm{Aut}(H)$ can be characterized as the subcomplex of $\mathbf{hom}(BH, BH)$ which consists of those n-simplices (functors) $H \times \mathbf{n} \to H$ whose vertices are automorphisms. Observe that the evaluation

$$\mathbf{hom}(BH, BH) \times BH \to BH$$

restricts to an action

$$\mathrm{Aut}(H) \times BH \to BH$$

of the simplicial group $\mathrm{Aut}(H)$ on the nerve BH.

Suppose that G and G' are presheaves of groupoids. The simplicial set $\mathbf{equi}(G,G')$ is the subobject of $\mathbf{hom}(BG,BG')$ consisting of all functors $G \times \mathbf{n} \to G'$ such that all restrictions to vertices

$$G \cong G \times \mathbf{0} \xrightarrow{1\times i} G \times \mathbf{n} \to G'$$

are local equivalences of presheaves of groupoids. Any $f : G \to G'$ which is homotopic to a local weak equivalence must be a local weak equivalence, so it suffices that there is some restriction to a vertex which is a local weak equivalence. It follows also that the simplicial set $\mathbf{equi}(G,G')$ is the nerve of a groupoid whose objects are the local weak equivalences $G \to G'$ and whose morphisms are the homotopies between them.

The simplicial presheaf $\mathbf{Equi}(G,G')$ is defined by

$$\mathbf{Equi}(G,G')(U) = \mathbf{equi}(G|_U, G'|_U)$$

for each object U of the underlying site $\mathcal{C}$. Then $\mathbf{Equi}(G,G')$ is the nerve of a presheaf of simplicial groupoids, in an obvious way. Write also

$$\mathbf{Aut}(G) = \mathbf{Equi}(G,G),$$

and

$$\mathbf{aut}(G) = \mathbf{equi}(G,G).$$

Then

$$\mathbf{Aut}(G)(U) = \mathbf{aut}(G|_U)$$

for each object U of the site $\mathcal{C}$.

Lemma 22. *Suppose that G is a sheaf of groups with associated stack morphism $j : G \to \mathrm{St}(G)$. Then the map j induces local weak equivalences*

$$\mathbf{Equi}(G,G) \xrightarrow{j_*} \mathbf{Equi}(G,\mathrm{St}(G)) \xleftarrow{j^*} \mathbf{Equi}(\mathrm{St}(G),\mathrm{St}(G)).$$

Proof. Note first of all that any local weak equivalence $G \to G'$ of fibrant presheaves of groupoids is a homotopy equivalence, since the associated map $BG \to BG'$ is a homotopy equivalence. It follows that if $G \to G'$ is a different choice of fibrant model for G, then there is a map $G' \to \mathrm{St}(G)$ which induces a homotopy equivalence

$$\mathbf{equi}(G,G') \to \mathbf{equi}(G,\mathrm{St}(G))$$

It follows that the induced map

$$\mathbf{Equi}(G,G') \to \mathbf{Equi}(G,\mathrm{St}(G))$$

is a sectionwise weak equivalence.

We can therefore assume that $\mathrm{St}(G)$ is a sheaf of groupoids. Write $*$ for the image of the unique object of G under j (the trivial torsor). Then since j is a local weak equivalence and $\mathrm{St}(G)$ is a sheaf of groupoids the induced map

$$j_* : G \to \hom(*, *)$$

is an isomorphism of sheaves of groups.

Suppose that $f : G \to \mathrm{St}(G)$ is a local equivalence of sheaves of groupoids, and let $x = f(*)$ be the image of $*$ in global sections. Then $\mathrm{St}(G)$ is locally connected, so there is a covering family of objects $U \to *$ of $\mathcal{C}$, and a morphism $x|_U \to *$ for each member of the covering family. Then $f|_U$ is homotopic to a composite of the form

$$G|_U \xrightarrow{f'} G|_U \xrightarrow{j} \mathrm{St}(G)|_U$$

for each U in the covering family. This is true in all sections, so it follows that the induced sheaf map

$$\tilde{\pi}_0 \mathbf{Equi}(G, G) \to \tilde{\pi}_0 \mathbf{Equi}(G, \mathrm{St}(G))$$

is an epimorphism. This map is also a monomorphism, on account of the sheaf isomorphism $G \cong \hom(*, *)$ which is induced by j. That same sheaf isomorphism induces a sheaf of fundamental groups isomorphism

$$\tilde{\pi}_1(\mathbf{Equi}(G, G), \alpha) \cong \tilde{\pi}_1(\mathbf{Equi}(G, \mathrm{St}(G), j\alpha)$$

for all (local) choices of base points α. The map

$$\mathbf{Equi}(G, G) \xrightarrow{j_*} \mathbf{Equi}(G, \mathrm{St}(G))$$

is induced by a morphism of presheaves of groupoids, and is therefore a map of presheaves of Kan complexes. It follows that j_* is a local weak equivalence.

The diagram

$$\begin{array}{ccc} \mathbf{Equi}(\mathrm{St}(G), \mathrm{St}(G)) & \longrightarrow & \mathbf{Hom}(B(\mathrm{St}(G)), B(\mathrm{St}(G))) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\ \mathbf{Equi}(G, \mathrm{St}(G)) & \longrightarrow & \mathbf{Hom}(BG, B(\mathrm{St}(G))) \end{array}$$

is a pullback since j is a local equivalence. The map j_* of function complex presheaves is a trivial fibration since j is a trivial cofibration and the simplicial presheaf $B(\mathrm{St}(G))$ is fibrant. It follows that the map

$$j^* : \mathbf{Equi}(\mathrm{St}(G), \mathrm{St}(G)) \to \mathbf{Equi}(G, \mathrm{St}(G))$$

is a trivial fibration, and is therefore a local weak equivalence. □

Suppose that H is a simplicial group and that simplicial sets X and Y are chosen such that X has a right H-action and Y has a left H-action. Then H acts on the left on $X \times Y$ via

$$(g, (x, y)) \mapsto (xg^{-1}, gy),$$

and the resulting bisimplicial set $EH \times_H (X \times Y)$ has horizontal path components isomorphic to $X \times_H Y$ (balanced product), where $X \times_H Y = (X \times Y)/$ and the indicated equivalence relation is generated by the relation $(xg, y) \simeq (x, gy)$.

Note that $H \times_H Y \cong Y$ in the special case where the group H is interpreted as having a right H-action by the group multiplication. In this case, the canonical map

$$d(EH \times_H (H \times Y)) \to H \times_H Y \cong Y$$

is a weak equivalence. In effect, the path component in $EH \times_H (H \times Y)$ corresponding to a fixed vertex (e, x) has objects consisting of all pairs $(g, g^{-1}x)$, and the map $g : (g, g^{-1}x) \to (e, x)$ is uniquely determined. The function $H \to H \times Y$ which is defined by $g \mapsto (g, g^{-1}x)$ induces an isomorphism of categories of $EH \times_H H$ (right action) with the path component of (e, x). It follows that if H acts freely on X, then the map

$$d(EH \times_H (X \times Y)) \to X \times_H Y$$

is a weak equivalence.

The corresponding (opposite) simplicial group H^o is obtained by reversing all arrows in H all simplicial degrees. A right (aka. contravariant) action of the simplicial group H on a simplicial set X corresponds to a left action $H^o \times X \to X$.

Suppose that G is a sheaf of groups and that F is a right $\mathbf{Aut}(G)$-torsor, meaning (see Remark 5) that F is a cofibrant $\mathbf{Aut}(G)^o$-object, and the map $F/\mathbf{Aut}(G) \to *$ is a local weak equivalence.In particular, F has a free right $\mathbf{Aut}(G)$-action. Remark 4 and Lemma 9 together imply that there are bijections

$$[*, dB(\mathbf{Aut}(G)^o)] \cong [*, \overline{W}(\mathbf{Aut}(G)^o)] \cong \pi_0(\mathbf{Aut}(G)^o - \mathbf{Tors}).$$

The remainder of this section consists of the proof of Theorem 23, which asserts that these objects are in bijective correspondence with the set $\pi_0(G - \mathbf{gerbe})$ of path components (ie. local equivalence classes) of G-gerbes.

The simplicial sheaf of groups $\mathbf{Aut}(G)$ acts on the simplicial sheaf BG via the composition

$$\mathbf{Aut}(G) \times BG \to \mathbf{Hom}(BG, BG) \times BG \xrightarrow{ev} BG$$

where ev is the evaluation map. The canonical map

$$d(E\mathbf{Aut}(G) \times_{\mathbf{Aut}(G)} (F \times BG)) \to F \times_{\mathbf{Aut}(G)} BG \tag{4}$$

is a local weak equivalence by the previous paragraphs.

Since the map $F/\mathbf{Aut}(G) \to *$ is a local weak equivalence, there is a covering family of maps $U \to *$ with $U \in \mathcal{C}$ such that there are sections

$$\begin{array}{ccc} & & F \\ & \overset{\sigma}{\nearrow} & \downarrow \\ U & \longrightarrow & * \end{array}$$

These sections induce $\mathbf{Aut}(G)$-equivariant equivalences (maps of right torsors) $\sigma_* : \mathbf{Aut}(G|_U) \to F|_U$ for all maps $U \to *$ in the covering family. The induced maps of balanced products

$$BG|_U \cong \mathbf{Aut}(G|_U) \times_{\mathbf{Aut}(G|_U)} BG|_U \to F|_U \times_{\mathbf{Aut}(G|_U)} BG|_U$$

are local weak equivalences for all $U \to *$ in the covering family by the previous paragraphs, so that $F \times_{\mathbf{Aut}(G)} BG$ is locally equivalent to BG. It follows that the stack completion $\mathrm{St}(\pi(F \times_{\mathbf{Aut}(G)} BG))$ of the corresponding fundamental groupoid is a G-gerbe. The fact that the maps (4) are weak equivalences for all right $\mathbf{Aut}(G)$-torsors also implies that any map $F \to F'$ of $\mathbf{Aut}(G)$-torsors induces a local weak equivalence

$$\mathrm{St}(\pi(F \times_{\mathbf{Aut}(G)} BG)) \to \mathrm{St}(\pi(F' \times_{\mathbf{Aut}(G)} BG))$$

of G-gerbes.

Suppose that E is a G-gerbe, interpreted as a stack which is locally equivalent to G. Then there is a covering family $U \to *$ by objects $U \in \mathcal{C}$ such that there are equivalences $\alpha_U : G|_U \to E|_U$ for all $U \to *$ in the covering family. Since $E|_U$ is a stack there are equivalences $G|_U - \mathbf{Tors} \to E|_U$ such that the diagrams

$$\begin{array}{ccc} G|_U & \xrightarrow{\alpha_U} & E|_U \\ {\scriptstyle j}\downarrow & \overset{\alpha'_U}{\nearrow} & \\ G|_U - \mathbf{Tors} & & \end{array}$$

commute. In the composite

$$\mathbf{Equi}(G|_U, G|_U) \xrightarrow{j_*} \mathbf{Equi}(G|_U, G|_U - \mathbf{Tors}) \xrightarrow{\alpha'_{U*}} \mathbf{Equi}(G|_U, E|_U)$$

the map α'_{U*} is a homotopy equivalence since α'_U is a weak equivalence of stacks, and the map j_* is a local weak equivalence by Lemma 22. These maps are equivariant for the action by $\mathbf{Equi}(G|_U, G|_U)$ on the right. It follows that the map

$$E\mathbf{Aut}(G) \times_{\mathbf{Aut}(G)} \mathbf{Equi}(G, E) \to *$$

is a local weak equivalence, so that $\mathbf{Equi}(G, E)$ represents a right $\mathbf{Aut}(G)$-torsor. The corresponding torsor is an $\mathbf{Aut}(G)$-cofibrant model

$$\pi : \mathbf{Equi}(G, E)_c \to \mathbf{Equi}(G, E).$$

Note that the cofibrant object $\mathbf{Equi}(G,E)_c$ has a free $\mathbf{Aut}(G)$-action, so that any restriction $\mathbf{Equi}(G,E)_c|_U$ has a free $\mathbf{Aut}(G|_U)$-action.

Suppose that there is a covering family $U \to *$ of objects $U \in \mathcal{C}$ such that there are local weak equivalences $\alpha_U : G|_U \to E|_U$ for all $U \to *$ in the covering family. Then there is a diagram

$$\begin{array}{ccc}
\mathbf{Equi}(G,G)_c|_U \times_{\mathbf{Aut}(G|_U)} BG|_U & \xrightarrow{\simeq} & \mathbf{Equi}(G,E)_c|_U \times_{\mathbf{Aut}(G|_U)} BG|_U \\
\simeq \downarrow & & \downarrow \\
\mathbf{Equi}(G|_U,G|_U) \times_{\mathbf{Aut}(G|_U)} BG|_U & \longrightarrow & \mathbf{Equi}(G|_U,E|_U) \times_{\mathbf{Aut}(G|_U)} BG|_U \\
\cong \downarrow & & \downarrow \\
BG|_U & \xrightarrow[\simeq]{} & BE|_U
\end{array}$$

The top horizontal map is a local weak equivalence since the map

$$\mathbf{Equi}(G,G)_c \to \mathbf{Equi}(G,E)_c$$

is a local weak equivalence of simplicial presheaves having free $\mathbf{Aut}(G)$-actions. Similarly, the map $\mathbf{Equi}(G,G)_c \to \mathbf{Equi}(G,G)$ is a weak equivalence of simplicial presheaves having free $\mathbf{Aut}(G)$ actions, so the corresponding vertical map is a weak equivalence. It follows that the induced composite

$$\mathbf{Equi}(G,E)_c \times_{\mathbf{Aut}(G)} BG \to \mathbf{Equi}(G,E) \times_{\mathbf{Aut}(G)} BG \to BE$$

determined by the evaluation map is a local weak equivalence. In particular, there is an induced natural local equivalence

$$\mathrm{St}(\pi(\mathbf{Equi}(G,E)_c \times_{\mathbf{Aut}(G)} BG)) \to E$$

of stacks.

Suppose that F is a right $\mathbf{Aut}(G)$-torsor. Then there is an $\mathbf{Aut}(G)$-equivariant map

$$F \to \mathbf{Equi}(G, \mathrm{St}(\pi(F \times_{\mathbf{Aut}(G)} BG))) \tag{5}$$

which is adjoint to the canonical map

$$F \times_{\mathbf{Aut}(G)} BG \to B\,\mathrm{St}(\pi(F \times_{\mathbf{Aut}(G)} BG)).$$

Locally, the map (5) has the form

$$\mathbf{Aut}(G) \to \mathbf{Equi}(G, \mathrm{St}(\pi(\mathbf{Aut}(G) \times_{\mathbf{Aut}(G)} BG))) \tag{6}$$

Thus, if we show that all instances of (6) are local weak equivalences, then all instances of (5) are local weak equivalences.

The evaluation isomorphism $\mathbf{Aut}(G) \times_G BG \to BG$ is adjoint to the isomorphism (identification) $\mathbf{Aut}(G) \to \mathbf{Equi}(G,G)$, and there is a commutative diagram of local weak equivalences

$$\begin{array}{ccc}
\mathbf{Aut}(G) \times_{\mathbf{Aut}(G)} BG & \longrightarrow & B(\mathrm{St}(\pi(\mathbf{Aut}(G) \times_{\mathbf{Aut}(G)} BG))) \\
\downarrow & & \downarrow \\
BG & \xrightarrow[j]{} & B(\mathrm{St}(\pi(BG)))
\end{array}$$

It follows that there is a commutative diagram

$$\begin{array}{ccc}
\mathbf{Aut}(G) & \longrightarrow & \mathbf{Equi}(G, \mathrm{St}(\pi(\mathbf{Aut}(G) \times_{\mathbf{Aut}(G)} BG))) \\
\cong \downarrow & & \downarrow \simeq \\
\mathbf{Equi}(G, G) & \xrightarrow[j_*]{} & \mathbf{Equi}(G, \mathrm{St}(\pi(BG)))
\end{array}$$

The map j_* is a local weak equivalence by Lemma 22, so the desired (top horizontal) map is a local weak equivalence.

We have therefore proved the following:

Theorem 23. *Suppose that G is a sheaf of groups on a small Grothendieck site $\mathcal{C}$. Then there are bijections*

$$[*, dB(\mathbf{Aut}(G)^o)] \cong [*, \overline{W}(\mathbf{Aut}(G)^o)] \cong \pi_0(G - \mathbf{gerbe}).$$

References

[1] A.K. Bousfield and D.M. Kan, *Homotopy Limits Completions and Localization*, Lecture Notes in Math. **304**, Springer-Verlag (1972).

[2] L. Breen, *Bitorseurs et Cohomologie Non Abélienne*, The Grothendieck Festschrift I, Progress in Mathematics **86**, Birkhäuser, Boston-Basel-Berlin (1990), 401-476.

[3] S.E. Crans, *Quillen closed model structures for sheaves*, J. Pure Applied Algebra **101** (1995), 35–57.

[4] J. Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[5] P.G. Goerss and J.F. Jardine, *Simplicial Homotopy Theory*, Progress in Math. **174**, Birkhäuser, Basel-Boston-Berlin (1999).

[6] J.F. Jardine, *Universal Hasse-Witt classes*, "Algebraic K-theory and Algebraic Number Theory", Contemp. Math. **199**, AMS (1989), 103–116.

[7] J.F. Jardine, *Stacks and the homotopy theory of simplicial sheaves*, Homotopy, Homology and Applications **3**(2) (2001), 361–384.

[8] A. Joyal and M. Tierney, *Classifying spaces for sheaves of simplicial groupoids*, J. Pure App. Algebra **89** (1993), 135–161.

[9] A. Joyal and M. Tierney, *On the homotopy theory of sheaves of simplicial groupoids*, Math. Proc. Camb. Phil. Soc. **120** (1996), 263–290.

[10] G. Laumon and L. Moret-Bailly, *Champs algb́riques*, Erg. der Mathematik 3. Folge, Band 39, Springer-Verlag, Berlin (2000).

[11] Z. Luo, *Closed model categories for presheaves of simplicial groupoids and presheaves of 2-groupoids*, Preprint (2002).

[12] Z. Luo, *Presheaves of simplicial groupoids*, Thesis, University of Western Ontario (2003).

[13] I. Moerdijk, *Bisimplicial sets and the group completion theorem*, "Algebraic K-Theory: Connections with Geometry and Topology", NATO ASI Series C, **279**, Kluwer (1989), 225–240.