The Hopf ring for Bockstein-nil homology of QS^n

Takuji Kashiwabara
Institut Fourier
UMR au CNRS 5582
BP 74 38402, St-Martin-d'Hères CEDEX FRANCE

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1 Introduction

Let QS^n denote the space $\Omega^\infty\Sigma^\infty S^n \cong colim_k \Omega^k\Sigma^k S^n$. It represents the stable cohomotopy which is a graded ring functor. Thus it is equipped with two pairings, the one coming from simply the loop space structure $\Omega^\infty\Sigma^\infty S^n \cong \Omega\Omega^\infty\Sigma^\infty S^{n+1}$, corresponding to the addition in the stable cohomotopy, and the other coming from the so-called composition pairing, which is nothing but the colimit of the composition of maps

$$\begin{aligned} \Omega^k\Sigma^k S^i \times \Omega^{k+i}\Sigma^{k+i} S^j &= Map_*(S^k, S^{k+i}) \times Map_*(S^{k+i}, S^{k+i+j}) \\ \to \quad & Map_*(S^k, S^{k+i+j}) = \Omega^k\Sigma^k S^{i+j} \end{aligned}$$

which corresponds to the multiplication in the stable cohomotopy. These pairings induce in mod p homology pairings which we note by $\star$ and $\circ$ respectively. These products together with the coproduct of $H_*(QS^0; Z/p)$ are related to each other via the distributivity law [24, 18], making it a ring object in the category of coalgebras (which is called a Hopf ring or a coalgebraic ring). It has been known since the results in [16] that as a coalgebraic ring, $H_*(QS^0; Z/p)$ is generated by elements $Q^i[1]$'s and $\beta Q^i[1]$'s, but it is only in [25, 6] that the complete set of relations as a coalgebraic ring for $H_*(QS^0; Z/2)$ and for $H_*(QS^n; Z/2)_{n\in Z^+}$ was given. In [11] an alternative proof for $H_*(QS^0; Z/2)$ was given. The purpose of this note is to generalize these results to the odd prime case. Of course, the arguments in [11] show that these results are consequences of the fact that the Quillen's homomorphism for the symmetric group is an isomorphism for mod 2-cohomology, which is not true for odd primes according to [7]. Thus some modifications are necessary. It turnes out that the notion of the "Bockstein-nil cohomology" introduced in [8] gives us a better understanding of Quillen's homomorphism. We show also that the modified object is of interest in view of its relationship with the BP-theory. We also obtain an odd prime "Bockstein-nil" analogue of results of [6]. However, our results rely on the knowledge of $H_*(QS^n; Z/p)$, so unlike in [6], we don't obtain a "new computation" of the Bockstein-nil homology of QS^n.

The problem of determining the complete set of relations for $H_*(QS^n; Z/p)$ as a coalgebraic ring remains open.

The author acknowledges the influence of the following works which are not directly used in the arguments : the philosophy of [19] that one can study $H_*(QS^0)$ by studying the Burnside rings of elementary abelian groups; the point of view in [4, 14] that relations among operations can be obtained in a simple way by consideration of $GL_2(Z/p)$ coinvariants in $H_*(B(Z/p \times Z/p))$; and, of course, the relationship between the Dyer-Lashof algebra and Dickson algebra [17, 20], even though only the $p = 2$ case is treated in theses works.

2 The colimit over a category and the ring object

The purpose of this section is to generalize the result of [10, section 2] and show that the colimit over a category with certain structures becomes a (semi-)ring object. Throughout the paper, a semi-ring will mean a commutative and associative semi-ring with unit.

First we define :

Definition 2.1 An external bipermutative (symmetric bimonoidal, respectively) category is a category R equipped with two bifunctors : $\boxplus$, $\boxtimes$ $\mathcal{R} \times \mathcal{R} \to \mathcal{R}$ each of which makes R a permutative (symmetric monoidal, resp.) category and such that there is a natural transformation (not necessarily isomorphisms) δ :

$$A \boxtimes (B \boxplus C) \to (A \boxtimes B) \boxplus (A \boxtimes C)$$

A functor $\tau : \mathcal{R} \to \mathcal{R}$ is called a conjugation if there is a functor $\mathcal{N}$ and there are natural transformations $\mathcal{N} \to 0$ and $\mathcal{N} \to id \boxplus \tau$, where 0 is the constant functor that sends every object to 0, the unit with respect to $\boxplus$.

Example 2.1.1 A bipermutative category is an external bipermutative category. In particular, if R is a semi-ring, the discrete category whose objects are elements of R, which will be noted $[R]$, becomes an external bipermutative category. If, furthermore, R is a ring, then the multiplication by (-1) on R induces a conjugation on $[R]$.

Example 2.1.2 Let $\mathcal{C}$ be a category that admits finite limits, $G : \mathcal{C} \to Semi - Rings$ be a contravariant functor, where $Semi - Rings$ is the category of semi-rings. Denote by $\mathcal{C}/G$ the category whose objects are pairs (X, s) where X is an object of C, and $s \in G(X)$, and whose morphisms from (X, s) to (Y, t) are just morphisms $\theta : X \to Y$ in $\mathcal{C}$ such that $G(\theta)(t) = s$. Then $\mathcal{C}/G$ becomes an external bipermutative category in the following manner :

$$\begin{aligned} (X_1, s_1) \boxplus (X_2, s_2) &= (X_1 \times X_2, \pi_1^* s_1 + \pi_2^* s_2) \\ (X_1, s_1) \boxtimes (X_2, s_2) &= (X_1 \times X_2, \pi_1^* s_1 \times \pi_2^* s_2) \end{aligned}$$

where π_i is the projection to the i-th factor $X_1 \times X_2 \to X_i$. The natural transformation δ is given by the diagonal map $X_1 \times X_2 \times X_3 \to X_1 \times X_2 \times X_1 \times X_3$.

Furthermore, if G takes its values in the category of rings, one can define a conjugation τ on $\mathcal{C}/G$ by :

$$\tau(X, s) = (X, -s), \ \mathcal{N}(X, s) = (X, 0).$$

The construction $\pi_1^* s_1 \times \pi_2^* s_2$ is what is usually called the external product, which explains the name of the external bipermutative (symmetric bimonoidal) category.

Definition 2.2 Let $\mathcal{R}$ be an external symmetric bimonoidal category, $\mathcal{C}$ a category admitting any finite product, F a functor from $\mathcal{R}$ to $\mathcal{C}$ that is strict monoidal with respect to each of the monoidal structures of $\mathcal{R}$ and that of $\mathcal{C}$ induced by the product. F is called a compatible functor if the following diagramme (where p and m's are the natural isomorphisms for a strict monoidal functor) is commutative.

$$\begin{tikzcd}
F(A) \times F(B) \times F(C) \arrow[r, "p_{B,C}"] \arrow[d, "\Delta_{F(A)}"] & F(A) \times F(B \boxplus C) \arrow[r, "p_{A,B\boxplus C}"] & F(A \boxtimes (B \boxplus C)) \arrow[d, "F(\delta)"'] \\
F(A) \times F(B) \times F(A) \times F(C) \arrow[r, "m_{A,B} \times m_{A,C}"] & F(A \boxtimes B) \times F(A \boxtimes C) \arrow[r, "p_{A\boxtimes B, A\boxtimes C}"] & F(A \boxtimes B) \boxplus F(A \boxtimes C)
\end{tikzcd}$$

Furthermore, when $\mathcal{R}$ is equipped with a conjugation, we also require that there exists a natural transformation $t : F \to F \circ \tau$ such that the following triangle commutes.

$$\begin{tikzcd}
F \arrow[d, "id\times t"] \arrow[dr, "F\mathcal{N}"] & \\
F \times F \circ \tau \arrow[r] & F(id \boxplus \tau)
\end{tikzcd}$$

Then we get immediately from the definition :

Proposition 2.3 *Let $\mathcal{R}$, $\mathcal{C}$ be as above, $\mathcal{D}$ a category admitting any finite product, F a compatible functor from $\mathcal{R}$ to $\mathcal{C}$, and H a functor from $\mathcal{C}$ to $\mathcal{D}$ that preserves the product. Then the composition $H \circ F$ is a compatible functor.*

Example 2.3.1 Let $\mathcal{C}$, G be as in Example 2.1.2. Then the functor $Source : \mathcal{C}/G \to \mathcal{C}, Source(X, s) = X$ is a compatible functor. Furtermore, if $\mathcal{D}$ is a category admitting any finite product, and H is a functor from $\mathcal{C}$ to $\mathcal{D}$ that respectes the product, then $H \circ Source$ is a compatible functor.

Now we can state the main result of the section

Theorem 2.4 *Let $\mathcal{R}$ be an external bipermutative category, $\mathcal{C}$ a category admitting any finite product, and F a compatible functor from $\mathcal{R}$ to $\mathcal{C}$. Then $Colim_{\mathcal{R}}F(-)$ (if it exists) is a semi-ring object in $\mathcal{C}$. Furthermore, if $\mathcal{R}$ is equipped with a conjugation, then $Colim_{\mathcal{R}}F(-)$ is a ring object.*

Proof. As F is strict monoidal with respect to two monoidal structures of $\mathcal{R}$, $Colim_{\mathcal{R}}F(-)$ has two monoidal structures. The compatibility condition guarantees the distributivity, so that the colimit becomes a semi-ring object in $\mathcal{C}$. When R is equipped with the conjugation, the natural transformatin t induces the conjugation in $Colim_{\mathcal{R}}F(-)$ making it a ring object.

Example 2.4.1 Let k, R be rings, $\mathcal{R} = [R]$, $\mathcal{C}$ the category of k-coalgebras, F the constant functor that sends every object of $[R]$ to k. Then $Colim_{\mathcal{R}}F(-)$ is nothing but the “ring-ring” $k[R]$.

3 The main relations

In this section, we prove "the main relations" for colimit model coalgebraic rings. We follow more or less the treatment in [2]. First of all we need to fix notations.

Definition 3.1 *Let $\mathcal{I}$, $\mathcal{C}$, $\mathcal{E}$ be categories, $G : \mathcal{I}^{op} \to \mathcal{C}$, $E : \mathcal{I} \to \mathcal{E}$ be functors. Suppose $\beta \in E(X)$ with $X \in obj(\mathcal{I})$, $x \in G(X)$. Then we denote by (x, β) the image of β by the canonical map $E(X) \to Colim_{\mathcal{I}/G} E \circ Source(-)$.*

By the definition of the colimit, in a suitable sense, $Colim_{\mathcal{I}/G} E \circ Source(-)$ is generated by the elements (x, β), subject only to relations generated by $(f^*x, \beta) = (x, f_*(\beta))$ where $f \in Mor_{\mathcal{I}}(X, Y)$, $x \in G(Y)$, $\beta \in E(X)$. In some cases this description is good enough. However in other cases we may need a more concrete description. It turns out that often we get these explicit relations in terms of equality between certain formal power series. For this purpose, we need to be more concrete about our categories and functors.

Hypothesis 3.2 *$\mathcal{I}$ admits arbitrary finite products, $\mathcal{C}$ is the category of coalgebras over a ring k, F is a functor from $\mathcal{I}^{op}$ to Semirings. E is a functor from $\mathcal{I}$ to $\mathcal{C}$. $X \in obj(\mathcal{I})$ such that $E(X^i) \cong E(X)^{\otimes i}$, $E(X)$ is a free module over $E_* = E(X^0)$, and $E^*(X) = Hom_{E_*}(E(X), E_*)$ has an element t such that $E^*(X)$ is (topologically) free over a set of generators t^i, $i \in J$.*

From now on throughout the section we assume that this hypothesis is satisfied. Now we can introduce some notations. We denote by $\langle \beta_i \rangle$ the basis of $E(X)$ that is dual to $\langle t^i \rangle$'s. Let x^F be an element of $F(X)$ (we don't require any particular property on this element, but it will have to be fixed once and for all). We denote

$$b_i = (x^F, \beta_i)$$

and

$$b(t) = \Sigma_{i \in J} b_i t^i.$$

Also, given an element α of $F^* = F(X^0)$, we denote $[\alpha] = (1, \alpha)$. Let σ be an element of $F^*[T_1, T_2, \cdots, T_k]$. Then we define

$$b_{[\rho]}(t_1, t_2, \cdots, t_k) = \star_I [\alpha_I] \circ b(t_1, t_2, \cdots, t_k)^{\circ I}$$

if

$$\rho(T_1, T_2, \cdots T_k) = \Sigma_I \alpha_I T^I$$

where

$$T^I = T_1^{I_1} T_2^{I_2} \cdots T_k^{I_k}$$

and

$$b(t_1, t_2, \cdots, t_k)^{\circ I} = b(t_1)^{\circ I_1} \circ \cdots \circ b(t_k)^{\circ I_k}.$$

This might sound quite scary, however, it is quite simple, we just replace the multiplication by the circle multiplication, sum by the star product, and α by $[\alpha]$, then apply the resulting “polynomial” to the ”variables” $b(t_1), \cdots, b(t_k)$. For example, if $\rho(T) = T^2 + 3T$, then $b_\rho(t) = b(t) \circ b(t) \star [3] \circ b(t)$. In terms of this notation, the Ravenel-Wilson’s main relations [24] can be expressed as :

Theorem 3.3 *Let $E_*(-)$, $F^*(-)$ be complex oriented cohomology theories. Then we have, in $Colim_{\mathcal{CP}/F_*}(E_* \circ Source(-))[[x_1^E, x_2^E]]$, we have*

$$b_{[x_1^F +_F x_2^F]}(x_1^E, x_2^E) = b_{[x^F]}(x_1^E +_E x_2^E)$$

where x^G $(G = E, F)$ denotes the orientation class of the cohomology theory G, $+_G$ denotes the formal sum in $G^*(-)$, and $\mathcal{CP}$ is the category whose objects are finite products of CP^∞’s and whose morphisms are homotopy classes of all continuous maps among them.

Note that we followed the treatment in [2] and replaced the “formal indeterminates” in [24] by the orientation classes. We now will prove the following generalisation of this formula.

Theorem 3.4 *Let $f : X^j \to X^k$ be a morphism in $\mathcal{I}$ such that*

$$F(f)(\psi(x_1^F, \cdots, x_k^F)) = \phi(x_1^F, \cdots x_j^F)$$

where $(\psi(x_1^F, \cdots, x_k^F)) \in F(X^k)$. Then we have

$$b_{[\phi]}(x_1^E, \cdots, x_j^E) = b_{[\psi]}(E^*(f)(x_1^E), \cdots E^*(f)(x_k^E)).$$

We could symbolically write this formula as $b_{[f^*\psi]}(x) = b_{[\psi]}(f^*(x))$.
Proof. Consider the following diagram :

$$\begin{tikzcd}
F(X^k) \arrow[r] \arrow[d, "F(f)"'] & Hom(E(X^k), R) \arrow[r, "\cong"] \arrow[d, "(E(f))^*"'] & R \otimes_{E_*} E^*(X^k) \arrow[r, "\cong"] \arrow[d, "E^*(f)"'] & R[[x_1^E, \cdots, x_k^E]] \arrow[d, "E^*(f)"'] \\
F((X^j) \arrow[r] & Hom(E(X^j), R) \arrow[r, "\cong"] & R \otimes_{E_*} E^*(X^k) \arrow[r, "\cong"] & R[[x_1^E, \cdots, x_j^E]]
\end{tikzcd}$$

where $R = Colim_{\mathcal{I}/F}(E \circ Source(-))$. The vertical maps are defined in such a way that the squares commute, with the exception of the leftmost one. The leftmost square commutes by the definition of the colimit.

Now, the leftmost horizontal arrows transform $+$ and x into $\star$ and $\circ$, as they are induced by the sum and product maps $F \times F \to F$. Similary, the multiplication by a becomes the circle multiplication by $[a]$. Furthermore, x_i^F in $F^*(X^k)$ maps to X_i^* in $Hom(E(X^k), R)$ which then maps to $b(x_i^E)$ by the definition of the dual basis. Thus the image of $\psi \in F(X^k)$ by the horizontal map is $b_{[\psi]}(x_1^E, \cdots x_k^E)$ which maps down to $b_{[\psi]}(E*(f)(x_1^E), \cdots, E*(f)(x_k^E))$, where as by the left vertical map, it maps down to ϕ whose image by the horizontal map is nothing but $b_{[\phi]}(x_1^E, \cdots x_j^E)$.

4 The main results and proof

First of all, we recall what is already known about the object of our study. Denote by CS^0 the combinatorial model for QS^0, $CS^0 = \coprod_n B\Sigma_n$. It is a semi-ring object in the homotopy category, the multiplication induced by maps $\Sigma_m \times \Sigma_n \to \Sigma_{mn}$ and the addition induced by maps $\Sigma_m \times \Sigma_n \to \Sigma_{m+n}$. These structures are compatible with those of QS^0, and $H_*(QS^0; Z/p)$ is obtained from $H_*(CS^0; Z/p)$ by the group completion [1, 21]. In our language, the pairings on CS^0 makes $H_*(CS^0; Z/p)$ a semi-ring object in the category of coalgebras, and $H_*(QS^0; Z/p)$ is the universal coalgebraic ring containing $H_*(CS^0; Z/p)$. Both CS^0 and QS^0 are E^∞ spaces so that the Dyer-Lashof operations act on their mod p homology. For a sequence (allowable or not) $I = (\epsilon_1, I_1, \cdots \epsilon_n, I_n)$, $\epsilon_i = 0, 1$, $I_i > 0$, I_i's are integers, we note Q_I the Dyer-Lashof operation $\beta^{\epsilon 1} Q_{I_1} \cdots \beta^{\epsilon n} Q_{I_n}$. We define $\beta(I)$ to be the sum $\epsilon_1 + \cdots \epsilon_n$. When $\beta(I) = 0$, we say that the sequence I is allowable if and only if all I_j's are multiple of $2(p-1)$ and $I_{j-1} \leq I_j$. (This agrees with the usual definition when we only consider Q_I's acting on even degree elements.) If x has degree d, then $Q_i(x)$ has the degree $pd + i$.

We also note BP the p-local Brown-Peterson spectrum [3, 22], $\Omega^\infty \Sigma^n BP$ its associated 0-th infinite loop space, i. e., the space that represents the functor $BP^n(-)$. There is a unit map $S \to BP$, which gives rise to a map $QS^0 \to \Omega^\infty \Sigma^n BP$. By composing it with the canonical map $CS^0 \to QS^0$, we get a map $CS^0 \to \Omega^\infty BP$. They induce respectively a map $H_*(QS^n; Z/p) \to H_*(\Omega^\infty \Sigma^n BP; Z/p)$ and $H_*(CS^0; Z/p) \to H_*(\Omega^\infty \Sigma^0 BP; Z/p)$. Furthermore there is a Thom map $BP \to HZ/p$ which gives rise to maps $BP^*(X) \to H^*(X; Z/p)$ for any space X.

Theorem 4.1 *The following subobjects of $H_*(CS^0;Z/p)$ from i) to iii), quotients of $H_*(CS^0;Z/p)$ from iv) to vi) are all isomorphic and dual to the subobjects of $H^*(CS^0;Z/p)$ from vii) to ix).*

- *(i) The subalgebra generated by the elements of the form $Q_I([1])$ with $\beta(I)=0$.*
- *(ii) The polynomial subalgebra generated by the elements of the form $Q_I([1])$ with $\beta(I)=0$, I allowable.*
- *(iii) The coalgebraic subsemiring generated by the elements of the form $Q_i[1]$.*
- *(iv) The quotient by the ideal generated by the elements of the form $Q_I([1])$ with $\beta(I)\geq 1$.*
- *(v) The quotient by the "coalgebraic ideal" generated by the elements of the form $\beta Q_i[1]$.*
- *(vi) The image of $H_*(CS^0;Z/p)$ in $H_*(\Omega^\infty BP;Z/p)$*
- *(vii) The image of $H^*((\Omega^\infty BP;Z/p)$ in $H^*(CS^0;Z/p)$.*
- *(viii) The image of $BP^*(CS^0;Z/p)$ in $H^*(CS^0;Z/p)$.*
- *(ix) $f^{*(-1)}(Im(BP^*(Y)\to H^*(Y)))$, where Y is a certain disjoint union of copies of $(BZ/p)^m$'s and $f:Y\to CS^0$ has the property such that f^* is injective.*

Theorem 4.2 *The following subobjects of $H_*(QS^n;Z/p)$ from i) to iii), quotients of $H_*(QS^n;Z/p)$ from iv) to vi) are all isomorphic and dual to the subobjects of $H^*(QS^n;Z/p)$ from vii) to ix). Here σ_n notes the fundamental class of $H_n(QS^n;Z/p)$ ($\sigma_0=[1]$).*

- *(i) The subalgebra generated by the elements of the form $Q_I(\sigma_n)$ with $\beta(I)=0$ (and [-1] for $n=0$).*
- *(ii) The polynomial subalgebra generated by the elements of the form $Q_I(\sigma n)$ with $\beta(I)=0$, I allowable (tensored with $Z/p[Z]$ for $n=0$) if n is even. The exterior subalgebra generated by single suspension of these elements if n is odd.*
- *(iii) The n-th spacelike degree part of the coalgebraic subring generated by the elements of the form $Q_i[1]$ and σ_n.*

(iv) *The quotient by the ideal generated by the elements of the form* $Q_I(\sigma n)$ *with* $\beta(I) \geq 1$.

(v) *The n-th spacelike degree part of the quotient by the "coalgebraic ideal" generated by the elements of the form* $\beta Q_i[1]$.

(vi) *The image of* $H_*(QS^n; Z/p)$ *in* $H_*(\Omega^\infty \Sigma^n BP; Z/p)$

(vii) *The image of* $H^*(\Omega^\infty \Sigma^n BP; Z/p)$ *in* $H^*(QS^n; Z/p)$.

(viii) *The image of* $BP^*(QS^n; Z/p)$ *in* $H^*(QS^n; Z/p)$.

(ix) $f^{*(-1)}(Im(BP^*(Y) \to H^*(Y)))$, *where* Y *is a certain disjoint union of copies of* $(BZ/p)^m \times (S^n)^{\times k}$*'s and* $f : Y \to QS^n$ *has the property such that* f^* *is injective.*

Remark 4.3 By the result of Dyer-Lashof in [5], i),ii) and iv) are isomorphic one another. It was shown by Wilson [26] that they are isomorphic to vi). The equality between i) and iii) was essentially proved by May [16]. vi) is obviously dual to viii). The equality between vii) and viii) were proved by author [12][Proposition 4.3]. The equality between iv) and v) is nothing but the first step of the proof of Proposition 5.1 in [13] (second property with $X = S^n$, $N = 0$). Finally the equality between all of these and ix) is the dual of the second step of the proof of Proposition 5.1 in [13]. See also Theorem 3.1 in [9]

Let's denote by M_0 the subobject (or quotient) of $H_*(CS^0; Z/p)$ in Theorem 4.1, by M_0^* its dual, $M_{1,n}$ the subobject (or quotient) of $H_*(QS^n; Z/p)$ in Theorem 4.2, and by $M_{1,n}^*$ its dual. We also denote by $\mathcal{BV}$ the category whose objects are the classifying groups of elementary abelian p-groups, and whose morphisms are homotopy classes of maps among them. For a space X, we note by $\mathcal{BV}/X$ the category whose objects are homotopy classes of maps from an object of $\mathcal{BV}$ to X, and whose morphisms are homotopy commutative triangles. Finally, for a left module over a Steenrod algebra M, its Bockstein-nil part as defined in [8] will be noted $M^{\beta N}$, and dually if M is a right module over a mod p Steenrod algebra, then $M_{\beta N}$ will denote the "Bockstein-nil quotient" of M, that is $M/(\Sigma_i P^i_* \beta(M))$. Then our main results are

Theorem 4.4 (local version) *(i)* M_0 *is a polynomial algebra generated by the elements of the form* $E_{i_0} \circ E_{p(i_0+i_1)} \circ E_{p^l(i_0+i_1+\cdots+i_l)}$ *with* $i_0 >$

0, $i_j \geq 0 (j \neq 0)$ and [1]. *Here E_i is an element of the image of $H_{2i(p-1)}(B\Sigma_p; Z/p) \subset H_{2i(p-1)}(CS^0; Z/p)$.*

(ii) $M_{1,0}$ is obtained from M_0 just by inverting [1]. *$M_{1,n} (n > 0)$ is a free commutative algebra generated by the elements of the form $\sigma_1^{\circ n} \circ E_{i_0} \circ E_{p(i_0+i_1)} \circ E_{p^l(i_0+i_1+\cdots+i_l)}$ with $n < 2i_0$.*

Theorem 4.5 (global version) *We have following isomorphisms*

(i) $M_0 \cong H_*(CS^0; Z/p)_{\beta N}$, $M_{1,n} \cong H_*(QS^n; Z/p)_{\beta N}$.

(ii) $M_0 \cong Colim_{\mathcal{BV}/CS^0} H_*(-; Z/p)_{\beta N}$, $M_{1,0} \cong Colim_{\mathcal{BV}/QS^0} H_*(-; Z/p)_{\beta N}$.

(iii) $M_0 \cong H_*^T(CS^0; Z/p)$, $M_{1,n} \cong H_*^T(QS^n; Z/p)$ where $H_*^T(CS^0; Z/p)$ is the coalgebraic semiring generated over $Z/p[Z^+]$ by elements E_i's of degree $2i(p-1)$, with the diagonal

$$\Delta(E_i) = \Sigma_{j+k=i} E_j \otimes E_k,$$

subject to the relations

(a) $E_0 = [p]$.

(b) $E(s^{p-1}) \circ E(t^{p-1}) = E(s^{p-1}) \circ E((s+t)^{p-1})$ in

$$H_*^T(CS^0; Z/p)[s,t] \cong Hom(H_{\beta N}(BZ/p \times BZ/p), H_*^T(CS^0; Z/p)[s,t]$$

where s, t are standard polynomial generators of $H^{\beta N}(BZ/p \times BZ/p)$ abd $E(X)$ denotes the formal sum $\Sigma_i E_i X^i$.

and $\{H_*^T(QS^n; Z/p)\}_{n \in Z^+}$ is the coalgebraic ring generated over $Z/p[Z]$ by same elements together with σ_1 (with bidegree (1,1)) subject to the relations (a), (b) and the following :

(c) $\sigma_1^{\circ 2n} \circ E_n = (\sigma_1^{\circ 2n})^{\star p}$

Proof of Theorem 4.3. This uses the knowledge of $H_*(QS^n; Z/p)$. However, it is also possible to derive i) and ii) for $n = 0$ from Theorem 3.4, which would give a "new" computation of $H_*(CS^0; Z/p)_{\beta N}$ and $H_*(QS^0; Z/p)_{\beta N}$. It is easy to see that it is suffices to prove the results for n even. First we show that these elements are linealy independant in the module of indecomposable. We proceed by induction on l for all n at once, assuming the identity (c) (which will be proved later without using Theorem 3.3). As a matter of fact, by (c) and the distributivity law, we get :

$$\sigma_1^{\circ 2i_0}\circ E_{i_0}\circ E_{p(i_0+i_1)}\circ\cdots\circ E_{p^l(i_0+i_1+\cdots+i_l)} = (\sigma_1^{\circ 2i_0}\circ E_{i_1}\circ\cdots\circ E_{p^{(l-1)}(i_1+\cdots+i_l)})^{\star p}.$$

Since M_0 and $M_{1,2n}$'s are known to be polynomial algebras, the p-th power map is injective. Thus by induction on l, we get the linear independence of elements of this form.

To prove that they span, it suffices to show that they span the vector space with the correct dimension. For that purpose, we establish an isomorphism of graded sets between our proposed basis and a known basis. Now note that

$$\phi : (i_0, i_1, \cdots i_l) \to (2(p-1)(i_0-n), 0, 2(p-1)(i_0+i_1-n), 0, \cdots 2(p-1)(i_0+i_1\cdots+i_l-n), 0)$$

gives a bijection between the set of sequences of non-negative numbers with $i_0 > n$ and the set of allowable sequences. Thus it only remains to show the equality

$$\begin{aligned} & deg(\sigma_1^{\circ 2n}\circ E_{i_0}\circ E_{p(i_0+i_1)}\circ\cdots\circ E_{p^l(i_0+i_1+\cdots+i_l)}) \\ = \quad & deg(Q_{2(p-1)(i_0-n)}Q_{2(p-1)(i_0+i_1-n)}\cdots Q_{2(p-1)(i_0+i_1\cdots+i_l-n)}(\sigma_1^{\circ 2n})). \end{aligned}$$

However, the latter is equal to

$$2(p-1)(i_0-n)+pdeg(Q_{2(p-1)(i_0+i_1-n)}\cdots Q_{2(p-1)(i_0+i_1\cdots+i_l-n)}(\sigma_1^{\circ 2n})),$$

and by induction on l, we see that this is equal to

$$\begin{aligned} & 2(p-1)(i_0-n)+pdeg(\sigma_1^{\circ 2n}\circ E_{(i_0+i_1)}\circ\cdots\circ E_{p^{l-1}(i_0+i_1+\cdots+i_l)}) \\ = \quad & 2(p-1)(i_0-n)+2np+pdeg(E_{(i_0+i_1)}\circ\cdots\circ E_{p^l(i_0+i_1+\cdots+i_l)}) \\ = \quad & 2n+deg(E_{i_0}\circ E_{p(i_0+i_1)}\circ\cdots\circ E_{p^l(i_0+i_1+\cdots+i_l)}) \end{aligned}$$

Proof of Theorem 4.4.

i). We use the notation of Theorems 4.1, 4.2 ix). Note that for the space Y we have $Im(BP^*(Y)\to H^*(Y;Z/p)) = H^*(Y;Z/p)^{\beta N}$. Since f^* is injective, an element of $H^*(CS^0;Z/p)$ or $H^*(QS^n;Z/p)$ respectively is in $H^*(CS^0;Z/p)^{\beta N}$ or $H^*(QS^n;Z/p)^{\beta N}$ respectively if and only if its image by f^* is in $H^*(Y;Z/p)^{\beta N}$. Thus we get the desired result.

ii) By the group completion theorem, it is enough to prove the isomorphisms just for M_0. Since the Quillen's category of a finite group G [23] is cofinal in $\mathcal{BV}/BG$, by dualizing the result of Lesh and Ha [8], one sees that

$$Colim_{\mathcal{BV}/B\Sigma_m} H_*(B(-); Z/p)_{\beta N} \cong H_*(B\Sigma_m; Z/p)_{\beta N}.$$

By assembling together these isomorphisms for different m, one gets

$$Colim_{\mathcal{BV}/CS^0} H_*(B(-); Z/p)_{\beta N} \cong H_*(CS^0; Z/p)_{\beta N}.$$

This together with i) prove ii).

iii). First we prove it for M_0. Again the group completion theorem implies that it holds for $M_{1,0}$. Now we modify the arguments in [11] to fit to odd prime case. First we need some notations.

Definition 4.6 We denote

- $\mathcal{C}$ the category of Z/p-coalgebras (graded, cocommutative and counitary).
- $\mathcal{V}$ the category of finite dimensional Z/p-vector spaces.
- $\mathcal{E}$ the category of sets,
- $\mathcal{F}$ the category of contravariant functors from $\mathcal{V}$ to $\mathcal{E}$ (which we consider as covariant functors from $\mathcal{V}^{op}$ to $\mathcal{E}$.

Let G be an object of $\mathcal{F}$. Then $\mathcal{V}/G$ will denote the category whose objects are the pairs (W, g) where W is an object of $\mathcal{V}$, g an element of $G(W)$, and whose morphisms frome $(W_1, g_1,)$ to (W_2, g_2) are linear maps $\phi : W_1 \to W_2$ such that $G(\phi)(g_1) = g_2$.

Then as in [11], we have

Proposition 4.7 *Let c denote the functor which sends G to the colimit of the composite $\mathcal{V}/G \to \mathcal{V} \overset{\Gamma_*}{\to} \mathcal{C}$ where $\Gamma_*(-)$ denotes the (graded) divided power coalgebra functor (with the elements of the vector space having degree 2). If G is a semi-ring object in $\mathcal{F}$, then $c(G)$ is a semi-ring object in $\mathcal{C}$. Furthermore, if G is a ring-object, then so is $c(G)$.*

Proof. One sees from Example 2.3.1 that the composite $\mathcal{V}/G \to \mathcal{V} \overset{\Gamma_*}{\to} \mathcal{C}$ is a compatible functor. The result follows from Theorem 2.4.

Definition 4.8 For a Z/p-vector space W, let $a(W)$ be the quotient of the free unitary commutative semiring generated by symbols a_u, $u \in W^*$ where * denotes the Z/p-dual, by the following relations :

(i) $a_0 = p$

(ii) $a_u a_v = a_{u+v} a_v \forall u, v$

(iii) $a_{qu} = a_u$ if $q \in (Z/p)^\times$

If f is a linear map from W_1 to W_2, define the map $a(f)$ simply by $a(f)(a_u) = a_{u \circ f}$.

$A(W)$ will mean the Burnside semi-ring (and not the Burnside ring), i.e., the semi-ring of all isomorphism classes of W-sets.

With these preparations, the proof of Theorem 4.5 iii) will be done in three steps. First we show that $c(a)$ is isomorphic to the algebric model $H^T_*(CS^0)$ (counterpart of Proposition 2.7 in [11]), then we show that $c(A)$ is isomorphic to the real object $H_*(CS^0; Z/p)_{\beta N}$ (counterpart of Proposition 2.9 in [11]) and finally we show that the functors A and a are isomorphic (counterpart of Proposition 2.10 in [11]), thus $c(A)$ isomorphic to $c(a)$.

The first step : we show that $c(a)$ is isomorphic to $H^T_*(CS^0)$. For this we introduce several more definitions.

Definition 4.9 An object X of $\mathcal{V}/a$ is a *standard decomposable* if it is of the form

$$X \cong (X_{1,1} \boxtimes \cdots \boxtimes X_{1,r_1}) \boxplus (X_{2,1} \boxtimes \cdots \boxtimes X_{2,r_2}) \cdots \boxplus (X_{n,1} \boxtimes \cdots \boxtimes X_{n,r_n}),$$

where each of the factor $X_{i,j}$'s is isomorphic to the pair $(Z/p, a_{id})$. For an object Z of $\mathcal{V}/a$, a *decomposable morphism* of Z is a morphism in $\mathcal{V}/a$ whose target is a standard decomposable object.

Now note that if $Y = (V, v)$, there exists a decomposable morphism $\theta : Y \to X$ where X is as above if and only if $v = \Sigma_{i=1}^n \Pi_{j=1}^{r_i} a_{\theta_{i,j}}$ with $\theta_{i,j}$ the map $V \to Z/p$ corresponding to the component $X_{i,j}$. Since a_u is a pull-back by u of $a_{id:Z/p \to Z/p}$, one sees that any object of $\mathcal{V}/a$ admits at least one decomposable morphism. If such decomposable morphism were unique (up to permutation of factors), then $c(a)$ would be just the direct sum of $\otimes_i \otimes_j \Gamma_*(Z/p)_{i,j}$ where $\Gamma_*(Z/p)_{i,j}$ coming from $X_{i,j}$ as above, which is nothing but the free (commutative) coalgebraic ring generated by $\Gamma_*(Z/p)$. Thus $c(a)$ is a coalgebraic ring generated by $\Gamma_*(Z/p)$, subject only to relations coming from the failure of the uniqueness of decomposable morphisms. Now, this failure, in turn, arises from the relations in $a(W)$'s. Note e_i a generator of $\Gamma_{2i}(Z/p)$. Denote by $\tilde{E}_i$ the element $(a_{id:Z/p\to Z/p}; e_i)$. The relation (iii) $a_{qu} = a_u$ is a pull-back by u of the relation $a_{q:Z/p\to Z/p} = a_{id:Z/p\to Z/p}$, which in turn can be written as

$$q^* a_u = a_u.$$

Therefore, Theorem 3.4 implies that we have

$$\tilde{E}(qs) = \tilde{E}(s).$$

As $q^i = 1$ if and only if $2(p-1)|i$ this means

$$\tilde{E}(s) = E(s^{(p-1)}).$$

The relation (ii) in $a(W)$ in general is a pull-back of the particular case $u = u_0$, $v = v_0$, $W = Z/p \times Z/p$, where u_0 and v_0 are the projections to the first and the second factor Z/p. This particular relation can be written as

$$T^*(a_{u_0} a_{v_0}) = a_{u_0} a_{v_0},$$

where $T : Z/p \times Z/p \to Z/p \times Z/p$ is the homomorphism given by $T(x, y) = (x + y, y)$. Therefore Theorem (3.4) implies

$$\tilde{E}(s) \circ \tilde{E}(t) = \tilde{E}(s + t) \circ \tilde{E}(t).$$

combining with the precedent equality, we get

$$E(s^{p-1}) \circ E(t^{p-1}) = E((s + t)^{p-1}) \circ E(t^{p-1})$$

which is the second defining relation in $H^T_*(CS^0)$.

The relation

$$a_0 = p$$

in the general case is the pull-back of the same relation in $a(0)$ by the map $W \to 0$. The particular case can be written as

$$j^* a_{id:Z/p\to Z/p} = p a_{id:0\to 0}$$

which corresponds to the relation $E_0 = [p]$.

As there are no other relations in $a(V)$, we get thus a complete set of relations in $c(a)$, which proves the isomorphism

$$c(a) \cong H^T_*(CS^0; Z/p).$$

The second step : we show that $c(A)$ is isomorphic to $H_*(CS^0; Z/p)_{\beta N}$. Note first that we have a natural isomorphism $A(V) \cong [BV, CS^0]$. This gives rise to an equivalence of categories $\mathcal{BV}/CS^0 \cong \mathcal{V}/A$. On the other hand, we have natural isomorphisms $H_*(BV)_{\beta N} \cong \Gamma_*(V)$, so we obtain an isomorphism $c(A) \cong colim_{\mathcal{BV}/CS^0} H_*(B(-); Z/p)_{\beta N}$. We have already seen that this latter is isomorphic to $H_*(CS^0; Z/p)_{\beta N}$.

The third step : we show that the functors A and a are naturally isomorphic. Let u be an element of V^*. Note by α_u Z/p considered as a V- set via the homomorphism $u : V \to Z/p$. Define a semi-ring map $\phi : a(V) \to A(V)$ by $\phi(a_u) = \alpha_u$. We need to show ϕ is well-defined. First of all we have $\alpha_0 = p$. To show that $\alpha_u \alpha_v = \alpha_u \alpha_{u+v}$, it suffices to prove it when $V = Z/p \times Z/p$ and u, v are projections. (The general case is just a pull-back of this particular case.) In this case $\alpha_u \alpha_v$ is V considered as a V-set by the identity homomorphism, and $\alpha_{u+v}\alpha_u$ is V considered as a V-set by the homomorphism T. Since both homomorphisms are injective, the resulting V-sets are isomorphic. Thus ϕ is well-defined. Next we construct a monoid homomorphism $\psi : A(V) \to a(V)$ such that $\psi(1) = 1$, $\psi(V/W) = a_{u_1} \cdots a_{u_r}$, where u_i's are linearly independent elements of V^* such that $W = \cup_i Ker(u_i)$. Now the relations $a_u a_v = a_u a_{u+v}$ and $a_{au} = a_u$ guarantee that this is a well-defined map. It is now easy to show that ϕ and ψ are inverse to each other.

Now it remains to prove the case for $M_{1,n}$ with $n > 0$. The relation c) comes from the property of Dyer-Lashof operations and that the fact that circling with σ_1 is just suspending, noting that $E_i = Q_{2(p-1)i}[1]$. Furthermore Theorem 4.4 ii) shows that in positive spacelike degrees, there is no relation other than the relation c). This completes the proof of Theorem 4.5.

Remark 4.10 It is probably possible to find $n > 0$ version of ii), replacing $\mathcal{BV}/QS^0$ by an appropriate category. However, for the relation c) to hold in the colimit, the category must contain the Borel construction $\overbrace{S^n \times \cdots \times S^n}^{p-factors} \times_{Z/p} EZ/p$. This suggests that such a category is rather complicated.

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