

Intermediate model structures for simplicial presheaves

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Suppose that \mathcal{C} is a small Grothendieck site, and let $s\text{Pre}(\mathcal{C})$ denote the category of simplicial presheaves on the site \mathcal{C} . It has been known for some time [5] that the category of simplicial presheaves carries a proper closed simplicial model structure for which a *cofibrations* is an inclusion of simplicial presheaves, the weak equivalences are the local weak equivalences, and the fibrations are the global fibrations.

In the presence of an adequate supply of stalks, a local weak equivalence is a map $f : X \rightarrow Y$ of simplicial presheaves which induces a weak equivalence $f_* : X_x \rightarrow Y_x$ of simplicial sets in all stalks. Alternatively, f is a *local weak equivalence* if

- 1) the induced map $f_* : \tilde{\pi}_0 X \rightarrow \tilde{\pi}_0 Y$ of sheaves of path components is an isomorphism, and
- 2) the comparison diagram

$$\begin{array}{ccc} \tilde{\pi}_n X & \xrightarrow{f_*} & \tilde{\pi}_n Y \\ \downarrow & & \downarrow \\ \tilde{X}_0 & \xrightarrow{f_*} & \tilde{Y}_0 \end{array}$$

of sheaves of group objects is a pullback, for $n \geq 1$.

A *global fibration* is a map which has the right lifting property with respect to all morphisms which are simultaneously cofibrations and local weak equivalences.

The indicated structure has become the standard tool for calculational applications of the homotopy theory of simplicial presheaves.

Recently, however, it has become clear that there is another useful model structure on $s\text{Pre}(\mathcal{C})$ which has the same weak equivalences and hence describe the same homotopy category, but has different cofibrations and fibrations. This is the local projective model structure, which was introduced by Blander [1] as a tool for investigating some of the models for the motivic stable category.

Specifically, a cofibration $i : A \rightarrow B$ is said to be *projective* if it has the left lifting property with respect to all maps $p : X \rightarrow Y$ which are trivial fibrations

of simplicial sets $f : X(U) \rightarrow Y(U)$, $U \in \mathcal{C}$, in all sections. A map $p : X \rightarrow Y$ is then said to be a *local projective fibration* if it has the right lifting property with respect to all maps which are simultaneously projective cofibrations and local weak equivalences. Blander [1] proves the following:

Theorem 1. *The category of simplicial presheaves $s\text{Pre}(\mathcal{C})$ on a small Grothendieck site \mathcal{C} , with the classes of projective cofibrations, local weak equivalences and local projective fibrations, satisfies the axioms for a proper closed simplicial model category.*

The closed model structure given by this result is the *local projective model structure* for the category $s\text{Pre}(\mathcal{C})$ of simplicial presheaves on \mathcal{C} .

Say that a map $f : X \rightarrow Y$ of simplicial presheaves is a *sectionwise weak equivalence* (respectively *sectionwise fibration*) if all of its component maps $f : X(U) \rightarrow Y(U)$, $U \in \mathcal{C}$, are weak equivalences (respectively fibrations) of simplicial sets. One sometimes sees these objects referred to as *pointwise weak equivalences* or *pointwise fibrations* in the literature [7].

The special case of Theorem 1 corresponding to the trivial (or chaotic) topology on \mathcal{C} , for which the local projective fibrations are the sectionwise fibrations, was proved long ago by Bousfield and Kan [2]. Explicitly, the *chaotic topology* on a site \mathcal{C} is the topology for which every covering family of an object U contains the identity map $1 : U \rightarrow U$. Every presheaf is a sheaf for the chaotic topology.

Observe that the definition of projective cofibration makes no reference to an underlying topology, and thus coincides with the notion arising in the Bousfield-Kan result. That result is easy to prove: if L_U denotes the left adjoint to the U -sections functor $s\text{Pre}(\mathcal{C}) \rightarrow \mathbf{S}$ taking values in simplicial sets, then the cofibrations $L_U(\partial\Delta^n) \rightarrow L_U(\Delta^n)$ generate the projective cofibrations in the usual sense, and the while the maps $L_U(\Lambda_k^n) \rightarrow L_U(\Delta^n)$ generate the acyclic projective cofibrations for the Bousfield-Kan theory. The function spaces $\mathbf{hom}(X, Y)$ have the form that we expect, and there is a minor amount of fussing (in the proof of the simplicial model axiom) to show that if $i : A \rightarrow B$ is a projective cofibration and $j : K \rightarrow L$ is an inclusion of simplicial sets then the induced map

$$(i, j)_* : (B \times K) \cup_{(A \times K)} (A \times L) \rightarrow B \times L$$

is a projective cofibration. One resolves this, however, by showing that the collection of all inclusions i for which the cofibration $(i, j)_*$ is projective is saturated, and includes all $L_U(\partial\Delta^n) \rightarrow L_U(\Delta^n)$.

The point of this note is to show that there are other intermediate model structures on the simplicial presheaf category $s\text{Pre}(\mathcal{C})$ which are intermediate between the local projective and standard theories.

Sufficient conditions are then given for these structures to be cofibrantly generated. These conditions apply in particular to the local projective structure, giving a new proof of a result of Blander. It remains to be seen, however, whether any of the new intermediate structures outside of either the standard or local projective cases are cofibrantly generated.

To start the discussion, let \mathbf{C}_P denote the class of projective cofibrations, and let \mathbf{C} denote the full class of cofibrations in $s\text{Pre}(\mathcal{C})$. Then there is clearly an inclusion relation $\mathbf{C}_P \subset \mathbf{C}$. Recall that \mathbf{C}_P has a generating family of cofibrations $S_0 = \{L_U(\partial\Delta^n) \rightarrow L_U(\Delta^n)\}$.

Suppose that $S = \{A_j \rightarrow B_j\}$ is some other set of cofibrations which contains S_0 , and let \mathbf{C}_S denote the saturation of the set of cofibrations

$$(B_j \times \partial\Delta^n) \cup_{(A_j \times \partial\Delta^n)} (A_j \times \Delta^n) \subset B_j \times \Delta^n$$

where $n \geq 0$, and the cofibrations $A_j \rightarrow B_j$ belong to the set S . Note that the case $n = 0$ reduces to maps $A_j \rightarrow B_j$ in the generating set for \mathbf{C}_S . The class \mathbf{C}_S will also be called the class of S -cofibrations. If S is a generating set for the full class of cofibrations, then $\mathbf{C}_S = \mathbf{C}$ and in that case the set of S -cofibrations is the full set of cofibrations.

Say that a map $p : X \rightarrow Y$ is a *local S -fibration* if p has the right lifting property with respect to all maps which are S -cofibrations and local weak equivalences.

Theorem 2. *With these definitions, the category $s\text{Pre}(\mathcal{C})$ and the classes of S -cofibrations, local weak equivalences and local S -fibrations together satisfy the axioms for a proper closed simplicial model category.*

Proof. Every map $f : X \rightarrow Y$ of simplicial presheaves has a factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow j & \nearrow p \\ & Z & \end{array}$$

where the map j is a member of \mathbf{C}_S and p has the right lifting property with respect to all morphisms of \mathbf{C}_S . It follows that p has the right lifting property with respect to all projective cofibrations, so that p is a projective fibration and a sectionwise (hence local) weak equivalence. The map p is also a local S -fibration.

The map $f : X \rightarrow Y$ can be factored

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow i & \nearrow q \\ & W & \end{array}$$

where q is a global fibration (hence a local S -fibration) and i is a cofibration and a local weak equivalence. The map i has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ & \searrow j & \nearrow p \\ & Z & \end{array}$$

where p is a local S -fibration and a local weak equivalence and j is an S -cofibration. Then j must also be a local weak equivalence, and the composite qp is a local S -fibration.

Suppose that the map $p : X \rightarrow Y$ is a local S -fibration and a local weak equivalence. Then p has a factorization

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ & \searrow j & \nearrow q \\ & & Z \end{array}$$

where j is an S -cofibration and q has the right lifting property with respect to all S -cofibrations. As before, q is therefore a local weak equivalence as well as a local S -fibration. The S -cofibration j is thus a local weak equivalence, and so the indicated lifting exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ j \downarrow & \nearrow \text{dotted} & \downarrow p \\ Z & \xrightarrow{q} & Y \end{array}$$

The map p is therefore a retract of q , and thus has the right lifting property with respect to all S -cofibrations.

The function complex is the standard one, and the model structure satisfies Quillen's axiom **SM7**, because the class C_S was cooked up so that it would do so: it includes all maps

$$(B_i \times \partial\Delta^n) \cup_{(A_i \times \partial\Delta^n)} (A_i \times \Delta^n) \subset B_i \times \Delta^n$$

All local S -fibrations are sectionwise fibrations and pullback along sectionwise fibrations preserves local weak equivalences, by a Boolean localization argument. "Dually", all S -cofibrations are cofibrations and the standard structure is proper, so that pushout along S -cofibrations preserves local weak equivalences. \square

The case $S = S_0$ for Theorem 2 is the Blander result Theorem 1, and the proof of Theorem 2 is an abstraction of Blander's proof.

The meaning of the term "local" in the statement of Theorem 2 can also vary wildly. In particular, if the ambient topology is the chaotic topology, then local weak equivalences are sectionwise weak equivalences. In this case the corresponding local S -fibrations will just be called S -fibrations; thus an S -fibration is a map which has the right lifting property with respect to all maps which are S -cofibrations and sectionwise weak equivalences.

Corollary 3. *With these definitions, the category $s\text{Pre}(\mathcal{C})$ and the classes of S -cofibrations, sectionwise weak equivalences and S -fibrations together satisfy the axioms for a proper closed simplicial model category.*

It's a rather silly way to prove the result, but the Bousfield-Kan theorem is the case $S = S_0$ of Corollary 3.

Remark 4. Some question remains about whether or not the S -structure just coincides with either the projective of the standard structure, in the absence of good examples.

Remark 5. If one is willing to forego the simplicial model structure, one can define \mathbf{C}_S to simply be the saturation of the class $\mathbf{C}_P \cup S$ in the full class of cofibrations \mathbf{C} . Then one just gets back a proper closed model structure, by the same argument.

A condition, most properly expressed in Proposition 9 below, for the closed model structure given by Theorem 2 to be cofibrantly generated will now be developed. The problem is to find a generating set for the class of maps which are simultaneously S -cofibrations and local weak equivalences, and Proposition 9 says that one can solve this problem for any given topology on the site \mathcal{C} if one can first solve it for the chaotic topology.

The method of proof of Proposition 9 involves an application of a *bounded approximation* technique which appears here as Lemma 7 and Corollary 8. This technique made its first appearance in [8].

The S -cofibrant replacements of local trivial cofibrations have a special role in this development. Explicitly, suppose that $i : A \rightarrow B$ is a cofibration. Then an *S -cofibrant replacement* for i is a commutative diagram

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\pi_A} & A \\ \tilde{i} \downarrow & & \downarrow i \\ \tilde{B} & \xrightarrow{\pi_B} & B \end{array}$$

where π_A and π_B are sectionwise weak equivalences, and \tilde{i} is an S -cofibration. We require further that the simplicial presheaf \tilde{A} is S -cofibrant. We can further assume that the maps π_A and π_B are trivial S -fibrations, and hence are sectionwise fibrations as well as sectionwise weak equivalences — this is how one shows that all cofibrations have S -cofibrant replacements.

There is a recognition principle for local S -fibrations which is based on the globally fibrant replacement for the chaotic topology. In general (for any topology) a *globally fibrant replacement* for a map $p : X \rightarrow Y$ of simplicial presheaves is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{j_X} & \overline{X} \\ p \downarrow & & \downarrow \overline{p} \\ Y & \xrightarrow{j_Y} & \overline{Y} \end{array} \tag{1}$$

where \overline{Y} is globally fibrant, \overline{p} is a global fibration, and the maps j_X and j_Y are local weak equivalences.

Lemma 6. *Suppose that the map $p : X \rightarrow Y$ is an S -fibration and that in some globally fibrant replacement (1) for the chaotic topology the map \bar{p} is a local S -fibration. Then p is a local S -fibration. In particular, if \bar{p} is a global fibration for the topology on \mathcal{C} , then p is a local S -fibration.*

Proof. Recall that the maps j_X and j_Y are sectionwise weak equivalences. It follows from properness for the simplicial set category that the induced map $(p, j_X) : X \rightarrow Y \times_{\bar{Y}} \bar{X}$ is a sectionwise weak equivalence. One also sees that the induced map $\tilde{p}_* : Y \times_{\bar{Y}} \bar{X} \rightarrow Y$ has the right lifting property with respect to all maps which are S -cofibrations and local weak equivalences. Find a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ & \searrow (p, j_X) & \downarrow q \\ & & Y \times_{\bar{Y}} \bar{X} \end{array}$$

where q is a sectionwise weak equivalence and an S -fibration, and i is an S -cofibration. Then q has the right lifting property with respect to all S -cofibrations, and i is sectionwise weak equivalence as well as an S -cofibration. The indicated lifting therefore exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ i \downarrow & \nearrow & \downarrow p \\ Z & \xrightarrow{\tilde{p}_* q} & Y \end{array}$$

It follows that the map p is a retract of a map $\tilde{p}_* q$ which has the right lifting property with respect to all maps which are both S -cofibrations and local weak equivalences. The map p is therefore a local S -fibration. \square

Pick an S -cofibrant replacement $\tilde{i} : \tilde{A} \rightarrow \tilde{B}$ for each $i : A \rightarrow B$ in a set of α -bounded local trivial cofibrations. The cardinal α is chosen sufficiently large such the set of all such maps i generates the class of locally trivial cofibrations for the standard model structure on the simplicial presheaf category $s\text{Pre}(\mathcal{C})$. Let S_p denote the set of all such S -cofibrant replacements.

We have the following simple consequences of the bounded cofibration condition [3] that will be used below.

Lemma 7. *Suppose given cofibrations $A \rightarrow B \rightarrow X$ such that the composite $A \rightarrow X$ is a weak equivalence and B is α -bounded. Then there is a trivial cofibration $C \subset X$ such that $B \subset C$ and C is α -bounded.*

Proof. This result is a consequence of the bounded cofibration condition.

Under the stated assumptions, and for the diagram of cofibrations

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ B & \longrightarrow & X \end{array}$$

there is an α -bounded object C with $B \subset C \subset X$ and such that the induced cofibration $C \cap A \rightarrow C$ is a weak equivalence. Then $A = C \cap A$ because $A \subset C$. \square

Corollary 8. *Suppose given a diagram of trivial cofibrations*

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ A & \longrightarrow & Y \end{array}$$

such that A is α -bounded. Then there is an α -bounded subobject $C \subset Y$ such that the inclusions $A \subset C$ and $C \cap X \rightarrow C$ are trivial cofibrations.

Proof. By the bounded cofibration condition, there is an α -bounded subobject $D_1 \subset Y$ with $A \subset D_1$ and such that $D_1 \cap X \rightarrow D_1$ is a trivial cofibration. By the second lemma, there is a trivial cofibration $C_1 \subset Y$ with C_1 α -bounded and such that $D_1 \subset C_1$. Inductively form the string of cofibrations

$$A \subset D_1 \subset C_1 \subset D_2 \subset C_2 \subset \dots$$

such that all objects are α -bounded, all maps $A \rightarrow C_i$ and $D_i \cap X \rightarrow D_i$ are trivial cofibrations. Let $C = \overline{\lim} C_i \subset Y$. Then C is α -bounded, the map $A \rightarrow C$ is a trivial cofibration, and $C \cap X \rightarrow C$ is a trivial cofibration. \square

Proposition 9. *Suppose that a map $p : X \rightarrow Y$ is an S -fibration which has the right lifting property with respect to all maps $\tilde{i} : \tilde{A} \rightarrow \tilde{B}$ appearing in the set S_p . Then p is a local S -fibration.*

Proof. Suppose that $\bar{p} : \bar{X} \rightarrow \bar{Y}$ is a global fibrant replacement for p in the chaotic topology as above. We want to show that the map \bar{p} is a global fibration for the original topology on \mathcal{C} , and then invoke Lemma 6.

We have to solve all lifting problems

$$\begin{array}{ccc} A & \longrightarrow & \bar{X} \\ \downarrow i & \nearrow \text{dotted} & \downarrow \bar{p} \\ B & \longrightarrow & \bar{Y} \end{array} \quad (2)$$

where $i : A \rightarrow B$ is an α -bounded locally trivial cofibration. This amounts to showing that the vertex in the base of the simplicial set fibration

$$(i^*, \bar{p}_*) : \mathbf{hom}(B, \bar{X}) \rightarrow \mathbf{hom}(A, \bar{X}) \times_{\mathbf{hom}(A, \bar{Y})} \mathbf{hom}(B, \bar{Y})$$

represented by the commutative square is the image of a vertex of the total space. The strategy is to replace the map $i : A \rightarrow B$ by another cofibration up to weak equivalence for which the lifting exists, and then compare fibrations.

Factorize the commutative square (2) as the composite diagram

$$\begin{array}{ccccc} A & \xrightarrow{i_A} & A_1 & \xrightarrow{p_A} & \overline{X} \\ \downarrow i & & \downarrow i_1 & & \downarrow \overline{p} \\ B & \xrightarrow{i_B} & B_1 & \xrightarrow{p_B} & \overline{Y} \end{array}$$

where i_A and i_B are sectionwise trivial cofibrations, p_A and p_B are sectionwise fibrations, and i_1 is a cofibration. To do this, factor the map $B \rightarrow Y$ as a composite

$$B \xrightarrow{j} U \xrightarrow{q} Y$$

where q is a sectionwise fibration and j is a sectionwise trivial projective cofibration. Factor the induced map $A \rightarrow U \times_{\overline{Y}} \overline{X} \rightarrow U$ as a composite

$$A \xrightarrow{j^A} A_1 \xrightarrow{\pi} U \times_{\overline{Y}} \overline{X},$$

where i_A is a sectionwise trivial projective cofibration and π is a sectionwise fibration. There is a cofibration i_1 and a trivial global fibration q' such that the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\pi} & U \times_{\overline{Y}} \overline{X} \\ \downarrow i_1 & & \downarrow \\ B_1 & \xrightarrow{q'} & U \end{array}$$

commutes. Then the lifting i_B in the diagram

$$\begin{array}{ccc} A & \xrightarrow{i_1 j^A} & B_1 \\ \downarrow i & \nearrow i_B & \downarrow q' \\ B & \xrightarrow{j} & U \end{array}$$

The map i_B is a cofibration since j is a cofibration. Also the composites

$$A_1 \xrightarrow{\pi} U \times_{\overline{Y}} \overline{X} \rightarrow \overline{X}$$

and

$$B_1 \xrightarrow{q'} U \xrightarrow{q} \overline{Y}$$

are sectionwise fibrations and are therefore candidates for p_A and p_B , respectively.

Pull back the chaotic cofibrant replacement diagram (1) along the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{p_A} & \overline{X} \\ \downarrow i_1 & & \downarrow \overline{p} \\ B_1 & \xrightarrow{p_B} & \overline{Y} \end{array}$$

to obtain the commutative cube

$$\begin{array}{ccccc}
A_2 & \xrightarrow{\quad} & X & & \\
\downarrow & \searrow \alpha_A & \downarrow & \searrow j_X & \\
& & A_1 & \xrightarrow{p_A} & \bar{X} \\
& & \downarrow i_1 & \downarrow p & \downarrow \bar{p} \\
B_2 & \xrightarrow{\quad} & Y & & \\
\downarrow & \searrow \alpha_B & \downarrow & \searrow j_Y & \\
& & B_1 & \xrightarrow{p_B} & \bar{Y}
\end{array}$$

Then the induced cofibrations α_A and α_B are sectionwise weak equivalences, since they are obtained by pulling back the sectionwise weak equivalences j_X and j_Y along the sectionwise fibrations p_A and p_B , respectively.

I claim that there are α -bounded objects A_3 and B_3 such that there is a diagram of cofibrations

$$\begin{array}{ccccc}
A & \longrightarrow & A_3 & \longrightarrow & A_1 \\
\downarrow & & \downarrow & & \downarrow \\
B & \longrightarrow & B_3 & \longrightarrow & B_1
\end{array}$$

and such that the induced maps $A_3 \cap A_2 \rightarrow A_3$ and $B_3 \cap B_2 \rightarrow B_2$ are sectionwise weak equivalences, and such that the maps $A \rightarrow A_3$ and $B \rightarrow B_3$ are sectionwise weak equivalences. In effect, A_3 is found by Corollary 8 applied to the sectionwise trivial cofibration $\alpha_A : A_2 \rightarrow A_1$ and the α -bounded subobject $A \subset A_1$. Then B_3 is found by applying Corollary 8 to the sectionwise trivial cofibration $\alpha_B : B_2 \rightarrow B_1$ and an α -bounded subobject D of B_1 which contains $i_1(A_3) \cup B$ and is sectionwise equivalent to B — the object D exists by Lemma 7.

Observe that the composite

$$A_2 \xrightarrow{\alpha_A} A_1 \xrightarrow{i_1} B_1$$

is a cofibration, so that the map $A_2 \rightarrow B_2$ is a cofibration. It also follows that the induced map $A_3 \cap A_2 \rightarrow B_3 \cap B_2$ is a cofibration, which is of course α -bounded, and is a local weak equivalence because it's sectionwise equivalent to the local weak equivalence $i : A \rightarrow B$.

Suppose that the map $\tilde{i} : \tilde{A} \rightarrow \tilde{B}$ is a projective cofibrant replacement for the α -bounded locally trivial cofibration $A_3 \cap A_2 \rightarrow B_3 \cap B_2$. Then by construction, the fibration

$$(i^*, \bar{p}_*) : \mathbf{hom}(B, \bar{X}) \rightarrow \mathbf{hom}(A, \bar{X}) \times_{\mathbf{hom}(A, \bar{Y})} \mathbf{hom}(B, \bar{Y})$$

is weakly equivalent to the fibration

$$(\tilde{i}^*, \tilde{p}_*) : \mathbf{hom}(\tilde{B}, \bar{X}) \rightarrow \mathbf{hom}(\tilde{A}, \bar{X}) \times_{\mathbf{hom}(\tilde{A}, \bar{Y})} \mathbf{hom}(\tilde{B}, \bar{Y})$$

and the element in

$$\pi_0(\mathbf{hom}(A, \overline{X}) \times_{\mathbf{hom}(A, \overline{Y})} \mathbf{hom}(B, \overline{Y}))$$

corresponding the original diagram (2) maps to an element of

$$\pi_0(\mathbf{hom}(\tilde{A}, \overline{X}) \times_{\mathbf{hom}(\tilde{A}, \overline{Y})} \mathbf{hom}(\tilde{B}, \overline{Y}))$$

which lifts to an element of $\pi_0 \mathbf{hom}(\tilde{B}, X)$ by the assumption for the sectionwise fibration $p : X \rightarrow Y$, and hence to an element of $\pi_0 \mathbf{hom}(\tilde{B}, \overline{Y})$. \square

Corollary 10. *Suppose that the class consisting of all S -cofibrations and sectionwise weak equivalences has a set of generators. Then the closed model structure given by Theorem 2 is cofibrantly generated.*

Proof. The set of S -cofibrations already has a set of generators, namely S , by construction.

According to Proposition 9 a map $p : X \rightarrow Y$ is a local S -fibration if and only if it has the right lifting property with respect to all trivial local S -cofibrations appearing in the set S_P and is an S -fibration. According to the assumption, p is an S -fibration if and only if it has the right lifting property with respect to some set S' of sectionwise trivial S -cofibrations. Thus, p is a local S -fibration if and only if it has the right lifting property with respect to the S -cofibrations in the set $S_P \cup S'$. \square

Corollary 11. *The local projective model structure for simplicial presheaves on a small Grothendieck site \mathcal{C} is cofibrantly generated.*

Proof. The S_0 -fibrations are the sectionwise fibrations, and a map $p : X \rightarrow Y$ is a sectionwise fibration if and only if it has the right lifting property with respect to the set of maps $L_U(\Lambda_k^n) \rightarrow L_U(\Delta^n)$, $U \in \mathcal{C}$. \square

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